Proceedings of the

International Congress of Mathematicians

Rio de Janeiro 2018

VOLUME I
Invited Lectures

Boyan Sirakov
Paulo Ney de Souza
Marcelo Viana
Editors

SBM

World Scientific

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International Congress of Mathematicians

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Editors
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Preface

The Proceedings of the International Congress of Mathematicians are a time honored tradition, and an invaluable service to the mathematical community at large. The files of all the previous Proceedings going back to the first ICM in 1897, and even before, to the preparatory 1893 International Mathematical Congress, are freely available at the IMU website: one cannot overestimate the importance of such a scientific and historical treasure.

At the same time, each Proceedings publication is a little different, slowly adjusting to a world in constant mutation.

The Proceedings for the Rio de Janeiro ICM 2018 consist of four volumes, adding up to over 5,000 pages.

Volumes 2 to 4 contain the papers from the invited speakers (all but five have contributed), who lectured in the 19 sections of the Congress. In keeping with a well-established tradition, those three volumes were finalized before the Congress. Their contents, together with the articles from the plenary speakers, were made available to all the ICM participants for download, and all the material will soon be posted permanently on the Congress' website.

The present Volume 1 contains several different important materials.

It starts with the speeches and reports presented during the ICM's three main social occasions: the opening, on August 1, which included the award ceremony for the Fields medal and the Nevanlinna prize; the social dinner, on August 4, which started with the Gauss prize and Chern medal award ceremony; and the closing, on August 9, which featured the Leelavati prize award ceremony.

Volume 1 also contains the articles from the plenary lecturers, the special (Abel and Emmy Noether) lecturers, the prize winners, and the laudators, as well as the technical reports from the Section 18 (mathematical education and popularization) invited panels, and the IMU contributed discussion panels.

We are proud that all the plenary lecturers, special lecturers and prize winners accepted to contribute, and so did the majority of laudators and panels.

Concerning the prize winners, a few explanations are in order. Caucher Birkar submitted a paper as a Section 4 invited speaker, and later declined to write yet another one as a Fields medalist. It was decided to keep his paper in volume 2, as part of Section 4. Similarly, we kept inside the plenary lectures series, in this volume, the article Peter Scholze wrote for his plenary lecture, after this became also a Fields medal lecture. Finally, Akshay Venkatesh had initially declined to submit a paper as a Section 7 invited speaker, but was fortunately able to send us an article related to his Fields medal lecture, which also appears in this volume.

The ICM 2018 Proceedings could not exist without the enthusiasm, commitment and competence of my colleagues Boyan Sirakov and Paulo Ney de Souza.

Boyan conducted all the interactions with potential publishers, established our fruitful and trustworthy relation with World Scientific, and was also responsible for successfully obtaining the articles from our 300 contributors. Paulo Ney was in charge of the actual editing of the papers, and the building of these four fine volumes, which he accomplished with his usual combination of creativity and virtuosity.

Above all, I am grateful to all the speakers and panel members for graciously providing their articles: theirs is the substance of these Proceedings.

Best regards,
Marcelo Viana
Chair, OC of ICM 2018

International Mathematical Congresses

1897	Zürich
1900	Paris
1904	Heidelberg
1908	Roma
1912	Cambridge, UK
1920	Strasbourg
1924	Toronto
1928	Bologna
1932	Zürich
1936	Oslo
1950	Cambridge, USA
1954	Amsterdam
1958	Edinburgh
1962	Stockholm
1966	Moscow
1970	Nice
1974	Vancouver
1978	Helsinki
1983	Warsaw
1986	Berkeley
1990	Kyoto
1994	Zürich
1998	Berlin
2002	Beijing
2006	Madrid
2010	Hyderabad
2014	Seoul

2018 Rio de Janeiro

Winners of the IMU Prizes

The Fields Medalists

2018
Caucher Birkar
Alessio Figalli
Peter Scholze
Akshay Venkatesh

2014
Artur Avila
Manjul Bhargava
Martin Hairer
Maryam Mirzakhani

2010
Elon Lindenstrauss
Ngô Bảo Châu
Stanislav Smirnov
Cédric Villani

2006
Andrei Okounkov
Grigori Perelman*
Terence Tao
Wendelin Werner

2002
Laurent Lafforgue
Vladimir Voevodsky

1998
Richard E. Borcherds
W. Timothy Gowers
Maxim Kontsevich
Curtis T. Mcmullen

1994
Jean Bourgain
Pierre-Louis Lions
Jean-Christophe Yoccoz
Efim Zelmanov

1990
Vladimir Drinfeld
Vaughan F. R. Jones
Shigefumi Mori
Edward Witten

1986
Simon K. Donaldson
Gerd Faltings
Michael H. Freedman

1982
Alain Connes
William P. Thurston
Shing-Tung Yau

1978
Pierre René Deligne
Charles Louis Fefferman
Gregori A. Margulis
Daniel G. Quillen

1974
Enrico Bombieri
David Bryant Mumford

1970
Alan Baker
Heisuke Hironaka
Sergei Novikov
John Griggs Thompson

1966
Michael Francis Atiyah
Paul Joseph Cohen
Alexander Grothendieck
Stephen Smale

1962
Lars Hörmander
John Willard Milnor

1958
Klaus Friedrich Roth
René Thom

1954
Kunihiko Kodaira
Jean-Pierre Serre

1950
Laurent Schwartz
Atle Selberg

1936
Lars Valerian Ahlfors
Jesse Douglas

*Grigori Perelman declined to accept the Fields Medal.

Rolf Nevanlinna Prize

2018
Constantinos Daskalakis

2014
Subhash Khot

2010
Daniel Spielman

2006
Jon Kleinberg

2002
Madhu Sudan

1998
Peter W. Shor

1994
Avi Wigderson

1990
Alexander Razborov

1986
Leslie Valiant

1982
Robert Tarjan

Carl Friedrich Gauß Prize

2018
David L. Donoho

2014
Stanley Osher

2010
Yves Meyer

2006
Kiyosi Itô

Shiing-Shen Chern Medal

2018
Masaki Kashiwara

2014
Phillip Griffiths

2010
Louis Nirenberg

Leelavati Prize

2018
Ali Nesin

2014
Adrián Paenza

2010
Simon Singh

Organizing Committee

Chair
Marcelo Viana (IMPA)

Proceedings
Boyan Sirakov (PUC-Rio)

Satellite Events
Eduardo Teixeira (UCF)

Travel Grants
Henrique Bursztyn (IMPA)

Organization and Coordination
Juliana Bressan (IMPA)

IMU General Assembly
Paolo Piccione (USP)

Electronic Infrastructure
Roberto Beauclair (IMPA)

Short Communications and Posters
Alexander Arbieto (UFRJ)

Women in Mathematics
Carolina Araujo (IMPA)

Scientific Program
Emanuel Carneiro (IMPA)

Rio Intelligencer
José Espinar (IMPA)

Volunteers
Maria João Resende (UFF)
Nelly Carvajal (IMPA)

Digital Platform
Paulo Ney de Souza (UC Berkeley)

Website Planning and Supervision
Roberto Imbuzeiro Oliveira (IMPA)

Opening address

Marcelo Viana, Chairman of the Rio de Janeiro ICM 2018

Dear Minister of Education, Rossieli Soares da Silva,

Dear Colleagues,

Ladies and Gentlemen,

When the Federal Polytechnical School of Zurich opened its gates to the participants of the first International Congress of Mathematicians, on the 9th of August 1897, Brazil was nowhere to be seen.

Back then this was a rural country with just a handful of scientific institutions, none in mathematics. The first schools of engineering were being created in a few major cities. The first universities, with undergraduate courses in mathematics, were still a few decades into the future.

The nation's mathematical awakening took place in the 1950's, when the Institute for Pure and Applied Mathematics was created, Brazil joined Group 1 of the International Mathematical Union, and the biennial series of Brazilian Mathematical Colloquia was launched.

By 1962, Leopoldo Nachbin became the first Brazilian invited speaker at an ICM, in Stockholm. Seven years later, in 1969, the Brazilian Mathematical Society was created and became our adhering organization to the IMU.

The country you are visiting today is totally different, and so is its mathematics.

Early this year the IMU members voted Brazil's promotion to Group 5. That news was greeted as a matter of national pride, by our media and our authorities, as much as by the mathematical community. As it should be.

Four years ago, Artur Avila was distinguished with the Fields medal, the first winner born, raised and educated in a developing country.

And today, we are honored to welcome the delegates to the first International Congress of Mathematicians ever held in the southern hemisphere.

Such progress, in such a short time, was the doing of generations of mathematicians, some of whom are unfortunately no longer with us.

But it is fair to say that no one symbolizes the achievements of Brazilian mathematics better than Jacob Palis, without whose dreams and whose vision we would probably not be here today.

That is why I ask you to join me in electing Jacob the honorary president of the Congress, through a round of applause.

However, for all the merits of the past, bringing ICM 2018 to Rio de Janeiro has always been about the future.

Most of our efforts over the last six years went into taking advantage of this historical opportunity to promote mathematics amongst our youth.

The Biennium of Mathematics, proclaimed by the Brazilian national parliament. The Math Festival we organized last year, with over 18,000 visitors of all ages. The International Mathematical Olympiad, also last year and also a huge success. And all the other initiatives we organized or triggered during these two years, under the umbrella of the Biennium.

As a token of all that, we have here today a group of very special guests: the 576 gold medalists of the Brazilian Mathematical Olympiad for Public Schools. This is the largest and, I dare say, most impressive school competition in the world. It is held by IMPA every year, with the participation of over 18 million students from all over Brazil.

This year, we have invited the gold medalists to attend the ICM opening. In a few minutes they will participate with us in the award ceremony for the Fields medal and the Nevanlinna prize. And they will receive their own medals in this same room, tomorrow afternoon. I don't think this is an experience they will ever forget.

I would like to add a more personal note. Four years ago, in Seoul, I pledged that we would put the best of Brazilian ingenuity to the service of this Congress. Of course,

I did not expect that we would be put to the test the way we were, just two days before this ceremony.

That we passed the test is the definite proof that we have a tremendous team, enormously competent and fully dedicated to this dream of having an ICM in Brazil. I want them to know how grateful I am for what they achieved here in the last 48 hours.

I want to finish by thanking you for your presence here today, and by welcoming you to Brazil and to the Marvelous City.

Enjoy Rio, enjoy ICM 2018, and, most of all, enjoy mathematics.

Thank you!

Welcoming Address

Shigefumi Mori, President of the International Mathematical Union

Minister of Education, distinguished guests, Prize Winners and families, every attendant of this event:

Watching the scenes from past ICMs, I recalled myself the ICM in Seoul four years ago. Perhaps you also did Hyderabad in 2010, Madrid in 2006, Beijing in 2002, or even further.

I even recall ICM 1990 Kyoto, when I received the Fields Medal. During that congress I was so tense and could not enjoy the congress and celebrate the occasion. That was a regret of mine if any. So I wish you especially the winners, plenary and invited speakers enjoy the congress as well as learn through it.

Back to this ICM 2018 at Rio de Janeiro, we are familiar with the beautiful Brazil and its history and culture. During the Congress, you will learn about the recent developments in mathematics and celebrate IMU Prize Winners. You will surely meet old friends, make new friends, since this is a unique opportunity to do so once every four years; thousands of mathematicians came here to participate in this ICM from all over the world.

I also hope that you will find time to explore this magnificent place, its history, culture, and beauty, and food, since this is the first ICM ever held in the Southern hemisphere.

But above all, I think you will find that all this is made possible by the very smooth organization and devoted and sustained work of the Local Organizing Committee. I

have had many occasions already, in the preparations to the ICM, to witness their dedication and efforts, and we will continue to do so during the whole ICM. I would like to thank all of them on your behalf, for this work.

To end this welcome address to the ICM, I want to express again my hope that you will enjoy the remainder of this Opening Ceremony and of the whole Conference. Thank you.

Award Ceremony

Shigefumi Mori

The Gauß Prize has been awarded at the ICM for the first time in 2006 and is now awarded at every ICM. The Prize was established by the IMU and the German Mathematical Society.

It honors a scientist whose mathematical research has had an impact outside mathematics either in technology, in business, or simply in people's everyday lives.

The 2018 Gauß Prize is awarded to David Donoho for his fundamental contributions to the mathematical, statistical and computational analysis of important problems in signal processing.

For the glory of the Opening Ceremony to be shared by later events of ICM, the award ceremony of Gauß Prize 2018 will take place at 7:30pm on Aug 4 (Sat) before the Social Dinner.

The Chern Medal Award was awarded for the first time at the 2010 ICM and is now awarded at every ICM. It was established by the IMU and the Chern Medal Foundation in cooperation with the Simons Foundation.

It is awarded to an individual whose accomplishments warrant the highest level of recognition for outstanding achievements in the fields of mathematics.

The 2018 Chern Medal Award is awarded to Masaki Kashiwara for his outstanding and foundational contributions to algebraic analysis and representation theory sustained over a period of almost 50 years.

For the glory of the Opening Ceremony to be shared by later events of ICM, the award ceremony of Chern Medal Award 2018 will take place at 7:30pm on Aug 4 (Sat) before the Social Dinner.

The Leelavati Prize was awarded for the first time during the Closing Ceremony of ICM 2010 in Hyderabad. The prize was established by the IMU and by the government of India; it is presently funded by Infosys, as a permanent IMU prize to be awarded at every ICM.

The Leelavati Prize accords high recognition for outstanding contributions to increasing public awareness of mathematics as an intellectual discipline and the crucial role it plays in diverse human endeavors.

The 2018 Leelavati Prize is awarded to Ali Nesin for his outstanding contributions towards increasing public awareness of mathematics in Turkey, in particular for his

tireless work in creating the *Mathematical Village* as an exceptional, peaceful place for education, research and the exploration of mathematics for anyone.

Following the precedent set at its first awarding at ICM 2010, it was decided that the Leelavati Prize 2018 winner be announced during the Opening Ceremony, but that the award ceremony itself is part of the Closing Ceremony of each ICM.

Award Ceremonies for the Fields Medals

We now start the ceremony for the Fields Medals.

The Fields Medals have been awarded by the IMU since 1936. They recognize outstanding mathematical achievement for existing work and for the promise of future achievement. From the start, they were meant for young mathematicians. The rule is now that to be eligible to receive a Fields Medal, mathematicians must have their 40th birthday after January 1st of the year in which the ICM is held.

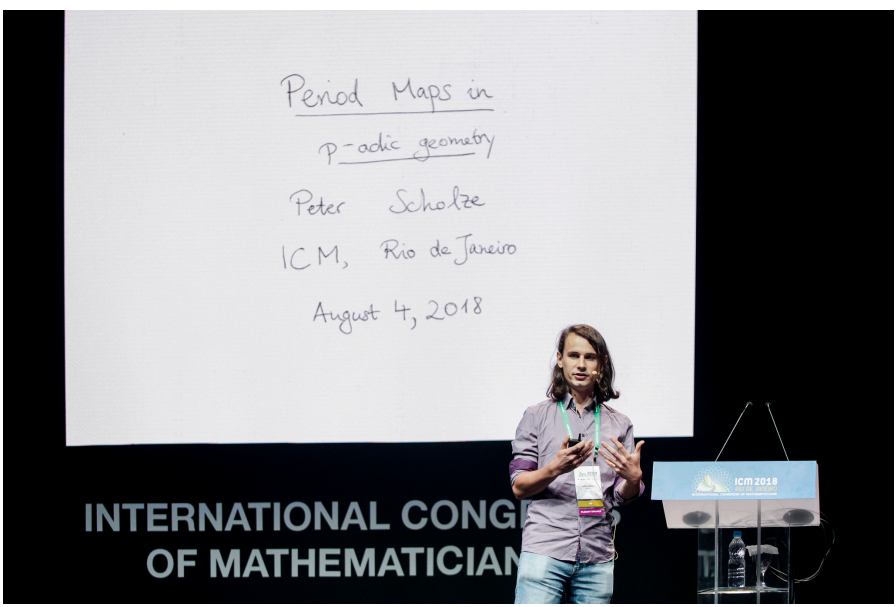
The Rolf Nevanlinna Prize has been awarded at every ICM since 1982. It was established by the IMU together with the Finnish Academy of Sciences. It recognizes outstanding contributions in mathematical aspects of information sciences. It is subject to the same age limit as the Fields Medal: to be eligible for the Nevanlinna Prize, the 40th birthday of the winner must be after January 1st of the ICM year.

The four 2018 Fields Medalists are, in the alphabetical order of last names of the laureates:

Caucher Birkar is awarded a Fields Medal – for the proof of the boundedness of Fano varieties and for contributions to the minimal model program.

Alessio Figalli is awarded a Fields Medal – for contributions to the theory of optimal transport and its applications in partial differential equations, metric geometry and probability.

Peter Scholze is awarded a Fields Medal – for his having transformed arithmetic algebraic geometry over p-adic fields.



Akshay Venkatesh is awarded a Fields Medal – for his synthesis of analytic number theory, homogeneous dynamics, topology, and representation theory, which has resolved long-standing problems in areas such as the equidistribution of arithmetic objects.

Award Ceremony for the Nevanlinna Prize

We now proceed to the Nevanlinna Prize.

Constantinos Daskalakis is awarded the Nevanlinna Prize for transforming our understanding of the computational complexity of fundamental problems in markets, auctions, equilibria, and other economic structures. His work provides both efficient algorithms and limits on what can be performed efficiently in these domains.

Announcements

Shigefumi Mori

I would like to make two remarks.

In this ICM 2018 Rio de Janeiro, each winner of the IMU Prizes, Gauß Prize, Chern Medal, and Leelavati Prize will be introduced to you by a short movie in addition to the Fields Medal and Nevanlinna Prize.

These videos are the result of a collaboration of the IMU and the Simons Foundation; the IMU is grateful to the Simons Foundation for having accepted to fund and produce these movies.

Brazil is a great country. When I visited Rio last year to examine the ICM preparation, it was in the year of the International Mathematical Olympiad and I saw people enjoying mathematics together as a team. Brazilians know how to foster through enjoyment talents of the next generation following Artur Avila the Fields Medalist 2014.

Brazil is famous and strong in mathematics with IMPA as one of its big centers.

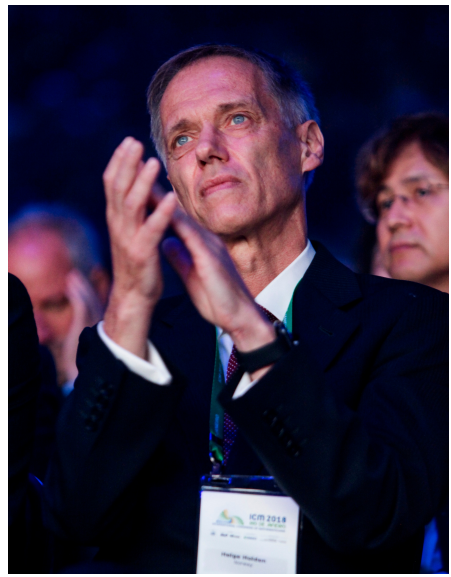
Just in case you are not familiar, ICM is held by the host country under the auspices of International Mathematical Union (IMU), which is why I am here at all. I served the executive committee (EC) of the IMU 2 terms (1995–2002) before I became President in 2015, Jacob Palis also served IMU as Secretary and President. Naturally I learned a lot from him. He is a great figure representing Brazil in my mind. We knew that he wanted to hold ICM in Brazil. So thank you, Jacob, and also Congratulations on fostering great talents who have made your dream come true.

IMU Status Report

Helge Holden, Secretary General of the International Mathematical Union

It is my pleasure to report to you about the activities of the International Mathematical Union (IMU) during the last four years since we assembled in Seoul, Republic of Korea, for the ICM 2014.

In addition to an Executive Committee (EC), the IMU operates five Committees and Commissions, namely the International Commission on Mathematical Instruction (ICMI), the Commission for Developing Countries (CDC), the International Commission on the History of Mathematics (ICHM), the Committee on Electronic Information and Communication (CEIC), as well as the Committee for Women in Mathematics (CWM). A major part of the operative work of the IMU is executed by our Committees and Commissions. For reasons of brevity, I refrain from reporting on their activities here, and I recommend that you consult the respective websites to get a better understanding of their work for the benefit of mathematics and mathematicians worldwide.



I would like to take this opportunity to thank all our colleagues who have spent endless hours working for the IMU during the last term. Your efforts are very well appreciated — even if rarely expressed in public — by the global mathematical community. In addition, I would like to express our deep gratitude for the generous donations and grants that the IMU has received from individuals and organizations worldwide. We very much depend on outside support to achieve our goals.

The IMU is also issuing a free electronic newsletter, the IMU-Net. Did you stop receiving it? Due to the GDPR (General Data Protection Regulation) directive introduced by the European Union in order to protect personal privacy, we had to cancel all current subscriptions. You will have to re-subscribe to the IMU-Net in order to continue receiving it. Please go to www.mathunion.org and resubscribe today!

Membership

Membership in the IMU is by countries. Regularly, I receive requests for individual membership, but the only way to adhere to the IMU is through an Adhering Organization (AO), which can be, e.g., a national academy, a national mathematical society, or a university. The IMU may be considered as the "United Nations of Mathematics", and the IMU itself is a member of one organization only, the *International Science Council* (ISC), which assembles international scientific unions and regional scientific organizations. The ISC was recently created as a result of the merger of the International Council of Science (ICSU) and the International Social Science Council (ISSC). It is our expressed goal to have as many countries worldwide as members. By increasing membership, our work to support mathematicians and mathematics worldwide becomes more forceful.

Membership in the IMU has been going through a positive development during the period 2015–2018. Even if some countries experience problems in terms of raising funds for settling the membership dues due to domestic problems, the IMU was able to increase its membership. In addition, four countries have been upgraded to a higher group of adherence. The following countries have applied for membership and been accepted as new members of the IMU (all in Group I): Malaysia, Kenya, Luxembourg, Morocco, Senegal, Thailand, Oman, and Kyrgyzstan. The following countries have been accepted as new Associate Members of the IMU for a four-year term: Paraguay, Bangladesh, and Uzbekistan. Madagascar, Gabon, and Papua New Guinea have been granted a second four-year term as an Associate Member of the IMU. Turkey has been

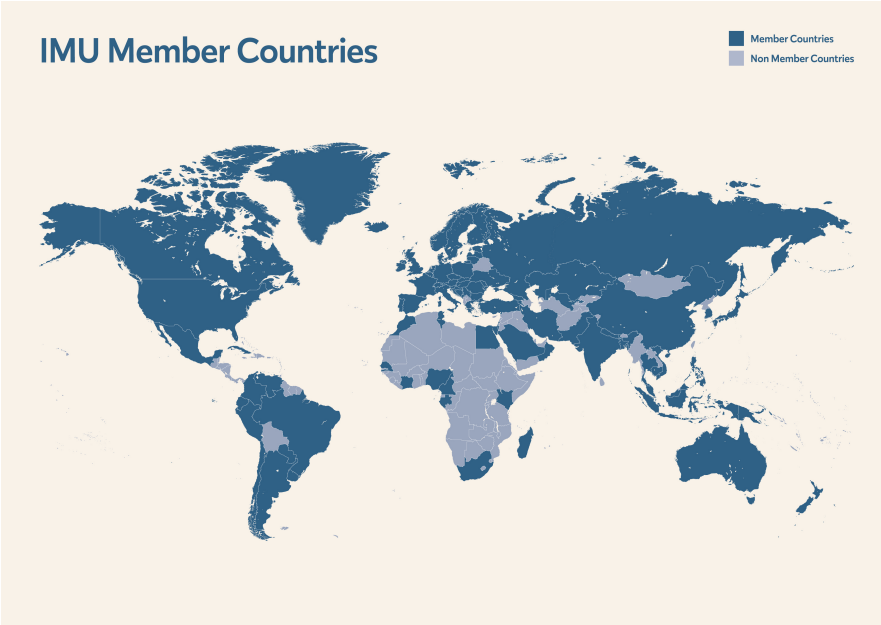


Figure 1: The world according to the IMU. Member countries are dark.

upgraded to Group II, Chile and Portugal have both been upgraded to Group III, and Brazil to Group V. The Mathematical Council of the Americas has been admitted as a new Affiliate member of the IMU. (Membership is listed in chronological order.)

Currently, the IMU has 88 regular and Associate Members and five Affiliate Members. While we are doing quite well compared to other scientific unions, we can and should do better. Note that 116 countries participated in the International Mathematical Olympiad in 2018, 193 countries are members of the UN, 206 countries participated in the Rio Olympic Games in 2016, 211 countries are recognized by FIFA, 215 countries are members of the IAAF! Thus there is ample room for improvement. I should add that each organization has its own interpretation of what constitutes a "country". In Figure 1 we can see IMU membership displayed on a world map.

It has always been a challenge for us to stay in contact with all IMU members. The means of communication have never been better, but yet it is not always easy to get feedback from all the AOs, probably because some countries have small and rather fragile national organizations. The majority of the members of the IMU are paying their membership dues properly, however, there are countries in arrears. Unpaid dues accumulate, and the IMU Statutes (Article 32) have clear regulations regarding IMU membership. Thus, the IMU EC tries to keep an active communication with the IMU members.

In the coming term, the IMU will seek to increase membership while keeping its current members.

International Day of Mathematics — March 14

The IMU has taken the initiative to have the UNESCO proclaim March 14 (*Pi Day*) as the International Day of Mathematics (IDM). This date was chosen in consultation with the IMU Adhering Organizations, which enthusiastically support the idea to create an IDM. March 14, often written 3/14 in English, is already celebrated as Pi Day in many countries (and not only English speaking countries) around the world. Hence, the date was proposed by many countries when IMU consulted its members. March 14 is convenient for all parts of the world, including the Southern hemisphere. Every year, all countries in the world will be invited to celebrate the IDM in the schools and with the public, under a (non mandatory) chosen theme for the year.

However, UNESCO still needs to be convinced that March 14 be proclaimed as the International Day of Mathematics. The ambition of the IMU is that the topic has gained necessary political support to be discussed at the General Conference of UNESCO in the fall of 2019, and that the official launch of the International Day of Mathematics will take place on March 14, 2020. The more countries support the proposal, the more likely it is to have UNESCO proclaim March 14 as the International Day of Mathematics.

Elections

The governing body of the IMU is its General Assembly, which meets every four years, just before the ICM. We have just held the 18th General Assembly of the IMU in São Paulo, Brazil. One of the important decisions taken at the General Assembly was to elect the IMU leadership for the term 2019–2022. The Executive Committee for the term 2019–2022 comprises President Carlos Kenig (USA), Secretary General Helge Holden (Norway), Vice Presidents Nalini Joshi (Australia) and Loyiso G. Nongxa (South Africa), and Members-at-Large are Luigi Ambrosio (Italy), Andrei Okounkov (Russia), Paolo Piccione (Brazil), R. T. Ramadas (India), Gang Tian (China), and Günter Ziegler (Germany). For the first time the EC has a representative from Africa, and also all five continents are represented in the EC.

The majority of the members of the Commission for Developing Countries are elected by the General Assembly. For the term 2019–2022 the President will be Dipendra Prasad (India), Secretary for Policy Olga Gil-Medrano (Spain) and Secretary for Grants Alf Onshuus (Colombia). Members-at-Large are Mama Foupougnigni (Cameroon, representing Africa), José Maria P. Balmaceda (Philippines, representing Asia), and Andrea Solotar (Argentina, representing Latin America). In addition, the EC will appoint two members and ICMI will appoint one member to the CDC.

For the International Commission on the History of Mathematics Isobel Falconer (UK) and Catherine Goldstein (France) were elected as Members-at-Large for the term 2019–2022.

The International Commission on Mathematical Instruction elects its own Executive Committee at the ICMI General Assembly, which takes place in years congruent to 0 modulo 4, while members of the IMU Committees are appointed by the EC.

Members of Committees

The Executive Committee of the IMU appointed some of the most important committees for this congress. The scientific program of the ICM here in Rio was the responsibility of the Program Committee with members János Kollár (chair), Ngô Bảo Châu, Irit Dinur, Jianshu Li, Hiraku Nakajima, Sorin Popa, Enrique Pujals, Laure Saint-Raymond, Stanislav Smirnov, Eitan Tadmor, Ulrike Tillmann, and Aad Van der Vaart. The responsibility of the Program Committee includes the decision of the scientific structure of the program, that is, how to split contemporary mathematics into sections and their relative sizes, decide the chair and members of a panel for each section, which in turn recommends speakers in its section, and finally the Program Committee decides all speakers, both plenary and sectional. All of this work is done within a time-span of two years, and in complete secrecy. A truly gigantic task! We will enjoy the result of their work during the coming ten days.

The selection of the winners of the IMU awards is decided by prize committees, again working in complete confidentiality.

The Fields Medal committee comprised Shigefumi Mori (chair), Hélène Esnault, Eduard Feireisl, Alice Guionnet, Nigel Hitchin, John Morgan, Hee Oh, Andrei Okounkov, M.S. Raghunathan, Kenneth A. Ribet, and Terence Tao.

The Nevanlinna Prize winner was selected by the following committee: Tony F. Chan (chair), Manindra Agrawal, Emmanuel Candès, Shafi Goldwasser, Nick Higham, and Jon Kleinberg.

The Gauß Prize committee comprised Björn Engquist (chair), Maria J. Esteban, Motoko Kotani, Martin Nowak, Andrew Stuart, Nick Trefethen, and Bin Yu.

The Chern Medal recipient was decided by the following committee: Caroline Series (chair), Jordan Ellenberg, Gerhard Huisken, Michio Jimbo, and Benoît Perthame.

The Leelavati Prize committee comprised Gert-Martin Greuel (chair), Shihoko Ishii, Andrés Navas Flores, Sun Sook Noh, and Simon Singh.

The ICM Emmy Noether Lecture committee consisted of Irene Fonseca (chair), Georgia Benkart, Kathryn Hess, Nalini Joshi, Izabella Laba, Dipendra Prasad, Fernando Rodriguez Villegas.

Permanent stable office of the IMU

Arguably the most important decision made at the 18th General Assembly was the decision regarding the permanent office of the IMU Secretariat.

Since January 2011, the Secretariat of the IMU has been based in the heart of the city of Berlin, Germany, generously supported by a grant from the Federal Republic of Germany and the State of Berlin and hosted by Weierstrass Institute for Applied Analysis and Stochastics (WIAS). The annual financial support exceeds the total membership dues received from the IMU members. The grant funds are spent on office rent, salaries for the staff, and to support various IMU activities. As of April 2018, the premises of the IMU Secretariat have been in Hausvogteiplatz 11A, near Gendarmenmarkt, see Figure 2. The venue consists of four office rooms, the IMU Archive, a lounge, a meeting room, as well as auxiliary rooms. The IMU staff comprises IMU Manager (100% position, responsible for overseeing IMU business and, in particular, the EC), CDC & ICMI Administrator (100% position, administration of CDC and ICMI), IMU Technician (50% position, in charge of email, internet, webpage, backup), IMU Accountant (100% position, membership dues, accounting), IMU Archivist (60% position, responsible for the IMU Archive). In addition, there is the Head of the Secretariat (unpaid), who is a senior scientist employed at WIAS and serves as a liaison between the IMU Secretariat, the IMU, and the WIAS. The IMU Secretariat staff members are all regular employees of WIAS, and all decisions regarding salaries, working conditions, pensions, sick leaves, vacations, etc., are handled by the WIAS, while the actual work that is carried out is the responsibility of the IMU.

The IMU Secretariat provides full administrative support for the IMU Executive Committee as well as the Commission for Developing Countries (CDC) and the International Commission on Mathematical Instruction (ICMI). In addition, the IMU Secretariat has full responsibility for all financial transactions of the IMU. The IMU Archive, containing the historical material from the early days of the IMU up to today, is physically located in the IMU premises. The IMU Secretariat is responsible for maintaining the IMU website as well as the mail servers.

In accordance with the decision taken at the General Assembly in 2010, the contract between WIAS and the IMU was made for a ten-year period. Thus it was on the agenda of the 2018 General Assembly to decide what would happen past this period. The matter was discussed and I am happy to report that the General Assembly unanimously decided to extend the contract with WIAS indefinitely. Thus the IMU Secretariat will remain in Berlin for the future. This gives the IMU stability and a reliable way to handle all financial issues of the complete slate of IMU activities. The fact that the IMU has a permanent Secretariat with a very able staff has resulted in a considerable increase in the total volume of activities of the IMU. Without the dedicated staff at the IMU Secretariat, and the generous financial support from Germany, the level of activity of the IMU would have to be drastically reduced.

The tradition of handing over the IMU "Secretariat" to the next President's and Secretary General's location and the "IMU Archive" consisting of a number of cardboard

Figure 2: The IMU Secretariat is located in the dark building on the 5th floor.

boxes, a tradition that dates from the inception of the IMU in the 1920's, has come to an end once and for all.

A new IMU website

After considerable work done by many individuals and a commercial company, the IMU as well as its Committees and Commissions have for the first time a uniform appearance on internet. The new website allows for the prominent exposure of news as well as the current requests to the Adhering Organizations. Now the IMU is able to display in real time on its homepage the names, including photos, full citations, brief videos, and popular write-ups, for the winners of all the IMU awards, that is, at the very same moment when the names are announced by the IMU President at the ICM Opening Ceremony. Please check www.mathunion.org and see for yourself.

The ICM in 2022

It may not feel right to talk about the ICM in 2022 before we have even started the scientific program of the current ICM in 2018 here in Rio de Janeiro, Brazil. But as we can see by looking around, an ICM requires meticulous long-term planning, and it is not too early to start the careful preparations for the next ICM four years from now.

By tradition, the Adhering Organizations — our members — were asked to submit bids for the ICM in 2022. We received two bids — one from France (with venues Paris for the ICM and Strasbourg for the IMU General Assembly) and one from Russia (with venue St. Petersburg both for the ICM and the IMU General Assembly). Both bids were of excellent quality and very professional. After site visits, the Site Selection Committee made its recommendation to the IMU General Assembly, which at its meeting in

São Paulo decided to accept the bid from Russia. Thus the ICM 2022 will take place in St. Petersburg, Russia.

The IMU wants to express its gratitude to our colleagues both in France and in Russia for preparing two excellent bids.

But right now we are in Rio de Janeiro, in Brazil, and we are looking forward with great expectations to the ICM 2018!

Award Ceremonies for Gauß Prize and Chern Medal Award

August 4, 2018

Opening words

Shigefumi Mori

Ladies and Gentlemen,

We have moved the Award Ceremonies of Gauß Prize and the Chen Medal Award to tonight to share their excitement after having described and announced the winners of those prize and award in the Opening Ceremony.

We will start with the Gauß Prize. It is my great pleasure to introduce Dr. Michael Röckner, President of the German Mathematical Society, who will award the prize. Dr. Röckner, please proceed to the stage.

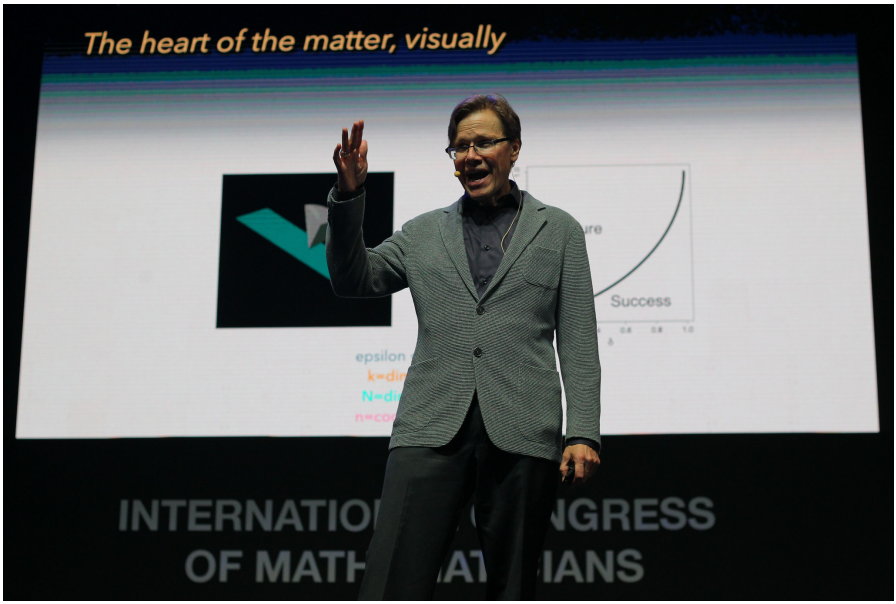
The Gauß Prize

Michael Röckner, President of the German Mathematical Society

It is a great honor for me to hand over the Carl Friedrich Gauß Prize here at the ICM in Rio de Janeiro. The Carl Friedrich Gauß Prize is awarded jointly by the IMU and the German Mathematical Society (DMV). It consists of a medal and a monetary award of 10.000 Euro. Its funding source is a surplus from the ICM 1998 in Berlin. The Gauß Prize Committee is chosen by the Executive Committee of the IMU. This time it was chaired by Professor Björn Engquist.

The Carl Friedrich Gauß Prize was created to raise awareness in the public that Mathematics is a driving force behind many modern technologies, and to honor scientists whose mathematical research has had an impact outside Mathematics – either in Technology, Business or simply in people's everyday lives.

It was awarded first at the ICM 2006 in Madrid to Kiyoshi Itô, then at the ICM 2010 in Hyderabad to Yves Meyer and at the ICM 2014 in Seoul to Stanley Osher. Today's awardee will certainly add a few more tons of scientific weight to this list of mathematical heavy weights and we are very proud that thus he will further enhance the reputation of this prize. Thank you very much for your attention.



Now is the turn for the Chern Medal Award.

It is my great pleasure to introduce Dr. Robert Bryant, Representative of Chern Medal Foundation (CMF), who will award the prize. Dr. Bryant, please proceed to the stage.

The Chern Medal

Robert Bryant, Chern Medal Foundation

The Chern Medal was founded to honor the memory of Shiing-Shen Chern, who passed away in December 2004 after having been a world leader in mathematics, especially differential geometry, for nearly 70 years. He made fundamental, essential contributions to differential geometry and topology. Mathematics and physics today would be unthinkable without the tools of Chern classes and the Chern-Simons invariants.

He was renowned for his personal efforts on behalf of mathematics and mathematicians, at all stages of their careers. In addition to having more than 1100 mathematical descendants, of which, I am pleased to say, I am one, he received numerous awards and recognition for his life's work, including the Wolf Prize and the inaugural Shaw Prize.

He founded three mathematical institutes, two in China and one in the US, the famous Mathematical Sciences Research Institute, and was that institute's founding director.

In 2010, in memory of Shiing-Shen Chern's profound influence on mathematics, the International Mathematical Union, jointly with the Chern Medal Foundation (a charitable organization supported by the family, friends, and admirers of Professor Chern),

established the Chern Medal whose purpose was to recognize “an individual whose accomplishments warrant the highest level of recognition for outstanding achievements in the field of mathematics”.

This medal, first awarded in 2010, is awarded once every 4 years at the International Congress of Mathematicians.

Each recipient receives a medal decorated with Chern’s likeness, a cash prize of $250,000 (USD), and the opportunity to direct an additional $250,000 of charitable donations to one or more organizations for the purpose of supporting research, education, or outreach in mathematics.

As you already know, the 2018 Chern Medal is awarded to Masaki Kashiwara – “for his outstanding and foundational contributions to algebraic analysis and representation theory, sustained over a period of almost 50 years”.

As I mentioned earlier, the awardee has the opportunity to direct an additional $250,000 of charitable donations to one or more organizations for the purpose of supporting research, education, or outreach in mathematics.

Professor Kashiwara has chosen RIMS, the justly famous Research Institute for Mathematical Sciences of the University of Kyoto, Japan, founded in 1963 to be the recipient of this contribution.

Some congratulatory words and an announcement

Shigefumi Mori

It is my great pleasure to congratulate Dr. Donoho on receiving the Gauß Prize, and Dr. Kashiwara on receiving the Chern Medal Award.

The Leelavati Prize 2018 winner, Dr. Ali Nesin was announced at the Opening Ceremony, and the award ceremony itself will take place during the Closing Ceremony on Aug 9. In addition the Prize Lecture will be given by the Winner Dr. Ali Nesin at 18:00 on Aug. 8.

Closing Ceremony

August 9, 2018

Report and Closing Remarks

Marcelo Viana, Chairman of the Rio de Janeiro ICM 2018

As the International Congress of Mathematicians draws to its end, I would like to offer some information about the Congress and what was involved in its organization.

Travel grants: The Open Arms program offered 630 international travel grants (500 from IMPA & SBM, plus 130 from the IMU) for mathematicians from the developing world to attend the Congress. In addition, 250 domestic travel grants were offered, for mathematicians based in Brazil.

A total 2,016 mathematicians applied for support, including 450 from Brazil. The breakdown of applications and grants by geographic region and by gender is detailed in the following table:

	Applications	Travel Grants
Asia and Pacific	903 (284 women)	200 (60 women)
Africa	280 (94 women)	130 (44 women)
Eastern Europe	138 (70 women)	100 (49 women)
Latin America	245 (76 women)	200 (76 women)
Total International	1566 (524 women)	630 (229 women)
Brazil	450 (110 women)	250 (60 women)
Total	2,016 (634 women, 31%)	880 (289 women, 33%)

Participation and attendance: There were a little over 3,000 participants, registered and distributed by geographic area as follows:

	Participants
Registered on-line	2,464
Registered on-site	181
Accompanying persons	238
Guests	135
Total	3,018

	Participants	
South America	1226	(40.6%)
Europe	615	(20.3%)
Asia	516	(17.1%)
North America	420	(13.9%)
Africa	186	(6.2%)
Oceania	55	(1.8%)
Total	3,018	

The total attendance of the Congress was much bigger, a little over 10,000, about half of which were school students and teachers of local schools participating in the public lectures program:

	Attendees
Congress participants	3,018
Public lectures program	5,790
Staff	1,301
Volunteers	228
Media	169
Total	10,506

Scientific program: The following table summarizes the lectures program:

	Scheduled	Cancelled	No-show	Presented
Plenary lectures	21	0	0	21
Invited lectures	181	5	0	176
Prize/special/laudations	18	0	0	18
Public lectures	5	0	0	5
Other lectures	5	0	0	5
Short communications	435	16	11	408
Posters	475	195	40	240
Total	1140	216	51	873

In addition, there were three Section 18 (mathematics education and popularization) invited panels and five IMU organized discussion panels.

The public lectures, by Étienne Ghys, Ingrid Daubechies, Cédric Villani, Tadashi Tokieda and Rogério Martins were attended by 5,790 people, including local school children and teachers.

Award ceremonies: There were three distinct award ceremonies. The first one, devoted to the Fields medal (Caucher Birkar, Alessio Figalli, Peter Scholze and Akshay Venkatesh) and the Nevanlinna prize (Constantinos Daskalakis), was part of the opening ceremony, on August 1. The Gauß prize (David Donoho) and Chern medal (Masaki Kashiwara) awards were handed on August 4, preceding the social dinner. The Leelavati prize (Ali Nesin) was awarded on the occasion of the closing ceremony, on August 9.

Infrastructure: A few figures about the services infrastructure and human resources involved in the organization:

Plenary lecture hall: 7,225 m^2 - 4,000 seats

Lecture rooms: 6,931 m^2 - 9 rooms for invited sections, 12 for short communications

Exhibition area: 5,050 m^2 - 29 stands and 2 special exhibits

Accreditation and cloakroom: 2,455 m^2 - 36 accreditation positions

Network lounge: 1,500 m^2 - panel sessions: 100 posters/day; Maryam Mirzakhani exhibit

Cocktail area: 1,100 m^2

Restaurant: 5,100 m^2 - 1280 seats

Additional catering: 3,920 m^2 – 10 food trucks, 3 snack bars, with 1,500 seats

Meals served: 2.4 million units

Beverages: 3.9 million units

Ceremonies & social events: 224 artists on stage

The organization of the Congress occupied a total area of 37,800 m^2 and involved a little over 1,500 professionals:

	Number
Services & suppliers	789
Temporary on-site staff	420
Volunteers	228
PCO staff	55
Local organizers	28
Secretariat office staff	8
Total	1,529

Website & communication: The Congress attained substantial visibility in the digital media:

Website www.icm2018.org	
Unique visitors	91,405
Page views	416,000
News	200

	Users
Twitter	1.8 million
Facebook	275,500
YouTube	269,400
Instagram	3,600
Total users reached	2.36 million

Brazil accounted for 31.0% of the visits, the USA for 13.4%, Germany for 6.2%, France for 4.8%, India for 4.6% and Japan for 3.5%.

The ICM 2018 mobile app was downloaded by nearly 2,000 users:

	Downloads
Android	1,249 (63 %)
iOS	733 (37%)
Total	1,982

Modules	Clicks
Program	83%
MyEvent	7%
Attendees	2%
Total clicks	548,877

Acknowledgements: I want to express my deep gratitude to my colleagues in the Organizing Committee:

- Alexander Arbieto (Rio de Janeiro) - Short Communications and Posters
- Boyan Sirakov (PUC-Rio) - Proceedings
- Carolina Araujo (IMPA) - Women in Mathematics
- Eduardo Teixeira (Central Florida) - Satellite Events
- Emanuel Carneiro (IMPA) - Scientific Program
- Henrique Bursztyn (IMPA) - Travel Grants
- José Espinar (IMPA) - Rio Intelligencer
- Juliana Bressan (IMPA) - Organization and Coordination
- Maria João Resende (Niterói), Nelly Carvajal (IMPA) – Volunteers
- Paolo Piccione (São Paulo) - General Assembly
- Paulo Ney de Souza (Berkeley) - Digital Platform
- Roberto Beauclair (IMPA) - Electronic Information and Communication
- Roberto Imbuzeiro Oliveira (IMPA) - Website Planning and Supervision

as well as to all the support from the staff of IMPA, led by Manuela Cervo, and the staff of the Brazilian Mathematical Society, led by Katia Coutinho.

Last but not least, I am deeply grateful to the work of our Volunteers.

Leelavati Prize Award Ceremony
Shigefumi Mori

It is quite exciting to start the Award Ceremony for the Leelavati Prize, after we have seen the winner's spectacular prize lecture last night.

The Leelavati Prize was awarded for the first time during the Closing Ceremony of ICM 2010 in Hyderabad. The prize was established by the IMU and by the government of India; it is presently funded by Infosys, as a permanent IMU prize to be awarded at every ICM.

The Leelavati Prize accords high recognition for outstanding contributions to increasing public awareness of mathematics as an intellectual discipline and the crucial role it plays in diverse human endeavors.

The 2018 Leelavati Prize is awarded to Ali Nesin for his outstanding contributions towards increasing public awareness of mathematics in Turkey, in particular for his tireless work in creating the *Mathematical Village* as an exceptional, peaceful place for education, research and the exploration of mathematics for anyone.

Infosys, the company funding this prize, hoped to be able to send an officer to represent them, but in the end schedule problems prevented this. Instead, Mr. Narayana Murthy, the founder of Infosys and the Chairman of their Board, asked me to read the following statement.

> "On behalf of all of us at Infosys, I'd like to congratulate Prof. Ali Nesin on winning the Leelavati Prize for 2018. Ali's love for the subject, his public spiritedness and his dogged perseverance in building something concrete that symbolizes it, is an example to us all. He has crystalized his passion for the subject by building the amazing Maths Village in his homeland, Turkey. This utopian Village allows students of different ages to congregate and learn what they love, not just from instructors but from each other

too. When they go back from the programs at the Village, I am sure their experience will inspire their peers and excite their curiosity about it. In this way, the effect of Prof. Nesin's work is amplified. He is therefore the perfect choice for the winner of the Leelavati Prize sponsored by Infosys, which is given not just for expertise in Mathematics but for public outreach. Congratulations again, Prof. Nesin!"

Report on the General Assembly and the future of IMU

Shigefumi Mori

The ICM 2018 has almost come to its end.

There will be still various IMU activities after the ICM and let me tell you a little bit about them, since IMU Secretary General Helge Holden already made a detailed status report in the Opening Ceremony.

In January 2019, the new Executive Committee (EC) will take over, and the rest of 2018 is the transition period when the old and new ECs will work together.

The new President Carlos Kenig of USA has participated in the IMU committee meetings as an observer as decided by IMU GA 2014. Watching the way Carlos participated in the IMU activities, I fully believe that he will make a great President in every respect together with the incumbent Secretary General Helge Holden.

At the General Assembly, the IMU created the Structure Committee chaired by former President László Lovász which decides the structure of the next ICM so that the Program Committee can concentrate on the selection of the Plenary and invited speakers. Taking advantage of the occasion of this ICM, we have already started the preparation.

Prior to the ICM, the CWM organized the World Meeting for Women in Mathematics $(\mathrm{WM})^2$ here in the ICM venue while the GA ended in São Paulo on the day before. For the year 2022, both the GA and the ICM take place in St. Petersburg, and the organization of the $(\mathrm{WM})^2$ will hopefully be easier.

That was what I wanted to tell you about IMU after the ICM.

Vote of Thanks

Shigefumi Mori

The ICM has come to its end. Could you enjoy or learn something from it?

Well, you cannot answer in this hall, but at least I myself enjoyed the great opportunity to listen to worldwide top researchers reaching out to us, general mathematical audience, in our language. I think this, especially feeling the atmosphere, is the essence of ICM, and I hope you share my feeling too.

I would like you to understand that ICM is a hand-made event, so to speak, although it is a gigantic meeting with thousands of participants;

Eight prize winners and about 200 plenary and invited speakers including the Noether lecturer greatly contributed to the academic success of the ICM. Public and special lectures and various Panels as well as short communications, posters, and exhibitions enriched the ICM. Chairs of the talks and panels made sure the event went smoothly and on time. Those are the people you see on surface, but behind them there are enormous number of people who made this ICM possible.

ICM is a fruit of the collaboration of all the people involved; starting with IMU EC with the administrative support of IMU Secretariat, Program Committee (PC) appointed Panel Committee and worked together to select plenary and panel speakers of about 200 in total.

Prize Committees (Fields, Nevanlinna, Gauß, Chern, Leelavati) and Noether Lecturer Committee carefully chose the recipients. I should not forget the nominators and referees who made the committees' work possible.

Please join me in thanking them for their splendid contribution.

These decisions were sent to the local organizers who practically planned and managed this wonderful congress.

Finally, the Organizing Committee of the ICM and IMU GA Chair, Marcelo Viana, spent several years for the organization of the ICM; he carefully chose this wonderful convention center RioCentro as the venue. They made the two years 2017 and 2018 the years of Brazilian Mathematics for International Mathematical Olympiad 2017 and ICM 2018. This way he managed to secure the planning of ICM including even financial matters.

I came here and witnessed the preparation of ICM in April 2017.

As you know, this ICM had a few surprising incidents, and the Local Organizers from various institutions of Brazil (and some from USA) have overcome and turned them into truly memorable events by their hard work; I will call it resilience.

Please allow me to list only the key members of the organizing team while there are many more behind who deserve to be mentioned.

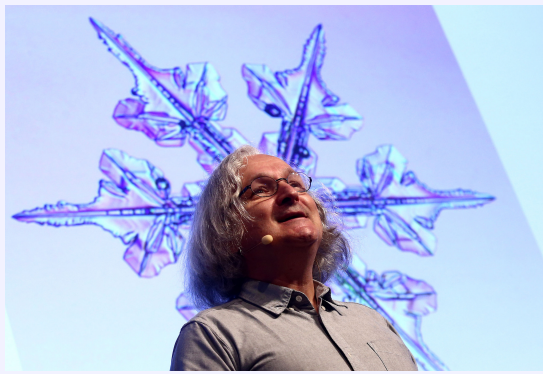
- Marcelo Viana (IMPA), Chair
- Alexander Arbieto (Uni. of Rio de Janeiro), Short Communications and Posters
- Boyan Sirakov (Catholic University of Rio de Janeiro), Proceedings
- Carolina Araujo (IMPA), Women in Mathematics
- Eduardo Teixeira (University of Central Florida), Satellite Events
- Emanuel Carneiro (IMPA), Scientific Program
- Henrique Bursztyn (IMPA), Travel Grants
- José Espinar (IMPA), Rio Intelligencer
- Juliana Bressan (IMPA), Organization and Coordination
- Maria João Resende (Uni. F. Fluminense) & Nelly Carvajal (IMPA), Volunteers

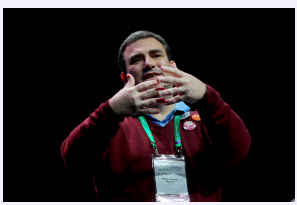
- Paulo Ney de Souza (U. C. Berkeley), Digital Platform
- Roberto Beauclair (IMPA), Electronic Information and Communication
- Roberto Imbuzeiro Oliveira (IMPA), Website Planning and Supervision

Last but not least, I would like to thank Chair of the Organizing Committee of IMU General Assembly Paolo Piccione (University of São Paulo)

Please join me again in thanking Marcelo Viana and his team for the splendid work they have accomplished for the ICM.

Finally, I want to thank all of you, participants who have come from all over the world. Your enthusiasm for mathematics was catalyst for the success of the 2018 International Congress of Mathematicians in Rio de Janeiro!





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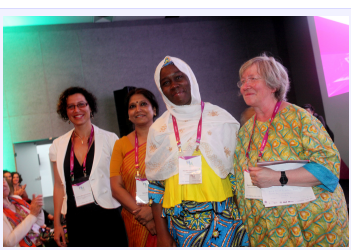

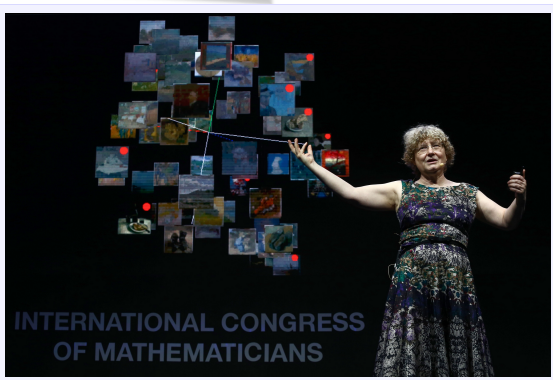
INTERNATIONAL CONGRESS
OF MATHEMATICIANS

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OF MATHEMATICIANS

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RIO DE JANEIRO
Rogério Martins

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RIO DE JANEIRO
MATH MUSEUMS
& WORLD MATH EXPLOSION

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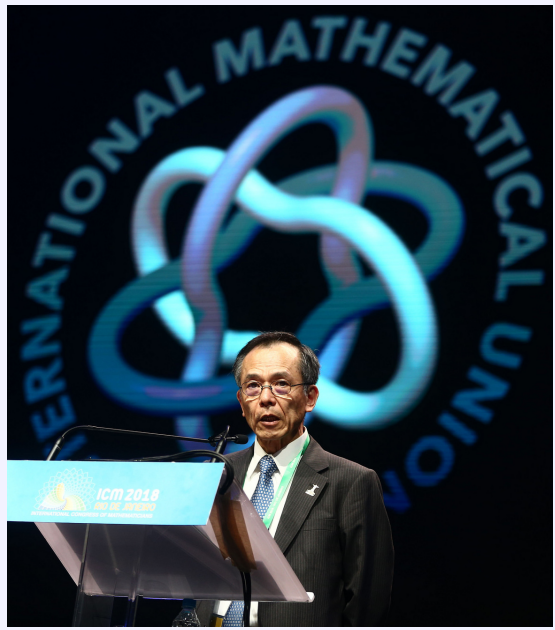
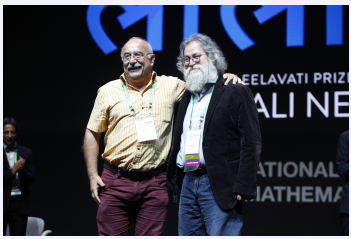

INTERNATIONAL MATHEMATICAL UNION
ICM 2018

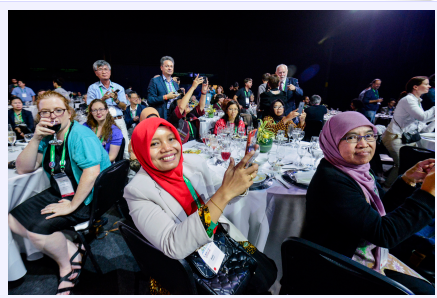
WORLD MEETING
FOR WOMEN
IN MATHEMATICS



ICM 2018
RIO DE JANEIRO

Masaki Kashiwara
Japan

DAVID
DONOHO



ICM 2018

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Orientação
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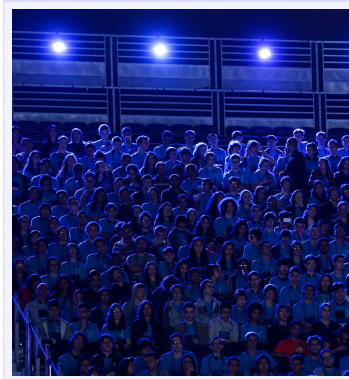

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ALESSIO FIGALLI: HIS CONTRIBUTIONS TO GEOMETRY AND ANALYSIS

Luis A. Caffarelli

Abstract

We present some highlights of Alessio Figalli mathematical work, which is impressive for its originality, breadth, innovative character and impact to various areas of mathematics. It focus mainly on his contributions to optimal transport theory and its connections to partial differential equations (PDE), the calculus of variations, geometric and functional inequalities and random matrices .

A recurring theme in Figalli's research is the interaction between the theory of optimal transport and other areas of mathematics. Optimal transport theory, although pioneered by G. Monge back in 1781, has emerged to paramount importance in many areas of mathematics only in the last two decades. One of the most valuable features of this theory is its unifying power, in the sense that key ideas and constructions in optimal transport have turned out to be useful in the most unexpected contexts. Figalli is currently one of the most authoritative experts in optimal transport and its many applications.

In may be one of his most striking papers, in collaboration with De Philippis, Figalli obtained a fundamental, groundbreaking result on the second order Sobolev regularity of optimal transport maps and their link with the Monge–Ampère equation. This result, a non linear, degenerate version of the classical Calderon Zygmund singular integral theory removes an obstruction that blocked, for years, the existence and regularity theory of solutions to the semigeostrophic equations, a system of PDE from meteorology (first derived by Eliassen, 1962, then rediscovered by Bretherton and Hoskins as an approximate model for frontogenes) deeply connected to the Monge–Ampère equation. In the same direction, they obtained a second order stability result for optimal transport maps, whose potential usefulness in the numerical computation of transport maps is evident.

One of the most important areas of application of mass transport theory is the analysis of geometric and functional inequalities in sharp form, which, in turn, is a central topic in mathematical physics, analysis and geometry.

Together with Maggi and Pratelli, Figalli obtained a sharp quantitative stability theorem for the Wulff inequality. This inequality is a most important mathematical result in our understanding of surface tension driven phase transitions, as it relates the microscopic structure of a given surface tension energy density to the macroscopic shape of the liquid/crystal observed at equilibrium. The Wulff inequality allows one to understand the structure of the absolute equilibrium state of a liquid/crystal, while its sharp stability version allows one to describe in quantitative terms every low energy state.

This is a result of clear physical importance, whose proof required several original and innovative mathematical ideas.

There are other important contributions by Figalli on the topic of quantitative stability inequalities to be mentioned. The first one concerns the Brunn–Minkowski (BM) inequality, easily introduced as the cornerstone of convex geometry and as the most pervasive inequality in mathematics. After addressing the quantitative stability of the BM inequality on convex sets by optimal transport methods, Figalli turned his attention to the formidable problem of extending this result to general non-convex sets. This is already highly non-trivial in one space dimension, and in this particular case, it has been known with a different formulation by number theorists for a long time. Figalli and Jerison succeeded in the quite challenging task of combining tools from additive combinatorics, affine geometry, harmonic analysis and optimal transport to obtain the first quantitative stability result for the BM inequality in arbitrary dimension and on generic sets. This work is quite impressive, both for the technical complexity the wealth of original ideas involved, and the mathematical beauty of the question under consideration.

It is well known that functional inequalities play an important role in the long-time asymptotic of many diffusion equations. The Keller–Segel (KS) equation is a classical model used in the macroscopic description of chemotaxis (i.e., the influence of chemical substances in the environment on the movement of mobile species.) In the KS model, the chemical attractant (say, a tumor) is produced by the cells themselves, thus the higher the amount of cells, the stronger is the effect of the chemical on it.

Further, there is a critical amount of cellular mass, above which solutions to KS blow-up, and below which the influence of the chemical attractant becomes negligible. In a remarkable joint work with Carlen, Figalli has addressed the delicate stability analysis of some Gagliardo–Nirenberg and logarithmic Hardy–Littlewood–Sobolev inequalities, which are then exploited to describe the global behavior in time, and in particular, to obtain a quantitative rate of convergence to a steady state for the critical mass KS equation.

Another field of interest for Figalli is random matrices. Large random matrices arise as a natural model in diverse fields such as quantum mechanics, quantum chaos, telecommunications, finance, and statistics. The central mathematical question in this area is how much their asymptotic properties depend on the fine details of the model. It dates back to the work of Wigner in the 1950's that proved that the macroscopic distribution of the spectrum converges under very mild assumptions to the so-called semi-circle law. Local properties of the spectrum took much longer to be revealed, as they are much harder to study. Recent remarkable breakthroughs, proved that under rather general assumptions, the local statistics of a Wigner matrix are indeed independent of the precise distribution of its entries. About the universality of the fluctuations of the eigenvalues in several-matrix models, little was known except in very particular situations. Figalli, in collaboration with Guionnet and Bekelman, developed a new approach to these questions by introducing particular approximate transport maps. These maps provide us with powerful changes of variables between different laws, and through their use, Figalli and Guionnet succeeded in showing universality in perturbative several-matrix models. This is a real breakthrough, which in addition, gives a firm mathematical grounding to the

widely held belief coming from physics that a universality of local fluctuations holds, at least until some phase transition occurs.

Transport equations with rough coefficients play a prominent role in mathematical physics and continuum mechanics. The celebrated Di Perna–Lions theory relates the well-posedness of transport equations to the almost-everywhere existence and uniqueness of flows for very rough vector fields. This is a crucial result in continuum mechanics, as it allows one to relate the Eulerian and the Lagrangian descriptions of continua. Figalli has been working on this topic since the beginning of his career, obtaining several important results related to the Di Perna–Lions theory, which led to a theory sufficiently robust and general to be useful in applications. In particular, in collaboration with Ambrosio, Friesecke, Giannoulis and Paul, Figalli has addressed the semiclassical limit of the Schrodinger equation with very rough potential, a problem of severe mathematical difficulty. They have shown that even in very singular situations such as for the Born–Oppenheimer quantum molecular dynamics, the evolution of particles converge to the classical Hamiltonian dynamics when the Planck constant tends to zero.

Figalli has fundamental contributions to the subject of the fractional obstacle problem arising from American options pricing. An American option gives an agent the possibility to buy a given asset at a fixed price at any time before the expiration date. The payoff of this option is a random variable that will depend on the value of this asset at the moment the option is exercised: If is a stochastic process which represents the price of the assets, the optimal choice of the moment to exercise the option corresponds to an optimal stopping problem for this process. The results that Figalli has obtained with his coauthors give for the first time, under natural assumptions on the payoff, a complete geometric description of the exercise region for a large class of stochastic processes. Currently, Figalli is applying to this problem very deep techniques coming from geometric measure theory, in order to understand the full geometric structure on the exercise region when no structural assumptions on the payoff are made. This is another case in which Figalli is able to employ techniques from very different areas of mathematics in order to obtain completely new and deep results in a new field.

Minimal surfaces, and more generally constant mean curvature surfaces, are one of the central objects of study in geometry. Some time ago, Souganidis and myself, in a paper on fractional diffusion processes, introduced new notions of minimal surface and of constant mean curvature surface by looking at critical points of fractional/nonlocal surface energies. The challenge of understanding the similarities and the differences between the classical and the nonlocal notions of minimal surface has become a central topic. Figalli and his collaborators have recently established a fundamental result in the nonlocal theory, by showing the validity of Alexandrov's theorem (bounded constant mean curvature implies being a sphere) in the nonlocal setting. Even more interestingly, they have quantitatively shown that surfaces with almost constant nonlocal mean curvature are close to be spheres, where the corresponding result in the classical setting is known to be false. This shows a remarkable rigidity property of the nonlocal theory in comparison to the local theory.

To wrap up, we just mention without entering in detail two very recent, beautiful pieces of work by Figalli:

The proof of the De Giorgi conjecture for the half Laplacian in spaces of dimension less or equal than four and the higher regularity description for the singular set of an obstacle problem.

To close, Figalli's work is of the highest quality in terms of originality, innovation and impact both on mathematics per se as well as on its applications. He is clearly a driving force in the global mathematics community today. His approach to research is lively, dynamic and effective, and without a doubt will lead him to achieve many more stunning discoveries in years to come.

Alessio Figalli is bound to be one of the most influential mathematicians of his generation.

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ALI NESIN AND THE NESIN MATHEMATICS VILLAGE

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Abstract

The Leelavati Prize of the International Mathematical Union was awarded during the ICM 2018 in Rio de Janeiro to Ali Nesin for his outstanding contributions towards increasing public awareness of mathematics in Turkey. We review Nesin's outreach activities, focusing on its founding and development of the "Mathematics Village" as an extraordinary, peaceful place for the exploration of mathematical knowledge dedicated to improving the understanding of mathematics of gifted students at all levels, in the inspiring environment and stimulating atmosphere of a camp.

Figure 1: Ali Nesin

Citation: Ali Nesin has been awarded with the Leelavati Prize 2018 in recognition of his outstanding contributions and great achievements towards increasing public awareness of mathematics in Turkey, especially because of his tireless work in creating and

Extended version of the Plenary Talk at ICM 2018 in Rio de Janeiro.
MSC2010: primary 97A80; secondary 97D40.

developing the "Mathematics Village" as an exceptional, peaceful place for education, research and the exploration of mathematics for a wide range of people.

Short Scientific CV (Research)

Hüseyin Ali Nesin was born in Istanbul in 1957; he is a Turkish citizen. After Junior high school in Turkey he went to high school in Switzerland and studied mathematics at Paris VII in France. In 1981 he went to the USA and studied as a research assistant at Yale University, where he made his PhD in 1985.

His early academic career is not unusual for a gifted mathematician with scientific ambitions. He held positions at several major universities in the United States, namely Berkeley, Notre Dame, and Irvine. He was an Associate Professor at the University of California at Irvine from 1991 through 1996. During this period his research focused on algebra, in particular on the interplay of mathematical logic, model theory and group theory, which has remained at the center of his research activity. Since 1996 until today he is Professor and Chair of the Mathematics Department at Bilgi University in Istanbul.

He has published 37 research articles in leading mathematical journals and the highly cited research monograph "Groups of Finite Morley Rank", together with Alexandre Borovik.

Return to Turkey

Ali Nesin's career took a significant turn on the death of his father, Aziz Nesin, a legendary Turkish writer of over 100 humorous and satirical books, in 1995. Aziz Nesin had assigned the earnings from all rights in his published works to a charitable foundation, the Nesin Foundation, which he had established for the education of orphans and children from destitute families. On the death of his father Aziz in 1995, Ali Nesin decided, in place of pursuing his academic career in the US, to return to Turkey, in the first instance to ensure the continuation of the work of the foundation. In that context he decided to take the opportunity to commit himself whole-heartedly to the enhancement of understanding of mathematics among the youth and to mathematical outreach in Turkey, with a view of mathematics as an essential force for the economic, social, and cultural development of the country.

His first step in this direction was the organization and direction of the department of mathematics at a newly established university in Istanbul, Bilgi university. In order to bring the undergraduate students in this department quickly to the international level, he instituted a program of instruction outside the regular university term that led in 2007 to the formal establishment of a Mathematics Village, which is now the publicly best known activity of the foundation. At the same time, he undertook to develop a number of outreach activities that were unprecedented in Turkey.

Since his return to Turkey Ali Nesin has devoted his life almost entirely to his mission, to developing the appreciation of mathematics as an element of modern Turkish culture.

Figure 2: Ali Nesin working with elementary school students

Nesin's Outreach Program

Ali Nesin's activities towards the awareness of mathematics in Turkish society are numerous. He initiated the following outreach activities, directed at the Turkish public generally as well as students of mathematics at the high school level and beyond.

- He was from 2003 to 2013 editor-in-chief of *Matematik Dünyası* ("The World of Mathematics"), a Turkish language monthly magazine for the popularisation of mathematics, mainly addressed to high school students. This wide-ranging publication has presented many fundamental notions of mathematics to its public, ranging from the foundations of mathematics, the theory of limits, the derivative, and the integral, to such topics as graph theory, the theory of groups, and p-adic numbers. Each issue sold around 20,000, something very rare for such a journal.

- He established his own publishing house, publishing a range of mathematical texts in Turkish, including nine of his own works of popularization (e.g., *Mathematics and Games, Mathematics and Nature, Mathematics and Infinity, Mathematics and Truth*) as well as an on-going series representing curriculum development at the Mathematics Village.

- He prepared open source courseware materials in Turkish available on-line through the Turkish Mathematical Society.

- He authored many popular mathematical articles in Turkish, appearing in periodicals aimed at a national audience, in addition to those in his own magazine.

- He founded, staffed, and chaired the Department of Mathematics at Bilgi University, Istanbul, and developed and taught a supplementary level enrichment program for undergraduates held outside the regular school term. It was designed to bring the students quickly to a competitive international level and developed into the nucleus of the Mathematics Village after 2007.

- Approximately 7000 videos relating to the activities of the Mathematics Village have been posted to YouTube, consisting in many cases of extended lectures by

Ali Nesin to middle school students participating in the Mathematics Village, as well as a public lecture given at Gezi park in central Istanbul. In total these have been viewed several million times.

Figure 3: Typesetting a number of *The World of Mathematics* (2009)

In order to reach its intended audience, the bulk of this material is in Turkish, though a graduate summer program at the Mathematics Village is conducted in English, with the active participation of instructors from the international community.

Ali Nesin's open source courseware received a prize from the Turkish Mathematical Society for expository excellence in 2010. This consists of six textbooks: *The Construction of Number Systems I, II; Axiomatic Set Theory I,II; Fundamentals of Analysis I,II.* He has also won prizes for the following textbooks: *Intuitive Set Theory (1st prize, 2011); Counting (honorable mention, 2011); Analysis IV (Topology), 2012.* Again, these materials are in Turkish and are aimed at high school students and undergraduates.

Nesin is currently developing a full high school level curriculum which will be based on 23 textbooks, of which three have been published to date.

His books, popular journals, editorial work, articles, public lectures, and internet activities had an enormous influence on enabling a new generation of Turkish mathematicians to flourish.

The idea of a Mathematics Village

However, what makes the work of Nesin unique and goes beyond all envisaged activities for the Leelavati Prize, is his creation, organization and development of the Mathematics Village, in face of considerable financial, bureaucratic, and practical difficulties and against ideologically motivated resistance.

Figure 4: The Nesin Mathematics Village - panoramic view

The Nesin Mathematics Village is located on a physical site south of Izmir, one kilometer from the village of Şirince in the region of the ancient Greek town of Ephesus. This site is owned by the Nesin Foundation and is dedicated entirely to teaching and learning of mathematics on a non-profit basis. It is fully devoted to the enhancement of understanding of mathematics of gifted students at all levels, from middle school to graduate level. However, it is not a competition to the official education system, but a supplement that gives students access to knowledge that is free from examinations, in the inspiring environment and stimulating atmosphere of a summer camp. Moreover, it has become an important venue for international conferences.

When Nesin first returned to Turkey, the newly founded Bilgi University in Istanbul offered him a position as the head of the Mathematics Department. He accepted with the idea to establish this department at the highest possible level and to create something extraordinary, comparable with the top centers of mathematics in the world. The courses were very advanced because they aimed to offer an undergraduate degree program in line with international standards, and in particular to ensure that graduates of the program would be strong candidates for graduate programs at major universities around the world.

Nesin quickly realized that the students in Turkey required additional preparation for such an education. He first organized additional lectures at night in his own house and at the Foundation on weekends. But soon he found it necessary to organize summer camps housed in tents in various rural locations in Turkey, which soon attracted talented undergraduates from all over Turkey in addition to those from his department.

Mathematically these lectures were very successful, but the organization of the workshops at varying venues was very difficult both financially and organizationally. Yet, seeing the results for his students and in view of his own understanding of the importance of mathematical sciences, Nesin came to the conclusion that a new model of education was a necessity for the proper development of Turkish mathematics, and indeed for the Turkish society. He has expressed the view that Turkey's existing strength in many aspects of engineering lacks a sufficient basis for technological innovation and that without fundamental sciences no innovation is possible[1].

At that time the idea of the village project slowly began to take shape.

Figure 5: Inside the village

Construction of the Village

In 2007, together with a close friend, the self-educated architect (and also a prominent Turkish linguist) Sevan Nişanyan, Ali undertook the construction of the village, initially on a small scale. He had bought the land in a nice but remote area adjacent to the small village Şirince near Selçuk, Izmir. It was a great idea but also a daring adventure to start such an endeavor. The architectural challenges were to build a village nestled organically in the landscape of a deserted area, and to transform the site into an inspiring environment aligned with the spirit of a new way of conveying mathematical understanding to Turkish youth.

Not counting the numerous practical problems, the bureaucratic difficulties to create the village were even greater. For example, the authorities refused official building permit because the land had no officially registered "street". Ali tried very hard to register the already existing respectable path into an official street, but he failed. The building

[1] Video at YouTube in Turkish with English subtitles

permit was never denied, but it was also not granted. After some time, he gave up the idea of taking the permission to build and started the construction. On the other hand he was accused of having founded an illegal educational institution and of "teaching without permission", contrary to freedom of teaching of sciences guaranteed by Turkish constitution. However, the Mathematics Village as a nonprofit organization is not an educational institution - the most important feature of the Mathematics Village is that it is *not* a school in any accepted meaning of the word. Young people come there only for a week or two. There are no examinations or grades, class failures or punishments, and no diplomas or transcripts of any kind are issued. It is a place for understanding, thinking and reflection, this is why there is no distraction by television, music and excessive noise. Nevertheless, the Mathematics Village was raided by Gendarmes and sealed. Fortunately the case was dropped and Ali was saved from prison, and the Village was reopened. Since then, the Mathematics Village flourished.

Part of the resistance to the Mathematics village stemmed perhaps from the antipathy to the name Nesin, for Ali's father was not only a famous man of letters, he was also a well-known leftist and an avowed atheist, a controversial view in the religious-conservative part of Turkish society. In 2014, Sevan Nişanyan, who is also a prominent atheist, was imprisoned on politically motivated trump-up charges. He fled the prison in 2017 and lives now in Greece. On the other hand, Ali just promotes teaching and learning of mathematics, all his actions are non-ideological.

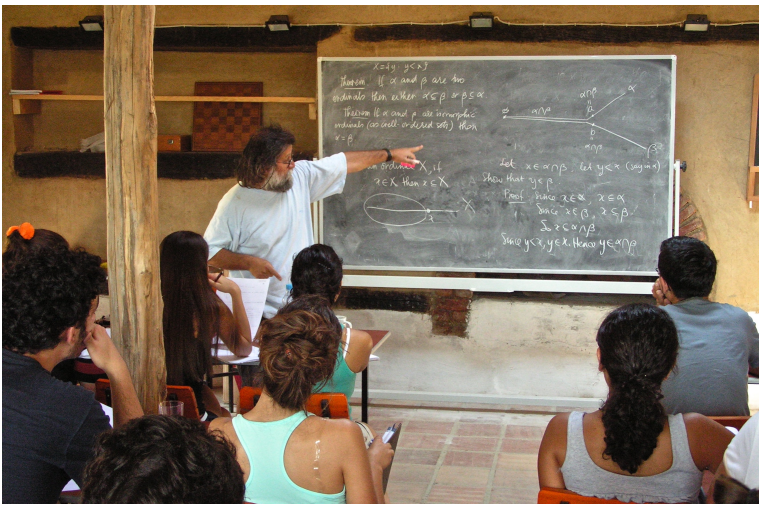

Figure 6: The Langlands Shed, 2007

The above mentioned difficulties could not stop Ali from realizing the dream of a place where mathematical sharing and learning was possible in an informal environment, free from all constraints and only dedicated to enhancing mathematical understanding. He was able from the start to enlist support from his colleagues and students not only within Turkey but within the international community. Lectures were given

Figure 7: The Sevan Nişanyan Library, today

initially in the so-called "Langlands Shed," named in honor of Robert Langlands, who had generously donated a significant portion of his Shaw prize to the foundation.[2] The continuing support of the international mathematics community in giving advanced intensive lectures on a voluntary basis during the graduate summer programs has also played an important role in the success of the project. In addition there was brief support from TUBITAK, the Turkish Scientific and Technological Research Council, and a very large number of mostly modest donations from the Turkish public have funded the construction and expansion of the Village.

In spite of the absence of external funding, the operation of the Mathematics Village is now self-sustaining, with public donations used only for the purpose of expansion and development.

Developing the Village

In the first year only about 100 students participated in the summer program, which then operated only at the undergraduate and graduate level but not at the high school level, with teaching provided by Ali and a few enthusiastic mathematicians, mostly Turkish, who volunteered their time and expertise. The Mathematics Village began to grow and prosper with the years. The building complex expanded and more and more students could be admitted. Turkish graduate students also volunteered to give courses in the village and in return the students benefited from discussions with colleagues and professors from all over, and from the relaxed and intense working atmosphere and the absence of distraction. In addition to university students, high school children from the age of fourteen were also admitted to the two-week camps during summer holidays, and soon the project enjoyed such a good reputation that it was considered an honor to be

[2]Langlands recently received the Abel Prize and again donated a significant portion of the prize to the Foundation.

admitted. Groups of younger children make short visits of one or two days and receive lectures by the Director. Many of these are now viewable as videos on YouTube.

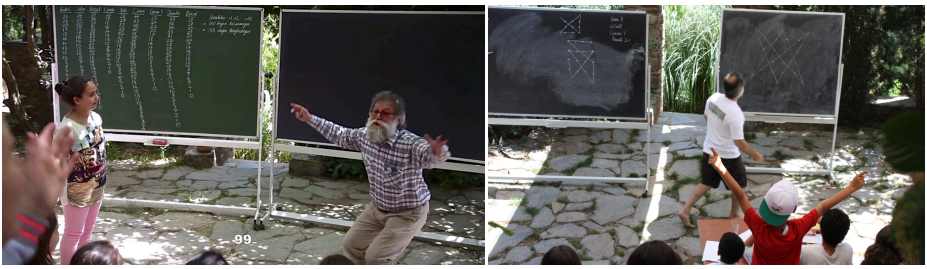

Figure 8: Ali's enthusiasm during open-air lectures is contagious

While normal high school education in Turkey is focused on university admission, with typical memorizing, the focus in the Mathematics Village is on communicating, understanding and independent thinking. This philosophy, which focuses entirely on the exchange of ideas between equals, makes up for the success of the Mathematics Village.

Teaching and Learning

The university level courses are organized according to topics. They enable concentration on research interests as well as cooperation between students and colleagues in the same or related fields. The range of subjects is large, ranging from Lie algebras to Fourier analysis, from measure theory to representation theory of groups, and includes interdisciplinary topics between mathematics, philosophy, physics and computer science.

Figure 9: Lectures in the Village

All teaching in the Village is voluntary and unpaid, in return the accommodation and meals for the lecturers are free. Most of the courses are in Turkish, but some are offered also in English. The Village attracts the best Turkish teachers and promoters

of mathematics, as well as mathematicians from all over the world. Many lecturers are former students of previous Summer Programs in the Village.

The two-week cycles begin and end on Sundays with the departure and arrival of hundreds of students. About fifteen paid employees and nearly one hundred volunteers work there every year.

Figure 10: Library conference hall and joint dining area

The Village comprises now more than 35,000 square meters, approximately half of it consists of olive groves. The complex of buildings consists at the moment of 16 bedrooms, two amphitheaters, four (including the library hall) closed and four open-air lecture halls, two Turkish baths, one for men and one for women, twenty-nine single or double rooms, a fully functional kitchen and cafeteria, a small shop and a wonderful two-story library with a cool, airy terrace in front of it. The village has the capacity to accommodate 150 people, but there is the option of pitching tents if more capacity is needed. Sometimes the place is overflowed with about 400 students.

The program for 2018 lists the following events on the English web page:

- 15-21 January: Week of Theoretical Physics
- 16-20 May: Antalya Algebra Days XX
- 21-25 May: Workshop on Integer Partitions
- 04-08 June: Randomness in Complex Geometry and Complex Analysis
- 18-29 June: Categories and Toposes and non-commutative Geometry
- 09 July-23 September: TMD Undergraduate and Graduate Summer School
- 30 July-05 August: High School Philosophy Summer School
- 02-04 November: International Aegean School of Human Rights

However, the Turkish web page lists another 46 (!) events, such as computer programming courses and physics schools but also in completely different areas like seminars on cinema, architecture, arts, history, politics and a freedom philosophy summer school.

Basic Principles

In 2014 Nesin had expressed his ideas to expand the Mathematics Village by constructing adjacent Philosophy and Art Villages with the words: “The whole valley should be dedicated to education, not a standard one but a ’pirate’ one”.

This idea has been put into action, and the Nesin Mathematics Village is now a cultural magnet, and even a tourist attraction whose architectural principles have become fashionable in Turkey. An independent Theatre School has also been established on an adjacent site, also with the assistance of Sevan Nişanyan.

The governing aim of all these events are essentially the same: access to knowledge, education, and freedom based on Nesin’s principles of

safety — independence — responsibility,

derived from his experiences as director of the Nesin Foundation. Quoting Ali: “It is not possible to have a proper education in an environment without freedom. You can give an average education in an environment with restricted freedom, but not a proper one.”[3]

Figure 11: Study in the Village

Many of the younger children arriving at the Village for the first time are excited to discover that the pursuit of mathematical understanding is a communal activity of international scope which may continue throughout one’s life. Genders and ages are freely mixed and a certain number of older students mentor the younger ones throughout their stay.

The program itself is very intense. The younger students receive from 2 to 4 hours of lectures from the Director in one day; he gives a total of over 1000 hours of lectures per year across all programs. Attendees at the summer programs take 8 hours of lecture per day, with each session lasting one or two weeks, depending on the topic.

A few numbers

- There were 10,379 visits by children at the age of 13 or older as part of organized groups in 2017.

[3] Video at loc. cit.

Figure 12: Lecture in an open-air theater

- The fees for the high school program are $25 per day, reduced or waived whenever there are financial difficulties.

- In 2017 scholarships to the total amount of $266,688 were awarded, in programs whose total cost was $528,364, and whose net receipts from fees were $400,054. This would appear to leave the Village running at a deficit, but the Village runs additional programs for preparation for university entrance exams which make up the difference.

- Over the 10 years that the high school program has been in operation, 50.6% of the participants have been female[4].

- The applications process is closed when the number of applications received amounts to three times the total capacity of the Village. The director reviews all applications up to this limit, a total of over 5,000 annually. The principal criterion for acceptance is a desire on the part of the student to study mathematics at the Village (as opposed, for example, to parental pressure, which is also common).

- The holdings of the Sevan Nişanyan Library at the Mathematics Village are currently 15,000 books at a wide range of levels.

[4]An OECD report from 2017 notes that among the countries studied, the most severe gender gaps in education are found in Turkey and in Japan (30% by their measure).

Figure 13: Sign my math book please!

Conclusion

Ali Nesin's achievements in terms of popularization and dissemination of mathematics in Turkey and his commitment to the mathematical training of the youth are unprecedented. The creation of the Mathematics Village as a place of undisturbed exchange of ideas in intellectual freedom and the imparting of mathematics as a fundamental science for innovation has a truly transformative impact on the perception of mathematics as a significant part of its culture in Turkish society.

According to the statutes of the Leelavati Prize, the prize is awarded to a person "in recognition of outstanding contributions for increasing public awareness of mathematics as an intellectual discipline and the crucial role it plays in diverse human endeavors." It is hard to imagine that someone else has earned this award more than Ali Nesin.

Acknowledgments. I would like to thank Alexandre Borovik and Gregory Cherlin for sharing their knowledge about Ali Nesin and the Mathematics Village and the administration of the Nesin Mathematics Village for providing recent financial and operational data. Furthermore I have made use of the articles Alladi and Nesin [2015] and Karaali [2014].

Photo credits.

Figs. 1, 2, 5-1, 8-1, 9-1, 11-1, 13: Screenshots from YouTube Video *Matematik Köyü (Mathematics Village)* by Ayser Sude Gök (2014), taken at minutes 0:08, 3:10, 0:54, 8:04, 4:11, 9:31, 10:58.

Figs. 4, 7, 9-2, 10-1, 10-2, 11-2: Photographs by Burak Barutcu, courtesy Aslı Can Korkmaz.

Figs. 3, 5-2, 6, 8-2, 12: Photographs by Alexandre Borovik.

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CAUCHER BIRKAR'S WORK IN BIRATIONAL ALGEBRAIC GEOMETRY

Christopher D. Hacon

Abstract

On Wednesday, August 1st 2018, Caucher Birkar was awarded the Fields Medal for his contributions to the minimal model program and his proof of the boundedness of ϵ-log canonical Fano varieties. In this note I will discuss some of Birkar's main achievements.

Algebraic geometry studies the solution sets of systems of polynomial equations. A complex projective variety is a subset of $\mathbb{P}^N_{\mathbb{C}}$ defined by homogeneous equations

$$X = V(P_1, \dots, P_r) \subset \mathbb{P}^N_{\mathbb{C}}$$

where $P_i \in \mathbb{C}[x_0, \dots, x_N]$ are homogeneous polynomials of degree d_i. Here $V(P_1, \dots, P_r)$ is the vanishing set of the polynomials P_i and

$$\mathbb{P}^N_{\mathbb{C}} := \left(\mathbb{C}^{N+1} \setminus \{(0, \dots, 0)\}\right)/\mathbb{C}^*$$

is N-dimensional complex projective space. Two non-zero vectors $\bar{c}$, $\bar{c}' \in \mathbb{C}^{N+1} \setminus \{(0, \dots, 0)\}$ are equivalent $\bar{c} \sim \bar{c}'$ if there is a non-zero constant $\lambda \in \mathbb{C}^*$ such that $\bar{c} = \lambda \bar{c}'$. If $\bar{c} = (c_0, \dots, c_N)$, then the corresponding equivalence class is denoted by $[c_0 : \dots : c_N] \in \mathbb{P}^N_{\mathbb{C}}$. Typically we assume that X is irreducible (i.e. not the union of two proper subvarieties) and reduced so that if $P \in \mathbb{C}[x_0, \dots, x_N]$ is an homogeneous polynomial vanishing along X, then P belongs to the ideal $(P_1, \dots, P_r) \subset \mathbb{C}[x_0, \dots, x_N]$. We say that a variety is non-singular (of dimension d) at a point $x \in X$ if locally analytically it is isomorphic to an open subset of $\mathbb{C}^d$. A variety X is smooth if it is non-singular at all of its points. In this case we can view X as a complex manifold. It is natural to try to classify smooth complex projective varieties up to isomorphism. The most natural invariant to consider is the dimension of a given variety. If $\dim X = 0$, then X is a point, so the first interesting case to consider is when $\dim X = 1$. In this case we say that X is a curve. Curves are compact orientable Riemann surfaces which are topologically classified by their genus $g = \dim H^0(\Omega^1_X)$. There are 3 main cases to consider:

- $g = 0$: In this case $X \cong \mathbb{P}^1_{\mathbb{C}}$.

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- $g = 1$: In this case X is an elliptic curve, belonging to a 1 parameter family $X_\lambda = V(x(x-z)(x-\lambda z) - zy^2) \subset \mathbb{P}^2_{\mathbb{C}}$ where $\lambda \in \mathbb{C} \setminus \{0, 1\}$.
- $g \geq 2$: In this case we say that X is a curve of general type. For fixed g there is a $3g-3$-dimensional algebraic family of these curves.

Interestingly, the above subdivision in to 3 cases captures many of the properties of curves in a variety of contexts such as topology, differential geometry, and number theory.

Naturally, one would like to extend these classification results to higher dimensions. One of the main tools used in the study of projective varieties is the canonical line bundle

$$\omega_X = \wedge^{\mathrm{dimX}}(T_X^\vee).$$

Note that if $\dim X = 1$, then $\omega_X = \Omega^1_X$ and so $\dim H^0(\omega_X) = g$ recovers the genus of the curve X. In higher dimensions it is better to consider the *canonical ring*

$$R(\omega_X) := \bigoplus_{m \geq 0} H^0(\omega_X^{\otimes m}).$$

For any $m > 0$ such that $H^0(\omega_X^{\otimes m}) \neq 0$ we obtain a rational map

$$\phi_m : X \dashrightarrow \mathbb{P}^N, \qquad x \dashrightarrow [s_0(x) : \ldots : s_N(x)]$$

where $s_0, \ldots, s_N$ is a basis of $H^0(\omega_X^{\otimes m})$ and the map is defined at every point such that $s_i(x) \neq 0$ for some $0 \leq i \leq N$.

When $\dim X = 1$ we have that:

- $g = 0$: $\omega_X = \omega_{\mathbb{P}^1_{\mathbb{C}}} \cong \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-2)$ so that $R(\omega_X) \cong \mathbb{C}$.
- $g = 1$: $\omega_X \cong \mathcal{O}_X$ so that $R(\omega_X) \cong \mathbb{C}[t]$.
- $g \geq 2$: ω_X is a line bundle of degree $2g-2 > 0$ and in fact $\omega_X^{\otimes k}$ is very ample for $k \geq 3$ which means that $\phi_k : X \hookrightarrow \mathbb{P}^N_{\mathbb{C}}$ is an embedding.

Not surprisingly, in higher dimensions, $\dim X \geq 2$, there are many additional difficulties. One important issue is that given any smooth variety X and any smooth subvariety $Z \subset X$ of codimension ≥ 2, one can define $f : X' := \mathrm{bl}_Z(X) \to X$ *the blow up of* X *along* Z which is an isomorphism over $X \setminus Z$ and

$$E := f^{-1}(Z) \cong \mathbb{P}(N_Z X) \to Z$$

is a fiber space whose fibers are projective spaces $\mathbb{P}^c_{\mathbb{C}}$ of dimension $c = \mathrm{codim}_X(Z) - 1$. We say that E is an *exceptional divisor*, i.e. a codimension 1 subvariety contained in the locus where f is not an isomorphism. Note that X and $\mathrm{bl}_Z(X)$ share many important geometric properties. For example, it is easy to see that they have isomorphic canonical rings $R(\omega_X) \cong R(\omega_{X'})$ and fundamental groups $\pi_1(X) \cong \pi_1(X')$, however they are not homeomorphic. For this reason, it is natural to consider the equivalence relation generated by this operation. We say that two varieties X and X' are *birational* if they

have isomorphic (non-empty) open subsets $U \subset X$ and $U' \subset X'$ such that $U \cong U'$. Equivalently one sees that X and X' are birational if and only if they have isomorphic fields of rational functions $\mathbb{C}(X) \cong \mathbb{C}(X')$. The main goal of birational geometry is to classify projective varieties up to birational isomorphism.

The strategy is as follows. Starting with a (irreducible and reduced) variety $X \subset \mathbb{P}^N_{\mathbb{C}}$ of dimension $d := \dim X$, by Hironaka's theorem, there exists $X' \to X$ a finite sequence of blow ups along smooth centers of X, such that X' is smooth and birational to X. Thus we may assume that X is smooth and consider the canonical ring $R(\omega_X) = \bigoplus_{m \geq 0} H^0(\omega_X^{\otimes m})$. The *Kodaira dimension* of X is given by

$$\kappa(X) := \text{tr.deg.}_{\mathbb{C}} R(\omega_X) - 1 \in \{-1, 0, 1, \dots, \dim X\}.$$

Equivalently, we have that $\kappa(X)$ is the maximum dimension of the image of X under the pluricanonical maps ϕ_k for $k \in \mathbb{N}$. For example if $\dim X = 1$, then $\kappa(X) = -1, 0,$ or 1 is equivalent to $g(X) = 0$, 1, or ≥ 2, which recovers the above subdivision in to three separate cases.

For a simple higher dimensional example, consider $X_k \subset \mathbb{P}^N_{\mathbb{C}}$ a smooth hypersurface of degree k. Then $\omega_{X_k} \cong \mathcal{O}_{\mathbb{P}^N_{\mathbb{C}}}(k - N - 1)|_{X_k}$ where $\mathcal{O}_{\mathbb{P}^N_{\mathbb{C}}}(l)$ is the line bundle on $\mathbb{P}^N_{\mathbb{C}}$ corresponding to homogeneous polynomials of degree l and $\mathcal{O}_{\mathbb{P}^N_{\mathbb{C}}}(l)|_X$ is its restriction to a subvariety $X \subset \mathbb{P}^N_{\mathbb{C}}$. It is then easy to see that there are three cases depending on whether ω_X is negative, zero or positive.

- $k \leq N$: we have $R(\omega_{X_k}) \cong \mathbb{C}$ so that $\kappa(X) = -1$,
- $k = N + 1$: we have $R(\omega_{X_k}) \cong \mathbb{C}[t]$ so that $\kappa(X) = 0$, and
- $k > N + 1$: then ω_{X_k} is very ample so that $\kappa(X) = \dim X$.

Further examples that include all possible values of the Kodaira dimension $\kappa(X) \in \{-1, 0, 1, \dots, \dim X\}$ can be easily constructed by considering the product of two varieties $X \times Y$. If $\min\{\kappa(X), \kappa(Y)\} = -1$, then $\kappa(X \times Y) = -1$. Otherwise, $\kappa(X \times Y) = \kappa(X) + \kappa(Y)$.

One of the most natural and important questions in birational algebraic geometry is, for any birational equivalence class, to produce a distinguished representative with good geometric properties. In dimension 2 this was achieved by the Italian school of algebraic geometry at the beginning of the 20-th century and in dimension 3 it was finally proven by S. Mori and others in the 1980's Mori [1988]. Around the same time the minimal model program (MMP) was established by work of Y. Kawamata, J. Kollár, S. Mori, M. Reid, V. Shokurov and others.

The MMP is a general framework that aims to extend the birational classification results in dimension 2 and 3 to all dimensions. This program has not been completed in full generality, but there has been some recent spectacular progress. Perhaps the most exciting result in this direction is the following result on the finite generation of canonical rings.

Theorem 1 (Birkar, Cascini, Hacon, and McKernan [2010], Hacon and McKernan [2010], and Siu [2008]). *If X is a smooth complex projective variety, then its canonical ring $R(\omega_X)$ is finitely generated.*

Of course this answers one of the most natural questions about the canonical ring, but it also has a number of important consequences that have greatly improved our understanding of birational geometry. For example we have the following.

Corollary 2 (Birkar, Cascini, Hacon, and McKernan [2010]). *If X is a smooth complex projective variety of general type (so that $\kappa(X) = \dim X$), then X has a canonical model X_{can} and a minimal model X_{min}.*

The canonical model is a unique distinguished representative of the birational equivalence class of X which is given by

$$X_{\text{can}} := \operatorname{Proj} R(\omega_X).$$

One can think of X_{can} as being defined by the generators and relations of the canonical ring $R(\omega_X)$.

Unfortunately, there are many examples where X_{can} is (mildly) singular. The singularities arising in this context are called canonical singularities. It is known that canonical singularities are rational so that in particular $H^i(\mathcal{O}_X) \cong H^i(\mathcal{O}_{X_{\text{can}}})$ for $i \geq 0$. It is also known that canonical singularities are $\mathbb{Q}$-Gorenstein. This means that even though $\omega_{X_{\text{can}}}$ is not necessarily a line bundle, it is a $\mathbb{Q}$-line bundle so that $\omega_{X_{\text{can}}}^{\otimes k}$ is a line bundle for some integer $k > 0$. Never the less we still abuse notation and refer to $\omega_{X_{\text{can}}}$ as the canonical line bundle. For example let V be the quotient of $\mathbb{C}^3$ via the involution $(x, y, z) \to (-x, -y, -z)$, then V has canonical singularities and $\omega_V^{\otimes m}$ is a line bundle if and only if m is an even integer.

Another important feature is that $\omega_{X_{\text{can}}}$ is ample so that $\omega_{X_{\text{can}}}^{\otimes k}$ is a very ample line bundle for any sufficiently big and divisible integer $k > 0$. This means that $\phi_k : X \to \mathbb{P}^N = \mathbb{P}(H^0(\omega_{X_{\text{can}}}^{\otimes k}))$ is an embedding and $\omega_{X_{\text{can}}}^{\otimes k} = \phi_k^* \mathcal{O}_{\mathbb{P}^N}(1)$. This gives an easy geometric interpretation of why $R(\omega_X)$ is finitely generated. In fact for any sufficiently big and divisible integer $k > 0$ we may assume that the homomorphism

$$\bigoplus_{r \in \mathbb{N}} H^0(\mathcal{O}_{\mathbb{P}^N}(r)) \to \bigoplus_{r \in \mathbb{N}} H^0(\omega_{X_{\text{can}}}^{\otimes kr})$$

is surjective. Since $\mathbb{C}[x_0, \ldots, x_N] = \bigoplus_{r \in \mathbb{N}} H^0(\mathcal{O}_{\mathbb{P}^N}(r))$ is clearly finitely generated, it follows that its quotient $\bigoplus_{r \in \mathbb{N}} H^0(\omega_{X_{\text{can}}}^{\otimes kr})$ is also finitely generated. This ring is the ring of $\mathbb{Z}/k\mathbb{Z}$ invariants in the graded ring $R(\omega_{X_{\text{can}}})$ and so, by a theorem of E. Noether on the finite generation of the integral closure, one sees that $R(\omega_{X_{\text{can}}})$ is finitely generated.

The existence of canonical models plays a pivotal role in our understanding of varieties of general type. For example it is crucial in the construction of a projective moduli space parametrizing canonically polarized varieties (with canonical singularities) and their natural degenerations (see eg. Kollár and Shepherd-Barron [1988], Kollár [2013a,b], Hacon and Xu [2013], Hacon, McKernan, and Xu [2018a,b], and Kovács and Patakfalvi [2017]).

When $0 \leq \kappa(X) < \dim X$, X can not have a canonical model, however it is still expected that X has a minimal model X_{min} which has mild (terminal) singularities, such that $\omega_{X_{\text{min}}}$ is a nef $\mathbb{Q}$-line bundle. This means that the intersection numbers with any

curve $C \subset X$ are non-negative $c_1(\omega_{X_{\min}})\cdot C \geq 0$. In fact, it is expected that $\omega_{X_{\min}}^{\otimes m}$ is base point free for any $m > 0$ sufficiently divisible. Thus that $\phi_m : X \to Z := \mathrm{Proj}(R(\omega_X))$ is a morphism (i.e. the map is everywhere defined on X). If F is a general fiber of ϕ_m, then it is known that $\kappa(F) = 0$ and in fact we have $\omega_F^{\otimes m} = \mathcal{O}_F$. Therefore, to understand the geometry of varieties of intermediate Kodaira dimension, it is important to understand minimal models of varieties of Kodaira dimension 0 and their moduli spaces. This is a very active area of research but unluckily our knowledge of these varieties is still incomplete.

It is conjectured that minimal models can be obtained by a finite sequence of elementary birational maps

$$X = X_1 \dashrightarrow X_2 \dashrightarrow \ldots \dashrightarrow X_n = X_{\min}.$$

These elementary birational maps are called flips and divisorial contractions.

Divisorial contractions are birational morphisms $f : X \to X'$ of normal varieties such that $\dim \mathrm{Ex}(f) = \dim X - 1$, $c_1(\omega_X) \cdot C < 0$ for any f-vertical curve and $\rho(X/X') = 1$. Recall that codimension 1 subvarieties are called divisors. It is easy to see that X' also has mild singularities. In dimension 2 the picture is very explicit. If ω_X is not nef, then there exists a -1 curve $E \subset X$ such that E is rational $E \cong \mathbb{P}^1$ and $c_1(\omega_X) \cdot E = -1$. By a theorem of Castelnuovo, these curves can always be contracted by a morphism $f : X \to X_1$ such that X_1 is smooth and if $P = f(E) \subset X_1$, then $X = \mathrm{bl}_P(X_1)$.

Flipping contractions are birational morphisms $f : X \to Z$ of normal varieties such that $\dim \mathrm{Ex}(f) < \dim X - 1$, $c_1(\omega_X) \cdot C < 0$ for any f-vertical curve and $\rho(X/Z) = 1$. It is easy to see that Z does not have mild singularities, in fact ω_Z is not even a $\mathbb{Q}$-line bundle and so we can not replace X by Z. In this case it is necessary to replace X by the flip $f^+ : X^+ \to Z$ of $f : X \to Z$. This operation is given by $X^+ = \mathrm{Proj}_Z(R(\omega_X/Z))$. It is well defined because, by the relative version of Theorem 1, $R(\omega_X/Z) = \oplus_{m\geq 0} f_*\mathcal{O}_X(\omega_X^{\otimes m})$ is finitely generated over Z. (Note that the question is local over Z and so we may assume that Z is affine. In this case we can identify $f_*\mathcal{O}_X(\omega_X^{\otimes m})$ with $H^0(\omega_X^{\otimes m})$.) It is easy to see that the singularities of X^+ are also mild and thus we may replace X by X^+.

Repeating this procedure, we obtain a sequence of flips and divisorial contractions $X = X_1 \dashrightarrow X_2 \dashrightarrow \ldots$. It is hoped that after finitely many flips and divisorial contractions we obtain a minimal model $X_{\min} = X_n$ (recall that we are assuming $\kappa(X) \geq 0$).

The most important feature of flips and divisorial contractions is that they improve the geometry of X by removing some ω_X negative curves and replacing them with ω_X positive curves. If $X \to X'$ is a divisorial contraction, then it is easy to see that $B_2(X') = B_2(X) - 1$ and since $B_2(X) \in \mathbb{N}$, any sequence of divisorial contractions is finite. On the other hand if $X \dashrightarrow X'$ is a flip, then $B_2(X) = B_2(X')$ and so it is not clear that any sequence of flips is finite and in fact this turns out to be one of the most important and difficult open questions of the minimal model program.

When X is of general type ($\kappa(X) = \dim X$), we are unable to show that arbitrary sequences of flips are always finite, however we can show that certain sequences of flips (for the minimal model program with scaling) are always finite and this implies the existence of minimal models in this case (cf. Corollary 2).

Birkar has proven some of the strongest known results on the finiteness of flips.

Theorem 3 (Birkar [2011]). *Assume the termination of flips for all $(d-1)$-dimensional varieties. If $\dim X = d$ and $\kappa(X) \geq 0$, then there are no infinite sequences of K_X-flips.*

We refer the reader to Birkar [ibid.] for further details and more precise statements. We also note that if one just considers the minimal model program with scaling, then there is a much stronger result (cf. Birkar and Hu [2014]):

Assume the non-vanishing conjecture so that if X is non-uniruled, then $\kappa(X) \geq 0$. Then there are no infinite sequences of flips with scaling.

Recall that X is *uniruled* if it is covered by rational curves so that there is a dominant (surjective on to an open subset) rational map $\mathbb{P}^1_{\mathbb{C}} \times W \dashrightarrow X$. We may assume that $\dim(\mathbb{P}^1_{\mathbb{C}} \times W) = \dim X$. It is then easy to see that $H^0(\omega^{\otimes m}_{\mathbb{P}^1_{\mathbb{C}} \times W}) \geq H^0(\omega^{\otimes m}_X)$ for $m \geq 0$. Since $\omega^{\otimes m}_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(-2m)$, it follows easily that $H^0(\omega^{\otimes m}_{\mathbb{P}^1_{\mathbb{C}} \times W}) = 0$ and hence also $H^0(\omega^{\otimes m}_X) = 0$ for all $m > 0$. The converse statement that *non-uniruled varieties have non-negative Kodaira dimension so that $H^0(\omega^{\otimes m}_X) \neq 0$ for some $m > 0$* is the non-vanishing conjecture. This is a very difficult and important open question closely related to the famous abundance conjecture.

Thus, conjecturally, the case of negative Kodaira dimension $\kappa(X) = -1$ corresponds to the case of uniruled varieties. There is a very precise structure theorem for these varieties.

Theorem 4 (Birkar, Cascini, Hacon, and McKernan [2010]). *Let X be a uniruled variety. Then there is a finite sequence of flips and divisorial contractions $X \dashrightarrow X'$ and a morphism $f : X' \to Z$ such that $\dim X' > \dim Z$, $\rho(X'/Z) = 1$, $c_1(\omega_{X'}) \cdot C < 0$ for any curve C contained in a fiber of f.*

The morphism $f : X' \to Z$ is called a *Mori fiber space*. The fibers F of f are *Fano varieties* with terminal singularities so that ω_F^{-1} is an ample $\mathbb{Q}$-line bundle. Fano varieties are well understood. They play an important role in algebraic geometry and many related subjects. For example it is known that they are simply connected.

By the above theorem we may think of Fano varieties as the building blocks for uniruled varieties. Therefore, to understand the geometry of uniruled varieties, it is important to understand the behaviour of families of Fano varieties. The most basic question is if Fano varieties with terminal singularities of a given dimension are *bounded* and hence are described by finitely many parameters. More precisely, do d-dimensional Fano varieties with terminal singularities appear as fibers of finitely many algebraic families of projective varieties? This is a very important intensely studied long standing question which is known to have many applications in birational geometry and other related areas. The boundedness of Fano varieties is known in many interesting cases, including the following cases.

1. In dimension 1, the only Fano variety is $\mathbb{P}^1$.

2. In dimension 2, terminal singularities are smooth and it is known that there are 10 families of smooth Fano varieties.

3. In dimension 3, it is known by work of Mori and Mukai and Iskovskih that there are 105 families of smooth Fano 3-folds (Iskovskih [1977] and Mori and Mukai [1981/82]) and that there are many more families of terminal Fano threefolds.

4. Smooth Fano varieties in any dimension are bounded by results of Nadel, Campana, Kollár, Mori and Miyaoka in the early 1990's (Nadel [1991], Campana [1991], and Kollár, Miyaoka, and Mori [1992]).

5. The boundedness of terminal Fano 3-folds was shown by Kawamata [1992] and for canonical Fano 3-folds by Kollár, Miyaoka, Mori, and Takagi [2000].

6. The boundedness of ϵ-log terminal toric Fano varieties was shown by A. A. Borisov and L. A. Borisov [1992].

7. The boundedness of ϵ-log terminal Fano surfaces was shown by Alexeev [1994].

8. The boundedness of ϵ-log terminal Fano varieties of bounded Cartier index, conjectured by Batyrev, was shown by Hacon, McKernan, and Xu [2014].

Based on their results, V. Alexeev, A. Borisov and L. Borisov conjectured that the set of ϵ-log terminal Fano varieties of dimension d is bounded. This conjecture is known as the BAB conjecture and it was probably the most important and influential conjecture in birational geometry concerning Fano varieties. Using many of the recent results related to the minimal model program and the existence of a moduli functor for log canonically polarized varieties, in his ground breaking papers Birkar [2016a,b], Caucher Birkar proves the BAB Conjecture in full generality.

Theorem 5 (Birkar)**.** *Fix $d \in \mathbb{N}$ and $\epsilon > 0$. The set of ϵ-log terminal projective Fano varieties of dimension d is bounded. In particular d-dimensional terminal Fano varieties are bounded.*

The proof of this result is a technical tour de force that also sheds light on many important techniques in birational geometry. In particular Birkar proves important results on the α-invariants of Fano varieties (answering a question of Tian with implications to K-stability and the existence of Kähler–Einstein metrics) as well as optimal results on the existence of complements that vastly generalize to arbitrary dimensions, results obtained by V. Shokurov in the case of surfaces. It is clear that Birkar [2016a] and Birkar [2016b] will have a major impact in birational geometry and related areas.

In conclusion, I hope to have conveyed that Birational Geometry is a classical subject which has undergone a dramatic transformation in recent years due to important and sustained contributions to the minimal model program by many distinguished researchers. Caucher Birkar has been at the forefront of this progress and has been a key player in several of the most important breakthroughs. It is a pleasure to congratulate him on his spectacular achievements.

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THE WORK OF PETER SCHOLZE

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Abstract

He has developed powerful methods in algebraic geometry over p-adic fields, and has proved striking theorems in this area.

My purpose here is to convey some idea of the accomplishments of Peter Scholze for which he was awarded the Fields medal.

Scholze has made ground-breaking contributions to fundamental problems in arithmetic geometry. Although his main results so far concern the geometry of algebraic varieties over p-adic fields, with important applications to the theory of automorphic forms, he has a much wider vision of mathematics. In particular, he has also contributed substantially to algebraic topology and has recently begun developing some fascinating ideas on arithmetic geometry beyond the p-adic setting.

Moreover, although Scholze has made major additions to the elaborate theoretic foundations of arithmetic geometry, at the same time his ideas have dramatically simplified and clarified our field. This is a characteristic feature of his universal approach to and vision of mathematics.

This report is structured as follows. In sections 1 and 2, we present Scholze's perfectoid spaces and some of their first applications. In section 3, his pro-étale topology is introduced. This technique will be used in the proofs in sections 4 and 5 on p-adic Hodge theory and then applied, in conjunction with the Hodge–Tate period map of section 6, to global problems in section 7. Section 8 is devoted to Scholze's theory of v-sheaves that extends the theory of diamonds from section 3. In sections 9 and 10, applications of these methods to local Shimura varieties and their cohomology, and the construction of Langlands parameters are presented. Thus the report is organized in three themes: *p-adic geometry* (sections 1, 2, 3, 8), *p-adic Hodge theory* (sections 4, 5, 6) and *(local and global) Shimura varieties and Langlands correspondences* (sections 7, 9, 10). Section 11 mentions some further results of Scholze. The report ends with a short summary assessment of Scholze as a mathematician.

1 Transferring the Frobenius map into mixed characteristic

Let p be a prime number. In various aspects, algebraic varieties in characteristic p, such as over $\mathbb{F}_p$, are easier to handle than in characteristic zero, such as over $\mathbb{Q}_p$. This may seem paradoxical to an analyst who works over fields of characteristic zero such as $\mathbb{R}$

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or $\mathbb{C}$. However, in characteristic p the *Frobenius map*, mapping an element to its p-th power, is compatible not only with multiplication but also with addition and therefore provides an extra symmetry which often simplifies algebraic problems.

One of the fundamental methods developed by Scholze is his theory of *perfectoid spaces*: this presents a general framework to reduce problems about algebraic varieties in characteristic zero to algebraic varieties in characteristic p.

Let us give an idea of this theory. Let us start by comparing the field $\mathbb{Q}_p$ of p-adic numbers with the field $\mathbb{F}_p((t))$ of Laurent series with coefficients in the finite field $\mathbb{F}_p$. In the first case, elements may be written as p-adic expansions $\sum_i a_i p^i$, where $0 \leq a_i < p$, and in the second case elements may be written as t-adic expansions $\sum_i a_i t^i$, where $a_i \in \mathbb{F}_p$. Thus they look superficially similar, but of course are quite different. However, by a theorem of Fontaine–Wintenberger, after adjoining successively higher and higher p-power roots of p, resp. t, these fields become more and more similar. In fact, after adjoining all the p-power roots, although they are not isomorphic, they have the same absolute Galois group. Scholze had the deep insight that this theorem is the manifestation of a much more general phenomenon.

The Fontaine–Wintenberger theorem may be reinterpreted as an equivalence between the category of finite extensions of $\mathbb{Q}_p(p^{1/p^\infty})$ and the corresponding category for $\mathbb{F}_p((t^{1/p^\infty}))$. Scholze's perception of this theorem is that this is merely the simplest, zero-dimensional case of a "tilting equivalence". He first introduces the notion of a *perfectoid field*: this is a complete topological field k whose topology is induced by a non-archimedean absolute value $|\ |\colon k \to \mathbb{R}_{\geq 0}$ with dense image, such that p is topologically nilpotent in k, and such that every element of O_k/pO_k admits a pth-root. Here $O_k \subseteq k$ denotes the subring of elements of absolute value ≤ 1. For example, the completions of the fields $\mathbb{Q}_p(p^{1/p^\infty})$ and $\mathbb{F}_p((t^{1/p^\infty}))$ are perfectoid. Taking this as a starting point, Scholze defines a whole class of *perfectoid algebras* over perfectoid fields: these are certain algebras equipped with a topology, again satisfying a certain p^{th}-root condition. And he constructs a *tilting functor* which associates to each perfectoid algebra of characteristic zero a perfectoid algebra of characteristic p. He shows that this is an equivalence of categories. Furthermore, he then geometrizes this construction by introducing *perfectoid spaces* over a perfectoid field k, obtained by gluing the adic spectra of perfectoid rings (just as schemes are obtained by gluing the spectra of rings). Here the *adic spectrum*, introduced by Huber in the 1990s, is a refinement of the usual notion of spectrum in algebraic geometry which takes into account a topology on the ring. The adic spectrum of a perfectoid ring is known as an *affinoid perfectoid*.

The fundamental theorem about perfectoid spaces is as follows:

Theorem 1.1. (i) *Let k be a perfectoid field, and denote by $k^\flat$ its tilt which is a perfectoid field of characteristic p. The tilting functor $X \mapsto X^\flat$ induces an equivalence of categories between the category of perfectoid spaces over k and the category of perfectoid spaces over $k^\flat$. Furthermore, the tilting functor induces an equivalence of étale sites, $X_{\text{ét}} \simeq X^\flat_{\text{ét}}$.*

(ii) *For any perfectoid space X, the structure pre-sheaf $\mathcal{O}_X$ is a sheaf and, if X is affinoid perfectoid, then*

$$H^i_{\text{ét}}(X, \mathcal{O}_X) = 0 \textit{ for } i > 0\,.$$

The first part of the theorem is Scholze's tilting equivalence for perfectoid spaces, which simultaneously extends the Fontaine–Wintenberger theorem and the *almost purity theorem* of Faltings, which was one of Faltings' key techniques in his work on p-adic Hodge theory. The second part of the theorem is the analogue for perfectoid spaces of Tate's acyclicity theorem for rigid-analytic spaces, or Cartan's theorem B for Stein complex spaces, or Serre's vanishing of higher cohomology on affine schemes. It is surprising in this highly non-noetherian situation.

There is also the notion of a perfectoid space without mention of a perfectoid ground field (Fontaine, Kedlaya). However, when the perfectoid ground field is not fixed, the tilting operation is not 'injective': in fact, the 'moduli' of all untilts over $\mathbb{Q}_p$ of a fixed complete algebraically closed field of characteristic p is the *Fargues–Fontaine curve* from p-adic Hodge theory, a 'compact p-adic Riemann surface' (in particular, a regular noetherian scheme of Krull dimension one) over $\mathbb{Q}_p$ whose geometric properties are closely tied to p-adic arithmetic.

2 First applications of perfectoid spaces

Scholze's first application of his theory of perfectoid spaces was a proof of Deligne's *weight monodromy conjecture* for a new class of algebraic varieties. Let F be a finite extension of $\mathbb{Q}_p$, and let X be a proper smooth variety over F. Deligne's conjecture is that for any degree i, the monodromy filtration on the étale cohomology group $H^i_{\text{ét}}(X_{\overline{F}}, \mathbb{Q}_\ell)$ is pure of weight i (essentially, that the associated graded pieces of the monodromy filtration afford an action by the Frobenius automorphism which is pure of a certain prescribed weight). This is undoubtedly the single most important open conjecture on the étale cohomology of algebraic varieties. Scholze proves:

Theorem 2.1. *Let X be a proper smooth algebraic variety over F such that X is a set-theoretic complete intersection in a projective smooth toric variety. Then the weight monodromy conjecture is true for X.*

The proof of this theorem uses the tilting functor to reduce subtly to the analogous conjecture in which F is replaced by a finite extension of $\mathbb{F}_p((t))$, which was proved earlier by Deligne. It is conceivable that any (projective smooth) algebraic variety over F satisfies the hypothesis of Scholze's theorem, but as long as this is not known, the monodromy conjecture has to be considered as still open in the general case.

The theory of perfectoid spaces has led to other applications. We mention a few of these in commutative algebra due to others.

- The proof of Hochster's direct summand conjecture (André and Bhatt).

- The proof of Hochster's conjecture on the existence and weak functoriality of big Cohen–Macaulay algebras (André, Heitmann–Ma).

- The p-adic analogue of Kunz's characterization of regular rings through their Frobenius endomorphism (Bhatt–Iyengar–Ma).

- The proof of the Auslander–Goldman–Grothendieck purity conjecture on the Brauer group of schemes (Česnavičius).

3 The pro-étale topology and diamonds

One of Grothendieck's main inventions was the introduction of the étale topology of schemes which lead him to a dramatic reworking of the concept of a topology. Scholze extends in several ways Grothendieck's concepts, with strong consequences. In this section we address Scholze's pro-étale topology; we will do this in the framework of perfectoid spaces, though there are also analogues for schemes.

A morphism $f : \mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ of affinoid perfectoids is *pro-étale* if (B, B^+) is a completed filtered colimit of perfectoid pairs (A_i, A_i^+) which are étale over (A, A^+); this definition is extended to morphisms $f : X \to Y$ of perfectoid spaces, so as to be local on the source and the target. In contrast to étale morphisms of schemes, pro-étale morphisms can have infinite degree. Another subtlety is that it may happen that the inclusion of a point in an affinoid perfectoid is a pro-étale morphism. Using pro-étale morphisms, Scholze defines the *pro-étale topology*. He proves the following analogue of Theorem 1.1.

Theorem 3.1. *Any perfectoid space is a sheaf for the pro-étale topology. Furthermore, for any perfectoid space X, the presheaf $\mathcal{O}_X$ on the pro-étale site is a sheaf and, if X is affinoid perfectoid, then*

$$H^i_{\text{pro-ét}}(X, \mathcal{O}_X) = 0 \textit{ for } i > 0 .$$

To work with the pro-étale topology, Scholze introduces the notion of a *totally disconnected* perfectoid space: this is a (quasi-compact and quasi-separated) perfectoid space X that is as close as possible to a profinite topological space, in the sense that each connected component has a unique closed point. He proves that any perfectoid space may be covered, in the sense of the pro-étale topology, by totally disconnected ones. This is somewhat reminiscent of the fact that any compact Hausdorff space is the continuous image of a pro-finite set. Moreover, when X is totally disconnected, he proves (roughly) that a morphism to X is pro-étale if and only if its geometric fibers are profinite sets. This result gives a fiberwise criterion to decide whether a morphism is pro-étale, locally for the pro-étale topology on the base, and makes the pro-étale topology manageable.

The pro-étale topology then leads to the notion of a *diamond*: a diamond is a sheaf for the pro-étale topology on the category of perfectoid spaces in characteristic p which can be written as a quotient of a perfectoid space by a pro-étale equivalence relation. This definition is analogous to Artin's definition of *algebraic spaces*, and expresses the intuitive idea that a diamond is obtained by glueing perfectoid spaces along pro-étale overlaps. Theorem 3.1 enables one to extend the tilting functor from perfectoid spaces to all rigid-analytic spaces: Scholze thus defines the *diamond functor*

$$\{\textit{adic spaces over } \mathbb{Q}_p\} \longrightarrow \{\textit{diamonds}\}, \quad X \longmapsto X^\diamond, \tag{3-1}$$

which, when restricted to the full subcategory of perfectoid spaces over $\mathbb{Q}_p$, induces the tilting functor $X \mapsto X^\flat$. In fact, for any non-archimedean field L, the functor $X \mapsto X^\diamond$ defines a fully faithful functor from the category of *seminormal* rigid-analytic spaces over L to the category of diamonds over $\mathrm{Spd}\, L = \mathrm{Spa}(L, O_L)^\diamond$.

The category of diamonds is much more flexible than the category of adic spaces, e.g., it allows one to take a product of diamonds $\mathrm{Spd}(\mathbb{Q}_p) \times \mathrm{Spd}(\mathbb{Q}_p)$. In this way, Scholze gives a meaning to the 'arithmetician's dream object' $\mathrm{Spec}(\mathbb{Z}) \times_{\mathbb{F}_1} \mathrm{Spec}(\mathbb{Z})$ after localization at (p, p), where $\mathbb{F}_1$ is the non-existent field with one element. Here the two copies of the prime number p have to be thought of as two independent variables.

Scholze uses the category of diamonds also as a method to construct objects in the category of rigid-analytic spaces by first showing that these objects exist as diamonds and then showing that they are in the essential image of the diamond functor. It appears that the concept of diamonds is just the right one to address topological questions in p-adic geometry.

4 Hodge theory of rigid-analytic spaces

The classical subject of Hodge theory is concerned with the singular cohomology and de Rham cohomologies of compact complex manifolds, and their relation. It applies not only to projective algebraic varieties over $\mathbb{C}$ but to the wider class of compact Kählerian manifolds. The analogous *p-adic Hodge theory* of p-adic algebraic varieties was initiated by Tate in the 1960s and subsequently completed by Fontaine–Messing, Faltings, Kato, Tsuji, Niziol and Beilinson. Tate asked in his original paper whether the theory worked not only for p-adic varieties but for the wider class of p-adic rigid-analytic spaces, which are the p-adic analogues of complex manifolds. The positive resolution of the main theorems of p-adic Hodge theory in this degree of generality is given by the following theorem of Scholze. Here the singular cohomology groups of the classical theory are replaced by the étale cohomology groups.

Theorem 4.1. *Let X be a proper smooth rigid-analytic space over a complete algebraically closed extension C of $\mathbb{Q}_p$.*

(i) *The Hodge-de Rham spectral sequence*

$$E_1^{ij} = H^j(X, \Omega^i_{X/C}) \Rightarrow H^{i+j}_{\mathrm{dR}}(X/C)$$

degenerates at the first page. Moreover, for all $i \geq 0$,

$$\sum_{j=0}^{i} \dim_C H^{i-j}(X, \Omega^j_{X/C}) = \dim_C H^i_{\mathrm{dR}}(X/C) = \dim_{\mathbb{Q}_p} H^i_{\text{ét}}(X, \mathbb{Q}_p).$$

(ii) *There is a* Hodge–Tate spectral sequence

$$E_2^{ij} = H^i(X, \Omega^j_{X/C}) \Rightarrow H^{i+j}_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} C$$

that degenerates at the second page.

The first part of the theorem implicitly includes the statement that the étale cohomology groups $H^i_{\text{ét}}(X, \mathbb{Q}_p)$ are finite-dimensional; this was for a long time conjectural. A key technique in the proof of this theorem is that any rigid-analytic space may be

covered, with respect to the pro-étale topology, by affinoid perfectoids. This allows one to then apply the vanishing theorems for the structure sheaves on such spaces, as in Theorem 1.1.

It is remarkable that, contrary to the complex case, the theorem holds without any Kähler type hypothesis on X.

5 Integral p-adic Hodge theory

When the rigid-analytic space X in Theorem 4.1 comes from a proper smooth formal scheme $\mathfrak{X}$ over the ring of integers O_C, one can refine the de Rham cohomology of X and prove comparison theorems with the étale cohomology of X, and also with the crystalline cohomology of the special fiber of $\mathfrak{X}$. Let $A_{\mathrm{inf}} = W(O_C^\flat)$ be Fontaine's ring, with its Frobenius automorphism φ and a fixed generator ξ of $\ker(A_{\mathrm{inf}} \to O_C)$. Also, let k denote the residue field of O_C.

In joint work with Bhatt and Morrow, Scholze proves:

Theorem 5.1. *There exists a perfect complex $R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X})$ of A_{inf}-modules together with a φ-linear endomorphism $\varphi\colon R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X}) \to R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X})$ that becomes an automorphism after inverting ξ, resp. $\varphi(\xi)$. Each cohomology group $H^i_{A_{\mathrm{inf}}}(\mathfrak{X})$ is a finitely presented A_{inf}-module that becomes free after inverting p. Furthermore, one has the following comparison isomorphisms.*

(i) de Rham: $R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} O_C \simeq R\Gamma_{\mathrm{dR}}(\mathfrak{X}/O_C)$.

(ii) étale: $R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X}) \otimes_{A_{\mathrm{inf}}} W(C^\flat) \simeq R\Gamma_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(C^\flat)$, *$\varphi$-equivariantly.*

(iii) crystalline: $R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} W(k) \simeq R\Gamma_{\mathrm{crys}}(\mathfrak{X}_k/W(k))$, *$\varphi$-equivariantly.*

As a consequence of this theorem, one gets bounds for the torsion in the étale cohomology in terms of the crystalline cohomology:

$$\mathrm{length}_{W(k)} H^i_{\mathrm{crys}}(\mathfrak{X}_k/W(k))_{\mathrm{tor}} \geq \mathrm{length}_{\mathbb{Z}_p} H^i_{\text{ét}}(X, \mathbb{Z}_p)_{\mathrm{tor}}.$$

In particular if the crystalline cohomology is torsion free then the étale cohomology is torsion free as well.

The proof of this theorem uses in an essential way the Faltings almost purity theorem, cf. section 1, enriched by a control of 'junk torsion' via the Berthelot–Ogus functor $L\eta$. Contrary to the crystalline theory, in which the action of Frobenius comes from the fact that cystalline cohomology only depends on the special fiber of $\mathfrak{X}$, the Frobenius action on A_{inf}-theory is much more subtle; it ultimately comes from the Frobenius action on the tilt of X.

The cohomology functor $R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X})$ is a new cohomological invariant which cannot be obtained by a formal procedure from other previously known cohomology theories.

There is a further refinement of this result. Let F be a finite extension of $\mathbb{Q}_p$ contained in C, and assume that $\mathfrak{X}$ comes by base change from a proper smooth scheme $\mathfrak{X}_{O_F}$ over O_F. In joint work with Bhatt and Morrow, Scholze constructs a cohomology theory $R\Gamma_{\mathfrak{S}}(\mathfrak{X}_{O_F})$ which recovers the A_{inf}-cohomology theory, i.e.,

$$R\Gamma_{\mathfrak{S}}(\mathfrak{X}_{O_F}) \otimes_{\mathfrak{S}} A_{\mathrm{inf}} \simeq R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X}).$$

Here $\mathfrak{S} = W(k)[[z]]$ is the ring considered by Breuil and Kisin. It is viewed as a subring of A_{inf} via the Frobenius on $W(k)$ and by sending z to the p-th power of a certain pseudo-uniformizer of O_C (one deduced from a compatible choice of successive p-power roots of a fixed uniformizer of O_F). The proof in loc. cit. is based on *topological Hochschild homology*. That theory was given new foundations by Scholze in joint work with Nikolaus, see section 11, c); the flexibility of this novel version of THH theory is essential to the proof.

Very recently, Scholze has constructed in joint work with Bhatt a new cohomology theory, *prismatic cohomology*, which clarifies the role of the Frobenius twist in the embedding of $\mathfrak{S}$ into A_{inf} and reproves some of the comparison isomorphisms in Theorem 5.1.

6 Period maps

By letting the rigid-analytic space in Theorem 4.1 vary, one obtains period maps. In classical Hodge theory, the trivialization of the local system defined by singular cohomology leads to a trivialization of de Rham cohomology and hence, via the Hodge-de Rham spectral sequence, to period maps in the sense of Griffiths. The Hodge–Tate spectral sequence of Theorem 4.1 leads to a new kind of period map. More precisely, Scholze proves:

Theorem 6.1. *Let $f : X \to Y$ be a proper smooth morphism of rigid-analytic spaces over a complete algebraically closed extension C of $\mathbb{Q}_p$.*

(i) *Let $\mathbb{L}$ be a lisse $\mathbb{Z}_p$-sheaf on X_{et}. Then for all $i \geq 0$, the higher direct image sheaf $R^i f_* \mathbb{L}$ is a lisse $\mathbb{Z}_p$-sheaf on $Y_{\text{ét}}$.*

(ii) *For any $i \geq 0$, there exists a perfectoid pro-étale cover $\tilde{Y} \to Y$ such that the pullback of $R^i f_* \mathbb{Z}_p$ becomes constant; consequently, the filtration induced by the Hodge–Tate spectral sequence defines a period map of adic spaces over C,*

$$\pi^i_{\mathrm{HT}} : \tilde{Y} \longrightarrow \mathfrak{F}^i_C,$$

where $\mathfrak{F}^i_C$ denotes a partial flag variety for a typical fiber of $R^i f_* \mathbb{Z}_p$.

As an example, consider the case of the universal elliptic curve $f : E \to M$ over the modular curve. In this case, we obtain a map $\pi_{\mathrm{HT}} : \tilde{M} \to \mathbb{P}^1$ from the pro-étale cover $\tilde{M}$ trivializing $R^1 f_* \mathbb{Z}_p$ to the projective line. The restriction of π_{HT} to $\mathbb{P}^1 \setminus \mathbb{P}^1(\mathbb{Q}_p)$ is a pro-finite étale cover, whereas the restriction to $\mathbb{P}^1(\mathbb{Q}_p)$ has one-dimensional fibers (p-adic lifts of Igusa curves). For an identification of $\tilde{M}$, comp. the remark after Theorem 7.2 below.

7 Existence of global Galois representations

Scholze has used perfectoid methods to prove a long-standing conjecture on the construction of representations of the absolute Galois group of number fields via the cohomology of locally symmetric spaces (conjecture of Grunewald–Ash). Let F be a

totally real field or a CM field. For a sufficiently small open compact subgroup $K \subset \mathrm{GL}_n(\mathbb{A}_{F,f})$, consider the locally symmetric space

$$X_K = \mathrm{GL}_n(F)\backslash[D \times \mathrm{GL}_n(\mathbb{A}_{F,f})/K],$$

where $D = \mathrm{GL}_n(F \otimes_{\mathbb{Q}} \mathbb{R})/\mathbb{R}_+ K_\infty$ is the symmetric space for $\mathrm{GL}_n(F \otimes_{\mathbb{Q}} \mathbb{R})$. Consider the singular cohomology groups with coefficients in $\mathbb{F}_p$, for some prime number p.

Theorem 7.1. *For any system of* Hecke eigenvalues ψ *appearing in* $H^i(X_K, \mathbb{F}_p)$*, there exists a continuous semi-simple representation* $\mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ *characterized by the property that for all but finitely many 'ramified' places* v *of* F*, the characteristic polynomial of the Frobenius* Frob_v *is described in terms of the Hecke eigenvalues* ψ *at* v.

In fact, a version of the theorem also holds with coefficients in $\mathbb{Z}/p^m$ instead of $\mathbb{F}_p$ and, passing to the limit over m, yields as a consequence the existence of Galois representations in $\mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ attached to *regular algebraic cuspidal representations* of $\mathrm{GL}_n(\mathbb{A}_F)$ related to *rational cohomology classes* proved earlier by Harris–Lan–Taylor–Thorne. However, rational cohomology classes are quite rare, whereas torsion classes in the cohomology as in Theorem 7.1 abound.

Like for that earlier result, the proof of Theorem 7.1 proceeds by realizing the cohomology of X_K as the boundary contribution of a (connected) Shimura variety of Hodge type. But by embedding the problem into the perfectoid world, Scholze goes much farther. Let S_K ($K \subset G(\mathbb{A}_f)$) be a Shimura variety of Hodge type, associated to the reductive group G over $\mathbb{Q}$ equipped with Shimura data. Let $\{\mu\}$ be the associated conjugacy class of cocharacters of $G_{\overline{\mathbb{Q}}}$, and E its field of definition (a finite extension of $\mathbb{Q}$ contained in $\overline{\mathbb{Q}}$). Scholze's main tool is the following fact.

Theorem 7.2. *Fix a prime number* p *and a place* $\mathfrak{p}$ *of* E *above it. For any open compact subgroup* $K^p \subset G(\mathbb{A}_f^p)$*, there exists a unique perfectoid space* S_{K^p} *which is the* completed *limit* $\varprojlim_{K_p \subset G(\mathbb{Q}_p)} S_{K^p K_p} \otimes_E E_{\mathfrak{p}}$*. Furthermore, there is a* $G(\mathbb{Q}_p)$*-equivariant Hodge–Tate period map* (*in the sense of section 6*),

$$\pi_{\mathrm{HT}} \colon S_{K^p} \longrightarrow \mathcal{F}_{G,\{\mu\}} \otimes_E E_{\mathfrak{p}}.$$

In the case of the modular curve we have $G = \mathrm{GL}_2$ and $E = \mathbb{Q}$. Then we obtain the Hodge–Tate period map mentioned at the end of section 6.

As an application of these methods, Scholze also proves the following vanishing theorem, conjectured by Calegari and Emerton. Recall the definition of the compactly supported *completed cohomology* groups for a fixed tame level $K^p \subset G(\mathbb{A}_f^p)$,

$$\tilde{H}_c^i(S_{K^p}, \mathbb{Z}_p) := \varprojlim_m \varinjlim_{K_p} H_c^i(S_{K^p K_p}, \mathbb{Z}/p^m).$$

Theorem 7.3. *For* $i > \dim S_K$*, the completed cohomology group with compact supports* $\tilde{H}_c^i(S_{K^p}, \mathbb{Z}_p)$ *vanishes.*

Even without passing to the limit, one has a vanishing theorem, proved by Scholze in joint work with Caraiani:

Theorem 7.4. *Let S_K be a* simple *Shimura variety associated to a fake unitary group (then S_K is compact). Let $\ell \neq p$. The localization $H^i(S_K, \mathbb{F}_\ell)_\mathfrak{m}$ at a p-*generic maximal ideal *of the Hecke algebra vanishes for $i \neq \dim S_K$.*

This result is a torsion analog of a well-known archimedean result that states that automorphic representations Π, contributing to the singular cohomology of S_K, with tempered archimedean component, only show up in the middle degree.

8 The v-topology and étale cohomology of diamonds

Scholze introduces another topology on the category of perfectoid spaces, besides the pro-étale topology of section 3. The *v-topology* is the topology obtained by declaring that any surjective map between affinoid perfectoids is an open cover. Even though it may appear at first sight that the v-topology admits far too many open covers to be useful, Scholze uses this topology to dramatic effect: in particular, it allows him to extend the diamond functor from rigid-analytic spaces to formal schemes. The basis of all applications is a descent theorem for the v-topology:

Theorem 8.1. (i) *Any diamond satisfies the sheaf axioms for the v-topology.*

(ii) *For any perfectoid space X the presheaf $\mathcal{O}_X$ on the v-site is a sheaf, and if X is affinoid perfectoid, then*

$$H^i_v(X, \mathcal{O}_X) = 0 \textit{ for } i > 0 \,.$$

(iii) *For any perfectoid space X, the category of locally free $\mathcal{O}_X$-modules of finite rank satisfies descent for the v-topology. The same holds for the category of separated étale morphisms.*

The statement (i) is the analogue of Gabber's theorem that any algebraic space is a fpqc-sheaf. The statement (iii) is a key tool in the work of Fargues–Scholze on the v-stack of vector bundles on the Fargues–Fontaine curve and its étale cohomology, comp. section 10.

Scholze also shows, under certain hypotheses, that any v-sheaf which is suitably covered by a perfectoid space is automatically a diamond. This is the analogue of Artin's theorem on algebraic spaces, reducing smooth, and even flat, groupoids to étale groupoids.

Using these concepts, Scholze has established an *étale cohomology theory of diamonds*, taking as a model Grothendieck's étale cohomology theory for schemes. In particular, he constructs the analogue of the 'six-operation calculus' and appropriate versions of the proper and smooth base change theorems. This theory is one of the key tools in the geometric construction of smooth representations of p-adic groups and in the geometric construction of Langlands parameters, cf. section 10. Remarkably, for perfectoid spaces the notion of smoothness is highly non-obvious (the usual characterizations, via differentials or via infinitesimal liftings, lose their sense in this context).

9 Local Shimura varieties

A *local Shimura datum* is a triple $(G, \{\mu\}, b)$ consisting of a reductive group G over $\mathbb{Q}_p$, a conjugacy class $\{\mu\}$ of minuscule cocharacters of $G_{\overline{\mathbb{Q}}_p}$, and an element $b \in G(\breve{\mathbb{Q}}_p)$ whose σ-conjugacy class lies in $B(G, \{\mu\})$, i.e. is *neutral acceptable* wrt. $\{\mu\}$. Here, for any finite extension F of $\mathbb{Q}_p$ contained in $\overline{\mathbb{Q}}_p$, we denote by $\breve{F}$ be the completion of the maximal unramified extension of F. Let E be the field of definition of $\{\mu\}$, a finite extension of $\mathbb{Q}_p$ contained in the fixed algebraic closure $\overline{\mathbb{Q}}_p$. Partly in joint work with Weinstein, Scholze proves:

Theorem 9.1. *There exists a* local Shimura variety *associated to* $(G, \{\mu\}, b)$*: a tower of rigid-analytic spaces* $\mathfrak{M}_{(G,\{\mu\},b),K}$ *over* $\breve{E}$*, parametrized by open compact subgroups* $K \subset G(\mathbb{Q}_p)$*, equipped with étale covering maps*

$$\mathfrak{M}_{(G,\{\mu\},b),K} \longrightarrow \breve{\mathfrak{F}}^{\mathrm{adm}}_{G,\{\mu\}} \subset \breve{\mathfrak{F}}_{G,\{\mu\}},$$

with geometric fibers $G(\mathbb{Q}_p)/K$.

Here $\breve{\mathfrak{F}}_{G,\{\mu\}}$ denotes the partial flag variety over $\breve{E}$ associated to G and $\{\mu\}$, and $\breve{\mathfrak{F}}^{\mathrm{adm}}_{G,\{\mu\}}$ denotes the open adic subset of *admissible points*. In the case that K is a parahoric subgroup, Scholze constructs a natural integral model over $O_{\breve{E}}$ of $\mathfrak{M}_{(G,\{\mu\},b),K}$ as a v-sheaf.

The proof of this theorem proceeds by first constructing the diamond over $\breve{E}$ associated to $\mathfrak{M}_{(G,\{\mu\},b),K}$ and then showing that it lies in the image of the fully faithful functor (3-1). The diamond is the moduli space of *p-adic shtukas*, the p-adic analogue of Drinfeld's shtukas in the function field case (except that here there is only one *leg*). Examples of local Shimura varieties are given by Rapoport–Zink moduli spaces of p-divisible groups inside a given quasi-isogeny class (and their integral models for parahoric level exist in this case as *formal schemes* and not merely as v-sheaves). This fact is highly non-trivial and is based on the following description of p-divisible groups due to Scholze and Weinstein which is reminiscent of Riemann's description of complex tori:

Theorem 9.2. *Let* C *be an algebraically closed complete extension of* $\mathbb{Q}_p$*, and* O_C *its ring of integers. There is an equivalence of categories*

$$\{p\text{-divisible groups over } O_C\} \simeq \{\text{pairs } (\Lambda, W), \text{ where } \Lambda \text{ is a finite free } \mathbb{Z}_p\text{-module and } W \subset \Lambda \otimes C \text{ is a } C\text{-subvector space}\}.$$

This description of p-divisible groups over O_C is closely related to Fargues' earlier description in terms of *integral p-adic Hodge theory* in the sense of section 5.

This new point of view of Rapoport–Zink spaces allows Scholze to establish isomorphisms between various such spaces (and their inverse limits over shrinking K) that have been conjectured for a long time:

- $\mathfrak{M}_{(G,\{\mu\},b),\infty} \simeq \mathfrak{M}_{(G^\vee,\{\mu^\vee\},b^\vee),\infty}$, where $(G^\vee, \{\mu^\vee\}, b^\vee)$ denotes the *dual local Shimura datum*, provided that b is *basic*. This was conjectured by Gross and Rapoport–Zink. The case $G = \mathrm{GL}_n$ was proved earlier by Faltings (and Fargues), but in a more

complicated indirect formulation. Furthermore, this *duality isomorphism* exchanges the Hodge–Tate period map with the de Rham period map, cf. section 6 (here the de Rham cohomology is trivialized).

- identification, in the 'fake' Drinfeld case, of a connected component of $\mathfrak{M}_{(G,\{\mu\},b),K}$ with Drinfeld's formal halfspace $\widehat{\Omega}^n_F \widehat{\otimes}_F \breve{F}$. Here the integral p-adic Hodge theory in the sense of section 5 plays a key role.

10 The cohomology of local Shimura varieties and smooth representations

Let I be a finite set with m elements. A *local Shtuka datum* with m legs is a triple $(G, \{\mu_i\}_i, b)$ consisting of a reductive group G over $\mathbb{Q}_p$, a collection $\{\mu_i\}_{i\in I}$ of cocharacters of $G_{\overline{\mathbb{Q}}_p}$ and $b \in G(\breve{\mathbb{Q}}_p)$. When $I = \{*\}$ and $\{\mu\} = \{\mu_*\}$ is minuscule, one recovers the definition of a local Shimura datum, cf. last section. Generalizing the case of local Shimura varieties, Scholze constructs a tower of diamonds (for varying $K \subset G(\mathbb{Q}_p)$),

$$f_K \colon \mathfrak{M}_{(G,\{\mu_i\},b),K} \longrightarrow \prod_{i\in I} \operatorname{Spd} \breve{E}_i,$$

which is a moduli space of *shtukas with m legs bounded by* $\{\mu_i\}$. Let J_b be the σ-centralizer group of b (an inner form of G over F when b is basic). Then $J_b(\mathbb{Q}_p)$ acts on each member $\mathfrak{M}_{(G,\{\mu_i\},b),K}$ of the tower, whereas $G(\mathbb{Q}_p)$ acts on the tower as a group of Hecke correspondences.

The tower is equipped with a *period map* to a Schubert variety inside a version of the Beilinson–Drinfeld affine Grassmannian. When $m = 1$, this Beilinson–Drinfeld Grassmannian can be identified with the B^+_{dR}-Grassmannian of Scholze, with point set $G\big(B_{\mathrm{dR}}(C)\big)/G\big(B^+_{\mathrm{dR}}(C)\big)$ over a complete algebraically closed extension C of $\mathbb{Q}_p$. Here $B_{\mathrm{dR}}(C)$ and $B^+_{\mathrm{dR}}(C)$ are Fontaine's rings associated to C.

Let Λ be the ring of integers in a finite extension of $\mathbb{Q}_\ell$. In their recent joint work, Fargues and Scholze associate to $\{\mu_i\}_{i\in I}$ a sheaf of Λ-modules $\mathscr{S}_{\{\mu_i\}}$ on $\mathfrak{M}_{(G,\{\mu_i\},b),K}$, to which the $J_b(\mathbb{Q}_p)$-action is lifted. This construction uses the period map mentioned above. When $I = \{*\}$ and $\{\mu\} = \{\mu_*\}$ is minuscule, then $\mathscr{S}_{\{\mu_*\}} = \Lambda$. They prove the following fundamental finiteness theorem.

Theorem 10.1. (i) *The complex $Rf_{K!}\mathscr{S}_{\{\mu_i\}}$ comes in a natural way from an object of $D(J_b(\mathbb{Q}_p)\times\prod_{i\in I} W_{E_i}, \Lambda)$, and its restriction to $D(J_b(\mathbb{Q}_p), \Lambda)$ is compact (i.e., lies in the thick triangulated subcategory generated by the ℓ-adic completions of $c\text{-}\mathrm{Ind}_K^{J_b(\mathbb{Q}_p)}\Lambda$ as K runs through open pro-p-subgroups of $J_b(\mathbb{Q}_p)$).*

(ii) *Let ρ be an admissible smooth representation of $J_b(\mathbb{Q}_p)$ with coefficients in Λ. Then*

$$\mathrm{RHom}_{J_b(\mathbb{Q}_p)}(Rf_{K!}\mathscr{S}_{\{\mu_i\}}, \rho)$$

is a representation of $\prod_{i\in I} W_{E_i}$ on a perfect complex of Λ-modules.

(iii) *Passing to the limit over K,*

$$\varinjlim_K \mathrm{RHom}_{J_b(\mathbb{Q}_p)}(Rf_{K!}\mathscr{S}_{\{\mu_i\}}, \rho)$$

gives rise to a complex of admissible $G(\mathbb{Q}_p)$-representations equipped with an action of $\prod_{i\in I} W_{E_i}$. If ρ is a compact object of $D(J_b(\mathbb{Q}_p),\Lambda)$, then so is this last complex of $G(\mathbb{Q}_p)$-representations.

One application of Theorem 10.1 is due to Fargues and Scholze and concerns *local L-parameters*. This application is inspired by the work of V. Lafforgue in the global function field case. Let us sketch it.

Let I be a finite set, and let $V \in \mathrm{Rep}_\Lambda((^LG)^I)$. Fargues and Scholze construct a variant $\mathfrak{M}_{(G,V,1),K}$ of $\mathfrak{M}_{(G,\{\mu_i\},b),K}$ for $b=1$ (then $J_b=G$): a space of shtukas bounded by V,

$$f_K\colon \mathfrak{M}_{(G,V,1),K} \longrightarrow (\mathrm{Spd}\,\breve{\mathbb{Q}}_p)^I,$$

which is equipped with a version of the period map. Furthermore, Fargues and Scholze construct a sheaf of Λ-modules $\mathscr{S}_V$ on $\mathfrak{M}_{(G,V,1),K}$, to which the action of $G(\mathbb{Q}_p)$ is lifted. Restriction to the diagonal

$$\Delta\colon \mathrm{Spd}\,\breve{\mathbb{Q}}_p \longrightarrow (\mathrm{Spd}\,\breve{\mathbb{Q}}_p)^I$$

yields a moduli space of shtukas with one leg, $f_K^\Delta\colon \mathfrak{M}_{(G,\Delta^*V,1),K} \to \mathrm{Spd}\,\breve{\mathbb{Q}}_p$, with a sheaf $\mathscr{S}_{\Delta^*V}$.

Let $i\colon G(\mathbb{Q}_p)/K = \mathfrak{M}_{(G,\Lambda,1),K} \hookrightarrow \mathfrak{M}_{(G,\Delta^*V,1),K}$ be the subspace of shtukas with no legs. Let $\alpha\colon \Lambda \to \Delta^*V$ and $\beta\colon \Delta^*V \to \Lambda$ be maps of ^LG-modules. Then α, resp. β, induce maps $\alpha\colon i_*\Lambda \to \mathscr{S}_{\Delta^*V}$, resp. $\beta\colon \mathscr{S}_{\Delta^*V} \to i_*\Lambda$. Let $(\gamma_i)_{i\in I} \in W_{\mathbb{Q}_p}^I$, and let $\overline{x}$ be a geometric point of $\Delta(\mathrm{Spd}\,\breve{\mathbb{Q}}_p)$. Then we obtain the endomorphism

$$\begin{aligned} c\text{-}\mathrm{Ind}_K^{G(\mathbb{Q}_p)}\Lambda &\xrightarrow{\alpha} (Rf_{K!}^\Delta \mathscr{S}_{\Delta^*V})_{\overline{x}} = (Rf_{K!}\mathscr{S}_V)_{\overline{x}} \xrightarrow{(\gamma_i)} \\ &\longrightarrow (Rf_{K!}\mathscr{S}_V)_{\overline{x}} = (Rf_{K!}^\Delta \mathscr{S}_{\Delta^*V})_{\overline{x}} \xrightarrow{\beta} c\text{-}\mathrm{Ind}_K^{G(\mathbb{Q}_p)}\Lambda\,. \end{aligned}$$

Here the action of (γ_i) is given by Theorem 10.1. Fargues and Scholze prove that this endomorphism is given by a central element of the Hecke algebra $\mathcal{H}(G,K) = \mathrm{End}_{G(\mathbb{Q}_p)}(c\text{-}\mathrm{Ind}_K^{G(\mathbb{Q}_p)}\Lambda)$. Passing to the limit over all K, they define thus an element of the Bernstein center of $G(\mathbb{Q}_p)$.

The following theorem associates L-parameters to representations of $G(\mathbb{Q}_p)$.

Theorem 10.2. *For any irreducible smooth $\overline{\mathbb{Q}}_\ell$-representation π of $G(\mathbb{Q}_p)$ which admits an invariant $\overline{\mathbb{Z}}_\ell$-lattice, there is a unique (up to conjugation by $G^\vee(\overline{\mathbb{Q}}_\ell)$) continuous semisimple map*

$$\phi_\pi\colon W_{\mathbb{Q}_p} \longrightarrow {}^LG(\overline{\mathbb{Q}}_\ell),$$

compatible with the projection of $^LG(\overline{\mathbb{Q}}_\ell)$ to $W_{\mathbb{Q}_p}$, with the following property. Let I be a finite set, let $V \in \mathrm{Rep}_\Lambda((^LG)^I)$, let $\alpha\colon \Lambda \to \Delta^*V$ and $\beta\colon \Delta^*V \to \Lambda$, and let $(\gamma_i)_{i\in I} \in W_{\mathbb{Q}_p}^I$. Then the action on π of the element of the Bernstein center above is given by the scalar that arises as the composition

$$\overline{\mathbb{Q}}_\ell \xrightarrow{\alpha} \Delta^*V = V \xrightarrow{(\phi_\pi(\gamma_i))} V = \Delta^*V \xrightarrow{\beta} \overline{\mathbb{Q}}_\ell.$$

Much work remains to be done to better understand this construction of Fargues–Scholze: which L-parameters arise in this way, how the corresponding L-packets are related to those known by explicit representation-theoretic methods, how these L-parameters can be used to construct stable distributions, etc.

Another application of Theorem 10.1 is due to Kaletha and Weinstein, and concerns the Kottwitz conjecture on the cohomology of local Shimura varieties. Let $(G, \{\mu\}, b)$ be a local Shimura datum such that b is *basic*, and let $\mathfrak{M}_{(G,\{\mu\},b),K}$ be the associated local Shimura variety, cf. section 9. As mentioned above, the Fargues–Scholze sheaf $\mathscr{S}_{\{\mu\}}$ on $\mathfrak{M}_{(G,\{\mu\},b),K}$ is in this case the constant sheaf Λ. Let ρ be an irreducible admissible smooth representation of $J_b(\mathbb{Q}_p)$ with coefficients in Λ, and form the virtual smooth representation of $G(\mathbb{Q}_p)$ from Theorem 10.1, (iii),

$$H^*(G, \{\mu\}, b)[\rho] = \sum\nolimits_i (-1)^i \big(\varinjlim_K \operatorname{Ext}^i_{J_b(F)}(Rf_{K!}\Lambda, \rho)\big).$$

In the following theorem, d denotes the dimension of $\mathfrak{M}_{(G,\{\mu\},b),K}$.

Theorem 10.3 (Kaletha, Weinstein)**.** *Let $\phi\colon W_F \to^L G$ be a* discrete Langlands parameter *for G, and let $\rho \in \Pi_\phi(J_b)$. Then the following identity holds in the quotient* $\operatorname{Groth}(G(\mathbb{Q}_p))^{\mathrm{ell}}$ *of the Grothendieck group of admissible smooth representations of $G(\mathbb{Q}_p)$ on $\overline{\mathbb{Q}}_\ell$-vector spaces by the subgroup generated by* non-elliptic *representations,*

$$H^*(G, \{\mu\}, b)[\rho] = (-1)^d \sum_{\pi \in \Pi_\phi(G)} \delta(\pi, \rho)\pi.$$

Here $\Pi_\phi(G)$, resp. $\Pi_\phi(J_b)$, denotes the L-packet associated to the parameter ϕ, assuming that this concept is defined (and satisfies some natural properties), as e.g., for $G = \mathrm{GSp}_4$. There is good hope that there is a definition for any tamely ramified group G and p sufficiently large wrt. G. Whether the Fargues–Scholze definition of $\Pi_\phi(G)$, sketched above, can be used is an open problem.

The multiplicities ocurring here were defined earlier by Kottwitz under more restrictive hypotheses. This theorem is the confirmation of a conjecture of Kottwitz in a weaker form (weaker because the Weil group action is disregarded, and because only the image in $\operatorname{Groth}(G(F))^{\mathrm{ell}}$ is considered). There is also an extension of this theorem to the case when $\{\mu\}$ is no longer assumed to be minuscule: instead of a local Shimura variety one uses the moduli space of shtukas with one leg (a diamond, but no longer a rigid-analytic space). This uses the *geometric Satake equivalence for the B^+_{dR}-Grassmannian* of Fargues and Scholze.

11 Further achievements

a) One of Scholze's first accomplishments was a new proof of the *local Langlands conjecture* for GL_n over a p-adic field F. This conjecture states that continuous representations (on finite-dimensional $\mathbb{C}$-vector spaces) of the absolute Galois group $\mathrm{Gal}(\overline{F}/F)$ correspond to irreducible admissible representations of $\mathrm{GL}_n(F)$. It was first proved by Harris–Taylor and by Henniart. These earlier proofs are similar and use in an essential

way Henniart's *numerical local Langlands conjecture*; this result in turn is based on a complicated reduction modulo p method and relies ultimately on Laumon's results on the Fourier–Deligne transform and Kazhdan's construction of exotic ℓ-adic Galois representations for function fields. Scholze's proof is purely in characteristic zero and structurally much simpler, based instead on a geometric argument via the *nearby cycles sheaves* of certain moduli spaces of p-divisible groups.

b) Let G denote a reductive group over the Laurent series field $k((t))$, where k is a field. Then for any parahoric group scheme $\mathcal{G}$ over $k[[t]]$, one has the construction of the associated partial affine flag variety $\mathcal{F}_{\mathcal{G}} = LG/L^+\mathcal{G}$, an ind-projective ind-scheme over k. In the case $G = \mathrm{GL}\big(k((t))^n\big)$, $\mathcal{G} = \mathrm{GL}\big(k[[t]]^n\big)$, this yields the *affine Grassmannian* $\mathrm{Gr}^{\mathrm{aff}} = \mathcal{F}_{\mathcal{G}}$ which parametrizes $k[[t]]$-lattices in $k((t))^n$. X. Zhu has transposed this "equal characteristic" theory to the unequal characteristic: he constructs a *Witt vector Grassmannian* $\mathrm{Gr}^{W,\mathrm{aff}}$ which is an inductive limit of perfections of algebraic spaces and whose R-valued points, for perfect rings R of characteristic p, parametrize $W(R)$-lattices in $\big(W(R)[1/p]\big)^n$. Scholze, in joint work with Bhatt, shows that $\mathrm{Gr}^{W,\mathrm{aff}}$ is an ind-projective scheme, by constructing an analogue of the natural ample line bundle on $\mathrm{Gr}^{\mathrm{aff}}$. The main tool in this construction is the v-descent of vector bundles, cf. Theorem 8.1, (iii). Scholze interprets $\mathrm{Gr}^{W,\mathrm{aff}}$ as the special fiber of an integral model of his B^+_{dR}-Grassmannian, cf. section 10.

c) *Cyclic homology* was introduced in the early eighties to serve as an extension of de Rham cohomology to a non-commutative setting. It relies on the *algebraic theory* of Hochschild homology. *Topological Hochschild homology* (THH) is Hochschild homology *relative to the sphere spectrum* $\mathbb{S}$. In joint work with Nikolaus, Scholze gives a definition of *Topological Cyclic homology* (TCH) in terms of a Frobenius operator on THH. This approach avoids the ad hoc methods used earlier to define TCH, by staying strictly within the realm of homotopy theory. In particular, it constructs a Frobenius map in stable homotopy theory that lives inherently in mixed characteristic, whereas the classical Frobenius map is restricted to characteristic p. The relevance for algebraic geometry is furnished by the work of Bhatt–Morrow–Scholze which defines "motivic filtrations" on THH and related theories, and relates the graded pieces with p-adic cohomology theories such as crystalline cohomology and the A_{inf}-cohomology, cf. section 5.

e) I refer to Scholze's write-up of his plenary lecture at this congress for his recent ideas which go far beyond the p-adic world for a fixed p.

12 Summary

Scholze has proved a whole array of theorems in p-adic geometry. These theorems are not disjoint but, rather, are the outflow of a theoretical edifice that Scholze has created in the last few years.

There is no doubt that Scholze's ideas will keep mathematicians busy for many years to come. What is remarkable about Scholze's approach to mathematics is the ultimate simplicity of his ideas. Even though the execution of these ideas demands great technical power (of which Scholze has an extraordinary command), it is still true that the initial

key idea and the final result have the appeal of inevitability of the classics, and their elegance. We surely can expect more great things of Scholze in the future, and it will be fascinating to see to what further heights Scholze's work will take him.

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ON THE WORK OF AKSHAY VENKATESH

Peter Sarnak

Abstract

Number theory while one of the oldest subjects in mathematics, continues to be one of the most active areas of research. One reason for this is that it borrows from and contributes to many other fields, sometimes quite unexpectedly. Uncovering deeper number theoretic truths often involves advancing techniques across these disciplines resulting in new avenues of research and theories. Venkatesh's work fits squarely into this mold. His resolution and advancement of a number of long standing problems start with fundamental new insights connecting theories and have led to active research areas. He is both a problem solver and a theory builder of the highest caliber. Due to space limitation and the nature of this report, I have chosen to describe three examples of Venkatesh's works and end by simply listing a number of his other striking results. Also, in this brief description of some of the results I have omitted some technicalities and the reader should go to the references for precise statements.

1 Subconvexity for $GL(2)$, equidistribution and Hilbert's 11-th Problem

The Riemann Hypothesis for the zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \quad , \tag{1}$$

implies an optimally sharp upper bound on the critical line.

$$\zeta(\frac{1}{2} + it) \underset{\epsilon}{\ll} (1 + |t|)^{\epsilon} \quad , \text{ for any } \quad \epsilon > 0.^{1} \tag{2}$$

The bound of $(1 + |t|)^{\frac{1}{4}}$ follows from a simple convexity argument and Weyl's method for estimating exponential sums led to the first subconvex bound for $\zeta(s)$ namely with the exponent $\frac{1}{6}$ replacing $\frac{1}{4}$ Titchmarsh [1986]. While for $\zeta(s)$ itself, the applications and interest in this technical estimation problem might be limited, the question of estimating general automorphic L-functions $L(s, \pi)$ on their critical lines has many far reaching applications (we explicate one below). Here π is an automorphic cusp form for GL_n over a number field F and $L(s, \pi)$ is its standard L-function [J]. They

[1] $A \underset{\epsilon}{\ll} B$ means $|A| \leq C_\epsilon B$, with a constant depending only on ϵ

are the complete generalizations of Riemann's, Dirichlet's, and Hecke's L-functions all of which correspond to GL_1. Associated to such a π is its analytic conductor $C(\pi,t)$ (Iwaniec and Sarnak [2000]) which measures its complexity. The convex bound is

$$L\left(\frac{1}{2}+it,\pi\right) \underset{\epsilon}{\ll} C(\pi,t)^{\frac{1}{4}+\epsilon} \quad ,\epsilon>0. \tag{3}$$

Any improvement in the exponent of $\frac{1}{4}$ is called a subconvex bound and very often it is decisive in applications Iwaniec and Sarnak [ibid.]. For GL_1 L-functions subconvex bounds were established in the 1970's while for various L-functions of modular forms on the upper half plane the problem was resolved in a series of papers Duke, Friedlander, and Iwaniec [2002] by Duke–Friedlander–Iwaniec in the 1980's and 1990's. In Venkatesh [2010] Venkatesh made a major breakthrough in both techniques and generality. These include a direct use of ergodic theoretic mixing rates for equidistribution of orbits in homogeneous spaces $G(F)\backslash G(\mathbb{A}_F)$,[2] and in particular passing these to smaller orbits "sparse equidistribution" and systematically exploiting the device of varying test vectors of local representations in various period formula for special values of L-functions. This led him to many new cases of subconvexity culminating in the paper Michel and Venkatesh [2010] where he and Michel establish subconvexity in full generality for GL_2:

There is a universal $\delta>0$ such that for F fixed and π an automorphic cuspidal representation of $GL_2(\mathbb{A}_F)$

$$L\left(\frac{1}{2},\pi\right) \underset{F}{\ll} C(\pi)^{\frac{1}{4}-\delta} \quad . \tag{4}$$

Venkatesh followed this work with a series of results concerning equidistribution in arithmetic. Hilbert's 11-th problem asks about solvability of quadratic equations in the integers or rationals of a number field F;

$$X^tAX=B \tag{5}$$

Here A is an $n\times n$ symmetric integral (or F-rational) matrix, B is $m\times m$ integral (rational) and (5) is to be solved for an $n\times m$ integral (rational) X. The rational question was resolved by Hasse; the Hasse–Minkowski Theorem asserting that (5) has a solution in F iff it can be solved at each localization F_v of F. The integral case is much more difficult and Siegel made substantial progress on it with his mass formula Siegel [1935]. The most difficult case is when A is definite at each archimedean place of F (the class number of the spin genus of A is then typically large) in which case there are only finitely many solutions to (5) and one seeks a local to global principle; that is if (5) is solvable integrally locally at all localizations, is it solvable integrally globally? The most studied case is the $m=1$ scalar case. One has to allow here for a finite number of exceptions, that is an integral local to global principle for Norm(B) large (and also some other mild assumptions on the factors of B when $n\leq 4$), and this "stable integral local to global principle" has been proven for $n\geqslant 3$ (see Duke and Schulze-Pillot [1990] and Blomer

[2]Here G is a linear algebraic group defined over a number field and $\mathbb{A}_F$ its ring of adèles.

and Harcos [2010]). The key ingredient in the proof of the threshold $n = 3$ case is GL_2 subconvexity for certain π's.

Prior to 2008 the best stable integral local global principle for $m \geqslant 2$ was established in Hsia, Kitaoka, and Kneser [1978] and requires $n \geqslant 2m + 3$. In collaboration with Ellenberg, Venkatesh (Ellenberg and Venkatesh [2008]) made dramatic progress, establishing the principle for $n \geqslant m+5$. [3] They treat the case of $F = \mathbb{Q}$, and in Schulze-Pillot [2009] Schulze–Pillot extends their arguments to general F and also relaxes the side conditions on the factors of Norm(disc(B)). Their breakthrough is the introduction of an ergodic theoretic action involving orthogonal groups associated to (5), specifically $SO_{n-m}(\mathbb{Q}_p)$ acting on $SO(\mathbb{Z}[\frac{1}{p}])\backslash SO_n(\mathbb{Q}_p)$. This technique has its roots in Linnik's work on the $m = 1$, $n = 3$ case, but unlike that case where $SO_2(\mathbb{Q}_p)$ is a torus, the $SO_{n-m}(\mathbb{Q}_p)$'s are generated by unipotents for many of these auxiliary p's and this allows them to invoke decisively the powerful measure classification theorems of Ratner [1995], for such actions.

The explicit formulae such as those of Waldspurger [1985] which relate periods of automorphic forms to special values of L-functions, are used both in the applications as well as in the proofs of subconvexity. Understanding these formulae in general is a fundamental problem and the recent monograph of Sakellaridis and Venkatesh [2017] lays out a far reaching theory for the periods associated with spherical varieties.

2 Class Groups and Cohen–Lenstra Heuristics

The class group of a number field is one of its fundamental invariants and one that is notoriously difficult to study. Cohen and Lenstra Jr. [1984] put forth a probabilistic model for the statistics of invariants of these groups as one varies over quadratic extensions $\mathbb{Q}(\sqrt{D})$ of $\mathbb{Q}$. Their probability laws on the set of finite abelian groups are based on the heuristic that one should put in weights of $|Aut(A)|^{-1}$ for such a group A. The numerical evidence for their conjectures is very convincing. While there are many works and extensions of these conjectures, very little has been proven. They avoid the 2-part of the class group since the number of elements of order 2 is determined by Gauss' genus theory and the rest of the 2-part of the class group can be addressed partially using class field theory Rédei [1939]. Smith [2017] recently announced a proof that the rest of the 2-part of the class groups does in fact follow the Cohen–Lenstra law.

The function field analogue is to replace $\mathbb{Q}$ by $\mathbb{F}_q(t)$, the rational functions in t with coefficients in the finite field $\mathbb{F}_q$, and to consider the class groups of its quadratic extensions; that is $\mathbb{F}_q(t)(\sqrt{D(t)})$ where $D(t)$ is square-free in $\mathbb{F}_q[t]$, or more geometrically $y^2 = D(t)$ is a hyperelliptic curve over $\mathbb{F}_q$. If one allows q to get large first and $\deg D$ after, then well developed tools from arithmetic geometry of curves and varieties over finite fields allow one to tackle many of these problems and their variants Achter [2006]. However, if q is fixed then the problem appeared to be as difficult as the case over $\mathbb{Q}$. In Ellenberg, Venkatesh, and Westerland [2016] they all but settle this problem in the function field. They show that for $\ell > 2$ a prime and A a finite abelian ℓ-group, the upper

[3] The least n for which this might hold is $m + 2$ and this brings us close to it.

(resp lower) densities of imaginary quadratic extensions of $\mathbb{F}_q(t)$ with ℓ-part isomorphic to A as $\deg D \to \infty$, converge to $\prod_{i\geq 1}(1-\ell^{-i})/|\mathrm{Aut}(A)|$ as $q \to \infty$ with $q \not\equiv 1(\ell)$. This density is exactly that predicted by the Cohen–Lenstra law. Note that here one first establishes bounds for the densities as deg D goes to infinity and these approach the prediction as q is increased at the end (conjecturally one should allow q to be fixed). Their methods also give the densities when $q \equiv 1(\ell)$ and these can be different which can be explained by there being ℓ-th roots of 1 in $\mathbb{F}_q$ in this case Garton [2015].

The striking and novel starting point for this work is that even though q is fixed, as $\deg(D)$ goes to infinity one can exploit a stable range of the cohomology of certain Hurwitz moduli spaces of curves attached to A, specifically applying the Lefshetz fixed point theorem to execute an effective count! For this to work one needs a rather large stability range for the Betti numbers for these moduli spaces of growing dimension. While such stability has been established in recent years for the moduli space M_g of curves of genus g (see Madsen and Weiss [2007] which perhaps was one inspiration for the present work?) the requisite stability for these Hurwitz spaces poses substantial problems since these spaces aren't even connected. A large portion of the paper is devoted to the proof of the stability range using tools from topology and combinatoral group theory. The Cohen–Lenstra law emerges from this analysis in a canonical and convincing fashion. This breakthrough has inspired a number of further works on both topological and arithmetic aspects, including non-abelian versions Boston and Wood [2017].

3 Homology of Arithmetic Groups of positive defect

Let G be a noncompact real semi-simple Lie group, K a maximal compact subgroup and $S = G/K$ the corresponding Riemannian symmetric space. For Γ an arithmetic subgroup of G (in fact a congruence subgroup of a $\mathbb{Q}$-group, here and in what follows we are a little loose with terminology) the cohomology groups $H^*(\Gamma, R)$ with R various coefficient rings, have been studied widely. These come with an action of Hecke correspondences and setting aside various technicalities $H^*(\Gamma, \mathbb{C})$ or $H^*(\Gamma\backslash S, \mathbb{C})$ can be computed using automorphic forms using $\mathfrak{g}$, K cohomology ("Matsushima's formula" Borel and Wallach [2000]). In particular one can use the trace formula to show that if $\delta(G) := \mathrm{rank}(G) - \mathrm{rank}(K)$, then as Γ varies over congruence subgroups (de George and Wallach [1978])[4]

$$\lim_{\mathrm{Vol}(\Gamma\backslash S)\to\infty} \frac{\dim H^j(\Gamma\backslash S, \mathbb{C})}{\mathrm{Vol}(\Gamma\backslash S)} \neq 0, \quad \text{iff } \delta(G) = 0 \text{ and } j = \frac{\dim S}{2}. \tag{6}$$

The 'defect' $\delta(G)$ being zero is precisely Harish-Chandra's condition for G to carry discrete series representations. In this case the maximal growth rate of Betti numbers is in the middle dimension. The much studied algebro-geometric arithmetic Shimura

[4] various recent works allow for this general formulation see Finis and Lapid [2018].

varieties all fall into this zero defect setting. Venkatesh's work is concerned with the positive defect cases for which much less is known about $H^*(\Gamma, \mathbb{Z})$ and in particular its torsion.

Ray and Singer defined analytic torsion as regularized determinants of Laplacians on differential forms and conjectured that it computes the topological torsions in the same way that the Hodge theorem computes Betti numbers. Their conjecture was proved independently by Cheeger [1977] and Müller [1978]. In work with Bergeron (Bergeron and Venkatesh [2013]) and the book with Calegari (Calegari and Venkatesh [2012]), Venkatesh introduced analytic torsion as an effective tool to study torsion in this arithmetic setting. In the first work the torsion analogue of (6), namely the maximal growth rate of torsion is conjectured and for certain cases of strongly acyclic arithmetic Γ-modules M (i.e. ones for which the eigenvalues of the Laplacians are bounded away from zero uniformly) proved:[5]

$$\lim_{\mathrm{Vol}(\Gamma\backslash S)\to\infty} \frac{\log |H_j(\Gamma\backslash S, M)_{\mathrm{tor}}|}{\mathrm{Vol}(\Gamma\backslash S)} \neq 0 \quad \text{iff } \delta(G) = 1 \text{ and } j = (\dim S - 1)/2. \tag{7}$$

For example for arithmetic hyperbolic 3-manifolds $\Gamma\backslash\mathbb{H}^3$, $\mathbb{H}^3 = SL_2(\mathbb{C})/SU(2)$, $\delta = 1$ and $H_1(\Gamma, \mathbb{Z})$ should be primarily torsion growing exponentially (with a universal exponent) in the volume. The second work goes deeper into the arithmetic. The well known Jacquet–Langlands correspondence Jacquet and Langlands [1970] gives an explicit relation between the automorphic spectrum of a compact arithmetic 3-manifold $Y = \Gamma\backslash\mathbb{H}^3$ defined over an imaginary quadratic field F and non-compact arithmetic 3-manifolds $Y' = \Gamma'\backslash\mathbb{H}^3$ which are congruence quotients corresponding to $GL_2(F)$. One may interpret the analysis in the Calegari–Venkatesh discovery as using the trace formula to compute the analytic torsion of Y and of Y' in terms of orbital integrals involving conjugacy classes of Γ and Γ', and comparing these to show that aspects of the Jacquet–Langlands correspondence extend to torsion homology!

Matsushima's transcendental formula shows that when the defect δ is positive, the tempered (these being the representations that occur in the decomposition of $L^2(G)$) cohomology lives in degrees $j_0 = \frac{\dim S - \delta}{2}$, $j_0 + 1, \ldots, j_0 + \delta = \frac{\dim S + \delta}{2}$, and moreover the Betti numbers are a fixed multiple of those of a δ-dimensional torus. Understanding this relation algebraically directly in terms of cohomology classes and operations that raise degrees is central to developing analogues of the powerful tools that have been developed in the Shimura variety (and specifically $\delta = 0$) setting. Venkatesh raised and answered this question brilliantly for $H^j(\Gamma, \mathbb{Q}_p)$ in his derived Hecke algebra paper Venkatesh [2016]. One would like to cup classes in $H^j(\Gamma, \mathbb{Q})$ with a class in $H^1(\Gamma, \mathbb{Q})$ to raise the degree, however the latter group is typically 0. On the other hand, one can produce elementary torsion classes in $H^1(\Gamma, \mathbb{Z}/p^n\mathbb{Z})$ by passing to suitable congruence subgroups of Γ. Using these via cup product he defines derived Hecke correspondences which act on $H^*(\Gamma, \mathbb{Q}_p)$ and raise degrees. The construction is reminiscent of the Taylor–Wiles patching method and aspects of that technique are used to show conditionally on expected properties of Galois representations, that this derived

[5] formulated for homology rather than cohomology

Hecke algebra acts non-trivially and that the full range δ of degrees is generated from the lowest degree in this range.

This inspiring paper was followed by a flurry of activity, specifically Galatius and Venkatesh [2018] introduce derived deformation rings associated to Galois representations in this positive defect setting. Prasanna and Venkatesh [2016] developed a derived action in the context of $H^*(\Gamma, \mathbb{R})$, using differential forms, and study the rationality properties of this action. Their conjectures have concrete implications that can be checked and proven in special cases. Not surprisingly the latter involves explicit period formulae in terms of special values of L-functions and computations with analytic torsion. Suffice it to say that these ideas, techniques and conjectures of Venkatesh are already a central part of a rapidly developing arithmetic cohomology theory in the positive defect setting.

The following is a list of some further striking works by Venkatesh:

1. Established a Weyl law for the cuspidal spectrum of non co-compact congruence locally symmetric spaces, joint with Lindenstrauss (Lindenstrauss and Venkatesh [2007]).

2. Equidistribution of torus orbits in $PGL_3(\mathbb{Z})\backslash PGL_3(\mathbb{R})$ corresponding to ideal classes of totally real cubic number fields F as $|\mathrm{disc}(F)|$ goes to infinity, joint with Einsiedler, Lindenstrauss and Michel (Einsiedler, Lindenstrauss, Michel, and Venkatesh [2011]).

3. Effective equidistribution of large periodic orbits in locally symmetric spaces, joint with Einsiedler and Margulis (Einsiedler, Margulis, and Venkatesh [2009]).

4. Integral points on elliptic curves and the 3-torsion of class groups, joint with Helfgott (Helfgott and Venkatesh [2006]).

5. Upper bounds for the number of extensions of number fields of fixed degree and discriminant below X, with Ellenberg (Ellenberg and Venkatesh [2006]).

6. First improvement in forty years on the lower bound for sphere packing in high dimensions (Venkatesh [2013]).

7. Functoriality, Smith theory and the Brauer homomorphism, with Truemann (Treumann and Venkatesh [2016]).

Thanks to his innovative use of modern tools to study the theory of numbers, Venkatesh is shaping fields ranging from the theory of L-functions, Diophantine analysis, automorphic forms, representation theory, topology and locally symmetric spaces, to homogeneous dynamics and especially the very fruitful interactions between these areas.

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FIFTY YEARS OF MATHEMATICS WITH MASAKI KASHIWARA

Pierre Schapira

Abstract

Professor Masaki Kashiwara is certainly one of the foremost mathematicians of our time. His influence is spreading over many fields of mathematics and the mathematical community slowly begins to appreciate the importance of the ideas and methods he has introduced.

Mikio Sato. Masaki Kashiwara was a student of Mikio Sato and I will begin with a few words about Sato (see Andronikof [2007] and Schapira [2007] for a more detailed exposition). The story begins long ago, in the late fifties, when Sato created a new branch of mathematics now called "Algebraic Analysis" by publishing two papers on hyperfunction theory M. Sato [1959, 1960] and then developed his vision of analysis and linear partial differential equations (LPDE) in a series of lectures at the university of Tokyo in the 60s. Sato's idea is to define hyperfunctions on a real analytic manifold M as cohomology classes supported by M of the sheaf $\mathcal{O}_X$ of holomorphic functions on a complexification X of M. One can then represent hyperfunctions as "boundary values" of holomorphic functions defined in tuboids in X with wedge on M. To understand where the boundary values come from leads naturally Sato (see M. Sato [1970]) to define his microlocalization functor and, as a byproduct, the analytic wave front set of hyperfunctions. This is the starting point of microlocal analysis. Indeed, Lars Hörmander immediately understood the importance of Sato's ideas and adapted them to the C^∞-setting by replacing boundary values of holomorphic functions with the Fourier transform (see Hörmander [1971]).

In these old times, trying to understand real phenomena by complexifying a real manifold and looking at what happens in the complex domain was a totally new idea. And using cohomology of sheaves in analysis was definitely a revolutionary vision.

Master's thesis and the SKK paper. Then came Masaki. In his master's thesis, dated 1970 and published in English in Kashiwara [1995], he introduces and develops the theory of $\mathfrak{D}$-modules. Of course, a $\mathfrak{D}$-module is a module (right or left) over the non commutative sheaf of rings $\mathfrak{D}_X$ of holomorphic finite order differential operators on a given complex manifold X. And, as it is well known, a finitely presented module M

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Figure 1: Mikio Sato and Masaki Kashiwara

over a ring R is the intrinsic way to formulate what is a finite system of R-linear equations with finitely many unknowns. Hence, a coherent $\mathfrak{D}_X$-module $\mathfrak{M}$ on X is nothing but a system of linear partial differential equations with holomorphic coefficients. Locally on X, it can be represented, non uniquely, by a matrix of differential operators.

In this thesis, Masaki defines the operations of inverse or direct images for $\mathfrak{D}$-modules. Roughly speaking, these operations describe the system of equations satisfied by the restriction or the integral of the solutions of a system of equations. In particular he extends the classical Cauchy–Kowalevski theorem to general systems of LPDE. Consider the contravariant functor $\mathcal{S}ol$, which to a $\mathfrak{D}_X$-module $\mathfrak{M}$ on the manifold X, associates the complex (in the derived category of sheaves of $\mathbb{C}$-vector spaces) $R\mathcal{H}om_{\mathfrak{D}_X}(\mathfrak{M}, \mathcal{O}_X)$ of its holomorphic solutions. The Cauchy–Kowalevski–Kashiwara theorem essentially asserts that the functor $\mathcal{S}ol$ commutes with the functor of taking the inverse image, under a non characteristic hypothesis.

Hence Masaki Kashiwara may be considered as the founder of analytic $\mathfrak{D}$-module theory, in parallel with Joseph Bernstein (see Bernstein [1971]) for the algebraic case, a theory which is now a fundamental tool in many branches of mathematics, from number theory to mathematical physics.

The seventies are, for the analysts, the era of microlocal analysis. As mentioned above, the starting point was the introduction by Sato of the microlocalization functor and the analytic wave front set. These ideas were then systematically developed in the famous paper M. Sato, Kawai, and Kashiwara [1973] by Mikio Sato, Takahiro Kawai and Masaki Kashiwara. Two fundamental results are proved here.

First, the involutivity of characteristics of microdifferential systems. This was an open and fundamental question which had, at that time, only a partial answer due to Guillemin,

Quillen, and Sternberg [1970] (later, a purely algebraic proof was given by Gabber [1981]).

The second result is a classification at generic points of any system of microdifferential equations. Roughly speaking, it is proved that, generically and after a so-called quantized contact transform, any such system is equivalent to a combination of a partial De Rham system, a partial Dolbeault system and a Hans Lewy's type system.

This paper has had an enormous influence on the analysis of partial differential equations (see in particular Hörmander [1983, 1985] and Sjöstrand [1982]).

The Riemann–Hilbert correspondence (regular case). Since the characteristic variety of a coherent $\mathfrak{D}$-module is involutive (one better say nowadays "co-isotropic"), it is natural to look at the extreme case, when this variety is Lagrangian. One calls such systems "holonomic". They are the higher dimensional version of classical ordinary differential equations (ODE). Among ODE, there is a class of particular interest, called the class of Fuchsian equations, or also, the equations with regular singularities. Roughly speaking, the classical Riemann–Hilbert correspondence (R-H correspondence, for short) is based on the following question: given a finite set of points on the Riemann sphere and at each point, an invertible matrix of complex numbers (all of the same size), does there exist a unique Fuchsian ODE whose singularities are the given points and such that the monodromy of its holomorphic solutions are the given matrices.

From 1975 to 1980, Masaki Kashiwara gives a precise formulation of the conjecture establishing this correspondence in any dimension and eventually proves it.

In 1975 (see Kashiwara [1975]) he proves that the contravariant functor $\mathcal{S}ol$, when restricted to the derived category of $\mathfrak{D}$-modules with holonomic cohomology, takes its values in the derived category of sheaves with $\mathbb{C}$-constructible cohomology. In the same paper, he also proves that if one starts with a $\mathfrak{D}$-module "concentrated in degree 0", then the complex one obtains satisfies, what will be called five years later by Beilinson, Bernstein, and Deligne [1982], the perversity conditions .

Moreover, already in 1973, he gives in Kashiwara [1973] a formula to calculate the local index of the complex $\mathcal{S}ol(\mathfrak{M})$ in terms of the characteristic cycle of the holonomic $\mathfrak{D}$-module $\mathfrak{M}$ and his formula contains the notion of "local Euler obstruction" introduced independently by MacPherson [1974].

Classical examples in dimension 1 show that the functor $\mathcal{S}ol$ cannot be fully faithful and the problem of defining the category of "regular holonomic $\mathfrak{D}$-modules", the higher dimensional version of the Fuchsian ODE, remained open. For that purpose, Masaki introduces with Toshio Oshima (see Kashiwara and Oshima [1977]) the notion of regular singularities along a smooth involutive manifold and then formulates precisely in 1978 the notion of regular holonomic $\mathfrak{D}$-module and what should be the Riemann–Hilbert correspondence (see Ramis [1978]), namely an equivalence of categories between the derived category of $\mathfrak{D}$-modules with regular holonomic cohomology and the derived category of sheaves with $\mathbb{C}$-constructible cohomology. He solves this conjecture in 1980 (see Kashiwara [1980, 1984]) by constructing a quasi-inverse to the functor $\mathcal{S}ol$, the functor $\mathrm{T}\mathcal{H}om$ of tempered cohomology. For a constructible sheaf F, the object

$T\mathcal{H}om(F, \mathcal{O}_X)$ is represented by applying the functor $\mathcal{H}om(F, \bullet)$ to the Dolbeault resolution of $\mathcal{O}_X$ by differential forms with distributions as coefficients.

Of course, Kashiwara's paper came after Pierre Deligne's famous book Deligne [1970] in which he solves the R-H problem for regular connections. This book has had a deep influence on the microlocal approach of the R-H correspondence, elaborated by Masaki jointly with T. Kawai (see Kashiwara and Kawai [1981]). Finally note that a different proof of this theorem was obtained later by Zogman Mebkhout [1984].

Other results on $\mathfrak{D}$-modules and related topics.

(i) Besides his proof of the R-H correspondence, Masaki obtains fundamental results in $\mathfrak{D}$-module theory. He proves in Kashiwara [1976] the rationality of the zeroes of the b-function of Bernstein–Sato by using Hironaka's theorem and adapting Grauert's direct image theorem to $\mathfrak{D}$-modules.

(ii) Motivated by the theory of holonomic $\mathfrak{D}$-modules, Masaki proves in Kashiwara [1982] the codimension-one property of quasi-unipotent sheaves.

(iii) Masaki gives a fundamental contribution to the theory of "variation of (mixed) Hodge structures" (see for example Kashiwara [1985b, 1986]).

(iv) In Barlet and Kashiwara [1981], Masaki and Daniel Barlet endow regular holonomic $\mathfrak{D}$-modules with a "canonical" good filtration.

(v) A classical theorem of complex geometry (Frisch–Guenot, Siu, Trautmann) asserts that, on a complex manifold X, any reflexive coherent sheaf defined on the complementary of a complex subvariety of codimension at least 3 extends as a coherent sheaf through this subvariety. The codimension 3 conjecture is an analogue statement for holonomic microdifferential modules when replacing X with a Lagrangian subvariety of the cotangent bundle. This extremely difficult conjecture was recently proved by Masaki together with Kari Vilonen in Kashiwara and Vilonen [2014].

(vi) Kashiwara's book on $\mathfrak{D}$-modules Kashiwara [2003] contains a lot of original and deep results. In this book he defines in particular the microlocal b-functions and gives a tool, the "holonomy diagrams", to calculate them.

(vii) The book on category theory Kashiwara and Schapira [2006], written with the author, sheds new light on a very classical subject and contains a great deal of original results.

Mathematical physics.

(i) In collaboration with Takahiro Kawai and Henri Stapp, Masaki applied the theory of holonomic $\mathfrak{D}$-modules to the study of Feynman integrals. See in particular Kashiwara and Kawai [1977b,a, 1978] and Kashiwara, Kawai, and Stapp [1977].

(ii) In M. Sato and Y. Sato [1982], Mikio Sato and Yasuko Sato established that soliton equations are dynamical systems on the infinite Grassmann manifold. Based on this work, Kashiwara, with Etsuro Date, Michio Jimbo and Tetsuji Miwa (see Date, Jimbo, Kashiwara, and Miwa [1981, 1981–1982, 1982]), have found links between hierarchies of soliton equations and representations of infinite dimensional Lie algebras, *e.g.,* between the KP hierarchy and $\mathfrak{gl}_\infty$, the KdV hierarchy and the affine Lie algebra of type $A_1^{(1)}$, and so on. In terms of Hirota's dependent variable Hirota [1971], the set of soliton solutions of a hierarchy is identified with the group orbit of 1 in the space of the vertex operator representation of the corresponding infinite-dimensional Lie algebra[1].

Representation theory. Masaki Kashiwara also had an enormous influence in representation theory, harmonic analysis and quantum groups. His work has transformed the field, in its algebraic, categorical, combinatorial, geometrical and analytical aspects.

(i) In Kashiwara, Kowata, Minemura, Okamoto, Oshima, and Tanaka [1978], Kashiwara solves a conjecture of Helgason on non-commutative harmonic analysis.

(ii) At the same period, he proves a fundamental result on the Campbell–Hausdorff formula Kashiwara and Vergne [1978] in collaboration with Michèle Vergne. There is currently a lot of activity stemming from this paper.

(iii) In collaboration with Jean-Luc Brylinski, he solves in Brylinski and Kashiwara [1981] a major open problem in representation theory, the Kazhdan–Lusztig conjecture on infinite-dimensional representations of simple Lie algebras, a conjecture proved independently by Beilinson and Bernstein [1981]. This is one of the most influential paper in geometric representation theory.

(iv) These results are generalized to Kac–Moody algebras with Toshiyuki Tanisaki (see Kashiwara and Tanisaki [1990]): this was one of the key steps in the proof of Lusztig's conjecture on simple modules for algebraic groups in positive characteristic.

(v) Kashiwara has also obtained major results on representations of real Lie groups.

He reinterprets the Harish-Chandra theory in terms of $\mathfrak{D}$-module theory and obtains by this method important theorems on semi-simple Lie algebras with Ryoshi Hotta (see Hotta and Kashiwara [1984]), on real reductive groups with Wilfried Schmid (see Kashiwara and Schmid [1994]).

The final theory constructed by Kashiwara [2008] shows how to construct geometrically the Lie group representations coming from Harish-Chandra modules:

[1] I warmly thank Tetsuji Miwa for his help concerning this topic.

the first step is localization, turning Harish-Chandra modules into $\mathfrak{D}$-modules on the flag variety. Kashiwara's Riemann–Hilbert correspondence turns those into constructible sheaves. Via a sheaf theoretic version of the Matsuki correspondence, these become equivariant sheaves for the real Lie group, which lead to the correct representations of the real Lie group.

(vi) In Kashiwara and Rouquier [2008], using deformation quantization modules (see below), Kashiwara constructs with Raphaël Rouquier a microlocalization of rational Cherednik algebras. This is the first extension of classical localization methods to symplectic manifolds that are not cotangent bundles and opens a new direction in geometric representation theory.

Quantum groups and crystal bases[2].

(i) Finite-dimensional representations of compact Lie groups are some of the most fundamental objects in representation theory. The search for good bases in these representations, in relation with invariant theory and geometry, was a source of attention since the late 19th century. A change of paradigm occurred with Kashiwara's work in 1990. This is based on quantum groups, which are deformations of the enveloping algebras of Kac–Moody Lie algebras. Kashiwara discovered that, when the quantum group parameter goes to 0 (temperature zero limit in the solvable lattice models setting of statistical mechanics), the theory acquires a combinatorial structure, replacing the linear structure. That leads to a basis at parameter 0 (crystal basis), whose existence was proven by an extraordinary combinatorial tour-de-force Kashiwara [1991]. Note that George Lusztig also considered the bases at $q = 0$ given by the PBW bases in the ADE case, and constructed canonical bases (see Lusztig [1990]).

Crystal basis can be lifted uniquely to a basis (global basis) satisfying certain symmetry properties. Crystal bases are now a central chapter of representation theory and algebraic combinatorics.

(ii) Kashiwara used those crystal bases to solve a very basic problem of representation theory, the decomposition of tensor products of irreducible representations of simple Lie algebras.

(iii) Kashiwara has given with Yoshihisa Saito a geometric construction of the crystal basis in terms of Lagrangian subvarieties of Lusztig quiver varieties Kashiwara and Y. Saito [1997].

(iv) Kashiwara's recent work on higher representation theory has been fundamental. In Kang and Kashiwara [2012], he solves with Seok-Jin Kang a basic open problem: he proves that cyclotomic quiver Hecke algebras give rise to simple 2-representations of Kac–Moody algebras.

[2]I warmly thank Raphaël Rouquier for his help on this section.

(v) Kashiwara has obtained a number of key results on finite-dimensional representations of affine quantum groups, in particular on the irreducibility of tensor products. This has led to new directions in higher representation theory.

One such instance is the groundbreaking discovery by Kashiwara, together with Seok-Jin Kang and Myungho Kim (see Kang, Kashiwara, and Kim [2018]) of a new type of Schur–Weyl duality relating quantum affine algebras of arbitrary types and certain quiver Hecke algebras. Another one is the general construction of monoidal categorification of cluster algebras via quiver Hecke algebras with these two authors and Se-jin Oh (Kang, Kashiwara, Kim, and Oh [2018]).

Microlocal sheaf theory. From 1982 to 1990, with Pierre Schapira, he introduces and develops the microlocal theory of sheaves (see Kashiwara and Schapira [1982, 1985, 1990]). This theory emerged from a paper (see Kashiwara and Schapira [1979]) in which they solve the Cauchy problem for microfunction solutions of hyperbolic $\mathfrak{D}$-modules on a real analytic manifold. Indeed, the basic idea is that of microsupport of sheaves which gives a precise meaning to the concept of propagation. On a real manifold M, for a (derived) sheaf F, its microsupport, or singular support, is a closed conic subset of the cotangent bundle T^*M which describes the codirections of non extension of sections of F. The microsupport of sheaves is, in some sense, a real analogue of the characteristic variety of coherent $\mathfrak{D}$-modules on complex manifolds and the functorial properties of the microsupport are very similar to those of the characteristic variety of $\mathfrak{D}$-modules. The precise link between both notions is a result which asserts that the microsupport of the complex $\mathcal{S}ol(\mathfrak{M})$ of holomorphic solutions of a coherent $\mathfrak{D}$-module $\mathfrak{M}$ is nothing but the characteristic variety of $\mathfrak{M}$. Moreover, and this is one of the main results of the theory, the microsupport is co-isotropic. As a by-product, one obtains a completely different proof of the involutivity of characteristics of $\mathfrak{D}$-modules.

By using the microsupport, one can now localize the derived category of sheaves on open subsets of T^*M and the prestack (presheaf of categories) one obtains is a candidate to be a first step for an alternative construction of the Fukaya category, a program recently initiated by Dmitry Tamarkin [2015].

Figure 2: Masaki Kashiwara and Pierre Schapira, Italy 2009

Microlocal sheaf theory naturally leads Kashiwara to extend his previous work on complex analytic Lagrangian cycles to the real setting. In Kashiwara [1985a] he defines the characteristic cycle of an $\mathbb{R}$-constructible sheaf and gives a new index formula. He also gives in this context a remarkable and unexpected "local Lefschetz formula" with applications to representation theory (see Kashiwara and Schapira [1990]*Ch. IX § 6).

Microlocal sheaf theory has found applications in many other fields of mathematics, such as representation theory (see above) and symplectic topology with Tamarkin, Nadler, Zaslow, Guillermou and many others (see in particular Tamarkin [2008], Nadler

and Zaslow [2009], Nadler [2009], and Guillermou [2012]). It has also applications in knot theory thanks to a result of Guillermou, Kashiwara, and Schapira [2012] which implies that the category of simple sheaves along a smooth Lagrangian submanifold is a Hamiltonian isotopy invariant (see *e.g.,* Shende, Treumann, and Zaslow [2017]). Recently, Alexander Beilinson [2015] adapted the definition of the microsupport of sheaves to arithmetic geometry and the theory is currently being developed, in particular by T. Saito [2017].

Deformation quantization. In Kashiwara [1996], Masaki introduces the notion of algebroid stacks in order to quantize complex contact manifolds several years before such constructions become extremely popular.

Then, with P. Schapira, they undertook in the book Kashiwara and Schapira [2012] a systematic study of DQ-modules (DQ for "Deformation Quantization") on complex Poisson manifolds. This is a theory which contains both that of usual $\mathfrak{D}$-modules and classical analytic geometry (the commutative case). A perversity theorem (in the symplectic case) is obtained. This book also contains the precise statement of an old important conjecture of Masaki on the Todd class in the Riemann–Roch theorem, a conjecture recently proved by Julien Grivaux [2012] (see also Ajay Ramadoss [2008] for the algebraic case).

An illustration of the usefulness of DQ-modules is the quantization of Hilbert schemes of points on the plane, constructed in Kashiwara and Rouquier [2008].

Ind-sheaves and the irregular Riemann–Hilbert correspondence. As already mentioned, Masaki introduced the functor of tempered cohomology in the 80s, in his proof of the R-H correspondence. This functor is systematically studied with P. Schapira in Kashiwara and Schapira [1996] where a dual functor, the functor of Whitney tensor product, is also introduced. However, the construction of these two functors appears soon as a particular case of a more general notion, that of ind-sheaves, that is, ind-objects of the category of sheaves with compact supports. This theory is developed in Kashiwara and Schapira [2001].

Ind-sheaf theory is a tool to treat functions or distributions with growth conditions with the techniques of sheaf theory. In particular, it allows one to define the (derived) sheaf $\mathcal{O}_X^{tp}$ of holomorphic functions with tempered growth (a sheaf for the so-called subanalytic topology). Already, in the early 2000, it became clear that this ind-sheaf was an essential tool for the study of irregular holonomic modules and a toy model was studied in Kashiwara and Schapira [2003]. However, although the functor of tempered holomorphic solutions is much more precise than the usual functor $\mathcal{S}ol$, it is still not enough precise to be fully faithful on the category of irregular holonomic $\mathfrak{D}$-modules. Then, by adapting to ind-sheaves a construction of Dmitry Tamarkin [2008], Masaki and Andrea D'Agnolo introduced in D'Agnolo and Kashiwara [2016] the "enhanced ind-sheaf of tempered holomorphic functions" and obtained a fully faithful functor. This deep theory, which uses in an essential manner the fundamental results of Takuro Mochizuki [2009, 2011] (see also Claude Sabbah [2000] for preliminary results and Kiran Kedlaya

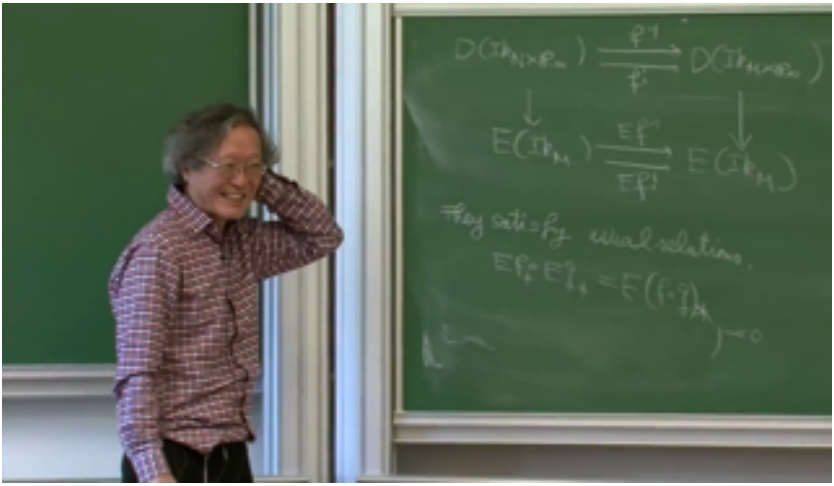

Figure 3: Masaki Kashiwara, IHES 2015

[2010, 2011] for the analytic case), has important applications, in particular in the study of the Laplace transform (see Kashiwara and Schapira [2016]).

Conclusion. Kashiwara's contribution to mathematics is really astonishing and it should be mentioned that his influence is not only due to his published work, but also to many informal talks. Important subjects such as second microlocalization, complex quantized contact transformations, the famous "watermelon theorem", etc. were initiated by him, although not published. Masaki is an invaluable source of inspiration for many people.

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ARITHMETIC PHYSICS – ABEL LECTURE

Michael Atiyah

1 Introduction

My Abel lecture in Rio de Janeiro was devoted to solving the riddle of the fine structure constant α, an outstanding problem from the world of fundamental physics.

Here is what the Nobel Laureate Richard Feynman had to say about α :*Where does α come from; is it related to π,or perhaps to e? Nobody knows, it is one of the great damn mysteries of physics: a magic number that comes to us with no understanding by man. You might say the hand of God wrote the number and we don't know how He pushed his pencil.*

Within the constraints of a one-hour lecture, before an audience of a thousand mathematicians, I sketched my explanation of where α came from. The actual title of my lecture was: The Future of Mathematical Physics: new ideas in old bottles.

During the ICM I listened to many brilliant lectures and talked with many clever people. As a result my ideas crystallised and, on the long flight back from Rio to Frankfurt, they took final shape. Rather than just expand on my Abel Lecture I will attempt to outline the broad programme that starts with the fine structure constant and aims to build a Theory of Arithmetic Physics.

A programme with a similar title was put forward by Yuri Manin in Bonn in 1984 Atiyah [1985]. In fact Manin was not able to come and I presented his lecture for him. This gave me the opportunity to suggest that Manin's programme was too modest, and that it should be lifted from the classical regime to the quantum regime with Hilbert spaces in the forefront.

Visionary programmes take decades to mature and it was 34 years later, on the flight from the new world to the old, that the programme revealed its true self. The title of my Abel lecture suddenly acquired a new meaning, with bottles replaced by continents.

In the next section I will survey the history of the "old bottles" and show how old ideas can be put to new use.

MSC2010: primary 11G05; secondary 11G40, 14G25, 14H52, 14K15.
Keywords: Elliptic curve.

2 Past, Present and Future

There are two stories from the past that I wish to resurrect. The first, and older of the two, is the Theory of von Neumann Algebras Murray and von Neumann [1936]. It arose from the study of operators in Hilbert space that von Neumann used to lay the foundations of quantum mechanics. It created a deep bridge between non-commutative Algebra and Analysis. In more recent years, in the hands of Alain Connes Connes [1994], it has been further refined to include Differential Geometry.

The second story from the past was initiated by Fritz Hirzebruch Hirzebruch [1966], as part of the new algebraic geometry centred on Paris and Princeton. Fritz was a magician, a true successor to Euler Dunham [1999], who found beautiful new applications of the Bernoulli numbers, creating a bridge between number theory and algebraic geometry. My own work in collaboration with Raoul Bott Atiyah and Bott [1966] and Is Singer Atiyah and Singer [1963] extended the bridge into differential geometry and topology.

The Hirzebruch era was a generation later than the von Neumann era, but by then the foundations of Arithmetic Physics were being laid, as Manin recognized. With subsequent developments in Physics led by Edward Witten, M-theory emerged and provided strong hints that Arithmetic Physics should indeed lift to the quantum world as I had speculated.

My paper on the fine structure constant, which provides the details behind the ideas sketched in my Abel Lecture, has been submitted to the Proceedings A of the Royal Society Atiyah [2018a]. It was recorded in the ArXiv on 15/8/18. As advertised in Atiyah and Kouneiher [n.d.] a second paper on Newton's constant is in preparation.

In this article I will explain that (quantum) Arithmetic Physics is essentially Algebraic Geometry over the Quaternions $\mathbb{H}$. I will show that it mimics algebraic geometry over the complex field $\mathbb{C}$. The only difference is that the complex field is replaced by the hyperfinite II-1 factor $A(\mathbb{C})$. Moreover an algebraic number field $\mathbb{Q}'$ in $\mathbb{C}$ is replaced by what I will call the Hirzebruch algebra $A(\mathbb{Q}')$.

A careful scrutiny of Hirzebruch's formalism shows that it essentially gives the Hilbert Nullstellensatz over the quaternions $\mathbb{H}$. Since Hilbert's theorem is the foundation stone of complex algebraic geometry, its counterpart over A will be the foundation stone of quaternionic algebraic geometry. $\mathbb{C}$ and its arithmetic subfields $\mathbb{Q}'$ get replaced by A and its arithmetic sub-factors $A(\mathbb{Q}')$.

These algebraic/analytic ideas enable us to extend classical algebra/analysis from the commutative world of $\mathbb{Q}'$ and $\mathbb{C}$ to the non-commutative world of $A(\mathbb{Q}')$ and $A(\mathbb{C})$. This converts large areas of classical mathematics into non-commutative counterparts. The traces of von Neumann factors, the holomorphic Euler characteristics of Hirzebruch and the index of Atiyah and Singer [1963] give stable commutative invariants from the

non-commutative world. This is the way π gets converted to $\gimel$.

Just as π is related to the circle via the equation

$$x^2 + y^2 = 1 \tag{1}$$

where x and y are commuting variables in $\mathbb{R} \subset \mathbb{C}$, $\gimel$ is related to the same equation with non-commuting variables in $A(\mathbb{R}) \subset A(\mathbb{C})$.

Keeping the same equation, but taking arithmetic subfields $\mathbb{Q}'$ gives interesting arithmetic invariants interpolating between π and $\gimel$. As Hirzebruch showed, the Bernoulli functions and the Todd polynomials play a key role in this story, which is precisely the topic of Hirzebruch [1966].

Hirzebruch had many variants and extensions of his Todd polynomials and they are all based on this one rational case.

Replacing the equation (2.1) by other polynomial equations leads to elliptic and higher genus curves whose counterparts give rise to invariants of elliptic fields or fields of higher genus.

It seems likely that this will provide a novel approach to algebraic number theory, yet to be explored. In the next section I will peer into this future and describe the vista I see.

3 A new programme for the future

I envisage Arithmetic Physics as a future programme ahead that will unify many fields. These include:

3.1 the Langlands programme (number theory)

3.2 Donaldson theory (differential geometry)

3.3 M–theory (theoretical physics)

3.4 Burnside problems (group theory)

3.5 Thompson theory (finite simple groups)

There are already many hints that these fields are connected in some way, via modular forms and L-functions. Examples of old problems that have been rejuvenated and extended by the first steps into the domain of Arithmetic Physics include:

3.6. Complex structures on the 6-sphere Atiyah [2016] Atiyah [2018b]

3.7 Groups of odd order Atiyah [n.d.]

3.8 The fine structure constant Atiyah [2018a]

Work in progress includes:

3.9 Newton's constant Atiyah and Kouneiher [n.d.], joint with J.Kouneiher

3.10 New results on the restricted Burnside problem Atiyah and Zuk [n.d.], joint with A. Zuk.

Going beyond the Quaternions $\mathbb{H}$, to the Octonions $\mathbb{O}$, deepens Arithmetic Physics in several ways:

3.11 It extends the physics to include Gravity and black holes

3.12 It extends group theory to include Exceptional groups

This indicates the potential scope of the theory showing that much remains to be done. In the next section I will comment on the technical tools that will be needed.

4 New Technology

The fundamental idea behind the new technology is that all the hard problems of mathematics or physics are non-linear, but our tools are linear. A familiar example is a CT scan which inputs linear waves and attempts to reconstruct your bones or your brain. Bombarding a nucleus with waves and analysing the scattering is another example. These are examples from biology and physics. Representation theory is an example from group theory, while classical modular forms provide an example from the theory of elliptic curves.

This was the bread and butter of Arithmetic Physics in the classical setting described by Manin in 1984. In the classical theories, as the subject developed, more elaborate processes were constructed. A simple example is the use of cohomology theory which began with H^1 but gradually moved on to H^2 and H^3.

The new version of Arithmetic Physics that I envisage, replacing classical by quantum ideas, is infinite-dimensional and is reached by an infinite iteration of steps. Addition and multiplication were handled with consummate skill, by Euler Dunham [1999] with infinite series and infinite products. But the hard operation, to iterate to infinity, is the exponential which converts addition into multiplication. In set theory the exponential 2^S is the set of all subsets of S, and it was Emil Artin who exploited this. But it was von Neumann who went one step further and considered an infinite iteration of exponentials as an analytical tool. This analysis was refined by Hirzebruch to be defined

over $\mathbb{Q}$ or over a number field $\mathbb{Q}'$. This is the fundamental tool for the new theory and it encapsulates linear information from Hilbert space, such as the spectrum.

Cohomology is the classical linear invariant and is the basis of much number theory, as in l-adic representations. But it was clear from the work of Grothendieck and Quillen that K-theory is a more sophisticated invariant, being filtered rather than graded. Moreover index theory extended it to differential geometry.

K-theory, built on families of linear objects, has myriad refinements, notably equivariant K-theory, which makes it the perfect tool when we want to extract stable linear invariants of non-linear objects. This is the ultimate key to understanding the non-linear world.

This is the philosophy, but the proof of the pudding is in the eating, and the examples given in section 3 demonstrate that the philosophy has genuine content. It is reminiscent of the early days of the Langlands programme where crucial examples were shown to lead to deep theorems.

5 Hard Analysis

In classical analysis there is an informal distinction between soft and hard analysis. Linear theory is "soft" and non-linear theory is "hard". Fourier Theory in L^2 is "soft" while L^p, for $p \neq 2$ (required for non-linear problems) is "hard". I am indebted to Jim Wright for a crash course in hard analysis explaining the sharp Hausdorff–Young inequality. I realized how varying p helped both analysts and number theorists move up the von Neumann factor $A(\mathbb{C})$ and the Hirzebruch algebra $A(\mathbb{Q})$

Large parts of mathematics and physics employ linear methods to attack non-linear problems. Soft mathematics comes to the help of hard mathematics. But this is a tricky process requiring much skill: the soft methods must be iterated, by feed-back, to reach the hard core. There are subtle points in how finite data converges to its various limits. Understanding this is important both logically and computationally.

The fundamental questions are always the same:

A) Does the iterated use of linear methods lead to complete information about the non-linear problem?

B) What collection of linear data can arise from the non-linear object?

Put more graphically in terms of shining light on a statue:

A') Can we see inside the statue?

B') What shadow does the statue cast?

More formally in terms of a functor from a non-linear category to a linear one :

A") What is the kernel of the functor; is it injective ?

B") What is the image of the functor; is it surjective (onto what we expect).

The technology of Arithmetic Physics which I envisage incorporates infinite iterations (von Neumann/Hirzebruch) and provides a powerful universal tool for answering the fundamental questions A) and B). There are many aspects of this process and they provide a great variety of approaches, often related by dualities. As seen by me, Arithmetic Physics is a branch of mathematics which has, *inter alia*, many applications in physics. Its foundations are the foundations of mathematics, where we encounter Kurt Gödel.

As seen from the other end of the vista, Arithmetic Physics is a branch of physics, based on experiments, which has many links with mathematics. Both perceptions are true and together form a deep truth in the sense of Pauli : a deep truth is one whose opposite is also true. Asking whether Pauli's dictum is itself a deep truth is the ultimate conundrum which only Gödel can answer.

It should cause no surprise if this programme of Arithmetic Physics solves many old problems or sheds light on their insolubility. Moreover old barriers disappear or are lowered; the barrier between the discrete and the continuous being just one. This is where Euler's constant γ appears as does its quaternionic counterpart Ч.

Because π is so important in classical mathematics it is inevitable that Ж will be equally important in "quantum" or "quaternionic" mathematics (perhaps q-mathematics is the right terminology). Later when we bring in "gravitational" or "octonionic" mathematics we will need to introduce a new Arabic symbol ȣ to move Ж to a higher level. A coloured table of symbols from different cultures is tabulated as an aide mémoire.

In this final summing up I have outlined the vision of Arithmetic Physics that inspired me as I flew back from Rio to Frankfurt. A vision or a programme is unfinished business, leaving subsequent generations to flesh it out. The Abel Foundation includes in its aims both the advancement of mathematics and its dissemination into the community. That was my goal.

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CONFORMAL GEOMETRY ON FOUR MANIFOLDS – NOETHER LECTURE

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Abstract

This is the lecture notes for the author's Emmy Noether lecture at 2018, ICM, Rio de Janeiro, Brazil. It is a great honor for the author to be invited to give the lecture.

Introduction

In the lecture notes, the author will survey the development of conformal geometry on four dimensional manifolds. The topic she chooses is one on which she has been involved in the past twenty or more years: the study of the integral conformal invariants on 4-manifolds and geometric applications. The development was heavily influenced by many earlier pioneer works; recent progress in conformal geometry has also been made in many different directions, here we will only present some slices of the development.

The notes is organized as follows.

In section 1, we briefly describe the prescribing Gaussian curvature problem on compact surfaces and the Yamabe problem on n-manifolds for $n \geq 3$; in both cases some second order PDE have played important roles.

In section 2, we introduce the quadratic curvature polynomial σ_2 on compact closed 4-manifolds, which appears as part of the integrand of the Gauss–Bonnet–Chern formula. We discuss its algebraic structure, its connection to the 4-th order Paneitz operator P_4 and its associated 4-th order Q curvature. We also discuss some variational approach to study the curvature and as a geometric application, results to characterize the diffeomorphism type of (S^4, g_c) and $(\mathbb{CP}^2, g_{FS})$ in terms of the size of the conformally invariant quantity: the integral of σ_2 over the manifold.

In section 3, we extend our discussion to compact 4-manifolds with boundary and introduce a third order pseudo-differential operator P_3 and 3-order curvature T on the boundary of the manifolds.

In section 4, we shift our attention to the class of conformally compact Einstein (abbreviated as CCE) four-manifolds. We survey some recent research on the problem of "filling in" a given 3-dimensional manifold as the conformal infinity of a CCE manifold.

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We relate the concept of "renormalized" volume in this setting again to the integral of σ_2.

In section 5, we discuss some partial results on a compactness problem on CCE manifolds. We believe the compactness results are the key steps toward an existence theory for CCE manifolds.

The author is fortunate to have many long-term close collaborators, who have greatly contributed to the development of the research described in this article – some more than the author. Among them Matthew Gursky, Jie Qing, Paul Yang and more recently Yuxin Ge. She would like to take the chance to express her deep gratitude toward them, for the fruitful collaborations and for the friendships.

1 Prescribing Gaussian curvature on compact surfaces and the Yamabe problem

In this section we will describe some second order elliptic equations which have played important roles in conformal geometry.

On a compact surface (M, g) with a Riemannian metric g, a natural curvature invariant associated with the Laplace operator $\Delta = \Delta_g$ is the Gaussian curvature $K = K_g$. Under the conformal change of metric $g_w = e^{2w} g$, we have

$$-\Delta w + K = K_{g_w} e^{2w} \ on \ M. \tag{1.1}$$

The classical uniformization theorem to classify compact closed surfaces can be viewed as finding solution of equation (1.1) with $K_{g_w} \equiv -1$, 0, or 1 according to the sign of $\int K_g dv_g$. Recall that the Gauss–Bonnet theorem states

$$2\pi \, \chi(M) = \int_M K_{g_w} \, dv_{g_w}, \tag{1.2}$$

where $\chi(M)$ is the Euler characteristic of M, a topological invariant. The variational functional with (1.1) as Euler equation for $K_{g_w} =$ constant is thus given by Moser's functional (Moser [1970/71, 1973])

$$J_g[w] = \int_M |\nabla w|^2 dv_g + 2\int_M K w dv_g - \left(\int_M K dv_g\right) \log \frac{\int_M dv_{g_w}}{\int_M dv_g}. \tag{1.3}$$

There is another geometric meaning of the functional J which influences the later development of the field, that is the formula of Polyakov [1981]

$$J_g[w] = 12\pi \ \log \left(\frac{\det \ (-\Delta)_g}{\det \ (-\Delta)_{g_w}}\right) \tag{1.4}$$

for metrics g_w with the same volume as g; where the determinant of the Laplacian $\det \ \Delta_g$ is defined by Ray–Singer via the "regularized" zeta function.

In Onofri [1982], (see also Hong [1986]), Onofri established the sharp inequality that on the 2-sphere $J[w] \geq 0$ and $J[w] = 0$ precisely for conformal factors w of the form $e^{2w} g_0 = T^* g_0$ where T is a Moebius transformation of the 2-sphere. Later Osgood, Phillips, and Sarnak [1988a,b] arrived at the same sharp inequality in their study of heights of the Laplacian. This inequality also plays an important role in their proof of the C^∞ compactness of isospectral metrics on compact surfaces.

On manifolds (M^n, g) for n greater than two, the conformal Laplacian L_g is defined as $L_g = -\Delta_g + c_n R_g$ where $c_n = \frac{n-2}{4(n-1)}$, and R_g denotes the scalar curvature of the metric g. An analogue of equation (1.1) is the equation, commonly referred to as the Yamabe equation (1.5), which relates the scalar curvature under conformal change of metric to the background metric. In this case, it is convenient to denote the conformal metric as $\hat{g} = u^{\frac{4}{n-2}} g$ for some positive function u, then the equation becomes

$$L_g u = c_n \hat{R} u^{\frac{n+2}{n-2}}. \tag{1.5}$$

The famous Yamabe problem to solve (1.5) with $\hat{R}$ a constant has been settled by Yamabe [1960], Trudinger [1968], Aubin [1976] and Schoen [1984]. The corresponding problem to prescribe scalar curvature has been intensively studied in the past decades by different groups of mathematicians, we will not be able to survey all the results here. We will only point out that in this case the study of $\hat{R} = c$, where c is a constant, over class of metrics $\hat{g}$ in the conformal class $[g]$ with the same volume as g, is a variational problem with respect to the functional $F_g[u] = \int_{M^n} R_{\hat{g}} dv_{\hat{g}}$ for any $n \geq 3$. Again the sign of the constant c agrees with the sign of the Yamabe invariant

$$Y(M, g) := \inf_{\hat{g} \in [g]} \frac{\int_M R_{\hat{g}} dv_{\hat{g}}}{(vol\ \hat{g})^{\frac{n-2}{n}}}. \tag{1.6}$$

2 σ_2 curvature on 4-manifold

2.1 Definition and structure of σ_2. We now introduce an integral conformal invariant which plays a crucial role in this paper, namely the integral of σ_2 curvature on four-manifolds.

To do so, we first recall the Gauss–Bonnet–Chern formula on closed compact manifold (M, g) of dimension four:

$$8\pi^2 \chi(M) = \int_M \frac{1}{4} |W_g|^2 dv_g + \int_M \frac{1}{6} (R_g^2 - 3|Ric_g|^2) dv_g, \tag{2.7}$$

where $\chi(M)$ denotes the Euler characteristic of M, W_g denotes the Weyl curvature, R_g the scalar curvature and Ric_g the Ricci curvature of the metric g.

In general, the Weyl curvature measures the obstruction to being conformally flat. More precisely, for a manifold of dimension greater or equals to four, W_g vanishes in a neighborhood of a point if an only if the metric is locally conformal to a Euclidean

metric; i.e., there are local coordinates such that $g = e^{2w}|dx|^2$ for some function w. Thus for example, the standard round metric g_c on the sphere S^n has $W_{g_c} \equiv 0$.

In terms of conformal geometry, what is relevant to us is that Weyl curvature is a *pointwise* conformal invariant, in the sense that under conformal change of metric $g_w = e^{2w}g$, $|W_{g_w}| = e^{-2w}|W_g|$, thus on 4-manifold $|W_{g_w}|^2 dv_{g_w} = |W_g|^2 dv_g$; this implies in particular that the first term in the Gauss–Bonnet–Chern formula above

$$g \to \int_M |W|_g^2 dv_g$$

is conformally invariant.

For reason which will be justified later below, we denote

$$\sigma_2(g) = \frac{1}{6}(R_g^2 - 3|Ric_g|^2) \tag{2.8}$$

and draw the conclusion from the above discussion of the Gauss–Bonnet–Chern formula that

$$g \to \int_M \sigma_2(g) dv_g$$

is also an integral conformal invariant. This is the fundamental conformal invariant which will be studied in this lecture notes. We begin by justifying the name of "σ_2" curvature.

On manifolds of dimensions greater than two, the Riemannian curvature tensor Rm can be decomposed into the different components. From the perspective of conformal geometry, a natural basis is the Weyl tensor W, and the Schouten tensor, defined by

$$A_g = Ric_g - \frac{R}{2(n-1)}g.$$

The curvature tensor can be decomposed as

$$Rm_g = W_g \oplus \frac{1}{n-2} A_g \owedge g.$$

Under conformal change of metrics $g_w = e^{2w}g$, since the Weyl tensor W transforms by scaling, only the Schouten tensor depends on the derivatives of the conformal factor. It is thus natural to consider $\sigma_k(A_g)$, the k-th symmetric function of the eigenvalues of the Schouten tensor A_g, as curvature invariants of the conformal metrics.

When $k = 1$, $\sigma_1(A_g) = Tr_g\ A_g = \frac{n-2}{2(n-1)}R_g$, so the σ_1-curvature is a dimensional multiple of the scalar curvature.

When $k = 2$, $\sigma_2(A_g) = \sum_{i<j} \lambda_i \lambda_j = \frac{1}{2}(|Tr_g\ A_g|^2 - |A_g|^2)$, where the λs are the eigenvalues of the tensor A_g. For a manifold of dimension 4, we have

$$\sigma_2(g) = \sigma_2(A_g) = \frac{1}{6}(R_g^2 - 3|Ric_g|^2).$$

When $k = n$, $\sigma_n(A_g) = determinant\ of\ A_g$, an equation of Monge–Ampère type.

In view of the Yamabe problem, it is natural to ask the question under what condition can one find a metric g_w in the conformal class of g, which solves the equation

$$\sigma_2(A_{g_w}) = constant. \tag{2.9}$$

To do so, we first observe that as a differential invariant of the conformal factor w, $\sigma_k(A_{g_w})$ is a fully nonlinear expression involving the Hessian and the gradient of the conformal factor w. We have

$$A_{g_w} = (n-2)\{-\nabla^2 w + dw \otimes dw - \frac{|\nabla w|^2}{2}\} + A_g.$$

To illustrate that (2.9) is a fully non-linear equation, we have when $n = 4$,

$$\begin{aligned} \sigma_2(A_{g_w})e^{4w} &= \sigma_2(A_g) + 2((\Delta w)^2 - |\nabla^2 w|^2 \\ &+ \Delta w|\nabla w|^2 + (\nabla w, \nabla|\nabla w|^2)) \\ &+ \text{lower order terms.} \end{aligned} \tag{2.10}$$

where all derivative are taken with respect to the g metric.

For a symmetric $n \times n$ matrix M, we say $M \in \Gamma_k^+$ in the sense of Gårding [1959] if $\sigma_k(M) > 0$ and M may be joined to the identity matrix by a path consisting entirely of matrices M_t such that $\sigma_k(M_t) > 0$. There is a rich literature concerning the equation

$$\sigma_k(\nabla^2 u) = f, \tag{2.11}$$

for a positive function f, which is beyond the scope of this article to cover. Here we will only note that when $k = 2$,

$$\sigma_2(\nabla^2 u) = (\Delta u)^2 - |\nabla^2 u|^2 = f, \tag{2.12}$$

for a positive function f. We remark the leading term of equations (2.12) and (2.10) agree.

We now discuss a variational approach to study the equation (2.9) for σ_2 curvature .

Recall in section 1 we have mentioned that the functional $\mathfrak{F}(g) := \int_{M^n} R_g dv_g$ is variational in the sense that when $n \geq 3$ and when one varies g in the same conformal class of metrics with fixed volume, the critical metric when attained satisfies $R_{\hat{g}} \equiv$ constant; while when $n = 2$, this is no longer true with R_g replaced by K_g and one needs to replace the functional $\mathfrak{F}(g)$ by the Moser's functional J_g.

Parallel phenomenon happens when one studies the σ_2 curvature. It turns out that when $n > 2$ and $n \neq 4$, the functional $\mathfrak{F}_2(g) := \int_{M^n} \sigma_2(g) dv_g$ is variational when one varies g in the same conformal class of metrics with fixed volume, while this is no longer true when $n = 4$. In section 2a below, we will describe a variational approach to study equation (2.9) in dimension 4 and the corresponding Moser's functional. Before we do so, we would like to end the discussion of this section by quoting a result of Gursky and Viaclovsky [2001].

In dimension 3, one can capture all metrics with constant sectional curvature (i.e. space forms) through the study of σ_2.

Theorem 2.1. (Gursky and Viaclovsky [2001]) On a compact 3-manifold, for any Riemannian metric g, denote $\mathfrak{F}_2(g) = \int_M \sigma_2(A_g) dv_g$. Then a metric g with $\mathfrak{F}_2(g) \geq 0$ is critical for the functional $\mathfrak{F}_2$ restricted to class of metrics with volume one if and only if g has constant sectional curvature.

2.2 4-th order Paneitz operator, Q-curvature. We now describe the rather surprising link between a 4-th order linear operator P_4, its associated curvature invariant Q_4, and the σ_2-curvature.

We first recall on (M^n, g), $n \geq 3$, the second order conformal Laplacian operator $L = -\Delta + \frac{n-2}{4(n-1)} R$ transforms under the conformal change of metric $\hat{g} = u^{\frac{4}{n-2}} g$, as

$$L_{\hat{g}}(\varphi) = u^{-\frac{n+2}{n-2}} L_g(u\,\varphi) \text{ for all } \varphi \in C^\infty(M^4). \tag{2.13}$$

There are many operators besides the Laplacian Δ on compact surfaces and the conformal Laplacian L on general compact manifold of dimension greater than two which have the conformal covariance property. One class of such operators of order 4 was studied by Paneitz [2008] (see also Eastwood and Singer [1985]) defined on (M^n, g) when $n > 2$; which we call the conformal Paneitz operator:

$$P_4^n = \Delta^2 + \delta\left(a_n R g + b_n \mathrm{Ric}\right) d + \frac{n-4}{2} Q_4^n \tag{2.14}$$

and

$$Q_4^n = c_n |Ric|^2 + d_n R^2 - \frac{1}{2(n-1)} \Delta R, \tag{2.15}$$

where a_n, b_n, c_n and d_n are some dimensional constants, δ denotes the divergence, d the deRham differential.

The conformal Paneitz operator is conformally covariant. In this case, we write the conformal metric as $\hat{g} = u^{\frac{4}{n-4}} g$ for some positive function u, then for $n \neq 4$,

$$(P_4^n)_{\hat{g}}(\varphi) = u^{-\frac{n+4}{n-4}} (P_4^n)_g(u\,\varphi) \quad \text{for all } \varphi \in C^\infty(M^4). \tag{2.16}$$

Properties of (P_4^n, Q_4^n) have been intensively studied in recent years, with many surprisingly strong results. We refer the readers to the recent articles (Case and P. Yang [2013], Gursky and Malchiodi [2015], Gursky, Hang, and Lin [2016], Hang and P. C. Yang [2015], Hang and P. C. Yang [2016] and beyond).

Notice that when n is not equal to 4, we have $P_4^n(1) = \frac{n-4}{2} Q_4^n$, while when $n = 4$, one does not read Q_4^4 from P_4^4; it was pointed out by T. Branson that nevertheless both $P := P_4^4$ and $Q := Q_4^4$ are well defined (which we named as *Branson's Q-curvature*):

$$P\varphi := \Delta^2 \varphi + \delta\left(\frac{2}{3} R g - 2\mathrm{Ric}\right) d\varphi.$$

The *Paneitz* operator P is conformally covariant of bidegree $(0, 4)$ on 4-manifolds, i.e.

$$P_{g_w}(\varphi) = e^{-4\omega} P_g(\varphi) \quad \text{for all} \quad \varphi \in C^\infty(M^4).$$

The Q curvature associated with P is defined as

$$2Q_g = -\frac{1}{6}\Delta R_g + \frac{1}{6}(R_g^2 - 3|Ric_g|^2)). \tag{2.17}$$

Thus the relation between Q curvature and σ_2 curvature is

$$2Q_g = -\frac{1}{6}\Delta R_g + \sigma_2(A_g). \tag{2.18}$$

The relation between P and Q curvature on manifolds of dimension four is like that of the Laplace operator $-\Delta$ and the Gaussian curvature K on compact surfaces.

$$P_g w + 2Q_g = 2Q_{g_w} e^{4w}. \tag{2.19}$$

Due to the pointwise relationship between Q and σ_2 in (2.18), notice on compact closed 4-manifolds M, $\int_M \Delta R_g dv_g = 0$, thus $\int_M Q_g dv_g = \frac{1}{2}\int \sigma_2(A_g) dv_g$ is also an integral conformal invariant. But Q curvature has the advantage of being prescribed by the linear operator P in the equation (2.19), which is easier to study. We remark the situation is different on compact 4-manifolds with boundary which we will discuss in later part of this article

Following Moser, the functional to study constant Q_{g_w} metric with $g_w \in [g]$ is given by

$$II[w] = \langle Pw, w\rangle + 4\int Qwdv - \left(\int Qdv\right) \log \frac{\int e^{4w} dv}{\int dv}.$$

In view of the relation between Q and σ_2 curvatures, if one consider the variational functional III whose Euler equation is $\Delta R = constant$,

$$III[w] = \frac{1}{3}\left(\int R_{g_w}^2 dv_{g_w} - \int R^2 dv\right),$$

and define

$$\mathfrak{F}[w] = II[w] - \frac{1}{12} III[w],$$

we draw the conclusion:

Proposition 2.2. *(Chang and P. C. Yang [2003], Chang, Gursky, and P. C. Yang [2002], see also Brendle and Viaclovsky [2004] for an alternative approach.)*
$\mathfrak{F}$ is the Lagrangian functional for σ_2 curvature.

We remark that the search for the functional $\mathfrak{F}$ above was originally motivated by the study of some other variational formulas, e.g. the variation of quotients of log determinant of conformal Laplacian operators under conformal change of metrics on 4-manifolds, analogues to that of the Polyakov formula (1.4) on compact surfaces . We

refer the readers to articles (Branson and Ørsted [1991], Branson, Chang, and P. C. Yang [1992], Chang, Gursky, and P. C. Yang [2002], Okikiolu [2001]) on this topic.

We also remark that on compact manifold of dimension n, there is a general class of conformal covariant operators P_{2k} of order $2k$, for all integers k with $2k \leq n$. This is the well-known class of GJMS operators Graham, Jenne, Mason, and Sparling [1992], where P_2 coincides with the conformal Laplace operator L and where P_4 coincides with the 4-th order Paneitz operator. GJMS operators have played important roles in many recent developments in conformal geometry.

2.3 Some properties of the conformal invariant $\int \sigma_2$. We now concentrate on the class of compact, closed four manifolds which allow a Riemannian metric g in the class $\mathcal{A}$, where

$$\mathcal{A} := \left\{ g|, Y(M,g) > 0, \int_M \sigma_2(A_g) dv_g > 0 \right\}.$$

Notice that for closed four manifolds, it follows from equation (2.18) that

$$2\int_M Q_g dv_g = \int_M \sigma_2(A_g) dv_g,$$

so in the definition of $\mathcal{A}$, we could also use Q instead of σ_2.

We recall some important properties of metrics in A.

Theorem 2.3. (Chang and P. C. Yang [1995], Gursky [1999], Chang, Gursky, and P. C. Yang [2002])

1. If $Y(M,g) > 0$, then $\int_M \sigma_2(g) dv_g \leq 16\pi^2$, equality holds if and only if (M,g) is conformally equivalent to (S^4, g_c).
2. $g \in \mathcal{A}$, then $P \geq 0$ with kernel (P) consists of constants; it follows there exists some $g_w \in [g]$ with $Q_{g_w} = constant$ and $R_{g_w} > 0$.
3. $g \in \mathcal{A}$, then there exists some $g_w \in [g]$ with $\sigma_2(A_{g_w}) > 0$ and $R_{g_w} > 0$; i. e. g_w exists in the positive two cone Γ_2^+ of A_g in the sense of Gårding [1959]. We remark $g \in \Gamma_2^+$ implies $Ric_g > 0$, as a consequence the first Betti number b_1 of M is zero.
4. $g \in \mathcal{A}$, then there exists some $g_w \in [g]$ with $\sigma_2(A_{g_w}) = 1$ and $R_{g_w} > 0$.
5. When (M,g) is not conformally equivalent to (S^4, g_c) and $g \in \mathcal{A}$, then for any positive smooth function f defined on M, there exists some $g_w \in [g]$ with $\sigma_2(A_{g_w}) = f$ and $R_{g_w} > 0$.

We remark that techniques for solving the σ_2 curvature equation can be modified to solve the equation $\sigma_2 = 1 + c|W|^2$ for some constant c, which is the equation we will use later in the proof of the theorems in section 2d.

As a consequence of above theorem we have

Corollary 2.1. *On (M^4, g), $g \in \mathcal{A}$ if and only if there exists some $g_w \in [g]$ with $g_w \in \Gamma_2^+$.*

A significant result in recent years is the following "uniqueness" result.

Theorem 2.4. (Gursky and Streets [2018])
Suppose (M^4, g) is not conformal to (S^4, g_c) and $g \in \Gamma_2^+$, then $g_w \in [g]$ with $g_w \in \Gamma_2^+$ and with $\sigma_2(A_{g_w}) = 1$ is unique.

The result was established by constructing some norm for metrics in Γ_2^+, with respect to which the functional $\mathfrak{F}$ is convex. The result is surprising in contrast with the famous example of Schoen [1989] where he showed that on $(S^1 \times S^n, g_{prod})$, where $n \geq 2$, the class of constant scalar curvature metrics (with the same volume) is not unique.

2.4 Diffeomorphism type. In terms of geometric application, this circle of ideas may be applied to characterize the diffeomorphism type of manifolds in terms of the the relative size of the conformal invariant $\int \sigma_2(A_g) dV_g$ compared with the Euler number of the underlying manifold, or equivalently the relative size of the two integral conformal invariants $\int \sigma_2(A_g) dv_g$ and $\int ||W||_g^2 dv_g$.

Note: In the following, we will view the Weyl tensor as an endomorphism of the space of two-forms: $W : \Omega^2(M) \to \Omega^2(M)$. It will therefore be natural to use the norm associated to this interpretation, which we denote by using $\| \cdot \|$. In particular,

$$|W|^2 = 4\|W\|^2.$$

Theorem 2.5. (Chang, Gursky, and P. C. Yang [2003])
Suppose (M, g) is a closed 4-manifold with $g \in \mathfrak{A}$.
(a). If $\int_M ||W||_g^2 dv_g < \int_M \sigma_2(A_g) dv_g$ then M is diffeomorphic to either S^4 or $\mathbb{RP}^4$.
(b). If M is not diffeomorphic to S^4 or $\mathbb{RP}^4$ and $\int_M ||W||_g^2 dv_g = \int_M \sigma_2(A_g) dv_g$, then (M, g) is conformally equivalent to $(\mathbb{CP}^2, g_{FS})$.

Remark 1: The theorem above is an L^2 version of an earlier result of Margerin [1998]. The first part of the theorem should also be compared to an result of Hamilton [1986]; where he pioneered the method of Ricci flow and established the diffeomorphism of M^4 to the 4-sphere under the assumption when the curvature operator is positive.

Remark 2: The assumption $g \in \mathfrak{A}$ excludes out the case when $(M, g) = (S^3 \times S^1, g_{prod})$, where $||W||_g = \sigma_2(A_g) \equiv 0$.

Sketch proof of Theorem 2.5.

Proof. For part (a) of the theorem, we apply the existence argument to find a conformal metric g_w which satisfies the pointwise inequality
(2.20)
$$||W_{g_w}||^2 < \sigma_2(A_{g_w}) \quad or \quad \sigma_2(A_{g_w}) = ||W_{g_w}||^2 + c \quad \text{for some constant } c > 0.$$

The diffeomorphism assertion follows from Margerin [1998] precise convergence result for the Ricci flow: such a metric will evolve under the Ricci flow to one with constant curvature. Therefore such a manifold is diffeomorphic to a quotient of the standard

4-sphere.

For part (b) of the theorem, we argue that if such a manifold is not diffeomorphic to the 4-sphere, then the conformal structure realizes the minimum of the quantity $\int |W_g|^2 dv_g$, and hence its Bach tensor vanishes; i.e.

$$B_g = \nabla^k \nabla^l W_{ikjl} + \frac{1}{2} R^{kl} W_{ikjl} = 0.$$

As we assume $g \in \mathcal{A}$, we can solve the equation

$$\sigma_2(A_{g_w}) = (1-\epsilon)||W_{g_w}||^2 + c_\epsilon, \tag{2.21}$$

where c_ϵ is a constant which tends to zero as ϵ tends to zero. We then let ϵ tends to zero. We obtain in the limit a $C^{1,1}$ metric which satisfies the equation on the open set $\Omega = \{x | W(x) \neq 0\}$:

$$\sigma_2(A_{g_w}) = ||W_{g_w}||^2. \tag{2.22}$$

We then decompose the $Weyl$ curvature of g_w into its self dual and anti-self dual part as in the Singer–Thorpe decomposition of the full curvature tensor, apply the Bach equation to estimate the operator norm of each of these parts as endormorphism on curvature tensors, and reduce the problem to some rather sophisticated Lagrange multiplier problem. We draw the conclusion that the curvature tensor of g_w agrees with that of the Fubini–Study metric on the open set Ω. Therefore $|W_{g_w}|$ is a constant on Ω, thus W cannot vanish at all. From this, we conclude that g_w is Einstein (and under some positive orientation assumption) with $W^-_{g_w} = 0$. It follows from a result of Hitchin (see Besse [1987], chapter 13) that the limit metric g_w agrees with the Fubini–Study metric of $\mathbb{CP}^2$ □

We now discuss some of the recent joint work of M. Gursky, Siyi Zhang and myself Chang, Gursky, and Zhang [2018] extending the theorem 2.5 above to a perturbation theorem on $\mathbb{CP}^2$.

For this purpose, for a metric $g \in \mathcal{A}$, we define the conformal invariant constant $\beta = \beta([g])$ defined as

$$\int ||W||_g^2 dv_g = \beta \int_M \sigma_2(A_g) dv_g.$$

Lemma 2.1. *Given $g \in \mathcal{A}$, if $1 < \beta < 2$, then M^4 is either homeomorphic to either S^4 or $\mathbb{RP}^4$* (hence $b_2^+ = b_2^- = 0$) *or M^4 is homeomorphic to $\mathbb{CP}^2$* (hence $b_2^+ = 1$).

We remark that $\beta = 2$ for the product metric on $S^2 \times S^2$.

An additional ingredient to establish the lemma above is the Signature formula :

$$12\pi^2 \tau = \int_M (||W^+||^2 - ||W^-||^2) dv,$$

where $\tau = b_2^+ - b_2^-$, $||W^+||$ is the self dual part of the Weyl curvature and $||W^-||$ the anti-self-dual part, b_2^+, b_2^- the positive and negative part of the intersection form; together with an earlier result of Gursky [1998].

In view of the statement of Theorem 2.5 above, it is tempting to ask if one can change the "homeomorphism type" to "diffeomorphism type" in the statement of the Lemma. So far we have not been able to do so, but we have a perturbation result.

Theorem 2.6. (Chang, Gursky, and Zhang [2018]) There exists some $\epsilon > 0$ such that if (M, g) is a four manifold with $b_2^+ > 0$ and with a metric of positive Yamabe type satisfying with $1 < \beta([g]) < 1 + \epsilon$, then (M, g) is diffeomorphic to standard $\mathbb{CP}^2$.

Sketch proof of Theorem 2.6.

Proof. A key ingredient is to apply the condition $b_2^+ > 0$ to choose a good representative metric $g_{GL} \in [g]$, which is constructed in the earlier work of Gursky [2000] and used in Gursky and LeBrun [1998] and Gursky and Lebrun [1999]). To do so, they considered a generalized Yamabe curvature

$$\tilde{R}_g = R_g - 2\sqrt{6}||W^+||_g,$$

and noticed that on manifold of dimension 4, due to the conformal invariance of $||W^+||_g$, the corresponding Yamabe type functional

$$g \to \mu_g := \inf_{g_w \in [g]} \frac{\int_M \tilde{R}_{g_w} dv_{g_w}}{(vol\, g_w)^{\frac{1}{2}}}$$

still attains its infimum; which we denote by g_{GL}. The key observation in Gursky [2000] is that $b_2^+ > 0$ implies $\mu_g < 0$ (thus $\tilde{R}_{GL} < 0$). To see this, we recall the Bochner formula satisfied by the non-trivial self dual harmonic 2-form ϕ at the extreme metric:

$$\frac{1}{2}\Delta(|\phi|^2) = |\nabla\phi|^2 - 2W^+ < \phi, \phi > + \frac{1}{3}R|\phi|^2,$$

which together with the algebraic inequality that

$$-2W^+ < \phi, \phi > + \frac{1}{3}R|\phi|^2 \geq \frac{1}{3}\tilde{R}|\phi|^2,$$

forces the sign of $\tilde{R}_{GL}$ when ϕ is non-trivial.

To continue the proof of the theorem, we notice that for a given metric g satisfying the conformal pinching condition $1 < \beta([g]) < 1+\epsilon$ on its curvature, the corresponding g_{GL} would satisfy $G_2(g_{GL}) \leq C(\epsilon)$, where for $k = 2, 3, 4$,

$$G_k(g) := \int_M \left((R - \bar{R})^k + |Ric^0|^k + ||W^-||^k + |\tilde{R}_-|^k\right) dv_g,$$

where $\bar{R}$ denotes the average of the scalar curvature R over the manifold, Ric^0 the denote the traceless part of the Ricci curvature and $\tilde{R}_-$ the negative part of $\tilde{R}$, and where $C(\epsilon)$ is a constant which tends to zero as ϵ tends to zero.

We now finish the proof of the theorem by a contradiction argument and by applying the Ricci flow method of Hamilton

$$\frac{\partial}{\partial t} g(t) = -2Ric_{g(t)}$$

to regularize the metric g_{GL}.

Suppose the statement of the theorem is not true, let $\{g_i\}$ be a sequence of metrics satisfying $1 < \beta([g_i]) < 1 + \epsilon_i$ with ϵ_i tends to zero as i tends to infinity. Choose $g_i(0) = (g_i)_{GL}$ to start the Ricci flow, we can derive the inequality

$$\frac{\partial}{\partial t} G_2(g(t)) \leq a G_2(g_t) - b G_4(g(t))$$

for some positive constants a and b, also at some fixed time t_0 independent of ϵ and i for each g_i. We then apply the regularity theory of parabolic PDE to derive that some subsequences of $\{g_i(t_0)\}$ converges to the g_{FS} metric of $\mathbb{CP}^2$, which in turn implies the original subsequence $(M, \{g_i(0)\})$ hence a subsequence of $(M, \{g_i\})$ is diffeomorphic to $(\mathbb{CP}^2, g_{FS})$. We thus reach a contradiction to our assumption. The reader is referred to the preprint Chang, Gursky, and Zhang [2018] for details of the proof. □

We end this section by pointing out there is a large class of manifolds with metrics in the class $\mathcal{A}$. By the work of Donaldson–Freedman (see Donaldson [1983], Freedman [1982]) and Lichnerowicz vanishing theorem, the homeomorphism type of the class of simply-connected 4-manifolds which allow a metric with positive scalar curvature consists of S^4 together with $k\mathbb{CP}^2 \# l\overline{\mathbb{CP}^2}$ and $k(S^2 \times S^2)$. Apply some basic algebraic manipulations with the Gauss–Bonnet–Chern formula and the Signature formula, we can show that a manifold which admits $g \in \mathcal{A}$ satisfies $4 + 5l > k$. The round metric on S^4, the Fubini–Study metric on $\mathbb{CP}^2$, and the product metric on $S^2 \times S^2$ are clearly in the class $\mathcal{A}$. When $l = 0$, which implies $k < 4$, the class $\mathcal{A}$ also includes the metrics constructed by LeBrun, Nayatani, and Nitta [1997] on $k\mathbb{CP}^2$ for $k \leq 2$. When $l = 1$, which implies $k < 9$, the class $\mathcal{A}$ also includes the (positive) Einstein metric constructed by Page [1978] on $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$, the (positive) Einstein metric by Chen, Lebrun, and Weber [2008] on $\mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2}$, and the Kähler Einstein metrics on $\mathbb{CP}^2 \# l\overline{\mathbb{CP}^2}$ for $3 \leq l \leq 8$ as in the work of Tian [2000]. It would be an ambitious program to locate the entire class of 4-manifolds with metric in $\mathcal{A}$, and to classify their diffeomorphism types by the (relative) size of the integral conformal invariants discussed in this lecture.

3 Compact 4-manifold with boundary, (Q, T) curvatures

To further develop the analysis of the Q-curvature equation, it is helpful to consider the associated boundary value problems. In the case of compact surface with boundary (X^2, M^1, g), where the metric g is defined on $X^2 \cup M^1$; the Gauss–Bonnet formula becomes

$$2\pi\chi(X) = \int_X K \, dv + \oint_M k \, d\sigma, \tag{3.23}$$

where k is the geodesic curvature on M. Under conformal change of metric g_w on X, the geodesic curvature changes according to the equation

$$\frac{\partial}{\partial n} w + k \;=\; k_{g_w}\, e^{w} \quad \text{on M.} \tag{3.24}$$

One can generalize above results to compact four manifold with boundary (X^4, M^3, g); with the role played by $(-\Delta, \frac{\partial}{\partial n})$ replaced by (P_4, P_3) and with (K, k) curvature replaced by (Q, T) curvatures; where P_4 is the Paneitz operator and Q the curvature discussed in section 2; and where P_3 is some 3rd order boundary operator constructed in Chang and Qing [1997a,b]. The key property of P_3 is that it is conformally covariant of bidegree $(0, 3)$, i.e.

$$(P_3)_{g_w} = e^{-3w} (P_3)_g$$

when operating on functions defined on the boundary of compact 4-manifolds; and under conformal change of metric $g_w = e^{2w} g$ on X^4 we have at the boundary M^3

$$P_3 w + T \;=\; T_{g_w}\, e^{3w}. \tag{3.25}$$

The precise formula of P_3 is rather complicated (see Chang and Qing [1997a]). Here we will only mention that on $(B^4, S^3, |dx|^2)$, where B^4 is the unit ball in $\mathbb{R}^4$, we have

$$P_4 = (-\Delta)^2, \;\; P_3 = -\left(\frac{1}{2}\,\frac{\partial}{\partial n}\,\Delta + \tilde{\Delta}\frac{\partial}{\partial n} + \tilde{\Delta}\right) \quad \text{and} \;\; T = 2, \tag{3.26}$$

where $\tilde{\Delta}$ the intrinsic boundary Laplacian on S^3. In general the formula for T curvature is also lengthy,

$$T = \frac{1}{12}\frac{\partial}{\partial n} R + \frac{1}{6} RH - R_{\alpha n \beta n} L_{\alpha\beta} + \frac{1}{9} H^3 - \frac{1}{3}\mathrm{Tr} L^3 - \frac{1}{3}\tilde{\Delta} H,$$

where L is the second fundamental form of M in (X, g), and H the mean curvature, and n its the outside normal. In terms of these curvatures, the Gauss–Bonnet–Chern formula can be expressed as:

$$8\pi^2 \chi(X) = \int_X (||W||^2 \,+\, 2Q)\; dv \;+ \oint_M (\mathfrak{L} + 2T)\; d\sigma. \tag{3.27}$$

where $\mathfrak{L}$ is a third order boundary curvature invariant that transforms by scaling under conformal change of metric, i.e. $\mathfrak{L} d\sigma$ is a pointwise conformal invariant.

The property which is relevant to us is that

$$\int_X Q\, dv + \oint_M T\, d\sigma$$

is an integral conformal invariant.

It turns out for the cases which are of interest to us later in this paper, (X, g) is with totally geodesic boundary, that is, its second fundamental form vanishes. In this special case we have

$$T = \frac{1}{12}\frac{\partial}{\partial n}R. \tag{3.28}$$

Thus in view of the definitions (2.13) and (3.28) of Q and T, in this case we have

$$2\left(\int_X Q dv + \oint_M T d\sigma\right) = \int_X \sigma_2 \, dv,$$

which is the key property we will apply later to study the renormalized volume and the compactness problem of conformal compact Einstein manifolds in sections 4 and 5 below.

4 Conformally compact Einstein manifolds

4.1 Definition and basics, some short survey. Given a manifold $(M^n, [h])$, when is it the boundary of a conformally compact Einstein manifold (X^{n+1}, g^+) with $r^2 g^+|_M = h$? This problem of finding "conformal filling in" is motivated by problems in the AdS/CFT correspondence in quantum gravity (proposed by Maldacena [1998]) and from the geometric considerations to study the structure of non-compact asymptotically hyperbolic Einstein manifolds.

Here we will only briefly outline some of the progress made in this problem pertaining to the conformal invariants we are studying.

Suppose that X^{n+1} is a smooth manifold of dimension $n+1$ with smooth boundary $\partial X = M^n$. A defining function for the boundary M^n in X^{n+1} is a smooth function r on $\bar{X}^{n+1}$ such that

$$\begin{cases} r > 0 \text{ in } X; \\ r = 0 \text{ on } M; \\ dr \neq 0 \text{ on } M. \end{cases}$$

A Riemannian metric g^+ on X^{n+1} is conformally compact if $(\bar{X}^{n+1}, r^2 g^+)$ is a compact Riemannian manifold with boundary M^n for some defining function r. We denote $h := r^2 g^+|_M$.

Conformally compact manifold (X^{n+1}, g^+) carries a well-defined conformal structure on the boundary $(M^n, [h])$ by choices of different defining functions r. We shall call $(M^n, [h])$ the conformal infinity of the conformally compact manifold (X^{n+1}, g^+).

If (X^{n+1}, g^+) is a conformally compact manifold and $\mathrm{Ric}[g^+] = -n\, g^+$, then we call (X^{n+1}, g^+) a conformally compact (Poincare) Einstein (abbreviated as CCE) manifold. We remark that on a CCE manifold X, for any given smooth metric h in the conformal infinity M, there exists a special defining function r (called the geodesic defining function) so that $r^2 g^+|_M = h$, and $|dr|^2_{r^2 g^+} = 1$ in a neighborhood of the boundary $[0, \epsilon) \times M$, also the metric $r^2 g^+$ has totally geodesic boundary.

Some basic examples. Example 1: On $(B^{n+1}, S^n, g_{\mathbb{H}})$

$$\left(B^{n+1}, \left(\frac{2}{1-|y|^2}\right)^2 |dy|^2\right).$$

We can then view $(S^n, [g_c])$ as the compactification of B^{n+1} using the defining function

$$r = 2\frac{1-|y|}{1+|y|}$$

$$g_{\mathbb{H}} = g^+ = r^{-2}\left(dr^2 + \left(1-\frac{r^2}{4}\right)^2 g_c\right).$$

Example 2: AdS-Schwarzchild space

$$On \ \ (R^2 \times S^2, g_m^+),$$

where

$$g_m^+ = Vdt^2 + V^{-1}dr^2 + r^2 g_c,$$
$$V = 1 + r^2 - \frac{2m}{r},$$

m is any positive number, $r \in [r_h, +\infty)$, $t \in S^1(\lambda)$ and g_c the surface measure on S^2 and r_h is the positive root for $1 + r^2 - \frac{2m}{r} = 0$. We remark, it turns out that in this case, there are two different values of m so that both g_m^+ are conformal compact Einstein filling for the same boundary metric $S^1(\lambda) \times S^2$. This is the famous non-unique "filling in" example of Hawking and Page [1982/83].

Existence and non-existence results. The most important existence result is the "Ambient Metric" construction by Fefferman and Graham [1985, 2012]. As a consequence of their construction, for any given compact manifold (M^n, h) with an analytic metric h, some CCE metric exists on some tubular neighborhood $M^n \times (0, \epsilon)$ of M. This later result was recently extended to manifolds M with smooth metrics by Gursky and Székelyhidi [2017].

A perturbation result of Graham and Lee [1991] asserts that in a smooth neighborhood of the standard surface measure g_c on S^n, there exist a conformal compact Einstein metric on B^{n+1} with any given conformal infinity h.

There is some recent important articles by Gursky–Han and Gursky–Han–Stolz (Gursky and Han [2017], Gursky, Han, and Stolz [2018]), where they showed that when X is spin and of dimension $4k \geq 8$, and the Yamabe invariant $Y(M, [h]) > 0$, then there are topological obstructions to the existence of a Poincaré–Einstein g^+ defined in the interior of X with conformal infinity given by $[h]$. One application of their work is that on the round sphere S^{4k-1} with $k \geq 2$, there are infinitely many conformal classes that have no Poincaré–Einstein filling in in the ball of dimension $4k$.

The result of Gursky–Han and Gursky–Han–Stolz was based on a key fact pointed out Qing [2003], which relies on some earlier work of Lee [1995].

Lemma 4.1. *On a CCE manifold (X^{n+1}, M^n, g^+), assuming $Y(M,[h]) > 0$, there exists a compactification of g^+ with positive scalar curvature; hence $Y(X,[r^2 g^+]) > 0$.*

Uniqueness and non-uniqueness results. Under the assumption of positive mass theorem, Qing [2003] has established (B^{n+1}, g_H) as the unique CCE manifold with $(S^n, [g_c])$ as its conformal infinity. The proof of this result was later refined and established without using positive mass theorem by Li, Qing, and Shi [2017] (see also Dutta and Dutta and Javaheri [2010]). Later in section 5 of this lecture notes, we will also prove the uniqueness of the CCE extension of the metrics constructed by Graham and Lee [1991] for the special dimension $n = 3$.

As we have mentioned in the example 2 above, when the conformal infinity is $S^1(\lambda) \times S^2$ with product metric, Hawking and Page [1982/83] have constructed non-unique CCE fill-ins.

4.2 Renormalized volume. We will now discuss the concept of "renormalized volume" in the CCE setting, introduced by Maldacena [1998] (see also the works of Witten [1998], Henningson and Skenderis [1998] and Graham [2000]). On CCE manifolds (X^{n+1}, M^n, g^+) with geodesic defining function r,
For n even,

$$\begin{aligned}\mathrm{Vol}_{g^+}(\{r > \epsilon\}) &= c_0\epsilon^{-n} + c_2\epsilon^{-n+2} + \cdots \\ &\quad + c_{n-2}\epsilon^{-2} + L\log\frac{1}{\epsilon} + V + o(1).\end{aligned}$$

For n odd,

$$\begin{aligned}\mathrm{Vol}_{g^+}(\{r > \epsilon\}) &= c_0\epsilon^{-n} + c_2\epsilon^{-n+2} + \cdots\cdots \\ &\quad + c_{n-1}\epsilon^{-1} + V + o(1).\end{aligned}$$

We call the zero order term V the renormalized volume. It turns out for n even, L is independent of $h \in [h]$ where $h = r^2 g^+|_M$, and for n odd, V is independent of $g \in [g]$, and hence are conformal invariants.

We recall

Theorem 4.1. *(Graham and Zworski [2003], Fefferman and Graham [2002])*
When n is even,

$$L = c_n \oint_M Q_h dv_h,$$

where c_n is some dimensional constant.

Theorem 4.2. *(Anderson [2001], P. Yang, King, and Chang [2006a], P. Yang, King, and Chang [2006b])*
On conformal compact Einstein manifold (X^4, M^3, g^+), we have

$$V = \frac{1}{6}\int_{X^4} \sigma_2(A_g) dv_g$$

for any compactified metric g with totally geodesic boundary. Thus

$$8\pi^2\chi(X^4, M^3) = \int ||W||_g^2 dv_g + 6V.$$

Remark: There is a generalization of Theorem 4.2 above for any n odd, with X^4 replaced by X^n and with $\int_{X^4}\sigma_2$ replaced by some other suitable integral conformal invariants $\int_{X^{n+1}} v^{n+1}$ on any CCE manifold (X^{n+1}, M^n, g^+); see (Chang, Fang, and Graham [2014], also P. Yang, King, and Chang [2006a]).

Sketch proof of Theorem 4.2 for $n = 3$.

Lemma 4.2. *(Fefferman and Graham [2002])*
Suppose (X^4, M^3, g^+) is conformally compact Einstein with conformal infinity $(M^3, [h])$, fix $h \in [h]$ and r its corresponding geodesic defining function. Consider

$$-\Delta_{g^+} w = 3 \quad on \quad X^4, \tag{4.29}$$

then w has the asymptotic behavior

$$w = log\, r + A + Br^3$$

near M, where A, B are functions even in r, $A|_M = 0$, and

$$V = \int_M B|_M.$$

Lemma 4.3. *With the same notation as in Lemma 4.2, consider the metric $g^* = g_w = e^{2w} g^+$, then g^* is totally geodesic on boundary with (1) $Q_{g^*} \equiv 0$, (2) $B|_M = \frac{1}{36}\frac{\partial}{\partial n} R_{g^*} = \frac{1}{3} T_{g^*}$.*

Proof of Lemma 4.3.

Proof. Recall we have g^+ is Einstein with $Ric_{g^+} = -3g^+$, thus

$$P_{g^+} = (-\Delta_{g^+}) \circ (-\Delta_{g^+} - 2)$$

and $2Q_{g^+} = 6$. Therefore

$$P_{g^+} w + 2Q_{g^+} = 0 = 2e^{2w} Q_{g^*}.$$

Assertion (2) follows from a straight forward computation using the scalar curvature equation and the asymptotic behavior of w. □

Applying Lemmas 4.2 and 4.3, we get

$$\begin{aligned} 6V &= 6\oint_{M^3} B|_M d\sigma_h = \frac{1}{6}\oint_{M^3} \frac{\partial}{\partial n} R_{g^*} d\sigma_h \\ &= 2(\int_X Q_{g^*} + \oint_M T_{g^*}) = \int_{X^4} \sigma_2(A_{g^*}) dv_{g^*}. \end{aligned}$$

For any other compactified metric g with totally geodesic boundary, $\int_{X^4}\sigma_2(g)dv_g$ is a conformal invariant, and V is a conformal invariant, thus the result holds once for g^*, holds for any such g in the same conformal class, which establishes Theorem 4.2.

5 Compactness of conformally compact Einstein manifolds on dimension 4

In this section, we will report on some joint works of Yuxin Ge and myself Chang and Ge [2018] and also Yuxin Ge, Jie Qing and myself Chang, Ge, and Qing [2018].

The project we work on is to address the problem of given a sequence of CCE manifolds $(X^4, M^3, \{(g_i)^+\})$ with $M = \partial X$ and $\{g_i\} = \{r_i^2(g_i)^+\}$ a sequence of compactified metrics, denote $h_i = g_i|_M$, assume $\{h_i\}$ forms a compact family of metrics in M, is it true that some representatives $\bar{g}_i \in [g_i]$ with $\{\bar{g}_i|_M\} = \{h_i\}$ also forms a compact family of metrics in X? Let me mention the eventual goal of the study of the compactness problem is to show existence of conformal filling in for some classes of Riemannian manifolds. A plausible candidate for the problem to have a positive answer is the class of metrics (S^3, h) with the scalar curvature of h being positive. In this case by a result of Marques [2012], the set of such metrics is path-connected, the non-existence argument of Gursky–Han, and Gursky–Han–Stolz (Gursky and Han [2017], Gursky, Han, and Stolz [2018]) also does not apply. One hopes that our compactness argument would lead via either the continuity method or degree theory to the existence of conformal filling in for this class of metrics. We remark some related program for the problem has been outlined in Anderson [1989, 2008].

The first observation is one of the difficulty of the problem is existence of some "non-local" term. To see this, we have the asymptotic behavior of the compactified metric g of CCE manifold (X^{n+1}, M^n, g^+) with conformal infinity (M^n, h) (Graham [2000], Fefferman and Graham [2012]) which in the special case when $n = 3$ takes the form

$$g := r^2 g^+ = h + g^{(2)} r^2 + g^{(3)} r^3 + g^{(4)} r^4 + \cdots.$$

on an asymptotic neighborhood of $M \times (0, \epsilon)$, where r denotes the geodesic defining function of g. It turns out $g^{(2)} = -\frac{1}{2} A_h$ and is determined by h (we call such terms local terms), $Tr_h g^{(3)} = 0$, while

$$g^{(3)}_{\alpha,\beta} = -\frac{1}{3}\frac{\partial}{\partial n}(Ric_g)_{\alpha,\beta}$$

where α, β denote the tangential coordinate on M, is a non-local term which is not determined by the boundary metric h. We remark that h together with $g^{(3)}$ determine the asymptotic behavior of g (Fefferman and Graham [ibid.], Biquard [2008]).

We now observe that different choices of the defining function r give rise to different conformal metric of $\hat{h}$ in $[h]$ on M. For convenience, In the rest of this article, we choose the representative $\hat{h} = h^Y$ be the Yamabe metric with constant scalar curvature in $[h]$ and denote it by h and its corresponding geodesic defining function by r. Similarly one might ask what is a "good' representative of $\hat{g} \in [g]$ on X? Our first attempt is to choose $\hat{g} := g^Y$, a Yamabe metric in $[g]$. The difficulty of this choice is that it is not clear how to control the boundary behavior of $g^Y|_M$ in terms of h^Y.

We also remark that in seeking the right conditions for the compactness problem, due to the nature of the problem, the natural conditions imposed should be conformally invariant.

In the statement of the results below, for a CCE manifold (X^4, M^3, g^+), and a conformal infinity $(M, [h])$ with the representative $h = h^Y \in [h]$, we solve the PDE

$$-\Delta_{g^+} w = 3 \tag{5.30}$$

and denote $g^* = e^{2w} g$ be the "Fefferman–Graham" compactification metric with $g^*|_M = h$.

We recall that $Q_{g^*} \equiv 0$, hence the renormalized volume of (X, M, g^+) is a multiple of

$$\int_X \sigma_2(A_{g*}) dv_{g^*} = 2 \oint_M T_{g*} d\sigma_h = \frac{1}{6} \oint_M \frac{\partial}{\partial n} R_{g^*} d\sigma_h.$$

Before we state our results, we recall formulas for the specific g^* metric in a model case.

Lemma 5.1. *On* $(B^4, S^3, g_{\mathbb{H}})$,

$$g^* = e^{(1-|x|^2)} |dx|^2 \ \ on \ \ B^4$$

$$Q_{g^*} \equiv 0, \quad T_{g^*} \equiv 2 \ \ on \ \ S^3$$

$$(g^*)^{(3)} \equiv 0$$

and

$$\int_{B^4} \sigma_2(A_{g*}) dv_{g^*} = 8\pi^2.$$

We will first state a perturbation result for the compactness problem.

Theorem 5.1. *Let* $\{(B^4, S^3, \{g_i{}^+\})\}$ *be a family of oriented CCE on* B^4 *with boundary* S^3. *We assume the boundary Yamabe metric* h_i *in conformal infinity* M *is of non-negative type. Let* $\{g_i^*\}$ *be the corresponding FG compactification. Assume*

1. *The boundary Yamabe metrics* $\{h_i\}$ *form a compact family in* C^{k+3} *norm with* $k \geq 2$*; and there exists some positive constant* $c_1 > 0$ *such that the Yamabe constant for the conformal infinity* $[h_i]$ *is bounded uniformly from below by* c_1*, that is,*

$$Y(M, [h_i]) \geq c_1;$$

2. *There exists some small positive constant* $\varepsilon > 0$ *such that for all i*

$$\int_{B^4} \sigma_2(A_{g_i^*}) dv_{g_i^*} \ \geq \ 8\pi^2 - \varepsilon. \tag{5.31}$$

Then the family of the g_i^* *is compact in* $C^{k+2,\alpha}$ *norm for any* $\alpha \in (0, 1)$ *up to a diffeomorphism fixing the boundary.*

Before we sketch the proof of the theorem, we will first mention that on (B^4, S^3, g), for a compact metric g with totally geodesic boundary, Gauss–Bonnet–Chern formula takes the form:

$$8\pi^2\chi(B^4, S^3) = 8\pi^2 = \int_{B^4} (||W||_g^2 + \sigma_2(A_g))dv_g,$$

which together with the conformal invariance of the L^2 norm of the Weyl tensor, imply that the condition (5.31) in the statement of the Theorem 5.1 is equivalent to

$$\int_{B^4} ||W||_{g_i^+}^2 dv_{g_i^+} \leq \varepsilon. \tag{5.32}$$

What is less obvious is that in this setting, we also have other equivalence conditions as stated in Corollary (5.1) below. This is mainly due to following result by Li, Qing, and Shi [2017].

Proposition 5.1. *Assume that (X^{n+1}, g^+) is a CCE manifold with C^3 regularity whose conformal infinity is of positive Yamabe type. Let $p \in X$ be a fixed point and $t > 0$. Then*

$$\left(\frac{Y(\partial X, [h])}{Y(\mathbb{S}^n, [g_c])}\right)^{\frac{n}{2}} \leq \frac{Vol(\partial B_{g^+}(p,t))}{Vol(\partial B_{g_{\mathbb{H}}}(p,t))} \leq \frac{Vol(B_{g^+}(p,t))}{Vol(B_{g_{\mathbb{H}}}(p,t))} \leq 1$$

where $B_{g^+}(p,t)$ and $B_{g_{\mathbb{H}}}(p,t)$ are geodesic balls.

Corollary 5.1. *Let $\{X = B^4, M = \partial X = S^3, g^+\}$ be a 4-dimensional oriented CCE on X with boundary ∂X. Assume the boundary Yamabe metric $h = h^Y$ in the conformal infinity of positive type and $Y(S^3, [h]) > c_1$ for some fixed $c_1 > 0$ and h is bounded in C^{k+3} norm with $k \geq 5$. Let g^* be the corresponding FG compactification. Then the following properties are equivalent:*

1. *There exists some small positive number $\varepsilon > 0$ such that*

$$\int_X \sigma(A_{g^*})dv_{g^*} \geq 8\pi^2 - \varepsilon. \tag{5.33}$$

2. *There exists some small positive number $\varepsilon > 0$ such that*

$$\int_X ||W||_{g^+}^2 dv_{g^+} \leq \varepsilon.$$

3. *There exists some small positive number $\varepsilon_1 > 0$ such that*

$$Y(S^3, [g_c]) \geq Y(S^3, [h]) > Y(S^3, [g_c]) - \varepsilon_1$$

   where g_c is the standard metric on S^3.

4. *There exists some small positive number $\varepsilon_2 > 0$ such that for all metrics g^* with boundary metric h same volume as the standard metric g_c on S^3, we have*

$$T(g^*) \geq 2 - \varepsilon_2.$$

5. *There exists some small positive number* $\varepsilon_3 > 0$ *such that*

$$|(g^*)^{(3)}| \leq \varepsilon_3.$$

Where all the ε_i *(i = 1,2,3) tends to zero when* ε *tends to zero and vice versa for each* i.

Another consequence of Theorem 5.1 is the "uniqueness" of the Graham–Lee metrics mentioned in section 4a.

Corollary 5.2. *There exists some* $\varepsilon > 0$*, such that for all metrics* h *on* S^3 *with* $||h - g_c||_{C^\infty} < \varepsilon$*, there exists a unique CCE filling in* (B^4, S^3, g^+) *of* h.

Sketch proof of Theorem 5.1. We refer the readers to the articles Chang and Ge [2018] and Chang, Ge, and Qing [2018], both will soon be posted on arXiv for details of the arguments, here we will present a brief outline.

We first state a lemma summarizing some analytic properties of the metrics g^*.

Lemma 5.2. *On a CCE manifold* (X^4, M^3, g^+)*, where the scalar curvature of the conformal infinity* (M, h) *is positive. Assume* h *is at least* C^l *smooth for* $l \geq 3$*. Denote* $g^* = e^{2w} g^+$ *the FG compactification. Then*
(1) $Q(g^*) \equiv 0$*,*
(2) $R_{g^*} > 0$*, which implies in particular* $|\nabla_{g^*} w| e^w \leq 1$*.*
(3) g^* *is Bach flat and satisfies an* ϵ*-regularity property, which implies in particular, once it is* C^3 *smooth, it is* C^l *smooth for* $l \geq 3$*.*

We remark that statement (2) in the Lemma above follows from a continuity argument via some theory of scattering matrix (see Case and Chang [2016a,b]), with the starting point of the argument the positive scalar curvature metric constructed by J. Lee which we have mentioned earlier in Lemma 4.1.

Sketch proof of Theorem 5.1. Proof of the theorem is built on contradiction arguments. We first note, assuming the conclusion of the theorem does not hold, then there is a sequence of $\{g_i^*\}$ which is not compact so that the L^2 norm of its Weyl tensor of the sequence tends to zero.

Our main assertion of the proof is that the C^1 norm of the curvature of the family $\{g_i{}^*\}$ remains uniformly bounded.

Assume the assertion is not true, we rescale the metric $\bar{g}_i = K_i^2 g_i{}^*$ where there exists some point $p_i \in X$ such that

$$K_i^2 = \max\{\sup_{B^4} |Rm_{g_i{}^*}|, \sqrt{\sup_{B^4} |\nabla Rm_{g_i{}^*}|}\} = |Rm_{g_i{}^*}|(p_i)(\text{ or } \sqrt{|\nabla Rm_{g_i{}^*}|(p_i)})$$

We mark the accumulation point p_i as $0 \in B^4$. Thus, we have

$$|Rm_{\bar{g}_i}|(0) = 1 \text{ or } |\nabla Rm_{\bar{g}_i}|(0) = 1 \tag{5.34}$$

We denote the corresponding defining function $\bar{w}_i$ so that $\bar{g}_i = e^{2\bar{w}_i} g_i^+$; that is $e^{2\bar{w}_i} = K_i^2 e^{2w_i}$ and denote $\bar{h}_i := \bar{g}_i|_{S^3}$. We remark that the metrics $\bar{g}_i$ also satisfy the conditions in Lemma 5.2.

As $0 \in \overline{B^4}$ is an accumulation point, depending on the location of 0, we call it either an interior or a boundary blow up point; in each case, we need a separate argument but for simplicity here we will assume we have $0 \in S^3$ is a boundary blow up point. In this case, we denote the (X_∞, g_∞) the Gromov–Hausdorff limit of the sequence $(B^4, \bar{g}_i)$, and $h_\infty := g_\infty|_{S^3}$.

Our first observation is that it follows from the assumption (1) in the statement of the theorem, we have $(\partial X_\infty, h_\infty) = (\mathbb{R}^3, |dx|^2)$.

Our second assertion is that by the estimates in Lemma 5.2, one can show $\bar{w}_i$ converges uniformly on compacta on X_∞, we call the limiting function $\bar{w}_\infty$. Hence, the corresponding metric g_∞^+ with $g_\infty = e^{2\bar{w}_\infty} g_\infty^+$ exists, satisfying $Ric_{g_\infty^+} = -3g_\infty^+$, i.e., the resulting g_∞ is again conformal to a Poincare Einstein metric g_∞^+ with $||W_{g_\infty^+}|| \equiv 0$.

Our third assertion is that (X_∞, g_∞^+) is (up to an isometry) the model space

$$(\mathbb{R}^4_+,\ g_{\mathbb{H}} := \frac{|dx|^2 + |dy|^2}{y^2})$$

where $\mathbb{R}^4_+ = \{(x, y) \in \mathbb{R}^4 | y > 0\}$. We can then apply a Liouville type PDE argument to conclude $\bar{w}_\infty = \log\ y$.

Thus g_∞ is in fact the flat metric $|dx|^2 + |dy|^2$, which contradicts the marking property (5.34). This contradiction establishes the main assertion.

Once the C^1 norm of the curvature of the metric $\{g_i^*\}$ is bounded, we can apply some further blow-up argument to show the diameter of the sequence of metrics is uniformly bounded, and apply a version of the Gromov–Hausdorff compactness result (see Cheeger, Gromov, and Taylor [1982]) for compact manifolds with totally geodesic boundary to prove that $\{g_i^*\}$ forms a compact family in a suitable C^l norm for some $l \geq 3$. This finishes the proof of the theorem.

We end this discussion by mentioning that, from Theorem 5.1, an obvious question to ask is if we can extend the perturbation result by improving condition (5.31) to the condition $\int_{B^4} \sigma_2 > 0$. This is a direction we are working on but have not yet be able to accomplish. Below are statements of two more general theorems that we have obtained in (Chang and Ge [2018], Chang, Ge, and Qing [2018]).

Theorem 5.2. *Under the assumption (1) as in Theorem 5.1, assume further the T curvature on the boundary $T_i = \frac{1}{12}\frac{\partial}{\partial n} R_{g_i*}$ satisfies the following condition*

$$\liminf_{r\to 0} \inf_i \inf_{S^3} \oint_{\partial B(x,r)} T_i \geq 0. \tag{5.35}$$

Then the family of metrics $\{g_i^*\}$ is compact in $C^{k+2,\alpha}$ norm for any $\alpha \in (0,1)$ up to a diffeomorphism fixing the boundary, provided $k \geq 5$.

Theorem 5.3. *Under the assumption (1) as in Theorem 5.1, assume further that there is no concentration of $S_i := (g_i^*)^{(3)}$-tensor defined on S^3 in L^1 norm for the g_i^* metric in the following sense,*

$$\lim_{r\to 0} \sup_i \sup_x \oint_{\partial B(x,r)} |S_i| = 0. \tag{5.36}$$

Then, the family of the metrics $\{g_i^*\}$ is compact in $C^{k+2,\alpha}$ norm for any $\alpha \in (0,1)$ up to a diffeomorphism fixing the boundary, provided $k \geq 2$.

The reason we can pass the information from the T curvature in Theorem 5.2 to the S tensor in Theorem 5.3 is due to the fact that for the blow-up limiting metric g_∞, $T_{g_\infty} \equiv 0$ if and only if $S_{g_\infty} \equiv 0$.

It remains to show if there is a connection between condition (5.35) in Theorem 5.2 to the positivity of the renormalized volume, i.e. when $\int_X \sigma_2(A_g) dv_g > 0$.

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EQUILIBRIA, FIXED POINTS, AND COMPUTATIONAL COMPLEXITY – NEVANLINNA PRIZE LECTURE

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Abstract

The concept of equilibrium, in its various forms, has played a central role in the development of Game Theory and Economics. The mathematical properties and computational complexity of equilibria are also intimately related to mathematical programming, online learning, and fixed point theory. More recently, equilibrium computation has been proposed as a means to learn generative models of high-dimensional distributions.

In this paper, we review fundamental results on minimax equilibrium and its relationship to mathematical programming and online learning. We then turn to Nash equilibrium, reviewing some of our work on its computational intractability. This intractability is of an unusual kind. While computing Nash equilibrium does belong to the well-known complexity class NP, Nash's theorem—that Nash equilibrium is guaranteed to exist—makes it unlikely that the problem is NP-complete. We show instead that it is as hard computationally as computing Brouwer fixed points, in a precise technical sense, giving rise to the complexity class PPAD, a subclass of total search problems in NP that we will define. The intractability of Nash equilibrium makes it unlikely to always arise in practice, so we study special circumstances where it becomes tractable. We also discuss modern applications of equilibrium computation, presenting recent progress and several open problems in the training of Generative Adversarial Networks. Finally, we take a broader look at the complexity of total search problems in NP, discussing their intimate relationship to fundamental problems in a range of fields, including Combinatorics, Discrete and Continuous Optimization, Social Choice Theory, Economics, and Cryptography. We overview recent work and present a host of open problems.

1 Introduction: 90 years of the minimax theorem

Humans strategize when making decisions. Sometimes, they optimize their decisions against a stationary, stochastically changing, or completely unpredictable environment. Optimizing a decision in such a situation falls in the realm of Optimization Theory. Other times, a decision is pit against those made by others who are also thinking strategically. Understanding the outcome of the interaction among strategic decision-makers falls in the realm of Game Theory, which studies situations of conflict, competition or cooperation as these may arise in everyday life, game-playing, politics, international

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relations, and many other multi-agent settings, such as interactions among species (Osborne and Rubinstein [1994]).

A founding stone in the development of Game Theory was von Neumann's minimax theorem (von Neumann [1928]), whose statement is as simple as it is remarkable: if $\mathcal{X} \subset \mathbb{R}^n$ and $\mathcal{Y} \subset \mathbb{R}^m$ are compact and convex subsets of Euclidean space, and $f(\mathbf{x}, \mathbf{y})$ is a continuous function that is convex in $\mathbf{x} \in \mathcal{X}$ and concave in $\mathbf{y} \in \mathcal{Y}$, then

$$\min_{\mathbf{x}\in\mathcal{X}} \max_{\mathbf{y}\in\mathcal{Y}} f(\mathbf{x}, \mathbf{y}) = \max_{\mathbf{y}\in\mathcal{Y}} \min_{\mathbf{x}\in\mathcal{X}} f(\mathbf{x}, \mathbf{y}). \tag{1}$$

To see how (1) relates to Game Theory, let us imagine two players, "Min" and "Max," engaging in a strategic interaction. Min can choose any strategy in $\mathcal{X}$, Max can choose any strategy in $\mathcal{Y}$, and $f(\mathbf{x}, \mathbf{y})$ is the payment of Min to Max when they choose strategies $\mathbf{x}$ and $\mathbf{y}$ respectively. Such interaction is called a *two-player zero-sum game*, as the sum of players' payoffs is zero for any choice of strategies. Let us now offer two interpretations of (1):

- First, it confirms an unusual fact. For some scalar value v, if for all strategies $\mathbf{y}$ of Max, there exists a strategy $\mathbf{x}$ of Min such that $f(\mathbf{x}, \mathbf{y}) \leq v$, that is Min pays no more than v to Max, then there exists a universal strategy $\mathbf{x}$ for Min such that for all strategies $\mathbf{y}$ of Max it holds that $f(\mathbf{x}, \mathbf{y}) \leq v$.
- In particular, if $\mathbf{x}^*$ is the optimal solution to the left hand side of (1), $\mathbf{y}^*$ is the optimal solution to the right hand side of (1), and v^* is the optimal value of both sides, then if Min plays $\mathbf{x}^*$ and Max plays $\mathbf{y}^*$ then Min pays v^* to Max and neither Min can decrease this payment nor can Max increase this payment by unilaterally changing their strategies. Such a pair of strategies is called a *minimax equilibrium*, and von Neumann's theorem confirms that it always exists!

One of the simplest two-player zero-sum games that were ever played is *rock-paper-scissors*, which every kid must have played in the school yard, or at least they used to before electronic games took over. In this game, both players have the same three actions, "rock," "paper," and "scissors," to choose from. Their actions are compared, and each player is rewarded or charged, or, if they choose the same action, the players tie. Figure 1 assigns numerical payoffs to the different outcomes of the game. In this tabular representation, the rows represent the actions available to Min, the columns represent the actions available to Max, and every square of the table is the payment of Min to Max for a pair of actions. For example, if Min chooses "rock" and Max chooses "scissors," then Min wins a dollar and Max loses a dollar, so the payment of Min to Max is -1, as the corresponding square of the table confirms. To map rock-paper-scissors to the setting of the minimax theorem, let us take both $\mathcal{X}$ and $\mathcal{Y}$ be the simplex of distributions over {rock, paper, scissors}, and let Z be the matrix of Figure 1 so that given a pair of distributions $\mathbf{x} \in \mathcal{X}$ and $\mathbf{y} \in \mathcal{Y}$, the expected payment of Min to Max is $f(\mathbf{x}, \mathbf{y}) = \mathbf{x}^{\mathrm{T}} Z \mathbf{y}$. The minimax equilibrium of the game thus defined is $(\mathbf{x}^*, \mathbf{y}^*)$, where $\mathbf{x}^* = \mathbf{y}^* = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, which confirms the experience of anyone who was challenged by a random student in a school yard to play this game. Under the uniform strategy, both players receive an expected payoff of 0, and no player can improve their payoff by unilaterally changing their strategy. As such, it is reasonable to expect a

	rock	*paper*	*scissors*
rock	0	1	−1
paper	−1	0	1
scissors	1	−1	0

Figure 1: Rock–paper–scissors

population of students to converge to the uniform strategy. After all, rock-paper-scissors is so symmetric that no other behavior would make sense.

The question that arises is how reasonable it is to expect minimax equilibrium to arise in more complex games $(\mathcal{X}, \mathcal{Y}, f)$, and how computationally burdensome it is to compute this equilibrium. Indeed, these two questions are intimately related as, if it is too burdensome computationally to compute a minimax equilibrium, then it is also unlikely that strategic agents will be able to compute it. After all, their cognitive abilities are also computationally bounded.

Shedding light on these important questions, a range of fundamental results have been shown over the past ninety years establishing an intimate relationship between the minimax theorem, mathematical programming, and online learning. Noting that all results we are about to discuss have straightforward analogs in the general case, we will remain consistent with the historical development of these ideas, discussing these results in the special case where $\mathcal{X} = \Delta_n \subset \mathbb{R}^n$ and $\mathcal{Y} = \Delta_m \subset \mathbb{R}^m$ are probability simplices, and $f(\mathbf{x}, \mathbf{y}) = \mathbf{x}^{\mathrm{T}} A \mathbf{y}$, where A is some $n \times m$ real matrix. In this case, (1) becomes:

$$\min_{\mathbf{x} \in \Delta_n} \max_{\mathbf{y} \in \Delta_m} \mathbf{x}^{\mathrm{T}} A \mathbf{y} = \max_{\mathbf{y} \in \Delta_m} \min_{\mathbf{x} \in \Delta_n} \mathbf{x}^{\mathrm{T}} A \mathbf{y}. \tag{2}$$

1.1 Minimax Equilibrium, Linear Programming, and Online Learning. It was not too long after the proof of the minimax theorem that Dantzig met von Neumann to show him the Simplex algorithm for linear programming. According to Dantzig [1982], von Neumann quickly stood up and sketched the mathematical theory of linear programming duality. It is not hard to see that (2) follows from strong linear programming duality. Indeed, one can write simple linear programs to solve both sides of (2), hence, as we came to know some thirty years after the Simplex algorithm, a minimax equilibrium can be computed in time polynomial in the description of matrix A (Khachiyan [1979]). Interestingly, it was conjectured by Dantzig [1951] and recently proven by Adler [2013] that the minimax theorem and strong linear programming duality are in some sense equivalent: it is not just true that a minimax equilibrium can be computed by solving linear programs, but that any linear program can be solved by finding an equilibrium!

As the connections between the minimax theorem and linear programming were sinking in, researchers proposed simple dynamics for solving min-max problems by having the Min and Max players of (2) run simple learning procedures in tandem. An early procedure, proposed by G. W. Brown [1951] and shown to converge by Robinson [1951],

was *fictitious play:* in every round of this procedure, each player plays a best response to the history of strategies played by her opponent thus far. This simple procedure converges to a minimax equilibrium in the following "average sense:" if $(\mathbf{x}_t, \mathbf{y}_t)_t$ is the trajectory of strategies traversed by the Min and Max players using fictitious play, then the average payment $\frac{1}{t}\sum_{\tau \le t} f(\mathbf{x}_\tau, \mathbf{y}_\tau)$ of Min to Max converges to the optimum of (2) as $t \to \infty$, albeit this convergence may be exponentially slow in the number of actions, $n + m$, available to the players, as shown in recent work with Pan (Daskalakis and Pan [2014]).

Soon after fictitious play was first proposed and analyzed, Blackwell's approachability (Blackwell [1956]) and Hannan's consistency theorem (Hannan [1957]) would propel the field of online learning in games, leading to the discovery of several learning procedures converging to minimax equilibrium in time that is inversely polynomial in the approximation accuracy, while also being somewhat robust to adversarial environments, situations where one of the players deviates from the prescribed learning procedure arbitrarily. These learning methods, called "no-regret," include the celebrated multiplicative-weights-update, follow-the-regularized-leader, and follow-the-perturbed-leader methods. See Cesa-Bianchi and Lugosi [2006], Shalev-Shwartz [2012], Bubeck and Cesa-Bianchi [2012], Shalev-Shwartz and Ben-David [2014], and Hazan [2016] for recent surveys of learning in games, and Daskalakis, Deckelbaum, and Kim [2011] and Rakhlin and Sridharan [2013] for no-regret methods with essentially optimal learning rates in zero-sum games.

All in all, ninety years of work in mathematical programming and online learning have delivered a great deal of understanding on the mathematical structure and computational complexity of minimax equilibria. A solution to (2) can be computed using linear programming in time polynomial in n, m and the description of A, and convex programming can similarly be used to solve (1) efficiently. Moreover, a broad range of learning algorithms employed by the players of the game lead to minimax equilibrium. These methods compute approximate minimax equilibria in a number of rounds and overall computational complexity that typically are inversely polynomial in the approximation, and their dependence on the game parameters is mild, depending on the details of the setting; see Cesa-Bianchi and Lugosi [2006], Shalev-Shwartz [2012], Bubeck and Cesa-Bianchi [2012], Shalev-Shwartz and Ben-David [2014], and Hazan [2016].

1.2 From Rock–Paper–Scissors to Chess, Go, and Poker. Evidently, many two-player zero-sum games cannot be written down explicitly, their payoff structure summarized in an explicit function $f(\mathbf{x}, \mathbf{y})$. Obvious examples can be found in board games. Take chess for instance. The number of legal positions is estimated to be of the order of 10^{43}, hence the strategy sets of the players have unfathomable dimensionality. The situation is even starker in Go, where the number of legal positions is of the order of 10^{170}, and Poker,[1] where the added complication is that players also have uncertainty about the cards dealt to their opponents and must therefore take expectations over their uncertainty to determine their expected payoff from different strategies.

[1] Let us consider two-player Poker right now to fit with the framework of two-player zero-sum games. We will come to multi-player Poker shortly.

In turn-based games of complete information, i.e. games where players have the same information about the state of the game at all times, such as Chess and Go, it is easy to see that minimax strategies can be found via a method known as *backward induction*. Let us present this method for deterministic games where there are no moves by nature through dice, shuffling, etc., although the method is easily extensible to this case. The method operates on the game tree, the tree whose root is the initial configuration of the game, and whose edges correspond to legal moves by one of the players from one configuration to another. The leaves of the game tree are terminal configurations, with an associated (possibly negative) payment from one player (Min) to the other (Max). Backward induction starts at these nodes, where the payment of Min to Max is determined by the rules of the game, and works its way up the tree, recursively computing for every node that it encounters the payment of Min to Max, if the two players where to play optimally starting at that configuration. It turns out that this payment can be derived simply from the payments that the method has already computed for the children of the node: it amounts to taking the minimum of the children, if it is Min's turn to play at this node, and taking the maximum, otherwise. Backward induction takes linear time in the size of the game tree, and for obvious reasons it is also called the "minimax algorithm." Importantly, the minimax payment computed by the algorithm can be realized via deterministic strategies: in every node controlled by Min, Min chooses an action that will transition to a child with minimum payment; in every node controlled by Max, Max chooses an action that will transition to a child with maximum payment!

In games of incomplete information, such as Poker, the situation gets more complex. First, the game tree representation is not adequate to describe the game as it does not reflect the information that players may have at different stages of the game. In particular, players cannot distinguish nodes of the tree whose difference is due to information that is not observable by them, such as cards held by others. There is a way to amend the game tree representation to capture player information by overlaying onto the game tree groupings of nodes that cannot be distinguished by different players using the information that they have. These are called "information sets." Indeed, the incomplete information of the players not only makes the representation of the game more involved, but also causes minimax strategies to use randomization, such as bluffing in Poker. Importantly the optimal strategy in a subtree may depend on the strategy in other subtrees. Thus a subgame cannot be solved in isolation using backward induction. Instead, linear programming can be intelligently employed to solve these games in time polynomial in the size of the game tree (Koller and Megiddo [1992] and von Stengel [1996]).

So turn-based games of both complete information, such as Chess and Go, and incomplete information, such as Poker, can be solved in time polynomial in the size of their game tree. However, for games as complex as Chess, Go and Poker the game tree is just too large for this to be any good news. Yet, current algorithms have risen to the level of top human game-playing. In 1997, Deep Blue famously beat Garry Kasparov in Chess (Hsu [2004]), while almost two decades later AlphaGo beat world champion Lee Sedol in Go (Silver et al. [2016]), and even more recently Libratus beat top Poker players in two-player no-limit Texas Hold'em (N. Brown and Sandholm [2017a,b]).

Arguably the success of algorithms, matching or exceeding top human performance in Chess, Go and Poker, is intimately related to the mathematical properties of minimax

equilibrium. While minimax strategies cannot be computed exactly in these large games, heuristics for computing good strategies have been guided by the mathematical properties of minimax equilibrium and the mechanics of algorithms that operate on the entire game tree. In particular, heuristics are often employed to compress the humongous set of legal positions into a smaller set of "compressed positions," effectively identifying different nodes of the game tree. Moreover, heuristics are employed to select parts of the game tree and approximately compute minimax equilibria in these subtrees, then progressively improve the approximation quality as the play proceeds. Importantly, the methods to compress the set of legal positions, and decide how to prune the tree and assign approximate values to unexplored parts of the game tree for the purposes of approximating the minimax strategies are often not hardwired a priori by the algorithm designer, but have flexible designs that are fine-tuned by studying past games between human experts and via hours of self-play, and are also adjusted as the play develops.

1.3 Minimax Equilibrium, Postmortem. All in all, two-player zero-sum games are well-understood from both a mathematical and an algorithmic perspective. Moreover, heuristics for playing large two-player zero-sum games have reached a high level of sophistication, delivering strategies that can comfortably beat humans in games that were considered intractable for computers before. This success is often heralded as an early sign of A.I.'s coming of age. Arguably, however, much of present success is due to the mathematical properties of minimax equilibrium, and a testament to the high sophistication that has been reached by algorithms for game tree solving, linear programming, and online learning. Outside the realm of two-player zero-sum games, the predictive power of Game Theory has been more limited. As asserted by Robert Aumann [2017], "strictly competitive games," a family of games shown to be equivalent to two-player zero-sum in work with Adler and Papadimitriou (Adler, Daskalakis, and Papadimitriou [2009]), "are one of the few areas in game theory, and indeed in social science, where a fairly sharp, unique prediction is made." Consistently, algorithmic progress towards playing multi-player games, e.g. multi-player Poker, has also been quite modest. In the next section, we seek to uncover the mathematical and computational reasons why non two-player zero-sum games are different.

1.4 Roadmap. In Section 2, we introduce Nash equilibrium and study its computational complexity. We argue that it is unlikely to be NP-complete, and develop the complexity-theoretic machinery to study its complexity, introducing the class PPAD, a subclass of total search problems in NP. We sketch a proof that both Nash equilibrium computation and Brouwer fixed point computation are PPAD-complete. These problems are thus computationally equivalent, and as hard computationally as any other problem in PPAD. We discuss the ramifications of these intractability results for Nash equilibrium, and introduce special families of games where Nash equilibrium is tractable. In Section 3, we take a broader look at the complexity of total search problems in NP, discussing their intimate relationship with fundamental problems in a range of fields, including Combinatorics, Discrete and Continuous Optimization, Social Choice Theory, Economics, and Cryptography. We overview recent work and present a host of

open problems. Finally, in Section 4, we revisit two-player zero-sum games, discussing how equilibrium computation has found use in adversarial training applications, such as training Generative Adversarial Networks (GANs). We present recent progress and important research challenges going forward.

2 Nash Equilibrium, Fixed Points, and Computational Complexity

Evidently two-player zero-sum games have limited capability in modeling strategic interactions (Osborne and Rubinstein [1994]). Most interactions involve multiple agents and are non zero-sum. Is equilibrium possible in multi-player or non zero-sum games? As it turns out, there is a natural generalization of the minimax equilibrium that does exist in very general settings. It is called *Nash equilibrium* after John Nash who proved its existence in a pair of elegant papers (Nash [1950, 1951]), which greatly influenced the course of Game Theory and Economics (Myerson [1999]).

We will define Nash equilibrium formally shortly, but let us define it in words right away. Let us consider a game with a finite number of players and a finite number of actions available to each. For any choice of actions by the players, each player receives some payoff as determined by the game. We assume that players also have access to private randomness, which they can use to randomize over actions in their strategies. Players are assumed to be risk-neutral and care about their expected payoffs as computed under the product measure defined by all players' strategies. Nash equilibrium is a collection of strategies under which no player can improve her expected payoff by unilaterally changing her own strategy.

Nash's celebrated theorem states that at least one Nash equilibrium exists in every game with a finite number of players and a finite number of actions available to each, as described in the previous paragraph. Generalizations of this theorem to more general settings abound (see e.g. Glicksberg [1952]), as do generalizations and refinements of Nash equilibrium (see e.g. Aumann [1974], Myerson [1978], and Osborne and Rubinstein [1994]). Nash equilibrium itself generalizes minimax equilibrium outside the realm of two-player zero-sum games, and is tautological to minimax equilibrium for two-player zero-sum games.

There are, however, fundamental differences in the mathematical and algorithmic nature of the two equilibrium concepts. Minimax equilibria in two-player zero-sum games are known to comprise a convex set, but it is easy to see that Nash equilibria comprise a non-convex set in general games. Moreover, as discussed earlier, minimax equilibria can be shown to exist via strong linear programming duality, and their computation is in fact tantamount to linear programming (Adler [2013]). The same is not known for Nash equilibrium. Nash showed the existence of Nash equilibrium using Kakutani's fixed point theorem (Nash [1950]), and soon after simplified his proof to use Brouwer's fixed point theorem (Nash [1951]). In particular, his proof was non-constructive, and, despite significant research on Nash equilibrium and fixed point computation (see e.g. Lemke and Howson [1964], Scarf [1967a], Wilson [1971], Rosenmüller [1971], Eaves [1972], and van der Laan and Talman [1982]), no efficient algorithm is known to date. Even

worse, most known algorithms have been shown to be inefficient (Hirsch, Papadimitriou, and Vavasis [1987], Savani and von Stengel [2006], and Goldberg, Papadimitriou, and Savani [2013]). A natural question arises:

> *Is there a lingering computational intractability for Nash equilibrium, and what computational complexity machinery can be used to establish it?*

We answer these questions in this section by reviewing work on the computational intractability of Nash equilibrium. This intractability is of an unusual kind. Computing Nash equilibrium does belong to the standard complexity class NP, but Nash's theorem that Nash equilibrium is guaranteed to exist makes it unlikely that computing it is NP-complete. We show instead that it is as hard computationally as computing Brouwer fixed points, in a precise technical sense, giving rise to the complexity class PPAD, which we will define shortly. This result implies that, from a computational point of view, the use of Brouwer's theorem to establish Nash's theorem is unavoidable, and we discuss the ramifications of this result.

Section outline: Our plan for the rest of this section is to first define Nash equilibrium formally, then discuss why the standard notion of NP-completeness does not appear adequate to study its computational complexity. This will motivate a deeper look into the complexity of total problems in NP, motivating the definition of the complexity class PPAD. We will define PPAD formally in Section 2.1, and motivate its definition by studying the proof of Sperner's lemma in Section 2.2. With this machinery in place, we will return to Brouwer fixed point and Nash equilibrium computation in Sections 2.3 and 2.4, discussing their intractability and what it means for Game Theory in Section 2.5. In Section 2.6, we discuss ways to alleviate our intractability results by considering approximation and special families of games.

Nash Equilibrium, formally. From now on, we will study games with a finite number of players and actions. A finite game can be specified by enumerating its players, the actions available to each player, and the payoffs resulting from every choice of actions by the players. For us, the players of a n-player game will be conveniently labeled $1, \ldots, n$. For every player $p \in \{1, \ldots, n\}$, we will denote by S_p the set of actions available to p. For every selection of actions by p and all other players, player p receives some *payoff*, or *utility*, as determined by some function $u_p : S_p \times \prod_{q \neq p} S_q \to \mathbb{R}$. We assume that players have access to private bits of randomness, and may randomize over their actions independently from the other players. In particular, player p may choose a probability distribution x_p over S_p, so that $x_p(s_p) \geq 0$, for all $s_p \in S_p$, and $\sum_{s_p} x_p(s_p) = 1$. A randomization over actions will be called a *mixed strategy*, or simply a *strategy*. A strategy might be deterministic, i.e. place all its probability mass on a single action. In this case it may be called a *pure strategy* to emphasize that it is deterministic.

With the above notation in place, a Nash equilibrium is a collection of strategies $(x_1, \ldots, x_n)$ such that, for all players p, x_p maximizes the expected payoff to player p when the other players choose an action from their strategies independently at random. Equivalently, it must be that for every player p and for every pair of actions $s_p, s'_p \in S_p$,

with $x_p(s_p) > 0$:

$$\mathbb{E}_{s_{-p} \sim x_{-p}}[u_p(s_p\ ;\ s_{-p})] \geq \mathbb{E}_{s_{-p} \sim x_{-p}}[u_p(s'_p\ ;\ s_{-p})], \tag{3}$$

where, for the purposes of the above expectations, $s_{-p} \in \prod_{q \neq p} S_q$ is chosen from the product measure, x_{-p}, defined by the strategies $\{x_q\}_{q \neq p}$.

The Apparent Inadequacy of NP-completeness. To begin our discussion of the computational complexity of Nash equilibrium, it is instructive to first consider Nash equilibrium computation in two-player games. From a computational complexity standpoint, a reassuring property of these games is that, when the players' utility functions range over rational numbers, then there always exists a Nash equilibrium whose strategies use probability values that are rational, and of bit complexity that is polynomial in the bit complexity required to specify the payoff functions, e.g. this follows from the algorith of Lemke and Howson [1964]. On the contrary, as shown by Nash in his 1951 paper (Nash [1951]), this property fails to hold for games with more than two players. Thus, studying Nash equilibrium computation in games with more than two players requires some care to accommodate the possibility that Nash equilibria may be irrational, and thus not straightforward to represent with a finite number of bits. It is a bit premature to address these issues, so let us stick to two-player games in order to build our intuition.

One way to see that there always must be some Nash equilibrium of polynomial bit complexity in the size of a two-player game is this: Suppose that some genie revealed to us what actions are assigned positive probability by the strategies of the two players in some Nash equilibrium of the game. If we knew the support of a Nash equilibrium, we could set up a system of inequalities expressing the constraints (3) that must be satisfied by a Nash equilibrium of that support. Because the game involves two players, these inequalities are linear in the probabilities used by the players' strategies. Since they are linear, a pair of mixed strategies satisfying these inequalities can be found using linear programming. We know that linear programs always have solutions in rational numbers and of polynomial bit complexity in the description of the linear program. It follows that some Nash equilibrium satisfying inequalities (3), for whatever support was provided by the genie, will have bit complexity polynomial in the bits required to specify these inequalities, which in turn are polynomial in the bits required to specify the game.

The existence of Nash equilibria with polynomial bit complexity implies that the problem of computing Nash equilibrium in two-player games belongs to FNP, the class of search problems in NP. For a search problem to lie in FNP it must be that there always exist solutions of bit complexity polynomial in the bit complexity of the input; there also must be a polynomial-time procedure that verifies the validity of a proposed solution for a given input to the problem. Computing Nash equilibrium in two-player games clearly has these properties. We already argued that there always exist Nash equilibria of bit complexity polynomial in the description of the game, and verifying that a pair of strategies are Nash equilibrium amounts to checking the equilibrium conditions (3), a computation that can be carried out efficiently.

Unfortunately, there are exponentially many possible pairs of supports that Nash equilibrium may have, so we cannot try them all out, using linear programming to check

whether a Nash equilibrium exists for each pair of supports. Is there a more clever, polynomial-time algorithm? Or, can we maybe show that the problem FNP-complete, i.e. polynomial-time equivalent to the hardest problems in FNP such as SATISFIABILITY, the problem of finding satisfying assignments to Boolean formulas? We do not know. But, we *do* know that the following, presumably harder, problems are FNP-complete: (i) finding a Nash equilibrium with a certain payoff-sum for the players, if one exists; and (ii) finding two Nash equilibria, unless there is only one (Gilboa and Zemel [1989] and Conitzer and Sandholm [2008]).

Indeed, NP and its search problem counterpart, FNP, draw much of their intractability from the difficulty in deciding whether a solution exists. This is the case in the variants (i) and (ii) of the Nash equilibrium computation problem presented in the previous paragraph. On the other hand, Nash equilibrium is guaranteed to exist, so the problem of finding a single equilibrium is unlikely to be FNP-complete; indeed, an attempt to establish this would have unexpected consequences such as as NP=co-NP (D. S. Johnson, Papadimitriou, and Yannakakis [1988] and Megiddo and Papadimitriou [1991]). Instead, Nash equilibrium computation belongs to what is believed to be a proper subclass of FNP, which contains search problems that always have solutions. This class is called TFNP, where 'T' in front of FNP stands for *total*, and it contains other familiar and important problems such as FACTORING. How hard is Nash equilibrium computation compared to other problems in TFNP? To answer this question we take a closer look at TFNP.

A look inside TFNP. A prophetic and elegant approach to classify problems in TFNP was proposed by Christos Papadimitriou [1994]. His idea was the following: If a problem is total, then there should be a mathematical proof showing that it always has a solution. If the problem is not known to be polynomial-time solvable, this existence argument should be *non-constructive* or *inefficiently constructive*; otherwise it would have been easy to turn it into an efficient algorithm, computing a solution by constructing it. It is thus reasonable to group TFNP problems into complexity classes according to the non-constructive/inefficiently constructive proof that is needed to establish their totality. As shown by Papadimitriou [ibid.], many important total problems owe their totality to the following existence arguments:

- "*If a finite directed graph has an unbalanced node—a node whose indegree is different from its outdegree—then it must have another unbalanced node.*" As we will see shortly, following Papadimitriou [ibid.] we can define a subclass of TFNP, containing problems that owe their totality to this parity argument on directed graphs. This class, called PPAD, contains—as we will see—Nash equilibrium computation, Brouwer fixed point computation, and other related problems in Topology and Combinatorics.
- "*If a finite undirected graph has an odd degree node, then it must have another odd degree node.*" This parity argument on undirected graphs gives rise to a broader subclass of TFNP, called PPA (Papadimitriou [ibid.]). This class contains fixed point problems on non-orientable surfaces, problems in Social Choice Theory, and interestingly FACTORING. We will discuss this class more in Section 3.

- "*Every finite directed acyclic graph has a sink.*" This argument gives rise to a subclass of TFNP called PLS, which was defined earlier in D. S. Johnson, Papadimitriou, and Yannakakis [1988]. This class contains many problems in Optimization that are solvable via local search, as well as equilibrium computation in an important class of games called "potential," as we discuss in Section 3.
- "*Any map from a finite set to itself is either onto or has a collision.*" This is the pigeonhole principle, which gives rise to another subclass of TFNP called PPP (Papadimitriou [1994]). This class also contains FACTORING, and several other problems in Cryptography and lattices, as we discuss in Section 3.

But, how does one turn an existence argument into a complexity class? And, relatedly, why are the above arguments inefficiently constructive? Isn't it trivial to find an unbalanced or odd degree node in a graph, a sink in a directed acyclic graph, or a collision in a function? Indeed, these problems are trivial if the graph or function are given explicitly. But, if the input, graph or function, is specified implicitly via a circuit, these problems are not as easy as they may seem. Consider, for instance, the following problem, which as we will see is intimately related to the parity argument in directed graphs:

END_OF_A_LINE: Given two circuits $S, P : \{0,1\}^n \to \{0,1\}^n$ with AND, OR, and NOT gates, such that $S(P(0^n)) \neq 0^n = P(S(0^n))$, find some $x \in \{0,1\}^n$ such that

$$P(S(x)) \neq x = S(P(x)) \text{ or } S(P(x)) \neq x = P(S(x)) \neq 0^n. \tag{4}$$

To understand the problem, let us think of P and S as specifying a directed graph with node set $\{0,1\}^n$ as follows: There is an edge from node u to node v iff $S(u) = v$ and $P(v) = u$. In particular, every node has in- and out-degree at most 1. Also, because $S(P(0^n)) \neq 0^n = P(S(0^n))$, the node 0^n has in-degree 0 and out-degree 1 and hence it is unbalanced. From the parity argument in directed graphs, there must exist some other unbalanced node. We seek to find any unbalanced node other than 0^n.[2]

Thus END_OF_A_LINE is total by the parity argument in directed graphs. It also clearly lies in FNP, as we can easily verify whether a given x is a solution by verifying the validity of (4). So the problem lies in TFNP. The obvious approach to compute a solution is to follow the path originating at 0^n until the sink node at the other end of this path is found. But this procedure may take exponential time, since there are 2^n nodes in the graph specified by S and P and, for all we know, these may all lie on this path. To solve the problem more efficiently, we would have to find a way to "telescope" the graph by looking at the inner workings of the circuits S and P, but we know no non-trivial ways for doing so.

We conjecture that END_OF_A_LINE cannot be solved in polynomial-time, and following Papadimitriou [ibid.] we use this problem as a basis to define complexity class

[2]The name END_OF_THE_LINE had originally been used for this problem, but as Rubinstein [2016] correctly observes END_OF_A_LINE is better suited to describe it.

PPAD via polynomial-time reductions, in the same way that FNP can be defined via reductions to SATISFIABILITY. We will thus carve out a subset of TFNP, which contains problems whose totality is due to the parity argument in directed graphs. The classes PPA, PLS, and PPP are defined similarly, via reductions to problems whose totality is due to the existence arguments presented above (Papadimitriou [1994]). We discuss these classes and the computational problems that they capture in more detail in Section 3. Figure 6 shows how these classes, and further classes introduced in Section 3 are related.

Our plan for the rest of this section is as follows. In Section 2.1, we define the class PPAD formally via polynomial-time reductions to END_OF_A_LINE. In Section 2.2, we motivate our definition of PPAD further by showing that SPERNER, a computational problem related to Sperner's lemma, lies in PPAD. Sperner's lemma is a combinatorial lemma that can be used to establish Brouwer's fixed point theorem, which can be used in turn to establish Nash's theorem. It will thus follow that both BROUWER, the problem of computing Brouwer fixed points, and NASH, the problem of computing Nash equilibria, also lie in PPAD, as we discuss in Sections 2.3 and 2.4 respectively. That these problems lie in PPAD means that PPAD contains important problems, and is thus an interesting subset of TFNP. What is remarkable though is that PPAD captures tightly the complexity of these problems in the form of PPAD-completeness results (Theorems 1, 2 and 3). In particular, not only SPERNER, BROUWER and NASH lie in PPAD, but also *any* problem in PPAD can be reduced to these problems. In particular, SPERNER, BROUWER and NASH are computationally equivalent to each other with respect to polynomial-time reductions.

2.1 The class PPAD. We define PPAD via efficient reductions to END_OF_A_LINE, in the same fashion that FNP can be defined in terms of reductions to SATISFIABILITY. Our notion of reduction is the following: A search problem $\mathcal{A}$ in TFNP is polynomial-time reducible to a search problem $\mathcal{B}$ in TFNP, if there exists a pair of polynomial-time computable functions (f, g) such that, for every instance x of problem $\mathcal{A}$, $f(x)$ is an instance of problem $\mathcal{B}$, and for every solution y of instance $f(x)$, $g(y, x)$ is a solution of instance x. With this terminology, we define PPAD as follows:

PPAD= {all problems in TFNP that are polynomial-time reducible to END_OF_A_LINE}.

We believe that PPAD contains problems that cannot be solved in polynomial time. But, since PPAD lies between FP and FNP, we cannot hope to show this without also showing that FP$\neq$FNP. In the absence of such proof, our reason to believe that PPAD is a hard class is the same reason of computational and mathematical experience that makes us believe that FNP is a hard class (even though our confidence is necessarily a little weaker for PPAD, since it lies inside FNP): PPAD contains many problems for which researchers have tried for decades to develop efficient algorithms; later in this article, we discuss some of these problems, including BROUWER the problem of computing approximate Brouwer fixed points. But, the problem END_OF_A_LINE is already a pretty convincingly hard problem: How could we hope to "telescope" exponentially large paths in a graph given implicitly via circuits?

2.2 Sperner's lemma, and its relationship to PPAD. To grasp the nature of PPAD better, let us consider the following elegant result in Combinatorics, known as *Sperner's Lemma* (Sperner [1928]), which can be used (Scarf [1967a]) to establish Brouwer's fixed point theorem (Hadamard [1910] and Brouwer [1911]). We will present its two-dimensional variant, but it generalizes to any dimension. To explain the lemma, let us take the unit square, subdivide it into smaller squares of size δ, and divide each of these little squares into two right triangles, as shown in Figure 2—ignore the colors at the moment. We are allowed to color the vertices of this subdivision with three colors, red, yellow, and blue, in an arbitrary fashion, except that our coloring must satisfy the following property:

(P): No vertex on the lower side of the square is colored red, no vertex on the left side is colored blue, and no vertex on the other two sides is colored yellow. The resulting coloring may look as in Figure 2.

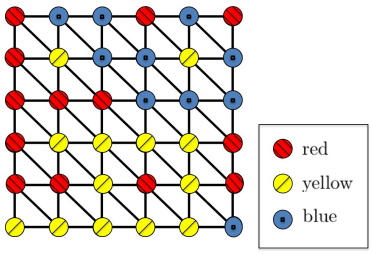

Figure 2: An illustration of the setup of Sperner's lemma. The vertices of the standard triangulation of a 2-dimensional grid are colored with red, yellow and blue arbitrarily, as long as the boundary coloring satisfies the following requirement: no vertex on the lower side of the square is colored red, no vertex on the left side is colored blue, and no vertex on the other two sides is colored yellow.

With this setup, Sperner's lemma asserts the following:

Lemma 1 (Sperner [1928]). *In all colorings satisfying property (P), there exists at least one triangle of the subdivision whose vertices have all three colors. In fact, there is an odd number of them.*

For the coloring shown in Figure 2, these trichromatic triangles are shaded in Figure 3.

Our plan next is to define a computational problem, SPERNER, which belongs to TFNP owing to Sperner's lemma. We will in fact show that SPERNER belongs to PPAD, and by doing so we will also obtain a proof of Sperner's lemma. In SPERNER, we imagine that the size of the subdivision is $\delta = 2^{-n}$, and that the colors of the vertices of the subdivision are not enumerated explicitly. Rather, a circuit is provided, which on input (x, y), $x, y \in \{0, 1, \ldots, 2^n\}$, outputs the color of the point $2^{-n} \cdot (x, y)$. The goal in SPERNER is finding a trichromatic triangle or a violation of the coloring requirement (P).

SPERNER	
Input:	A circuit C, which on input (x, y), $x, y \in \{0, 1, \ldots, 2^n\}$, outputs red, yellow, or blue for the vertex $2^{-n} \cdot (x, y)$ of the δ-triangulation of $[0, 1]^2$ with $\delta = 2^{-n}$.
Output:	A trichromatic triangle or a violation of the coloring requirement (P) on the boundary.

Clearly, SPERNER lies in TFNP. Indeed, it is easy to verify that a triangle is trichromatic, or that a vertex on the boundary violates (P). So the problem lies in FNP. Moreover, if (P) is satisfied, then there must exist a trichromatic triangle by Sperner's lemma. So the problem is total. Our next goal is to argue that the problem belongs to PPAD. Our proof will construct an instance of END_OF_A_LINE, whose solutions will correspond to trichromatic triangles, or violations of property (P) in the SPERNER instance.

As it turns out we can, without loss of generality, assume that the coloring circuit provided in the input of SPERNER satisfies property (P), and that there is only one change from yellow to red on the left side of the unit square, occurring at the bottom-most segment of the subdivision, as in Figure 2. Indeed, a simple preprocessing step we can do on a SPERNER instance is to embed the δ-subdivision of the square $[0, 1]^2$ into the δ-subdivision of a slightly bigger square $[-\delta, 1 + \delta]^2$. The coloring of the latter is chosen as follows. Any vertex of the subdivision that lies in $[0, 1]^2$ is colored according to the circuit of the given SPERNER instance. Any vertex that lies on the boundary of $[-\delta, 1 + \delta]^2$ is colored in a very canonical way:

- all vertices but the bottom-most vertex of the left side of $[-\delta, 1 + \delta]^2$ are colored red;
- all vertices but the right-most vertex of the bottom side of $[-\delta, 1 + \delta]^2$ are colored yellow;
- all other vertices of the boundary of $[-\delta, 1 + \delta]^2$ are colored blue.

It is clear that the coloring of $[-\delta, 1 + \delta]^2$ that we just described satisfies property (P). Moreover, there is a single change from yellow to red on the left side of the square at the bottom-most segment of the subdivision. Importantly, the original coloring of $[0, 1]^2$, as specified by the given SPERNER instance, was not modified. So any trichromatic triangle that lies in $[0, 1]^2$ existed already in the original SPERNER instance. Moreover, our afore-described choice of coloring of the boundary of $[-\delta, 1 + \delta]^2$ makes sure that any trichromatic triangle that lies outside of $(0, 1)^2$ must contain a vertex on the boundary of $[0, 1]^2$ that was violating (P) in the original SPERNER instance.

In summary, our preprocessing step took an instance of SPERNER and defined a modified instance of SPERNER[3] such that Property (P) is automatically satisfied, there is a single change from yellow to red on the left side of the square at the bottom-most segment of the subdivision, and such that a trichromatic triangle in the modified instance

[3]If you are worried that the new instance is not defined on $[0, 1]^2$, note that we can always shift and scale $[-\delta, 1 + \delta]^2$ so that it becomes $[0, 1]^2$. The resulting triangular subdivision of $[0, 1]^2$ will correspond to a slightly smaller δ.

gives us either a trichromatic triangle in the original instance or a violation of Property (P) by the original instance. Moreover, the preprocessing step can be carried out in polynomial time: Given the circuit of the original instance, we can easily define a circuit for the modified instance.

Now let us provide a polynomial-time reduction from our modified SPERNER instance to END_OF_A_LINE. We will define a directed graph G whose nodes have in- and out-degree at most 1. We will then define circuits S and P in a END_OF_A_LINE instance to induce that same graph G. We identify the nodes of graph G with the right triangles of the subdivision, so that node 0^n is identified with the bottom-left triangle T^*, which contains the unique, by construction of the modified SPERNER instance, segment of the subdivision on the left side of the square where the transition from yellow to red occurs. This triangle is marked in Figure 3 by a diamond. The edges of graph G are defined as follows. We introduce an edge from node u to node v of G, if the corresponding right triangles T_u and T_v of the SPERNER subdivision share a red-yellow edge, which goes from red to yellow clockwise in T_u and anti-clockwise in T_v. The edges of G are denoted by arrows in Figure 3. The triangles that have no incoming or outgoing arrows in the figure correspond to isolated nodes in G.

It is not hard to check that the graph G that we constructed has in-degree and out-degree at most 1. Indeed, no triangle can have more than one red-yellow transition clockwise and more than one red-yellow transition anti-clockwise. So graph G is comprised of directed paths, directed cycles and isolated nodes. Moreover, in our modified SPERNER instance, the bottom-left triangle T^* cannot be trichromatic, as the subdivision of a modified SPERNER instance has at least four vertices on the bottom boundary, and all but the right-most are colored yellow. In particular, by construction of the coloring in the modified SPERNER instance, T^* has two yellow vertices and one red vertex, and in particular it shares a red-yellow edge with its unique neighbor, and this edge corresponds to a clockwise red-yellow transition in T^*. As such, node 0^n in G, which corresponds to triangle T^*, is the source of a directed path in G. We claim that the sink of that path must correspond to a trichromatic triangle. Indeed, it follows from the definition of the edges in G and the boundary coloring of our modified SPERNER instance, that any sink in graph G must correspond to a trichromatic triangle. We can similarly argue that any source in G that is different from 0^n must also correspond to a trichromatic triangle. Moreover, no isolated node or node with total degree 2 corresponds to a trichromatic triangle. It follows from all of the above that there must exist an odd number of trichromatic triangles in the modified SPERNER instance. These are the shaded triangles in Figure 3, corresponding to all the sinks, and all the sources other than 0^n in G.

Our construction, outlined above, proves Sperner's lemma. Indeed, if a Sperner coloring satisfies boundary conditions (P), then the modified SPERNER instance that we described above introduces no further trichromatic triangles. And our construction of graph G shows that there is an odd number of those. To conclude our polynomial-time reduction from SPERNER to END_OF_A_LINE, observe that the circuit computing the coloring of the vertices of the subdivision in the modified SPERNER instance can be used to construct in polynomial-time circuits S and P that specify an instance of

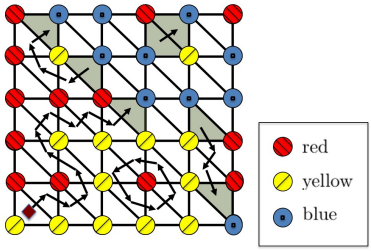

Figure 3: An illustration of the proof of Sperner's lemma.

END_OF_A_LINE whose graph is G. So our reduction from the modified SPERNER instance to END_OF_A_LINE can be carried out in polynomial time.

All in all, it follows from the above that SPERNER $\in$ PPAD. In particular, we saw how the directed parity argument can be used to prove Sperner's lemma, and how the proof can be turned into a polynomial-time reduction from SPERNER to END_OF_A_LINE, establishing that SPERNER lies in PPAD. What is even more remarkable, however, is the following result established by Papadimitriou [1994] for the three-dimensional variant of SPERNER and more recently established for the two-dimensional variant by Chen and Deng [2006a].

Theorem 1 (Papadimitriou [1994] and Chen and Deng [2006a])**.** SPERNER *is* PPAD-*complete.*

Theorem 1 states that SPERNER not only lies in PPAD, but every problem in PPAD can be polynomial-time reduced to it. This is what the PPAD-completeness of the problem means. In particular, PPAD not only contains SPERNER, but it tightly captures the computational complexity of SPERNER. SPERNER is exactly as hard as END_OF_A_LINE, which defines PPAD.

2.3 The Complexity of Brouwer Fixed Points. We have already mentioned that the existence of Nash equilibrium was established by Nash [1951] via an application of Brouwer's fixed-point theorem. On route to understanding the computational complexity of Nash equilibria, we now study the complexity of Brouwer fixed points. But, first, what is the statement of Brouwer's fixed point theorem? Even if you're not familiar with its statement, you are most certainly familiar with the applications of this theorem. Here is one: Take two identical sheets of graph paper with coordinate systems on them, lay one flat on the table and crumple up (without ripping or tearing) the other one and place it any fashion you want on top of the first so that the crumpled paper does not reach outside the flat one. There will then be at least one point of the crumpled sheet of paper that lies exactly on top of its corresponding point (i.e. the point with the same coordinates) of the flat sheet. The magic that guarantees the existence of this point is topological in nature, and this is precisely what Brouwer's fixed-point theorem captures. The statement formally is that any continuous map F from a *compact* (that is, closed and bounded) and *convex* (that is, without holes) subset of the Euclidean space to itself always has a *fixed point*, i.e. there is some point $x = F(x)$.

Brouwer's theorem suggests the following interesting computational problem: Given a continuous function F from some compact and convex subset of the Euclidean space to itself, find a fixed point. Of course, to make this problem computationally meaningful we need to address the following: How do we represent a compact and convex subset of the Euclidean space? How do we specify the continuous map from this set to itself? And how do we deal with the possibility that only irrational fixed points may exist?

For simplicity, let us fix the compact and convex set to be the unit hypercube $[0,1]^m$. We could allow the set to be any convex polytope (given as the bounded intersection of a finite set of half-spaces, or as the convex hull of a finite set of points in $\mathbb{R}^m$). We could even allow more general domains representable via a finite number of bits, e.g., spherical or ellipsoidal domains. Usually, the problem in a more general domain can be reduced to the hypercube by shrinking the domain, translating it so that it lies inside the hypercube, and extending the function to the whole cube so that no new fixed points are introduced. To avoid such technicalities, we fix our domain to be the hypercube.

To deal with the possibility that there might only be irrational fixed points, we turn our problem into an approximation one. Following Scarf [1967a], rather than a fixed point, we seek a point that is approximately fixed. In particular, for some $\varepsilon > 0$, we want to identify a point x such that $d(x, F(x)) \leq \varepsilon$, for some distance $d(\cdot,\cdot)$. For simplicity, let us take d to be the ℓ_∞ norm. Clearly, a rational such point exists, resolving the issue of irrationality. We could alternatively insist on a stronger type of approximation, seeking a point that is close to an exact fixed point. Which of the two notions of approximation is more relevant depends on the underlying application. Since our own motivation for considering Brouwer fixed points is their relationship to Nash equilibria, it is more relevant for us to target the weaker notion of approximation. A complexity theoretic treatment of the stronger notion can be found in Etessami and Yannakakis [2010], and is discussed briefly in Section 3.2.

The function F whose approximate fixed point is to be computed can be specified as a polynomial-time algorithm Π_F, which can be provided as a subroutine that computes $F(x)$ for each point $x \in [0,1]^m$.[4] To ensure that our problem is computationally meaningful, we assume that F is Lipschitz:

$$d(F(x_1), F(x_2)) \leq K \cdot d(x_1, x_2), \quad \forall x_1, x_2 \in [0,1]^m, \tag{5}$$

for some Lipschitz constant K that is provided in the input. This condition ensures that there exist approximate fixed points whose bit complexity is polynomial in the size of the input. Indeed, we have the following guarantee: for any ε, there is a point x such that $d(F(x), x) \leq \varepsilon$, whose coordinates are integer multiples of $2^{-\ell}$, where 2^ℓ is linear in K and $1/\varepsilon$. Without some control on the continuity of F, there would be no such guarantee.

Putting everything together, a computationally meaningful fixed point computation problem can be defined as follows.[5]

[4] As polynomial running time is not a property of a subroutine that is tractable to verify, if we want to be pedantic in our definition, we should require that the subroutine comes with a timer that stops the computation if it takes longer than some polynomial function of the input size, outputting a standard output if this happens.

[5] Again, because Lipschitzness cannot be tractably verified given Π_F, if we want to be pedantic in our definition, we could include in the Output line of the following definition "or a violation of Condition (5)."

BROUWER	
Input:	A polynomial-time algorithm Π_F that evaluates a function $F : [0,1]^m \to [0,1]^m$; a constant K for which F is claimed to satisfy Condition (5); and the desired accuracy ε.
Output:	A point x such that $d(F(x), x) \leq \varepsilon$.

As defined above, BROUWER can be used to capture many fundamental problems in TFNP whose existence of solutions is shown via Brouwer's fixed point theorem, including several problems in Game Theory and Economics (Scarf [1967a]). But what is the computational complexity of BROUWER? As we have already noted, Brouwer's theorem can be proved via Sperner's lemma (Scarf [ibid.]). This proof can be turned into a proof that BROUWER $\in$ PPAD, using that SPERNER $\in$ PPAD. What is more important though is that the opposite is also true, as was shown by Papadimitriou.

Theorem 2 (Papadimitriou [1994]). BROUWER *is* PPAD-*complete.*

Once again, PPAD not only contains BROUWER but it is equivalent to BROUWER, in the sense that any problem in PPAD can be reduced to BROUWER. Moreover, since SPERNER is also PPAD-complete, BROUWER and SPERNER are computationally equivalent.

Sketch of Proof of Theorem 2. That BROUWER $\in$ PPAD follows from the fact that Brouwer's theorem can be shown via Sperner's lemma. This can be turned into a polynomial-time reduction from BROUWER to SPERNER, and since the latter lies in PPAD, by Theorem 1, so does the former.

To show that BROUWER is PPAD-complete, it remains to establish that any problem in PPAD can be polynomial-time reduced to BROUWER. Indeed, it suffices to provide a polynomial-time reduction from END_OF_A_LINE to BROUWER. We sketch this reduction here. We will reduce END_OF_A_LINE to the 3-dimensional version of BROUWER, i.e. $m = 3$. Given an instance of END_OF_A_LINE, i.e. circuits S and P, we embed the graph G defined by these circuits into $[0,1]^3$. Recall that G is comprised of directed paths, directed cycles, and isolated vertices. We will embed this graph in $[0,1]^3$ so as to satisfy the following properties:

- isolated nodes of G have no image in $[0,1]^3$;
- non-isolated nodes v of G are mapped to segments in $[0,1]^3$ arranged close to the $(x,0,0)$ edge of $[0,1]^3$; in particular, node v is mapped to a segment from point v_1 to point v_1', which are respectively called the "in-probe" and the "out-probe" of the embedding of v;
- edges of G are mapped to rectilinear (single-dimensional) subsets of $[0,1]^3$; in particular, an edge from node u to node v is embedded as a rectilinear path from the out-probe of u to the in-probe of v;
- the embeddings of nodes and edges of G are non-intersecting.

Figure 4 gives a sense of the embedding by showing how non-isolated nodes u, v and the directed edge from u to v are embedded. In particular, segment u_1 to u_1' is the

embedding of node u, the segment from v_1 to v_1' is the embedding of node v, and the rectilinear path from u_1' to v_1 is the embedding of edge (u, v) of G.

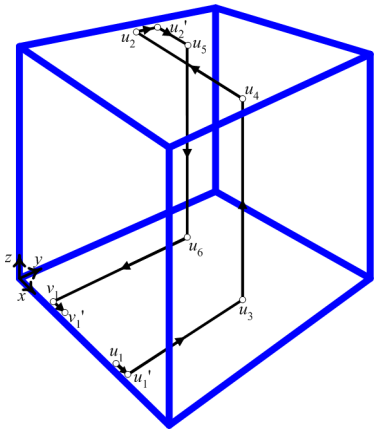

Figure 4: Embedding the END_OF_A_LINE graph in a cube. The embedding is used to define a continuous function F, whose approximate fixed points correspond to the unbalanced nodes of the END_OF_A_LINE graph.

Our continuous function F is defined in terms of this embedding. Roughly speaking, in most of $[0,1]^3$ the vector field $F(x) - x$ is parallel to $(-1,-1,-1)$. On the faces of $[0,1]^3$ the vector field is parallel to one of the following vectors $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(-1,-1,-1)$, as appropriate so that F does not point outside of the cube on those faces. Lastly, on the rectilinear embedding of graph G and its proximity the vector field $F(x) - x$ is a convex combination of vectors $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. With these constraints, F is defined in such a way that the only points of $[0,1]^3$ where the vector field may be a convex combination of all four vectors $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(-1,-1,-1)$, i.e. the only points of $[0,1]^3$ that may be fixed points of F, must be in the proximity of where the rectilinear embedding of G is "open," i.e. in the proximity of in-probes of embeddings of sources of G and out-probes of embeddings of sinks of G. Node 0^n of G, which is a source, is embedded in a special manner to avoid having a fixed point in the proximity of its in-probe. In particular, its in-probe is connected in a special manner to the point $(0,0,0)$. Lastly, the embedding of G and the function F are defined as above so that the value of F can be computed efficiently from S and P on a query point x, and the reduction is thus polynomial-time computable. □

2.4 The Complexity of Nash Equilibrium. We are now ready to come back to our original motivation, that of understanding the computational complexity of Nash equilibrium. In 2-player games, we argued that there always exists an equilibrium in rational probabilities, and of bit complexity polynomial in the description of the game. We have also noted that, as shown by Nash [1951], there exist 3-player games that only have irrational equilibria. In the absence of rational solutions, what does it mean to compute a Nash equilibrium?

For a meaningful computational question, we turn again to approximation. As in the computation of Brouwer fixed points, there are two obvious types of approximation. One is to seek a collection of strategies that is within distance ε from an exact Nash equilibrium, for some choice of distance. This certainly resolves the issue of irrationality. However, it is too strong a requirement. The players of a game are interested in their own payoff, which is what their strategies aim to maximize. The distance of the joint strategy profile of all players from an exact Nash equilibrium is instead a global property of the joint profile. Each individual player may not care about this distance, or may not even be in a position to measure it—e.g. she may not observe the payoffs of the other players.

Motivated by this discussion, we opt for a more meaningful notion of approximation, defined in terms of each player's individual payoffs. We call a collection $x_1, \ldots, x_n$ of strategies an *ε-Nash equilibrium* if, for every player p and for every pair of actions s_p, s'_p, with $x_p(s_p) > 0$:

$$\mathbb{E}_{s_{-p} \sim x_{-p}}[u_p(s_p\ ;\ s_{-p})] \geq \mathbb{E}_{s_{-p} \sim x_{-p}}[u_p(s'_p\ ;\ s_{-p})] - \varepsilon, \tag{6}$$

where, for the purposes of the above expectations, $s_{-p} \in \prod_{q \neq p} S_q$ is chosen from the product measure, x_{-p}, defined by the strategies $\{x_q\}_{q \neq p}$. That is, we allow in the support of the strategy of player p actions that approximately optimize p's expected payoff. If ε is sufficiently small, e.g., smaller than the quantization of the currency used by the players, or the cost of changing one's strategy, it may be reasonable to assume that approximate payoff optimality is good enough for the players. Note also that it is easy for a player to check the approximate optimality of the actions used by her strategy. So the notion of ε-Nash equilibrium is an appealing notion of approximation for Nash equilibrium computation. Indeed, this is the notion that is most commonly used in Game Theory and Economics (Scarf [1967a]). For a study of the former, stronger notion of approximation, discussed in the previous paragraph, we refer the reader to Etessami and Yannakakis [2010] and our brief discussion in Section 3.2.

It is not hard to see that every game has a rational ε-Nash equilibrium of polynomial bit complexity in the description of the game. Hence, the following is a computationally meaningful definition of the Nash equilibrium computation problem.

NASH: Given a game $\mathcal{G}$ and an approximation requirement ε, find an ε-Nash equilibrium of $\mathcal{G}$.

The main result in this article is the following result from joint work with Goldberg and Papadimitriou.

Theorem 3 (Daskalakis, Goldberg, and Papadimitriou [2006, 2009a,b]). NASH *is* PPAD-complete.

Idea of the Proof of Theorem 3. That NASH $\in$ PPAD follows from the fact that Nash's theorem can be shown via an application of Brouwer's fixed point theorem. This can be turned into a polynomial-time reduction from NASH to BROUWER, and since the latter lies in PPAD, by Theorem 2, so does the former.

To show that NASH is PPAD-complete, it remains to establish that any problem in PPAD can be polynomial-time reduced to NASH. Indeed, it suffices to provide a polynomial-time reduction from END_OF_A_LINE to NASH. We only provide a high-level idea of this reduction. For different takes on a high-level description of the reduction we refer the reader to Daskalakis, Goldberg, and Papadimitriou [2009b] and Daskalakis [2009]. For the complete proof, we refer the reader to Daskalakis, Goldberg, and Papadimitriou [2006, 2009a]. A major difference between NASH and the problems SPERNER and BROUWER, which were shown to be PPAD-complete earlier than NASH, is that the instances of both SPERNER and BROUWER contain circuits or Turing machines so it is easier to obtain a reduction to these problems from END_OF_A_LINE, a problem whose instances also contain circuits. On the other hand, no circuit or Turing machine is provided in an instance of NASH. So the complexity in identifying a source or a sink in an implicitly defined directed graph, which is what is driving the computational complexity of END_OF_A_LINE, must be somehow embedded into the payoff functions of the players in a game.

To obtain a reduction from END_OF_A_LINE to NASH, it is instructive to first consider the possible efficacy of a reduction with a particular structure, described below. By studying what fails in this structure, we will get an idea for a reduction that does work. Recall that, given circuits S and P, the goal in END_OF_A_LINE is to return some $x \in \{0,1\}^n \setminus \{0^n\}$ corresponding to a sink or a source of the graph G induced by S and P, i.e. some x satisfying:

$$P(S(x)) \neq x = S(P(x)) \text{ or } S(P(x)) \neq x = P(S(x)) \neq 0^n. \tag{7}$$

Given that S and P are Boolean circuits, and (7) is a Boolean constraint in terms of these circuits, we would like to define a game such that the Nash equilibrium strategies of some special group of players in this game encode a Boolean vector x satisfying (7). Inspired by the common CIRCUIT-SATISFIABILITY to SATISFIABILITY reduction, a reasonable way to reduce END_OF_A_LINE to NASH would be to define a multi-player game that has three layers:

- A group of special players $p_{x_1}, \ldots, p_{x_n}$ at the "bottom layer of the game" have action sets $\{0,1\}$. They interact with each other but with no other player of the game. Their payoff functions are defined in such a way that, regardless of whatever else is happening in the game, their strategies $x_1, \ldots, x_n$ in a Nash equilibrium of the game are deterministic, i.e. $x_1, \ldots, x_n \in \{0,1\}$. Moreover, we want that for all $z \in \{0,1\}^n$, there is a Nash equilibrium of this layer such that $x_i = z_i$, for all i.

- The strategies of the players in the bottom layer feed into the payoffs of a group of players in the "medium layer of the game" whose payoffs depend on each other's strategies and also the strategies of the players in the bottom layer. We would like to define these payoffs in such a way that some special player p^* in this layer computes the truth value of the Boolean expression (7). I.e. p^* has action set $\{0,1\}$ and for whatever strategies $x_1, \ldots, x_n \in \{0,1\}$ players $p_{x_1}, \ldots, p_{x_n}$ play, the Nash equilibrium strategy of p^* is either 1 or 0 as determined by the Boolean formula (7).

- Then we would like the strategy of player p^* to feed into the payoffs of a group of players in the "top layer of the game" whose payoffs depend on each other's strategies and also the strategy of p^*. We would like to define these payoffs in such a way that the following is guaranteed: if p^* plays 1, the players at the top layer have Nash equilibrium strategies; if p^* plays 0, they do not.

If we were able to define a game satisfying the above properties we would be done. Indeed, the only way for the game to be in Nash equilibrium would be for player p^* to play 1, which in turn would mean that Boolean formula (7) evaluates to 1 on the strategies $x_1, \ldots, x_n$ played by players $p_{x_1}, \ldots, p_{x_n}$, which are guaranteed to lie in $\{0, 1\}$ by the properties of the bottom layer of the game.

Alas, a game as described above can simply not exist. While the bottom and medium layers can indeed be implemented, there is no way to implement the top layer. We need that, if p^* plays 0, the players at the top layer cannot find Nash equilibrium strategies. However, by Nash's theorem, *every game has a Nash equilibrium*, so it is simply not possible to have the top layer not have a Nash equilibrium when player p^* plays 0. (Notice that we assumed that p^*'s strategy feeds into the payoffs of the players in the top layer, but p^*'s payoff does not depend on the strategies of the players in that layer.) Another important issue of a reduction with the afore-described structure is that, if it worked, it would also work to reduce SATISFIABILITY to NASH, as we never used any special structure of the Boolean formula (7) in our construction. As we have already noted, however, such a reduction would have unexpected consequences (D. S. Johnson, Papadimitriou, and Yannakakis [1988] and Megiddo and Papadimitriou [1991]).

The above discussion was to illustrate that, in order to reduce END_OF_A_LINE to NASH, we need to move away from the structure of reductions that are commonly employed to show NP-completeness. A salient feature of the structure described above was that information was flowing one way. The strategies of the players in the bottom layer were feeding into the payoffs of the players in the medium layer, whose strategies were in turn feeding into the payoffs of the players in the top layer. The goal of the bottom layer was to "guess" the solution to (7), but whether this guess was good or not was punted to the top layer, the players of which had no way to influence the strategies of players in the bottom layer, and could thus always find some way to reach Nash equilibrium regardless of what these bottom players played. To fix the reduction we must somehow "close the loop" in how strategic information flows inside the game.

A successful reduction from END_OF_A_LINE to NASH uses the following ideas. First, rather than pursuing a reduction from END_OF_A_LINE to NASH it is more convenient to pursue a reduction from BROUWER to NASH. Indeed, while the solutions to END_OF_A_LINE are combinatorial objects, Brouwer fixed points and Nash equilibria are continuous objects, so they are better suited to reduce one to the other. By proving his existence theorem, Nash already established a reduction from NASH to BROUWER. We are seeking a reduction in the opposite direction. While in our previous attempt we wanted to construct a game where Nash equilibrium strategies are deterministic, and thus simulate Boolean variables, we can now allow Nash equilibrium strategies to be randomized and thus simulate continuous variables.

To reduce an instance of BROUWER to an instance of NASH, the bulk of the work is to construct a multi-player game whose (approximate) Nash equilibria are in correspondence with the approximate fixed points of the function F provided in the input to BROUWER. We will then post-process this game to construct a game with a small number of players having the same property. All players in our multi-player game have action set $\{0, 1\}$. There are m special players $p_{x_1}, \ldots, p_{x_m}$, whose joint strategies $x_1, \ldots, x_m$ are interpreted as point in $[0, 1]^m$; indeed each x_i is a probability in $[0, 1]$. We want to construct the game so that, in any (approximate) Nash equilibrium, these strategies will correspond to an approximate fixed point of F. The game is roughly constructed as follows:

1. We have players $p_{x_1}, \ldots, p_{x_m}$ as described above in the bottom layer of the game.
2. The strategies, $x_1, \ldots, x_m$, of these players are fed into the middle layer of the game. They affect the payoffs of the players in that middle layer, but the payoffs of players $p_{x_1}, \ldots, p_{x_m}$ do not directly depend on the strategies of the players in this layer.
3. The payoffs of the players in the middle layer are set in such a way that an m-tuple, $p_{y_1}, \ldots, p_{y_m}$, of special players in this layer will play Nash equilibrium strategies, $y_1, \ldots, y_m \in [0, 1]$, which satisfy $y_i \approx F(x_1, \ldots, x_m)_i$, for all i, for whatever strategies $x_1, \ldots, x_m$ the players of the bottom layer play. The bulk of the difficulty in our construction is to define the payoffs in this layer so that all Nash equilibria of the game satisfy this property.
4. An important ingredient enabling the construction of the middle layer are "game-gadgets," small multiplayer games that implement real arithmetic and Boolean operations in the players' Nash equilibrium strategies. Let us illustrate one such game-gadget, implementing multiplication.

   The multiplication game has four players with action sets $\{0, 1\}$. It has two special players X and Y who are the "input players." The payoffs of these players do not depend on the strategies of the other players in the game-gadget so their strategies, however they are chosen, act as inputs to the game-gadget. In particular, we interpret the probabilities, x and y, that these players assign to action '1' as real inputs to the game-gadget. The game-gadget has another special player Z, called the "output player," and our goal is to define the payoffs of the players in the game-gadget in such a way that, in all Nash equilibria of the game-gadget, the probability z assigned by player Z to action '1' satisfies $z = x \cdot y$. In fact, we want that the same is true in all Nash equilibria of any game that contains our game-gadget. In particular, we want that this holds true if the payoffs of the input players depend in arbitrary ways on the strategies of the players of that larger game that contains our game-gadget (but the payoffs of the non-input players of our game-gadget should not depend on the strategies of any player not in our game-gadget and be exactly as we specify them below).

   Now let us briefly describe how the game-gadget is defined to implement multiplication. Besides the input players and the output player, the game-gadget also contains a fourth player W who intermediates between the input players and the

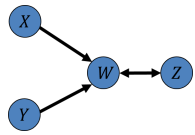

Figure 5: The players of the multiplication game. The graph shows how players' strategies affect each other's payoffs.

output player. Figure 5 illustrates the payoff dependencies among players, where a directed edge from some player to another means that the payoff function of the latter directly depends on the strategy of the former. Here is how we define the payoffs of the players W and Z to induce multiplication: We pay player W an expected payoff of $\$x \cdot y$ for playing '0' and an expected $\$z$ for playing '1.' We pay Z a payoff of $\$0.5$ for playing '0' and an expected payoff of $\$(1-w)$ for playing '1.' It is not hard to check that in any Nash equilibrium of the game thus defined, it must be that $z = x \cdot y$. Indeed, if $z > x \cdot y$, then W would prefer action '1', making $w = 1$, and therefore Z would prefer '0', making $z = 0$, which violates the assumption $z > x \cdot y$. We can similarly argue that it cannot be that $z < x \cdot y$. Hence, the payoffs in the game induce player Z to play mixed strategy $z = x \cdot y$ in any Nash equilibrium.

5. We can similarly define game-gadgets that implement the arithmetic operations of addition $(+)$, subtraction $(-)$, assignment $(:=)$, and setting a player's strategy to a desired constant $c \in [0, 1]$.[6]

6. We can also define game-gadgets implementing the Boolean operations AND $(\wedge)$, OR $(\vee)$, NOT $(\neg)$. These game-gadgets implement their intended Boolean operations ($z = x \wedge y$, etc.) whenever their input players play deterministic strategies (i.e. '0' or '1' with probability 1), which can be interpreted as Boolean values.

7. Lastly, we can define a *brittle comparator* game-gadget that compares the strategies $x, y \in [0, 1]$ of its input players X, Y. In all Nash equilibria of this game, the output player Z plays '1' with probability 1, if $x > y$, plays '0' with probability 1, if $x < y$, and it may play any mixed strategy in $[0, 1]$, if $x = y$; we call this latter property "brittleness." There is no way to force the output player to play some fixed strategy when $x = y$. Indeed, if such non-brittle comparator gadget existed, then we could construct a game that has no Nash equilibria, contradicting Nash's theorem.

8. We construct a circuit satisfying the property stated in Step 3 by pasting together copies of the afore-described game-gadgets in the right connectivity in order to implement the following computations:

[6]Of course, addition and subtraction are truncated to $[0, 1]$. E.g. the output player Z of the addition gadget plays Nash equilibrium strategy $z = \min(1, x + y)$, where x and y are the strategies of the input players.

(a) First, we extract the most significant bits from the binary expansion of $x_1, \ldots, x_m$. The number of bits ℓ to be extracted are determined by the approximation accuracy ε that we want to achieve in BROUWER and the Lipschitz constant K of function F that is also specified in the input to BROUWER, and is bounded by a polynomial in the bit complexity of the BROUWER instance—see the discussion following (5) in Section 2.3. Bit extraction can be performed via a circuit of game-gadgets that uses arithmetic and comparator gadgets. The circuit is constructed so that the Nash equilibrium strategies of special players $X_{i1}, \ldots, X_{i\ell}$ lie in $\{0, 1\}$ and correspond to the ℓ most significant bits of x_i, for all i. Let us pretend that we can achieve this, for now. We will return to this shortly.

(b) Next, we simulate the computation of F on $\lfloor x_1 \rfloor_\ell, \ldots, \lfloor x_m \rfloor_\ell$, where $\lfloor x_i \rfloor_\ell$ is the number resulting from keeping the ℓ most significant bits in the binary expansion of x_i. This is performed by combining the Boolean game-gadgets into a circuit that has players X_{ij} as its input players and computes F on inputs with ℓ bits. (Because F is specified via a polynomial-time Turing machine, its computation on inputs with ℓ bits can be described by a circuit of size polynomial in the bit complexity of the BROUWER instance—recall that ℓ itself is bounded by a polynomial in the bit complexity of the BROUWER instance.) If the equilibrium strategies of players X_{ij} lie in $\{0, 1\}$, the Boolean game-gadgets function well, and the circuit computes $F(\lfloor x_1 \rfloor_\ell, \ldots, \lfloor x_m \rfloor_\ell)$.

(c) We take the binary expansion of $F(\lfloor x_1 \rfloor_\ell, \ldots, \lfloor x_m \rfloor_\ell)$ as computed by the circuit described in the previous step, and use arithmetic gadgets to compute its real representation. Let us call $p_{y_1}, \ldots, p_{y_m}$ the output players of this computation, where p_{y_i}'s Nash equilibrium strategy equals the i-th coordinate of the real vector $F(\lfloor x_1 \rfloor_\ell, \ldots, \lfloor x_m \rfloor_\ell)$.

(d) Clearly, if Steps 8a–8c work as intended, the Nash equilibrium strategies of players $p_{y_1}, \ldots, p_{y_m}$ satisfy: $y_i \approx F(x_1, \ldots, x_m)_i$, for all i, where the approximation depends on how many bits ℓ we extracted and the Lipschitz constant of F. Unfortunately, there is an important issue that arises in the afore-described construction. Because our comparators are brittle, the bit extraction fails whenever some strategy x_i causes a comparator in the bit extraction circuit to make comparison of equal values.[7] If this happens we cannot guarantee that the Nash equilibrium strategies of players X_{ij} will be deterministic, and all bets are off about the computation that will take place in the circuit of Step 8b. We resolve this issue by running the circuits of Steps 8a–8c for several points near the point $(x_1, \ldots, x_m)$, and averaging the results. By selecting our points carefully we can show that most of the circuit computations will not run into the brittle comparator problem, and their average will indeed satisfy $y_i \approx F(x_1, \ldots, x_m)_i$, for all i.

[7] This happens for values x_i that are integer multiples of $2^{-\ell}$.

9. The last part of the construction is the "feedback loop." This part of the game uses assignment gadgets to guarantee $x_i \equiv y_i$ in every Nash equilibrium of the game, for all i.

In all Nash equilibria of the game described above, the strategies of players $p_{x_1}, \ldots, p_{x_m}$ satisfy $(x_1, \ldots, x_m) \approx F(x_1, \ldots, x_m)$. Indeed, this follows from the constraints $x_i = y_i$ and $y_i \approx F(x_1, \ldots, x_m)_i$, for all i. It can also be shown that $(x_1, \ldots, x_m) \approx F(x_1, \ldots, x_m)$ holds in all δ-Nash equilibria of the game, if δ is sufficiently small, and this concludes the bulk of the proof of Theorem 3.

The game $\mathcal{G}$ thus constructed has many players (their number depends mainly on how complicated the circuit for computing function F was), and two actions per player. Game $\mathcal{G}$ has a special structure, called a *graphical game* (Kearns, Littman, and Singh [2001]): the players can be viewed as vertices in a low-degree graph, and their payoffs only depend on the strategies of their neighbors in the graph. Indeed, our game was constructed by putting together game-gadgets, and the payoffs of the non-input players in a game-gadget only depend on the strategies of their few neighbors in the game-gadget. Our construction thus far has shown that finding approximate Nash equilibria in graphical games of bounded degree is PPAD-complete.

The final step in the reduction is to simulate this graphical game via a game with a small number of players. Roughly speaking, this can be accomplished by considering a game between *lawyers*. The players of $\mathcal{G}$ are assigned to lawyers, and each lawyer chooses a strategy for each player that he represents. Thus, a game between lawyers is defined whose Nash equilibria are in correspondence with the Nash equilibria of the graphical game. The assignment of players of $\mathcal{G}$ to lawyers must be done in such a way that lawyers have no "conflict of interest," i.e. they are not asked to choose a strategy for two players that interact with each other in $\mathcal{G}$. This determines the number of lawyers that are needed. Our original proof of Theorem 3, reported in Daskalakis, Goldberg, and Papadimitriou [2005], was using 4 lawyers, and thus showed the hardness of NASH for games with at least 4 players. Soon after our proof became available, we modified our construction to also show the hardness of 3-player games in Daskalakis and Papadimitriou [2005], and independently from us so did Chen and Deng [2005a]. A few weeks later Chen and Deng [2005b] had a clever observation that extended our construction to 2-player games. These results were published in the following papers: Daskalakis, Goldberg, and Papadimitriou [2006, 2009a,b], Chen and Deng [2006b], and Chen, Deng, and Teng [2009]. □

2.5 Implications of Theorem 3. Theorem 3 implies that NASH is at least as hard as any problem in PPAD. So, unless PPAD=FP, there is no polynomial-time algorithm for solving NASH. And this implies in turn that there exist games in which it would take players excruciatingly long to arrive at Nash equilibrium strategies, assuming that the computation that players can perform in one step of thinking/interacting is polynomial time. In such a game, it does not seem reasonable to expect the players to play a Nash equilibrium. Hence, Theorem 3 can be seen as a computational barrier to the universality of the Nash equilibrium: *While mathematically Nash equilibria always exist, they are*

not always attainable, assuming that players cannot perform super-polynomial-time computation instantaneously.

Another important implication of Theorem 3 is identifying precisely the inefficiently constructive argument that is needed to prove the existence of Nash equilibrium. That an algorithm for End_Of_A_Line is sufficient for computing (approximate) Nash equilibria was already implicit in the Nash equilibrium computation algorithms proposed since the 1960's (Lemke and Howson [1964], Scarf [1967a], Rosenmüller [1971], and Wilson [1971]). Theorem 3 formalizes this, but more importantly establishes the opposite implication, that Nash is at least as hard as End_Of_A_Line. Theorem 3 means that End_Of_A_Line is a barrier to Nash and it cannot be avoided. Moreover, since Brouwer and Sperner are also PPAD-complete, by Theorems 1 and 2, Theorem 3 implies that Nash, Sperner and Brouwer are computationally equivalent.

Lastly, Nash's proof used Brouwer's theorem to establish the existence of Nash equilibrium. Our proof of Theorem 3 reduces Brouwer to Nash. This reduction can be turned into a proof that the existence of Nash equilibrium implies the existence of Brouwer fixed points, providing a converse to Nash's result.

2.6 Nash Equilibrium, Way Forward. In view of Theorem 3, there are games where Nash equilibria are computationally unattainable. Does this mean that we should dismiss the concept of Nash equilibrium altogether? The answer is 'no,' but we should be cautious when and how we use Nash equilibrium as a method to predict or interpret strategic behavior. We discuss two lines of investigation that would help shed light on these considerations.

Study of Approximate Nash equilibria. While Theorem 3 shows that Nash equilibrium is intractable, it does not preclude that players might arrive in polynomial-time to an ε-Nash equilibrium (6) for some large enough value of ε. But how small might ε be? It follows from the proof of Theorem 3 that it is PPAD-complete to compute ε-Nash equilibrium for values of ε that scale inverse exponentially in the size of the game (number of players/actions). In fact, follow-up work of Chen, Deng, and Teng [2006] strengthens Theorem 3 for values of ε that even scale inverse polynomially in the size of the game. But how about even coarser approximations? What if we ask for a fixed constant ε that does not scale with the size of the game? If ε is chosen smaller than the cost of updating one's strategy, or the quantization of the currency in which the players payoffs are expressed, an ε-Nash equilibrium is almost as attractive as Nash equilibrium.

The complexity of approximate Nash equilibria has been researched extensively in the past fifteen years; see e.g. Lipton, Markakis, and A. Mehta [2003], Daskalakis, A. Mehta, and Papadimitriou [2006], Feder, Nazerzadeh, and Saberi [2007], Daskalakis, A. Mehta, and Papadimitriou [2007], Bosse, Byrka, and Markakis [2007], Tsaknakis and Spirakis [2007], Kannan and Theobald [2007], Tsaknakis, Spirakis, and Kanoulas [2008], Kontogiannis, Panagopoulou, and Spirakis [2009], Daskalakis and Papadimitriou [2009b], Minder and Vilenchik [2009], Kontogiannis and Spirakis [2010], Awasthi, Balcan, Blum, Sheffet, and Vempala [2010], Tsaknakis and Spirakis [2010], Hazan and Krauthgamer [2011], Austrin, Braverman, and Chlamtáč [2011], Daskalakis [2013],

Alon, T. Lee, Shraibman, and Vempala [2013], Braverman, Ko, and Weinstein [2014], Barman [2015], Babichenko [2016], Rubinstein [2016], Czumaj, Deligkas, Fasoulakis, Fearnley, Jurdziński, and Savani [2016], Harrow, Natarajan, and Wu [2016], Babichenko and Rubinstein [2017], Rubinstein [2018], and Kothari and R. Mehta [2018]. Most of this research has focused on two-player games where if the payoffs are scaled to lie in $[0, 1]$ the best approximation that is known to be achievable in polynomial time is $\varepsilon \approx 0.34$ as shown by Tsaknakis and Spirakis [2007]. If instead we are willing to give up on polynomial time, an elegant algorithm by Lipton, Markakis, and A. Mehta [2003] computes an ε-Nash equilibrium in time $m^{O(\log m/\varepsilon^2)}$, where m is the number of actions available to each player. This algorithm is based on the following interesting observation: for any ε, there exists an ε-Nash equilibrium in which the players randomize uniformly over a multiset of their actions of size $O(\log m/\varepsilon^2)$. Hence, we can exhaustively search over all pairs of multisets of that size, then check if the pair of uniform distributions over these multisets constitute an ε-Nash equilibrium. Checking that a pair of strategies is an approximate equilibrium can be performed efficiently, so we pay a quasi-polynomial running time in the number m of actions—i.e. a running time whose dependence on m is of the form $2^{\mathrm{poly}(\log m)}$ — due to the exhaustive search over pairs of multisets. A main open question in this area is the following:

Open Question 1. *Is it possible to compute ε-Nash equilibrium in two-player games with m actions per player and payoffs in $[0, 1]$ in time $m^{f(1/\varepsilon)}$ for some arbitrary function $f(\cdot)$? Or, more modestly, is there an algorithm whose dependence on m is asymptotically better than $m^{O(\log m/\varepsilon^2)}$ for every fixed ε?*

We have known for some while that we cannot hope to get an affirmative answer to these questions via an algorithm that is also *oblivious*. To explain what we mean by "oblivious," note that the algorithm of Lipton, Markakis and Mehta has a very special property: no matter what the input game is, the algorithm searches over a fixed set of candidate solutions—all pairs of uniform distributions on multisets of size $O(\log m/\varepsilon^2)$—and it only looks at the game to verify whether a candidate solution is indeed an ε-Nash equilibrium. In work with Papadimitriou (Daskalakis and Papadimitriou [2009b]), we show that any oblivious algorithm of this kind needs to run in time $m^{O_\varepsilon(\log m)}$. So the answers to both questions within Open Question 1 are negative, if we restrict attention to the family of oblivious algorithms. In fact, recent work of Kothari and R. Mehta [2018] has strengthened this impossibility result to a much broader class of algorithms that are based on the natural Sum-of-Squares relaxation of the problem. Moreover, recent breakthrough work by Rubinstein [2016] has shown that there are no affirmative answers to Open Question 1 for any type of algorithm, under an "exponential time hypothesis" for PPAD, which postulates that solving End_Of_A_Line requires time at least $2^{m/\log^c m}$ in the size m of the input for some c. Under this hypothesis, Rubinstein shows that there exists some constant ε for which any algorithm that computes ε-Nash equilibrium in 2-player games with m actions per player requires $m^{\log^{1-o(1)} m}$ time. In view of this result, it is important to investigate the plausibility of the exponential time hypothesis for PPAD.

Open Question 2. *Disprove the exponential time hypothesis for* End_Of_A_Line*, or relate this hypothesis to the exponential time hypothesis for* Satisfiability.

Finally, we have thus far focused on the computation of approximate Nash equilibria where the approximation guarantee for the actions included in the support of the strategies is additive, i.e. computing player strategies $x_1, \ldots, x_n$ satisfying (6). What happens if we turn the right hand side of (6) into a multiplicative approximation guarantee of the following form

$$\mathbb{E}_{s_{-p} \sim x_{-p}}[u_p(s_p\,;\,s_{-p})] \geq \mathbb{E}_{s_{-p} \sim x_{-p}}[u_p(s'_p\,;\,s_{-p})] - \varepsilon \cdot |\mathbb{E}_{s_{-p} \sim x_{-p}}[u_p(s'_p\,;\,s_{-p})]| \ ?$$

As it turns out approximate Nash equilibria with multiplicative approximation guarantees can be shown to be PPAD-complete for any value of $\varepsilon \in [0, 1)$ as shown in Daskalakis [2013] and Rubinstein [2018].

Study of Special Game Families. Given that even approximate Nash equilibrium is computationally intractable, as we just discussed, we should not expect strategic behavior to even approximate Nash equilibrium in general. On the other hand, it is possible that Nash equilibria arise in games with appropriate structure that turns Nash equilibrium, or approximate Nash equilibrium tractable. In Section 1, we discussed one such family of games, namely two-player zero-sum games. In these games, Nash equilibria are polynomial-time computable, and they exhibit several other properties that make them quite an attractive model of strategic behavior: they comprise a convex set, they all give the same payoffs to the players, and they are efficiently attainable via simple learning procedures that the players can run in tandem; in particular, any no-regret learning procedure works. Are there other interesting families of games with tractable (approximate) Nash equilibria? We discuss some of the game families that have been studied in the literature.

Constant-Rank Games: Extending two-player zero-sum games, Kannan and Theobald [2007] defined the *rank* of a two-player game as follows. A two-player game can be described by a pair of payoff matrices (A, B), whose rows are indexed by the actions of one of the players and whose columns are indexed by the actions of the other player, so that A_{ij} and B_{ij} are the payoffs of the two players if row i and column j are chosen. The game is zero-sum if $A + B = 0$. Generalizing, the game is *rank-r* if the rank of the matrix $A + B$ is r. Kannan and Theobald gave a fully polynomial-time approximation scheme, i.e. an algorithm that runs in time polynomial in $1/\varepsilon$ and the description of the game, for computing ε-Nash equilibria in games of constant rank. They also asked if there exists a polynomial time algorithm to compute exact Nash Equilibrium. Adsul, J. Garg, R. Mehta, and Sohoni [2011] answered this question affirmatively for rank-1 games, and later R. Mehta [2014] showed that Nash equilibrium computation in games with rank ≥ 3 is PPAD-complete. The complexity of rank-2 games remains an open question.

Open Question 3. *Is Nash equilibrium computation in rank-*2 *games* PPAD*-complete?*

Zero-Sum Polymatrix Games: Going beyond two-player games, joint work with Cai, Candogan and Papadimitriou (Daskalakis and Papadimitriou [2009a], Cai and Daskalakis [2011], and Cai, Candogan, Daskalakis, and Papadimitriou [2016]) studies Nash equilibrium computation in multi-player zero-sum games. While multi-player zero-sum games

with more than 2 players can be trivially shown to be PPAD-complete by using the results of Section 2.4, we consider multi-player zero-sum games with pairwise separable payoff functions, also known as *zero-sum polymatrix games*. A n-player polymatrix game $\mathcal{G}$ is described via a collection of matrices, $A^{p,q}$ for each pair of players $p, q \in \{1, \ldots, n\}$. The rows of $A^{p,q}$ are indexed by the actions of player p and the columns are indexed by the actions of player q, so that entry $A^{p,q}_{ij}$ of this matrix corresponds to the payoff derived by player p from her interaction with player q if the players choose actions i and j respectively. As such, the payoff derived by player p when the players choose actions $i_1, \ldots, i_n$ is

$$U_p(i_1, \ldots, i_n) = \sum_q A^{p,q}_{i_p i_q}.$$

The polymatrix game is zero-sum iff for all $i_1, \ldots, i_n$ the sum $\sum_p U_p(i_1, \ldots, i_n) = 0$.

Polymatrix zero-sum games are a natural generalization of two-player zero-sum games, capturing multi-player interactions where players interact pairwise and make decisions (for example, adopt one of many technologies, or choose one or more of their neighbors for preferential interaction), and which is a closed system of payoffs, in that it is impossible for payoffs to flow in or out of the system. These games were studied in early work by Bregman and Fokin [1987, 1998] who proposed algorithms for Nash equilibrium computation. In Daskalakis and Papadimitriou [2009a], Cai and Daskalakis [2011], and Cai, Candogan, Daskalakis, and Papadimitriou [2016] we develop polynomial-time algorithms based on linear programming. We also discover that these games inherit several of the properties that make Nash equilibrium an attractive model of strategic behavior in two-player zero-sum games. In particular, the set of Nash equilibria is convex, and Nash equilibrium is attainable via no-regret learning; see also Even-Dar, Mansour, and Nadav [2009]. On the other hand, and contrary to two-player zero-sum games, the payoffs of players need not be invariant with respect to different Nash equilibria. In particular, Nash equilibrium in zero-sum polymatrix games inherits the tractability of Nash equilibrium from two-player zero-sum games, but exhibits richer combinatorial structure arising from the larger number of players.

It is worth exploring relaxations of zero-sum polymatrix games that maintain the tractability of Nash equilibrium. We know, however, that moving away from pairwise separable payoff functions to allow even three-way separable functions already makes Nash equilibrium PPAD-complete. Similarly, maintaining pairwise separability but removing the zero-sum constraint also turns Nash equilibrium PPAD-complete.

Symmetric and Anonymous Games: Many strategic interactions, especially those involving multiple players, have a large number of symmetries. How do symmetries affect the structure and computation of Nash equilibria. Symmetries were already studied by Nash [1951], where an interesting relationship between the symmetries in a game and the structure of its Nash equilibria was established. To give a flavor of Nash's more general result, let us consider a multi-player game in which all players have the same action sets, and the same payoff functions, which depend on a player's own action and are symmetric functions of the other players' actions. Such a game is called a *symmetric game*. A trivial example of a symmetric game is the rock-paper-scissors game of Figure 1. The game is symmetric because, well, there are only two players, and their

payoff functions are the same functions of their own and their opponent's action. For a more elaborate example, consider a group of drivers driving from Boston to New York on a weekend: they all select from the same set of available routes, and their travel time will depend (at least to some approximation of reality) on their own choice of route and the number, but not the identity, of the other drivers who choose each available route.

Interestingly, Nash [ibid.] showed that every symmetric game has a Nash equilibrium where all players use the same strategy. This structural result was exploited by Papadimitriou and Roughgarden [2008] to propose an algorithm for computing Nash equilibria in symmetric games. Their algorithm runs in polynomial-time as long as the number of actions available to each player is small relative to the number of players, namely as long as the number of actions is $m = O(\log n / \log\log n)$, where n is the number of players. On the other hand, if a symmetric game has a large number of actions and a small number of players there is not much we can do. From a symmetrization construction of G. W. Brown and von Neumann [1950]—and another computationally efficient one by Gale, Kuhn, and Tucker [1950], it follows that two-player games can be reduced to symmetric two-player games. And, this quickly implies that Nash equilibrium computation in two-player symmetric games is PPAD-complete. On the other hand, the following is still open as far as we know.

Open Question 4. *Is Nash equilibrium computation in* 3*-player symmetric games* PPAD*-complete?*

Coming back to multi-player games, can we relax the symmetries satisfied by symmetric games while maintaining the computational tractability of Nash equilibrium? A natural and broad generalization of symmetric games are the so-called *anonymous games*. Like symmetric games, anonymous games involve players with the same action sets, and utility functions that depend symmetrically on the actions chosen by the other players. Contrary to symmetric games, however, the players need not have the same payoff functions. As a result, anonymous games have far broader applicability in modeling strategic interactions than symmetric games. Consider, e.g., a road network. It is reasonable to assume that the drivers do not care about the identities of the other drivers. On the other hand, drivers may have different source-destination pairs, and their own way to evaluate their driving experience. So, while their payoff functions are symmetric with respect to the actions chosen by the other players, these payoff functions might be different. Anonymous games have been used to model several other strategic interactions, including auctions and social norms; see e.g. Milchtaich [1996], Blonski [2000], and Kalai [2005].

Now, to what extent do the structural and computational properties of Nash equilibrium in multi-player symmetric games extend to the broader class of anonymous games? A line of research starting in joint work with Papadimitriou has shown that much of these properties is maintained as long as we shift attention to approximate Nash equilibria.

On the computational front, it was shown by Chen, Durfee, and Orfanou [2015] that ε-Nash equilibrium computation is PPAD-complete in multi-player anonymous games with seven actions per player, if ε is taken to scale inverse exponentially in the number of players. Yet, it has also been established that, for any constant $\varepsilon \in (0, 1]$, there is a

polynomial-time algorithm for ε-Nash equilibrium computation in multi-player anonymous games with payoffs in $[0, 1]$, as long as the number of actions is bounded by a constant (Daskalakis and Papadimitriou [2007, 2008], Daskalakis [2008], Daskalakis and Papadimitriou [2009b, 2015a,b], Diakonikolas, Kane, and Stewart [2016a], Daskalakis, De, Kamath, and Tzamos [2016], Diakonikolas, Kane, and Stewart [2016b], Goldberg and Turchetta [2017], and Cheng, Diakonikolas, and Stewart [2017]).

On the structural front, anonymous games have provided an interesting and fruitful point of interaction between Nash equilibria and distributions of sums of independent categorical random variables.[8] The reason why these distributions become relevant to the study of anonymous games is apparent. Consider the expected utility of some player p resulting from a particular strategy given the strategies of the other players. The strategies of players in an anonymous game can be viewed as categorical random variables $X_1, \ldots, X_n$ taking values in S, the common set of actions available to all players. Since the game is anonymous, the expected utility of player p depends on his own strategy X_p and the sum $\sum_{q \neq p} X_q$ of the strategies of the others. Thus, the behavior of sums of categorical random variables influences the behavior of expected utilities in anonymous games, which in turn influences the properties of Nash equilibrium in these games.

Indeed, virtually all progress in computing approximate Nash equilibria in anonymous games in the past decade or so owes itself to a progressively refined understanding of sums of independent categorical random variables. To give a flavor, let us state a couple of probability results from joint work with Papadimitriou (Daskalakis and Papadimitriou [2015b]) and reflect on their implications for Nash equilibrium in anonymous games (Daskalakis and Papadimitriou [2015a]). We will state structural results for sums of independent categorical random variables taking values in a binary set, i.e. sums of independent indicators, and discuss the implications of these results to Nash equilibria in anonymous games with two actions per player. The results stated below have been generalized to categorical variables taking values in larger sets, which find applications to anonymous games with larger action sets.

Theorem 4 (Daskalakis and Papadimitriou [2007, 2015b]). *There is some absolute constant $c > 0$ such that for all $n \in \mathbb{N}$, $\varepsilon \in (0, 1]$, and any collection of independent $0/1$ random variables $X_1, \ldots, X_n$, there exists some other collection of independent $0/1$ random variables $Y_1, \ldots, Y_n$ such that:*

1. *for all i, the expectation of Y_i is an integer multiple of $\lceil \varepsilon^{-1} \rceil^{-2}$;*
2. *for all i, $\mathbb{E}[X_i] \in \{0, 1\}$ implies $\mathbb{E}[Y_i] = \mathbb{E}[X_i]$;*
3. *the total variation distance between $\sum_i X_i$ and $\sum_i Y_i$ is at most $c \cdot \varepsilon$; and*
4. *for all i, the total variation distance between $\sum_{j \neq i} X_j$ and $\sum_{j \neq i} Y_j$ is at most $c \cdot \varepsilon$.*

Observe that the straightforward approach to proving Theorem 4 replaces each X_i by an indicator Y_i whose expectation is the closest multiple of $\lceil \varepsilon^{-1} \rceil^{-2}$ to X_i's expectation.

[8] When we say "sum of categorical variables" we mean it in the natural sense. Namely, if the variables take values in some finite set S, then their sum is a variable that records how many of these variables take each of the values in S.

This results in total variation distance error of $O(n \cdot \varepsilon^2)$ in the worst case. To get rid of the dependence of this error on n, we make instead a delicate use of Poisson and Binomial approximations. See Daskalakis and Papadimitriou [2015b] for more details.

Now, Theorem 4 is just a structural result for sums of independent indicators. It is easy to see, however, that it has immediate implications to anonymous games. It allows us to take a Nash equilibrium $X_1, \ldots, X_n$ and turn it into an approximate one $Y_1, \ldots, Y_n$ with extra structure. Indeed, note that the change in players' expected utilities, going from strategies $X_1, \ldots, X_n$ to strategies $Y_1, \ldots, Y_n$, is controlled by the total variation distance between sums of leave-one-out subsets of the variables $X_1, \ldots, X_n$ and the corresponding subsets of variables $Y_1, \ldots, Y_n$. Thus, an immediate corollary of Theorem 4 is the following.

Theorem 5 (Daskalakis and Papadimitriou [2007, 2015a]). *There is some absolute constant $c > 0$ such that, for all $\varepsilon \in (0, 1]$, in every two-action anonymous game with payoffs in $[0, 1]$, there exists an ε-Nash equilibrium in which all players use strategies that are integer multiples of $c \cdot \lceil \varepsilon^{-1} \rceil^{-2}$.*

In turn, Theorem 5 can be used to prune the search space for ε-Nash equilibrium in two-action anonymous games to strategy profiles that use probabilities which are integer multiples of $c \cdot \lceil \varepsilon^{-1} \rceil^{-2}$. This can be exploited to obtain a polynomial-time algorithm for ε-Nash equilibrium as follows: We perform a search over all *unordered* collections (i.e. multisets) of n integer multiples of $c \cdot \lceil \varepsilon^{-1} \rceil^{-2}$. For each such collection, we decide if there is an assignment of its elements to the players of the game that is an ε-Nash equilibrium. The latter task can be formulated as a max-flow problem and solved via linear programming. The overall running time of the algorithm is $n^{O(1/\varepsilon^2)}$, which is polynomial in n, for any fixed ε.

Theorem 6 (Daskalakis and Papadimitriou [2007, 2015a]). *For all $\varepsilon \in (0, 1]$, there is a polynomial-time algorithm for ε-Nash equilibrium computation in two-action anonymous games with payoffs in $[0, 1]$*

It turns out that the algorithm for computing ε-Nash equilibrium in two-action anonymous games that we just outlined can be improved by obtaining sharper approximation results for sums of independent $0/1$ random variables. Here is the one that we will use.

Theorem 7 (Daskalakis [2008] and Daskalakis and Papadimitriou [2015a,b]). *There is some absolute constant $c > 0$ such that for all $n \in \mathbb{N}$, $\varepsilon \in (0, 1]$, and any collection of independent $0/1$ random variables $X_1, \ldots, X_n$, there exists some other collection of independent $0/1$ random variables $Y_1, \ldots, Y_n$, which satisfy Properties 2–4 of Theorem 4, and at least one of the following:*

1. *at most $\lceil 1/\varepsilon \rceil^3$ of the Y_i's have expectations that do not equal 0 or 1, and their expectations are integer multiples of $\lceil \varepsilon^{-1} \rceil^{-2}$; or*
2. *there is some q that is an integer multiple of $\frac{1}{\lceil \varepsilon^{-1} \rceil n}$ such that all Y_i's have expectations in $\{0, q, 1\}$.*

Theorem 7 implies an interesting approximation result: an arbitrary sum of independent indicators is ε-close to a translated Binomial distribution or to a sum of indicators

with support size at most $\lceil 1/\varepsilon \rceil^3$. The proof of the theorem is delicate, but the intuition behind it is clear: If a sum of indicators has large enough variance, then it should be close to a Normal distribution and thus also to a Binomial distribution. If its variance is small, then it should be approximable by a sum of indicators with small support. See Daskalakis [2008] and Daskalakis and Papadimitriou [2015a,b] for more details.

Theorem 7 can easily be turned into a structural result for ε-Nash equilibria.

Theorem 8 (Daskalakis [2008] and Daskalakis and Papadimitriou [2015a])**.** *There is some absolute constant $c > 0$ such that, for all $\varepsilon \in (0, 1]$, in every two-action anonymous game with payoffs in $[0, 1]$, there is a $(c \cdot \varepsilon)$-Nash equilibrium in which at least one of the following holds:*

1. *at most $\lceil 1/\varepsilon \rceil^3$ players randomize, and their mixed strategies are integer multiples of $\lceil \varepsilon^{-1} \rceil^{-2}$;*
2. *all players who randomize use the same mixed strategy which is an integer multiple of $\frac{1}{\lceil \varepsilon^{-1} \rceil n}$.*

Observe that Case 2 of Theorem 8 is analogous to Nash's assertion for symmetric games, that there always exists a Nash equilibrium in which every player uses the same strategy. However, there is now another possibility, accounted by Case 1, that only a few players randomize, using potentially different strategies.

With the refined characterization of ε-Nash equilibria provided by Theorem 8, we can improve the running time of our previous algorithm to $\text{poly}(n) \cdot (1/\varepsilon)^{O(1/\varepsilon^2)}$. Roughly speaking, checking if there is some ε-Nash equilibrium that falls under Case 2 of Theorem 8 is cheap, and we only need to use our previous, expensive algorithm to check for ε-Nash equilibria that fall under Case 1 of Theorem 8. The gains come from the fact that the effective number of (mixing) players is now $\lceil 1/\varepsilon \rceil^3$.

To improve the running time even further, joint work with Papadimitriou (Daskalakis and Papadimitriou [2009b]) establishes a property of moments of sums of independent indicators, which we only state informally here (see Daskalakis and Papadimitriou [2009b, 2015b]):

"The total variation distance between two sums of independent indicators scales inverse exponentially with the number of moments of the two sums that are equal."

This property is combined in Daskalakis and Papadimitriou [2009b] with the afore-described algorithmic ideas to push the running time down to $\text{poly}(n) \cdot (1/\varepsilon)^{O(\log^2(1/\varepsilon))}$, which is latter sharpened to $\text{poly}(n) \cdot (1/\varepsilon)^{O(\log(1/\varepsilon))}$ by Diakonikolas, Kane, and Stewart [2016a] by looking at the Fourier spectrum of sums of independent indicators.

In the aftermath of about a decade of work on two-action anonymous games, the following remains an intriguing open question.

Open Question 5. *Develop an algorithm for ε-Nash equilibrium computation in two-action anonymous games with payoffs in $[0, 1]$ whose running time is polynomial in both the number of players and $1/\varepsilon$.*

We have thus concluded our discussion of approximation algorithms for multi-player anonymous games. We presented three results for sums of independent indicators. Each

of these gave new insight into the structure of ε-Nash equilibria in 2-action anonymous games, and a faster algorithm for their computation. Analogous approximation results have been obtained in higher dimensions, for sums of categorical random variables over larger sets, and these have been used to understand the structure and obtain algorithms for approximate equilibria in anonymous games with larger action sets. Going forward, analogous algorithmic challenges remain. See Daskalakis and Papadimitriou [2008, 2015a], Daskalakis, De, Kamath, and Tzamos [2016], Diakonikolas, Kane, and Stewart [2016b], and Cheng, Diakonikolas, and Stewart [2017].

3 A Broader Look at TFNP

Theorems 1, 2, and 3 reveal a profound connection between PPAD and central computational problems in Game Theory and Topology. In this section, we take a broader look at the class TFNP of total search problems in NP, revealing its intimate relationship to other fields, including Combinatorics, Discrete and Continuous Optimization, Economics, Social Choice Theory, and Cryptography.

The structure of this section is as follows. We start with a more detailed dive into the structure of TFNP, picking up from where we left it in the beginning of Section 2. We present several subclasses of TFNP based on different existence arguments, and discuss their known relationships. We then present an array of connections between these subclasses and problems in a diverse range of disciplines. For central problems in these disciplines we present completeness results, which serve to sharply identify computational obstacles towards efficient algorithms, and also serve to establish computational equivalence between problems that are complete for the same class.

Our exposition in this section will intentionally remain at a high level as our goal is to outline the lay of the land, providing pointers to the literature. We also state several interesting open problems towards a better understanding of the structure of TFNP and the problems that it contains.

3.1 The Structure of TFNP. The following subclasses of TFNP were presented alongside PPAD in Papadimitriou [1994]. The first was defined in D. S. Johnson, Papadimitriou, and Yannakakis [1988]. The rest were defined by Papadimitriou [1994]. We have already discussed the first three classes in the beginning of Section 2.

PLS. The class of problems whose totality is established via the following potential function argument.
Every finite directed acyclic graph has a sink.

PPP. The class of problems whose totality is proved via the pigeonhole principle.
Any map from a finite set to itself is either onto or has a collision.

PPA. The class of problems whose totality is proved via the following parity argument.
Any finite graph has an even number of odd-degree nodes.

PPADS. The class of problems whose totality is proved via the following directed parity argument.

All finite directed graphs of in- and out-degree bounded by one, which have a source, have a sink.

Two further subclasses of TFNP were proposed more recently in Daskalakis and Papadimitriou [2011] and Jeřábek [2016] respectively:

CLS. This class is a subset of PPAD ∩ PLS, the class of problems whose totality is established using both a potential function argument and a directed parity argument. We skip stating the precise existence argument that defines this class as this is more technical.

PWPP. This class is a subclass of PPP containing problems with guaranteed collisions. We skip stating the precise existence argument that defines this class as this is more technical.

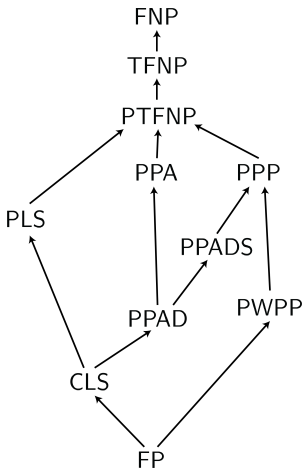

Figure 6: The TFNP world as it is known today. Arrows mean inclusion.

Figure 6 shows known inclusions between the afore-described classes, where arrows mean inclusion of a class into another. Proving that any of these arrows is a strict inclusion would separate FP from FNP, and as such is a huge open problem. Proving any inclusion that is not already shown in Figure 6 or implied by transitivity is also an intriguing open problem. On the other hand, we know that there are *oracles* under which essentially all of the missing inclusions do not hold (Beame, Cook, J. Edmonds, Impagliazzo, and Pitassi [1995], Simon [1998], Morioka [2001], and Buresh-Oppenheim and Morioka [2004]), with the exception of whether PLS is contained in any of the

classes PPADS and PPP, and how PWPP relates to PPADS, PLS and CLS.[9] For example, the following is an intriguing open problem.

Open Question 6. *Is PLS $\subseteq$ PPP or is there an oracle P such that $\mathrm{PLS}^P \not\subseteq \mathrm{PPP}^P$?*

Another interesting question is whether the above classes are closed under poilynomial-time (Cook) reductions.[10] Buss and A. S. Johnson [2012] show that PPA, PPAD, PPADS and PLS are all closed under Cook-reductions. The question of whether PPP is closed under Cook-reductions is still open. An affirmative answer to this question would imply that the problem of finding an n-approximate shortest vector of a lattice belongs to PPP.

Open Question 7. *Is PPP closed under Cook-reductions?*

In the next sections, we present several natural problems that are complete for the complexity classes presented above, with the exception of the class PPADS, which is the least understood subclass of TFNP. It is an interesting research direction to identify complete problems for PPADS that do not have a circuit or a Turing machine as part of their input.

Open Question 8. *Identify a natural PPADS-complete problem that does not have a circuit or a Turing machine as part of its input.*

The nature of TFNP makes it unlikely to have complete problems (D. S. Johnson, Papadimitriou, and Yannakakis [1988] and Megiddo and Papadimitriou [1991]). Motivated by this, Goldberg and Papadimitriou [2018] define a broad subclass of TFNP, called PTFNP, through a complete problem and show that this class contains all of the classes presented above.

3.2 PPAD: Beyond Nash, Brouwer, and Sperner. Section 2 was mainly devoted to the relationship between PPAD and the computation of Nash equilibrium, Brouwer fixed points, and Sperner triangles. More recent work has discovered connections between PPAD and other fundamental problems, spanning a diverse array of disciplines. We briefly outline this work here.

Kintali, Poplawski, Rajaraman, Sundaram, and Teng [2009] present a range of PPAD-complete problems with significance to Internet routing, Combinatorics, Cooperative Game Theory, Social Choice Theory, and Social networks. These problems include the computational version of Scarf's lemma (Scarf [1967b]), and other total problems whose totality is guaranteed by this lemma, such as computing fractional hypergraph

[9] Roughly speaking, if A is some complexity class defined in terms of circuits, such as PPAD, and P is some computational problem, then A^P is a complexity class defined in the same way as A, except that the circuits that define it can also use a special gate that instantaneously solves any instance of problem P that the circuit prepares. This gate is called an "oracle." We say that two complexity classes A and B are different relative to oracle P if $A^P \neq B^P$.

[10] A problem P is *polynomial-time reducible*, or *Cook-reducible*, to a problem P' iff any instance of problem P can be solved via a polynomial-time Turing machine that has access to a subroutine that instantaneously solves any instance of problem P'. Roughly speaking, a complexity class $\mathfrak{C}$ is closed under Cook reductions iff for any problem $P \in \mathfrak{C}$ and any problem P' that is Cook-reducible to P it holds that $P' \in \mathfrak{C}$.

stable matchings (Aharoni and Fleiner [2003]), fractional kernels in digraphs (Aharoni and Holzman [1998]), fractional stable paths in routing networks (Haxell and Wilfong [2008]), and the core of cooperative games with non-transferable utilities satisfying Scarf's conditions (Scarf [1967b]).

The problem of computing Arrow–Debreu equilibrium in exchange economies (Arrow and Debreu [1954]) has also been studied extensively. An early paper by Codenotti, Saberi, Varadarajan, and Ye [2006] shows that equilibrium computation in exchange economies with Leontief utilities is at least as hard as Nash equilibrium computation in two-player games. In particular, given a two-player game they present a corresponding exchange economy whose traders have Leontief utilities over goods and whose Arrow–Debreu equilibria are in one-to-one correspondence with the Nash equilibria of the game. Chen, Dai, Du, and Teng [2009] present a restricted family of exchange economies whose traders have utility functions that are additively separable and piecewise-linear concave over good quantities. They show that Arrow–Debreu equilibrium in these economies is guaranteed to exist, but computing it is PPAD-hard. Devanur and Kannan [2008] and independently Vazirani and Yannakakis [2011] prove that equilibrium computation in these economies also lies in PPAD, and is thus PPAD-complete. Vazirani and Yannakakis extend this result to the Fisher market model, showing that computing equilibrium in this model is also PPAD-complete under the same kind of utility functions and some mild conditions. The PPAD-hardness result for this model was also shown independently by Chen and Teng [2009]. More recently, Chen, Paparas, and Yannakakis [2017] prove that it is PPAD-hard to compute an approximate Arrow–Debreu equilibrium in exchange economies where agents' utilities are either linear or satisfy some type of non-monotonicity property, and it is PPAD-complete for utilities in the Constant Elasticity of Substitution (CES) family, as long as the elasticity of substitution parameter is any constant less than -1.

In Section 2, we defined Brouwer as the problem of computing a point x that is approximately fixed under a given map F, namely $d(F(x), x) \leq \varepsilon$, for some distance d and a given accuracy ε. As we also noted, a stronger notion of approximation arises when we seek a point x that is close to an exact fixed point, i.e. $d(x, x^*) \leq \varepsilon$ for some point x^* such that $F(x^*) = x^*$. Etessami and Yannakakis [2010] define the complexity class FIXP to capture the complexity of computing approximate fixed points of the latter kind for maps that are expressible as algebraic circuits (straight line programs) using the operations $\{+, *, -, /, \max, \min\}$ with rational constants; indeed, they show that the class remains the same if the operations are restricted to $\{+, *, \max\}$. Building on the techniques of Daskalakis, Goldberg, and Papadimitriou [2006, 2009a], they show that finding a mixed strategy profile that is close to an exact Nash Equilibrium in games with at least three players is FIXP-complete, complementing Theorem 3 that computing ε-Nash equilibrium in these games is PPAD-complete. They also show that identifying prices that are close to an Arrow–Debreu equilibrium in exchange economies described via demand functions that are expressible as algebraic circuits with the above operations is FIXP-complete. This result has been extended to other exchange economies; see e.g. J. Garg, R. Mehta, and Vazirani [2016] and J. Garg, R. Mehta, Vazirani, and Yazdanbod [2017]. On the other hand, they show that the piecewise-linear fragment of FIXP, in particular computing exact fixed points (or approximate fixed points of any kind) of

maps that are expressible as algebraic circuits using the operations $\{+, -, \max, \min\}$ and rational constants, equals PPAD. Otherwise, the relationship of FIXP and FNP remains unknown.

Finally, Daskalakis, Fabrikant, and Papadimitriou [2006] and Etessami [2014] study the complexity of Nash equilibrium refinements, showing PPAD and FIXP completeness results for respectively weak and strong approximations.

3.3 TFNP and Optimization. An interesting point of contact between combinatorial and continuous optimization and the class TFNP is through its subclasses PLS and CLS. The class PLS captures optimization problems solvable via local search algorithms. Important problems that have been shown to be PLS-complete include the Local Max-Cut Problem (Schäffer and Yannakakis [1991]), the Local Traveling Salesman Problem (Papadimitriou [1992]), and finding Nash equilibrium in deterministic strategies in network congestion games (Fabrikant, Papadimitriou, and Talwar [2004]) and in network coordination games (Cai and Daskalakis [2011]). Recently, (quasi-)polynomial-time algorithms were obtained for the Local Max-Cut problem under the smoothed analysis model where the edge-weights of the graph are randomly perturbed (Etscheid and Röglin [2017] and Angel, Bubeck, Peres, and Wei [2017]). The completeness of Local Max-Cut for PLS raises hope that similar smoothed analysis results might be attainable for other problems in PLS.

Open Question 9. *Study the smoothed complexity of PLS-complete problems.*

The class CLS was defined in joint work with Papadimitriou (Daskalakis and Papadimitriou [2011]) to capture the complexity of continuous optimization problems that are both solvable via local search algorithms but whose totality can also be established by invoking Brouwer's fixed point theorem. It was shown that the class contains intriguing problems that lie at the outskirts of FP yet are not known to be solvable in polynomial-time, including the P-matrix Linear Complementarity Problem, Computing KKT points, and finding Nash equilibrium (in mixed strategies) in congestion games, network coordination games, and Simple Stochastic Games. CLS was also shown to contain the problem of computing approximate fixed points of contraction maps, whose existence and uniqueness of fixed points is guaranteed by Banach's fixed point theorem.

Until recently no complete problem was known for CLS except for the problem provided in Daskalakis and Papadimitriou [ibid.] to define the class via reductions. In joint work with Tzamos and Zampetakis (Daskalakis, Tzamos, and Zampetakis [2018]), we show that the problem of computing approximate fixed points of maps that are contracting with respect to an input-provided metric is CLS-complete. To accomplish this, we establish a strong converse to Banach's fixed point theorem, showing that it is a universal argument for establishing global convergence of iterative maps to a unique fixed point and bounding their rate of convergence. In other words, we show that, whenever an iterative map globally converges to a unique fixed point, there exists a metric under which the iterative map is contracting and which can be used to bound the number of iterations until convergence. Concurrently and independently from our work, Fearnley, Gordon, R. Mehta, and Savani [2017, 2018] show that computing approximate fixed

points of maps that are contracting with respect to an input-provided *metametric* is CLS-complete. They also propose END_OF_POTENTIAL_LINE, a variant of END_OF_A_LINE, as a natural candidate for a CLS-complete problem, showing that the P-matrix Linear Complementarity Problem reduces to it, and so does the piece-wise linear fragment of FIXP from Section 3.2 for maps that are contracting with respect to ℓ_p norms. Thus they raise the following question.

Open Question 10. *Is* END_OF_POTENTIAL_LINE *CLS-complete?*

Finally, despite the recent progress in understanding the complexity of continuous local search problems, there is no known CLS-complete problem that does not contain a circuit or a Turing machine as part of its input. For example, the fixed point problems that were shown complete by Daskalakis, Tzamos, and Zampetakis [2018] and Fearnley, Gordon, R. Mehta, and Savani [2017, 2018] have in their inputs circuits that specify a map and the (meta)metric with respect to which it is contracting. The following remains a natural challenge.

Open Question 11. *Identify a natural CLS-complete problem that does not have a circuit or a Turing machine as part of its input.*

3.4 TFNP, Non-Orientable Spaces, and Social Choice Theory. Recall that PPA is the undirected generalization of PPAD. It is natural then that PPA-complete problems arise from unoriented/undirected analogs of PPAD-complete problems. For example, we have already seen in Theorem 1 that finding trichromatic triangles whose existence is guaranteed by Sperner's lemma on a triangulated, 3-colored, unit square is PPAD-complete, and as we have noted the same is true in higher dimensions with the right adaptations: the unit square becomes the d-dimensional hypercube, the triangulation becomes a simplicial subdivision whose vertices are now colored with $d+1$ colors, the boundary requirements (P) are appropriately generalized, and we are now seeking panchromatic simplices rather than triangles; see Papadimitriou [1994]. Complementing Papadimitriou's result on the PPAD-completeness of the three-dimensional version of SPERNER, Grigni [2001] shows that Sperner's lemma in a 3-manifold consisting of the product of the 2-dimensional Möbius band and a line segment is PPA-complete. Similarly, complementing Theorem 1, Deng, J. R. Edmonds, Feng, Liu, Qi, and Z. Xu [2016] show that Sperner's lemma on the 2-dimensional Möbius band is PPA-complete. As the Möbius band is non-orientable, the problems shown PPA-complete by Grigni and Deng et al. are unoriented analogs of the corresponding problems in the unit cube and respectively the unit square, which are PPAD-complete. Aisenberg, Bonet, and Buss [2015] and Deng, Feng, and Kulkarni [2017] show the PPA-completeness of problems whose totality is guaranteed by Tucker's lemma, the discrete analog of the Borsuk–Ulam theorem, while Belovs, Ivanyos, Qiao, Santha, and Yang [2017] show PPA-completeness of problems presented in terms of arithmetic circuits representing instances of the Chevalley-Warning Theorem, and Alon's Combinatorial Nullstellensatz.

Recently an interesting connection between social choice theory and PPA was established. Filos-Ratsikas and Goldberg [2018a] identify the first PPA-complete problem that does not require a circuit or a Turing machine as part of its input. This problem,

called CONSENSUS HALVING, arises from the Consensus Halving theorem for fair division proved by Simmons and Su [2003]. Filos–Ratsikas and Goldberg show that it is PPA-complete to compute a balanced partition whose existence is guaranteed by the Consensus Halving theorem. More recently, Filos-Ratsikas and Goldberg [2018b] show that DISCRETE HAM SANDWICH, the problem of finding a balanced partition whose existence is guaranteed by the discrete version of the Ham Sandwich theorem (Stone and Tukey [1942]), is also PPA-complete. Besides its application to social choice theory, the Ham Sandwich theorem has applications in topology and computational geometry. As such, a polynomial-time algorithm for DISCRETE HAM SANDWICH would resolve several algorithmic questions in these fields. The hardness result by Filos-Ratsikas and Goldberg [2018b] gives evidence against the existence of a polynomial-time algorithm.

Finally, an interesting open question regarding PPA dates back to Papadimitriou [1994]. The question pertains to the computational problem SMITH: given a Hamiltonian cycle in a cubic graph, find another Hamiltonian cycle. The existence of the second cycle is guaranteed by Smith's theorem. Papadimitriou [ibid.] shows that SMITH lies in PPA. Proving the PPA-completeness of SMITH is an interesting open question.

Open Question 12. *Is SMITH PPA-complete?*

3.5 TFNP and Cryptography. TFNP has interesting and promising connections to cryptography. The first was forged by Papadimitriou [ibid.] where he observes that, if PPP = FP, then *one-way permutations* cannot exist. In the same paper he proposes a total version of the SUBSET SUM problem, called EQUAL SUMS: given n positive integers whose sum is less than $2^n - 1$, find two subsets whose sum is equal. It is easy to see that EQUAL SUMS lies in PPP and it is interesting to study the following.

Open Question 13. *Is EQUAL SUMS PPP-complete?*

More recently, Ban, Jain, Papadimitriou, Psomas, and Rubinstein [2015] prove that central problems in lattice-based cryptography belong to PPP and they conjecture these problems to be PPP-complete. Even more recently, Sotiraki, Zampetakis, and Zirdelis [2018] identify the first complete problem for PPP that does not have a circuit or a Turing machine as part of its input. This problem is related to the Short Integer Solutions problem from lattice-based cryptography, and their result motivates the question of whether PPP has further connections to lattice-based cryptography.

There is a robust literature studying the average-case hardness of TFNP or its subclasses and their relationship to cryptography. This literature can be split into two strands of investigation. The first aims at proving the average-case hardness of problems in TFNP, based on average-case hardness assumptions for cryptographic primitives. The second seeks to identify cryptographic primitives that are based on the hardness of subclasses of TFNP. We briefly discuss some results in both directions.

In the first vein, a question that was already raised by Papadimitriou [1994] is whether FACTORING, which is known to lie in NP ∩ coNP, belongs to any of the subclasses of TFNP. In Buresh-Oppenheim [2006], a special case of FACTORING was shown to lie in PPA ∩ PPP. This was generalized in Jeřábek [2016], where it is shown that FACTORING lies in PPA ∩ PPP under randomized reductions. Combined with the work of Filos-Ratsikas and Goldberg [2018a,b], this result establishes the average-case hardness of

Consensus Halving and Discrete Ham Sandwich, assuming the average-case hardness of Factoring. That Factoring lies in PPA ∩ PPP raises the important question of whether Factoring also lies in PPAD. A positive answer to this question would provide, via Theorem 3, a family of games for which finding a Nash equilibrium is not only hard in the worst-case but also hard on average, assuming the average-case hardness of Factoring.

Open Question 14. *Is Factoring in PPAD?*

The average-case hardness of PPAD and thus Nash has been recently investigated in Bitansky, Paneth, and Rosen [2015] and S. Garg, Pandey, and Srinivasan [2016], where it is established under the strong cryptographic assumption of *indistinguishability obfuscation*. In fact, it was later shown that essentially the same construction can be used to prove the average-case hardness of CLS (Hubáček and Yogev [2017]). The average-case hardness of TFNP has been shown in Hubáček, Naor, and Yogev [2017] to be implied by the average-case hardness of decision problems in NP. Establishing the average-case hardness of PPAD and CLS using more standard cryptographic assumptions, such as the average-case hardness of factoring, discrete logarithm, or lattice-based problems remains an intriguing and important open problem.

Open Question 15. *Establish the average-case hardness of PPAD and CLS based on standard cryptographic assumptions.*

Finally, let us discuss the quest for cryptographic primitives whose hardness is based on the hardness of TFNP or its subclasses.

Open Question 16. *Identify cryptographic primitives whose hardness is based on the hardness of TFNP, PPAD, PPA, PPP or PLS.*

In this vein, Komargodski, Naor, and Yogev [2017] prove that the existence of *multi-collision resistant hash functions* is equivalent to the hardness of a variant of the total search problem Ramsey: Given a Boolean circuit encoding the edges of a graph with 2^{2n} nodes, find n nodes that are either a clique or an independent set. Ramsey is total by Ramsey's theorem, but not known to lie in any of the subclasses of TFNP presented in Section 3.1. Interestingly, Komargodski et al. prove that a variant of Ramsey, called *colorful*-Ramsey, is PWPP-hard. The inclusion or hardness of Ramsey with respect to any of the known subclasses of TFNP remains an important open question.

Open Question 17. *Does Ramsey belong to PPP (or some other subclass of TFNP)? Is it hard for any of these classes?*

4 Generative Adversarial Networks

In this section, we outline a recent and important application of Game Theory to Machine Learning. We start with a brief description of the problem that it aims to tackle and continue with an intriguing game-theoretic approach to solving it. We shall mainly focus on the challenges and open directions that this application reveals. In connection with the rest of this article, we will focus almost entirely on the optimization challenges, although there are interesting and important statistical challenges to consider as well.

4.1 Learning High-Dimensional Distributions. The archetypical challenge in Science involves collecting observations of some phenomenon of interest and using these observations to learn about the mechanism that drives this phenomenon. In probabilistic language, the challenge takes the form of the following learning task:

Given:	Samples $z_1, \ldots, z_n$ from a distribution $\mathcal{P}$ over some domain $S \subseteq \mathbb{R}^d$;
Goal:	Identify a model $\hat{\mathcal{P}}$ that well-approximates $\mathcal{P}$.

To make this task precise we should, of course, specify what assumptions we are willing to place on the distribution of interest $\mathcal{P}$, as well as what type of model $\hat{\mathcal{P}}$ we are seeking to identify, and what type of approximation we are willing to accept. Depending on these choices, a variety of learning problems with different domains of practical application arise.

A simple instance of our task arises when the samples $z_1, \ldots, z_n$ record measurements of some physical quantity, such as mass, on elements that are randomly drawn from a large homogeneous population. In this case, scientific argument might support the assumption that $z_1, \ldots, z_n$ are samples from a normal distribution $\mathcal{P}$ over $\mathbb{R}$. This suggests restricting our attention to models $\hat{\mathcal{P}} = \mathfrak{N}(\hat{\mu}, \hat{\sigma}^2)$ that are Normal distributions of some estimated mean $\hat{\mu}$ and variance $\hat{\sigma}^2$, and we may be interested to minimize the total variation distance between $\mathcal{P}$ and $\hat{\mathcal{P}}$. Observe that the model $\hat{\mathcal{P}}$ that we are after takes the form of an explicit probability density function, and as such is called an *explicit model*.

Let us now turn to a more complex scenario, and compare it to our simple example above. Imagine that the samples $z_1, \ldots, z_n$ are images of objects from some class, e.g. cats. For simplicity, let us assume that our images are gray-scale 32×32 images, so that they can be viewed as real vectors in $S = \mathbb{R}^{1024}$, recording the gray-scale color of every pixel in the image. In this case, $\mathcal{P}$ is a distribution over S, but it seems onerous to postulate a parametric form for $\mathcal{P}$. Rather, it may be easier to seek an *implicit model* $\hat{\mathcal{P}}$ of $\mathcal{P}$ that can be used to generate samples from the domain of interest S. An implicit model can be viewed as a deterministic function that maps a sample x from some simple source of randomness, e.g. an isotropic Gaussian, to a sample z in the domain of interest S. The goal for a good implicit model $\hat{\mathcal{P}}$ is that its output-distribution is close to the target $\mathcal{P}$. Given that implicit models allow for the easy generation of samples they are also called *implicit generative models*.

The pertinent questions are, of course, what type of functions one should use in an implicit generative model, and how to select from among these functions to obtain an implicit model whose output distribution is close to the target. A modern approach is to consider functions that are computable via a deep neural network of a certain architecture, and to use the samples $z_1, \ldots, z_n$ from the target distribution $\mathcal{P}$ to fine-tune the parameters of this architecture. Such implicit models are called *deep generative models*, and a popular approach to training them, called *adversarial training*, is by setting up a zero-sum game between two players as follows. One player aims to set the parameters of the neural net so that its output distribution is close to the target, while the other tries to distinguish between the network's output distribution and the target. Training occurs

by means of converging to an equilibrium of the game. There is, of course, much freedom in what neural network architecture to use, how to define the payoffs of the players, and what algorithmic procedure to use to near equilibrium. All these choices affect the quality of the resulting model.

Deep generative models that are trained adversarially are called *Generative Adversarial Networks*, or *GANs*, and were proposed in an influential paper by Goodfellow, Pouget-Abadie, Mirza, B. Xu, Warde-Farley, Ozair, Courville, and Bengio [2014]. They have since been the focus of intense research and experimentation effort, and have delivered impressive results; see e.g. Reed, Akata, Yan, Logeswaran, Schiele, and H. Lee [2016], Zhu, Park, Isola, and Efros [2017], Berthelot, Schumm, and Metz [2017], Isola, Zhu, Zhou, and Efros [2017], and Karras, Aila, Laine, and Lehtinen [2018]. Yet, training them poses substantial statistical and optimization challenges (Goodfellow [2016]), and there is much debate around the quality of GANs trained via current techniques; see e.g. Arora, Ge, Y. Liang, Ma, and Zhang [2017]. In the next couple of sections, we describe GANs and their training, and discuss some of the challenges that arise in their optimization. Consistent with the computational focus of this article, we will not discuss the statistical challenges that also arise.

4.2 Deep Generative Models. A deep neural network is a function $G_{\boldsymbol{w}} : \mathbb{R}^m \to \mathbb{R}^d$ that is parametrized by a vector $\boldsymbol{w}$ of real-valued parameters, commonly called *weights* of the neural net. Setting $G(\boldsymbol{w}; \boldsymbol{x}) \equiv G_{\boldsymbol{w}}(\boldsymbol{x})$ we should think of $G(\boldsymbol{w}; \boldsymbol{x})$ as a non-linear, almost-everywhere differentiable function in both its arguments. The precise details of this function depend on the architecture of the neural network, and will not be important for our discussion here. Its choice is very important though and should be done judiciously depending on the application. The deeper and wider the neural network is, the larger is its expressive power, but also the number of parameters that we need to tune.

A deep neural network can be used to obtain a generative model as follows.

Sample $\boldsymbol{x}$ from an isotropic normal distribution $\mathfrak{N}(0, I_{m\times m})$;
Output the vector $G_{\boldsymbol{w}}(\boldsymbol{x})$.

Assuming that there is a choice of weights $\boldsymbol{w}^*$ such that the distribution output by the above procedure well-approximates the target distribution $\mathcal{P}$, the challenge is to identify such weights. Fine-tuning the weights of the neural network using samples $z_1, \ldots, z_n$ from $\mathcal{P}$ is called *training*. In the next section, we focus on a training approach that uses some of the game-theoretic ideas that we have discussed.

4.3 Generative Adversarial Networks. The idea in adversarial training is to set up a game whose Nash equilibrium realizes a good choice for the weights $\boldsymbol{w}$ of a deep generative model $G_{\boldsymbol{w}}$. How this game is set up differs in different implementations of the idea. To give a taste, we will briefly discuss the perspective of Wasserstein-GANs by Arjovsky, Chintala, and Bottou [2017].

Suppose that we would like to select weights $\boldsymbol{w}$ to minimize the earth-mover distance (equivalently the 1st Wasserstein distance) between the output distribution of the deep

generative model $G_{\boldsymbol{w}}$ and the target distribution $\mathcal{P}$. To achieve this, we could set up a two-player zero-sum game as follows. We could have one of the players of the game select weights $\boldsymbol{w}$ for the deep generative model, and have the other player select a test to distinguish between the two distributions. Ideally, we would like to find an equilibrium of the following game:

$$\inf_{\boldsymbol{w}} \sup_{\substack{g:\mathbb{R}^d \to \mathbb{R},\\ g:\ 1\text{-Lipschitz}}} \left(\mathbb{E}_{Z\sim\mathcal{P}}[g(Z)] - \mathbb{E}_{X\sim\mathcal{N}(0,I)}[g(G_{\boldsymbol{w}}(X))]\right) \tag{8}$$

Notice that the maximization problem sitting inside the infimum is precisely the earth-mover distance between the output distribution of the deep generative model on Gaussian inputs and the target distribution $\mathcal{P}$, by Monge–Kantorovich duality. The sup-player aims to distinguish between the two distributions, while the inf-player wants to set the weights of the deep generative model to make the distributions as close as possible.

Now, there are several practical issues with Formulation (8). We discuss these issues and the attempts/compromises to address them in practice:

1. First, the strategy set of the sup-player is very high-dimensional even for moderate values of d. As optimizing over 1-Lipschitz functions seems onerous, a possible compromise is to optimize over functions that are themselves computable via a deep neural network of a certain architecture with weights $\boldsymbol{\theta}$. Let us denote by $D_{\boldsymbol{\theta}}$ the function computed by the neural network from a fixing of these weights, and call this network the *discriminator* neural network, to distinguish it from the generator neural network $G_{\boldsymbol{w}}$. Rather than aiming to minimize the Wasserstein distance between the target and the generated distribution, as a compromise we are now aiming to make the two distributions as indistinguishable as possible with respect to the discriminators $D_{\boldsymbol{\theta}}$ under different settings of $\boldsymbol{\theta}$. Namely, we target the following min-max optimization problem:

$$\inf_{\boldsymbol{w}} \sup_{\boldsymbol{\theta}} \left(\mathbb{E}_{Z\sim\mathcal{P}}[D_{\boldsymbol{\theta}}(Z)] - \mathbb{E}_{X\sim\mathcal{N}(0,I)}[D_{\boldsymbol{\theta}}(G_{\boldsymbol{w}}(X))]\right). \tag{9}$$

   In practice, we also restrict the weights $\boldsymbol{\theta}$ to impose some Lipschitzness on $D_{\boldsymbol{\theta}}$.

2. The second challenge is that the min-max objective (9) contains expectations, but we only have sample access to the target distribution $\mathcal{P}$, and the deep neural networks $G_{\boldsymbol{w}}$ and $D_{\boldsymbol{\theta}}$ will commonly have complex architectures that do not allow us to compute analytically even the second expectation in the objective, which does not depend on $\mathcal{P}$. For a fixed setting of $\boldsymbol{w}$ and $\boldsymbol{\theta}$, we can certainly approximate these expectations from finite sample averages, but we would like our finite sample averages to provide good approximations *uniformly* for all settings of $\boldsymbol{w}$ and $\boldsymbol{\theta}$. Proving such uniform convergence bounds is tricky for deep neural networks, but there are such bounds, even though they might be pessimistic in terms of the sample requirements. See e.g. Golowich, Rakhlin, and Shamir [2018] and its references. After replacing expectations by finite sample averages, the min-max optimization problem takes the following form, for some deterministic function

f, whose details are abstracted away.

$$\inf_{\boldsymbol{w}} \sup_{\boldsymbol{\theta}} f(\boldsymbol{w}, \boldsymbol{\theta}). \tag{10}$$

3. Third, the min-max objective $f(\boldsymbol{w}, \boldsymbol{\theta})$ will typically be non convex-concave in its variables $\boldsymbol{w}$ and $\boldsymbol{\theta}$, because the neural networks $G_{\boldsymbol{w}}$ and $D_{\boldsymbol{\theta}}$ will be typically taken to be non-linear, non convex/concave functions of their weights in order to be expressive enough. As such, the mathematical and computational properties of minimax equilibria discussed in Section 1 do not apply here. Combined with the fact that $\boldsymbol{w}$ and $\boldsymbol{\theta}$ are commonly very high-dimensional, solving (10) becomes a real challenge.

We spend the rest of this section discussing the third challenge, pertaining to min-max optimization of non convex-concave functions of multi-dimensional inputs. We should note that, while Formulation (10) was motivated by the training of Wasserstein-GANs, the same type of problem arises in training other kinds of GANs, and adversarial training more broadly. Our discussion applies to these applications as well.

So how may one attempt to solve (10)?

Given the high-dimensionality of $\boldsymbol{w}$ and $\boldsymbol{\theta}$, a common approach is to have the inf-player run some variant of online gradient descent with respect to $\boldsymbol{w}$ and have the sup-player run some variant of online gradient ascent with respect to $\boldsymbol{\theta}$, in tandem. It is known, however, that first-order methods encounter fundamental challenges in min-max optimization problems. The training dynamics are known to oscillate and the quality of the computed solutions is commonly poor; see e.g. Goodfellow [2016], Metz, Poole, Pfau, and Sohl-Dickstein [2016], and Arora, Ge, Y. Liang, Ma, and Zhang [2017] for examples and discussions. In view of these challenges, an important research direction presents itself.

Open Question 18. *Understand the behavior of first-order methods in min-max optimization problems. Design new methods that are more stable and converge to higher quality solutions.*

It should be noted that first-order methods exhibit oscillatory behavior even in the simpler case of min-max problems with convex-concave objectives. Perhaps the simplest example of such behavior occurs when both $\boldsymbol{w}$ and $\boldsymbol{\theta}$ are unconstrained scalars, $f(w, \theta) = w \cdot \theta$, and the the inf and sup players of (10) run respectively gradient descent and ascent in tandem. Starting from any initialization $(w_0, \theta_0) \neq (0, 0)$, the resulting dynamics of players' joint behavior $(w_t, \theta_t)_t$ spirals out to infinity. Similar oscillatory behavior for bilinear objectives also occurs if the players run common variants of gradient descent/ascent; see Daskalakis, Ilyas, Syrgkanis, and Zeng [2018].

As discussed in Section 1, when $f(\boldsymbol{w}, \boldsymbol{\theta})$ is convex in $\boldsymbol{w}$ and concave in $\boldsymbol{\theta}$ and these variables range over compact and convex sets, the minimax theorem holds and numerous dynamics converge to the optimum of (10). A generic way to obtain such dynamics is to have the inf and sup players of (10) run *no-regret learning procedures* in tandem. No-regret learning has been the focus of extensive study in online learning and online

optimization and numerous such procedures are known; see e.g. Cesa-Bianchi and Lugosi [2006], Shalev-Shwartz [2012], Bubeck and Cesa-Bianchi [2012], Shalev-Shwartz and Ben-David [2014], and Hazan [2016]. In fact, online gradient descent can be seen as an instantiation of a more general no-regret learning procedure, called follow-the-regularized-leader (FTRL),[11] when the regularization function is taken to be squared Euclidean norm; see e.g. McMahan [2011].

Despite the extensive literature on no-regret learning, an unsatisfactory feature of no-regret learning procedures is that their guarantees are commonly only attained in an *average sense*. As a result, when the players of a zero-sum game with a convex-concave f use no-regret learning, the average trajectory of the resulting dynamics converges to a minimax equilibrium, but the dynamics themselves may diverge from the equilibrium. More formally, if $(\boldsymbol{w}_t, \boldsymbol{\theta}_t)_t$ is the trajectory obtained when the players of (10) run some no-regret learning procedure, it is known that the average $\frac{1}{t}\sum_{\tau \le t} f(\boldsymbol{w}_\tau, \boldsymbol{\theta}_\tau)$ converges to the optimal value of the min-max problem, as $t \to \infty$. Moreover, if the solutions to (10) and its sup-inf dual problem are unique, then $\frac{1}{t}\sum_{\tau \le t}(\boldsymbol{w}_\tau, \boldsymbol{\theta}_\tau)$ converges to these unique solutions. Unfortunately that does not imply that the last iterate $(\boldsymbol{w}_t, \boldsymbol{\theta}_t)_t$ also converges to these unique solutions, as $t \to \infty$. Indeed, it commonly does not, even in the most basic scenarios; see e.g. Bailey and Piliouras [2018].

Motivated by this state of affairs, recent work of Daskalakis, Ilyas, Syrgkanis, and Zeng [2018], as well as subsequent work by T. Liang and Stokes [2018] examine whether last-iterate convergence can be realized when the inf and sup players of (10) run online gradient descent/ascent style no-regret learning procedures, in the simple unconstrained bilinear problem:

$$\inf_{\boldsymbol{w} \in \mathbb{R}^n} \sup_{\boldsymbol{\theta} \in \mathbb{R}^m} \boldsymbol{w}^{\mathrm{T}} A \boldsymbol{\theta}, \tag{11}$$

where A is a $n \times m$ matrix. Rather than having the players of the game run online gradient descent/ascent to update their strategies, we analyze what happens when they use *optimistic* online gradient descent/ascent; see Rakhlin and Sridharan [2013]. This can be viewed as an instantiation of *optimistic* FTRL with ℓ_2^2 regularization,[12] in the same way that online gradient descent/ascent can be viewed as an instantiation of FTRL with ℓ_2^2 regularization. OGDA is the dynamics resulting when the inf and sup players of a min-max problem run respectively optimistic online gradient descent and ascent procedures in tandem. GDA is the dynamics when they run online gradient descent and ascent respectively. OGDA boils down to GDA with *negative momentum*, as seen in the dynamics shown below, written for a generic min-max objective f and step size $\alpha > 0$:

$$\begin{aligned} \boldsymbol{w}_{t+1} &= \boldsymbol{w}_t - 2\alpha \nabla_{\boldsymbol{w}} f(\boldsymbol{w}_t, \boldsymbol{\theta}_t) + \alpha \nabla_{\boldsymbol{w}} f(\boldsymbol{w}_{t-1}, \boldsymbol{\theta}_{t-1}), \\ \boldsymbol{\theta}_{t+1} &= \boldsymbol{\theta}_t + 2\alpha \nabla_{\boldsymbol{\theta}} f(\boldsymbol{w}_t, \boldsymbol{\theta}_t) - \alpha \nabla_{\boldsymbol{\theta}} f(\boldsymbol{w}_{t-1}, \boldsymbol{\theta}_{t-1}). \end{aligned} \tag{12}$$

[11] FTRL is similar to fictitious play, explained in Section 1, with the following important modification: in every step of learning, rather than playing a best response to the history of strategies played by the opponent, the learner plays a regularized best response to that history.

[12] Optimistic FTRL is similar to FTRL except that the most recent strategy of the opponent is up-weighted in the history for the purposes of computing the regularized best response. This up-weighting is a reasonable thing to do in environments where the opponent is using a smooth learning procedure to update her strategies, hence it is expected that her next round strategy will be close to her most recent strategy.

In particular, in every update step t of the dynamics, the main update is according to the gradient at time t, but we also undo some fraction of the gradient at time $t-1$. If we were to remove the third term on the right-hand-side of the above updates we would get back GDA. While GDA dynamics exhibits oscillatory behavior, as we noted earlier, it is shown in Daskalakis, Ilyas, Syrgkanis, and Zeng [2018] and T. Liang and Stokes [2018] that OGDA exhibits last-iterate convergence to a solution of (11). That is, the dynamics $(\boldsymbol{w}_t, \boldsymbol{\theta}_t)_t$ shown above converges to a min-max solution.

In more recent work with Panageas (Daskalakis and Panageas [2019]), we generalize the results of Daskalakis, Ilyas, Syrgkanis, and Zeng [2018] and T. Liang and Stokes [2018] to the constrained min-max problem

$$\inf_{\boldsymbol{w}\in\Delta_n} \sup_{\boldsymbol{\theta}\in\Delta_m} \boldsymbol{w}^{\mathrm{T}} A\boldsymbol{\theta}, \tag{13}$$

which captures all of linear programming as noted in Section 1. As the dynamics should now remain inside $\Delta_n \times \Delta_m$, GDA and OGDA are not directly applicable to (13). In this context, a natural online gradient descent-style procedure for the players to run and remain in the simplex is the celebrated Multiplicative-Weights-Update (MWU) method, which is tantamount to FTRL with entropy regularization. Unsurprisingly, in the same way that online gradient descent results in dynamics with oscillatory behavior in (11), MWU results in dynamics with oscillatory behavior in (13); see e.g. Bailey and Piliouras [2018]. On the other hand, we prove that OMWU, the optimistic variant of MWU proposed by Rakhlin and Sridharan [2013], attains last-iterate convergence, answering an open question of Syrgkanis, Agarwal, Luo, and Schapire [2015]. Roughly speaking, when the players of (13) run OMWU, we show that the resulting dynamics monotonically improves its Kullback–Leibler divergence to minimax equilibrium, until it enters a neighborhood of minimax equilibrium where it becomes a contraction mapping.

So far, we reported on some recent effort towards gaining a better understanding of the behavior of first-order no-regret learning procedures in min-max optimization problems with convex-concave objectives. We expect this area to advance further in the next few years. Coming back to our original motivation, however, recall that the objective function in GAN training will typically be non convex-concave. GDA and its many variants are well-documented to exhibit oscillatory behavior in this setting as well; see e.g. Metz, Poole, Pfau, and Sohl-Dickstein [2016] for some nice examples.

Inspired by its better performance for convex-concave objectives, OGDA and variants of this dynamics are proposed in Daskalakis, Ilyas, Syrgkanis, and Zeng [2018] as a method for training GANs, and experimental evidence is provided that it performs well in practice. It is important to investigate further the behavior of optimistic variants of first-order methods in adversarial training applications.

From a theoretical standpoint, the computational study of min-max problems with non convex-concave objectives is wide open. As in, it is not even clear what solutions are reasonable targets for optimization procedures. The behavior of commonly used first-order methods is also poorly understood. Recent work with Panageas (Daskalakis and Panageas [2018]) studies the limiting behavior of GDA and OGDA. Our main focus is to relate the stable fixed points of GDA and OGDA dynamics to *local saddles* of the min-max objective f. A local saddle is a critical point $(\boldsymbol{w}^*, \boldsymbol{\theta}^*)$ of f that satisfies the

following: compared to value $f(\boldsymbol{w}^*, \boldsymbol{\theta}^*)$, if we fix $\boldsymbol{w}^*$ and perturb $\boldsymbol{\theta}^*$ infinitesimally, the value of f does not *increase* and similarly if we fix $\boldsymbol{\theta}^*$ and perturb $\boldsymbol{w}^*$ infinitesimally, the value of f does not *decrease*. Local saddles are robust under perturbations and are locally min-max solutions. Using techniques from Dynamical Systems, for step-size α sufficiently small and under mild assumptions on the second derivatives of f, we show that the set of stable fixed points of OGDA is a superset of those of GDA, which is a superset of the set of local saddles, and there do exist functions for which these inclusions are *strict*. In particular, for each local saddle point, there is a positive probability that GDA/OGDA will converge to it under random initialization. But there are functions with critical points that are not local saddles and where both GDA/OGDA dynamics converge with positive probability under random initialization.

In view of all the above, the following important research direction presents itself.

Open Question 19. *Identify a first or second-order method that converges almost surely to local saddle points (under random initialization according to a measure that is absolutely continuous with respect to Lebesgue measure).*

Finally, and from a more philosophical standpoint, it is important to study what might be the desirable limiting behavior of a method for GAN training in view of what the objective represents in terms of the ability of the sup player to discriminate between the distribution of the generator selected by the inf player and the target distribution. An intricate interplay between optimization and statistics reveals itself in this investigation.

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FROM BLACKBOARD TO BEDSIDE – GAUß PRIZE LECTURE

David Donoho

Abstract

(I) *Aide Mémoire.* We briefly review the contents of the 2018 Gauss Lecture ;
(II) *Esprit de l'Escalier.* Some mathematical work that ought to have been mentioned.
(II) *Meta Remarks.* Comments about the ICM and its awards.

1 *Aide Mémoire*

The 2018 Gauss Lecture had two parts.

Part I. Here, I review a June 2017 Congressional Briefing in Washington, organized by the American Mathematical Society and the Mathematical Sciences Research Institute. That briefing has been described at length in an article published in the January 2018 Donoho [2018]. The interested reader might consult that article before proceeding.

The Congressional Briefing took place in an atmosphere of uncertainty for US Federal policy, where there was some possibility that funding for US Mathematics research might see noticeable declines under the budgets soon to be put forward by the new administration and congressional majority.

David Eisenbud, director of MSRI, had in mind that I review for an audience of Congresspersons, staff and science officials, a recent transition from research to practice, showcasing the contribution that Mathematical research can make to issues of national concern.

I told the audience on Capitol Hill how the US Food and Drug Administration had very recently (Spring 2017) given bioequivalence certification to a new generation of Magnetic Resonance Imaging (MRI) scanners that the major manufacturers[1] were each bringing to market. Depending on the application, the new generation of scanners can get equivalent images in much less time – in some cases $1/10$ the time. Applications I mentioned included:

[1] There are 3 major manufacturers of MRI scanners for clinical use: GE, Philips, and Siemens. Siemens and GE obtained FDA bioequivalence certification in Spring 2017. Philips in Spring 2018.

- Imaging fidgety children in much less time, without sedation and without ionizing radiation[2].
- Commercializing new diagnostics, for example dynamic images (movies) of the beating heart.[3]
- Enabling new clinical procedures - for example MRI guided biopsy[4] - and advanced surgical interventions[5].

The individual examples that I showed really only scratch the surface; the industrial developers of the new technology envision it spreading, over time, throughout the whole range of MRI applications[6]. MRI imaging is today a large field: there are 40 million MRI scans annually in the US alone, and 80 million globally. We can envision that as the world grows more wealthy and developed there will eventually be billions of MRI scans yearly, all of which are made much less expensive and much more convenient by these speedups. While many of these scans might indeed be looking at knees, hips, and ankles of aging wealthy Americans and Europeans, many others will be peering deep into the brains of needy younger patients, as MRI imaging is also essential for modern Neurology and Neurosurgery.

The interested viewer might watch the documentary *The English Surgeon* about the UK Neurosurgeon Henry Marsh's travels to perform surgery in less advanced countries. In a wrenching pivotal scene, Marsh diagnoses what will inevitably become permanent blindness in a young girl who did not receive an MRI scan when it would still have been helpful. Marsh remarks that the young girl will live forever after with the consequences of lack of availability of prompt MRI scanning in her country. As MRI becomes more available and cheaper globally, we can expect such human tragedies to diminish in frequency.

In marketing the new generation of scanners, the manufacturers all say they are using *Compressed Sensing*, a term of art [7] introduced by mathematicians a little over 10 years ago.

[2]I showed examples developed by radiologist Shreyas Vasanawala (MD) Lucille Packard Children's Hospital (LCPH, Stanford), in collaboration with Michael Lustig (PhD) (EECS Prof. UC Berkeley) and John Pauly (Elec Eng Prof. Stanford).

[3]I showed movies illustrating commercial bioequivalence at 15X speedup, provided by Edgar Muller of Siemens Healthineers; and movies from the team led by Dr. Shreyas Vasanawala at Stanford LCPH showing the beating heart overlayed with measured flow fields visualizing flow through the arteries and veins supplying the beating heart. Data acquisition for this last example was sped up from more than an hour to under 10 minutes.

[4]I mentioned MRI-guided prostate biopsy, which might have saved my own father from aggressive prostate cancer, had it been available ten years ago. Dr. Faina Shtern of ADMeTech and her collaborators have shown that MRI-guided biopsies can have a 5x better success rate in cancer detection

[5]I mentioned MRI-guided surgery, giving examples from my own son's work as a Neurosurgeon at LA County Hopsital in Los Angeles, where a procedure conducted today with pre-existing technology takes an hour and a half of MRI scanning, during which time more than a half dozen skilled surgeons, anesthesiologists and nurses are sitting idle.

[6]For this, I quoted Jason Polzin of GE Research and Edgar Muller of Siemens.

[7]Siemens' corporate website used the tagline 'Compressed Sensing: Beyond Speed' in Spring 2017. Philips published an article in their 'FieldStrength' MRI publication, January 2018, with the title 'Faster MRI throughout the whole body with Compressed SENSE'; GE published an article in GESignaPulse magazine

In a series of papers submitted starting in 2004[8], mathematicians put forward theorems showing (under a variety of assumptions) that one does not need to actually make one million measurements to reconstruct a million-pixel image.

Such theorems inspired efforts by many engineers, doctors and physicists to develop new algorithms in some way related to the mathematics, to run experiments testing the new algorithms, and to present their work at major conferences. For an early example, see Lustig, Donoho, and Pauly [2007]. For a recent review article reflecting the evolving state-of-the-art, see Feng, Benkert, Block, Sodickson, Otazo, and Chandarana [2017]. In some individual research hospitals, such as Lucille Packard Children's Hospital (Vasanawala and Lustig [2011]), faster MRI scanning entered regular use within a few years. Rolling out a technology for widespread us in the US medical market entails regulatory scrutiny from the US Food and Drug Administration. In this present case, the transition to the marketplace has happened in not much more than 10 years.

In my talk, I made the point that in this instance, the Federal research funding system worked exactly as lawmakers had intended. I specifically mentioned Federal funding of individual investigators that supported the many mathematicians and engineers who worked to penetrate the mysteries of randomly-undersampled measurements; I pointed to Michael Lustig, who in 2004 was a Federally-supported graduate student in EE at Stanford but is today a Federally-supported EECS Professor at UC Berkeley; and to Emmanuel Candès, Justin Romberg, and Terence Tao, who in 2004 collaborated on some of the early and inspiring results, while Candès had an office on the UCLA campus at IPAM, one of the NSF-funded Mathematics Institutes, during an IPAM long program on multiscale geometric analysis.

I also tried to make the point that 'serious' mathematics was involved. I formulated the intellectual question in the following terms: *Consider a million-dimensional vector which has 10,000 entries scattered at random among many zeros.*[9] *Can one reconstruct this vector from only 100,000 measurements rather than 1,000,000?* If the measurements are in an appropriate sense random, and we reinstruct by ℓ_1-minimization of the reconstructed vector, this is the same thing as asking if a random 900,000-dimensional linear subspace typically intersects a certain simplicial cone transversely.

An instructive but very elementary version of the same problem is to ask if a random 1-dimensional linear subspace of R^N has nontrivial intersection with a simplicial cone. In the $N = 3$-D dimensional case, this amounts to asking if a point randomly distributed on the surface of a sphere lies inside a given spherical triangle. This of course is the ratio of the area of the triangle to the surface area of the sphere.

Autumn 2016 with the title 'HyperSENSE gives shorter scan times without compromising image quality'. The second sentence of the GE article reads '*Compressed Sensing (CS) is a technique similar to image compression ... [that] .. acquires less k-space data and uses a special reconstruction algorithm to recover the missing information without an appreciable impact on image quality*' and later says '*GE Healthcare has developed a CS technology called HyperSense which obtains randomly undersampled raw data thereby reducing the overall scan time with compromising overall image quality.*'.

[8]Examples include: Candès, Romberg, and Tao [2006a,b] and Donoho [2006a,b]

[9]We don't know the positions of the nonzeros, nor do we know the values of the nonzeros.

Gauss himself gave us the formula that solves this elementary problem; he showed that the area of a spherical triangle is given in terms of the sum of the 3 interior angles, which will not be 180 degrees owing to the curvature of the sphere[10].

Today there is a field called *combinatorial geometry* which develops generalizations of Gauss' formula, involving interior and exterior angles of polyhedral cones, Schneider and Weil [2008], and which allows us to solve the more ambitious problem involving $1,000,000$-dimensional simplicial cones, mentioned above. Such mathematical results allow us to see that, for the combination of parameters outlined above, the probability is effectively 1.0 that the subspace and cone intersect transversely! See for example: Donoho and Tanner [2005], Donoho [2006c], and Donoho and Tanner [2009] In short, with overwhelming probability, we can recover such a vector from 10x fewer measurements than one might have naïvely supposed.

Part II. Here we discuss issues raised by Part I. I mentioned two questions

- *Why did the transition from Theory to Practice happen so quickly?*
  Indeed, it seems that a dozen years is very fast for such a transition, especially in a regulated industry (Medicine).

- *What areas of mathematics contributed?*
  Indeed, attendees of the ICM2018 would no doubt be more interested in an understanding of which fields of mathematics contributed to speeding up MRI, than in knowing about the specific clinical applications (Pediatrics, Cardiology, Prostate Cancer Diagnosis) mentioned in Part I. Although in Part I, I was able to mention combinatorial geometry, in reality *many* subfields of the mathematical sciences played important roles.

1.0.1 Why did the transition from Theory to Practice happen so quickly? In my lecture, I pointed to several ingredients that combined to cause a rapid transition from Theory to Practice.

Everyone wanted MRI speedups. Everyone wants MRIs to go faster. Sitting confined in a tube is no fun, even when one is healthy and strong[11]

Experimental efforts were already succeeding. In my talk I pointed to many prior experimental works exploiting undersampling to in some way speed up MRI[12] In the lecture I called out the 2001 Leiden PhD Thesis of Frank Wajer, see Wajer

[10] In my lecture I pointed out that yes, I really had invoked Gauss' theorem in the Congressional Briefing, even illustrating it with computer graphics.

[11] I personally have been aware of the problem of speeding up MRI's since the early 1990's. I directed a *stage* by two students of *École Polytechnique* in the summer of 1995, hoping we could speed up MRI's by adaptive sampling. Stéphane Mallat was the examiner. And if even *I* knew of this problem you can be sure that this was a known and important problem in the MRI research community.

[12] In the cognate field of MR spectroscopy, Jeffrey Hoch, a Physical Chemist now at University of Connecticut Health Sciences Center, and co-authors showed already in the early/mid 1990's that random undersampling combined with nonlinear reconstruction could lead to notable speedups in MR spectroscopy.

[2001] who used random undersampling and nonlinear reconstruction to speed up MRI[13].

The MRI industry has assembled a massive collection of talent . The industry is servicing 80 million MRI scans annually. It has assembled a talent pool of thousands of MR physicists, electrical engineers, and computer scientists; this pool was available to be pressed into service at any sign of exciting new opportunities.

What, then, was preventing progress?

In lecture, I explained that experimental evidence of successful undersampling certainly existed, but many people didn't lend it much weight.

In general, purely computational results face an uphill battle to acceptance, partially because *computational results have historically not been not fully reproducible*. If someone told you of a computational result which required novel algorithms and data, it might involve a great deal of work for you to try the idea yourself. Under such circumstances you might simply find it easier to imagine that a reported success was due to a mistake in coding or in data preparation; and you might therefore view the result very warily.

In contrast, if someone tells you that there are mathematical theorems spelling out precise conditions under which certain effects can be observed, there is much less room for corrosive doubt.

On top of that, I mentioned in Lecture another factor: Terence Tao was awarded a Fields Medal in 2006, close to the beginning of the 'Compressed Sensing phenomenon'.

While in general the Fields Medal is not instrumental to any larger purpose, in this specific instance, at least, the Fields medal must have played an enabling role, lending charisma and intellectual authority to those striving for change in MRI.

It seems that mathematics broke a conceptual logjam, after which the MRI community really mobilized its talent to implement and study compressed sensing.

In lecture I mentioned that Michael Lustig - initially a graduate student at Stanford, now a Professor of EECS at UCBerkeley - in addition to being the first to develop and present many very influential compressed sensing applications to MRI, also developed a compelling approach to explaining Compressed Sensing for MRI researchers, in which he streamlined and recast some of the mathematician's assumptions and language in ways that MRI researchers could better understand, and developed a successful intuitive and heuristic framework that non-mathematicians found persuasive. Lustig himself points to decisive influences by many other MRI researchers[14].

1.0.2 What areas of Mathematics Contributed? I next pointed to several subfields of mathematics that had contributed substantially to the formulation and development of compressed sensing.

[13] Thanks to Michael Lustig for this reference.

[14] Lustig wrote me: "I would emphasize the contributions of Tobias Block, Uecker, and Frahm [2007] and Ricardo Otazo, Kim, Axel, and Sodickson [2010]. These guys have taken the clinical translation of Compressed Sensing orders of magnitude forward." He also mentioned key contributions of Zhi-Pei Lian of UIUC, Joshua Trzasko of Mayo Clinic, and of Alexey Samsonov and Julia Velikina of UW Madison

Computational Harmonic Analysis. In the 1980's and 1990's a variety of new tools for digital image processing - wavelet transforms and associated time-frequency transforms - were developed by applied mathematicians including Ingrid Daubechies and Yves Meyer, as well as Raphy Coifman, Stéphane Mallat, Michael Unser, and Martin Vetterli. The wavelet transforms developed by these researchers, when applied to digital signals and images, revealed a previously hidden sparsity – images were sparse in the wavelet domain; this sparsity can be helpful for data compression, and ultimately explains the adoption of wavelets as part of JPEG-2000 and in MPEG-4 standards.

A more ambitious 'sparsity-promoting' program arose from this: the thesis that all natural signals and images can be sparsely represented in *some* transform domain – we simply construct the appropriate transform. Attempts to realize this sparsity-promoting program led to Ridgelets, and from their to curvelets and shearlets and other 'X'-lets[15].

The idea of sparsity as a property to be sought after and enhanced led to the identification of ℓ_p balls with $0 < p < 1$ as mathematical models of sparsity[16]. While generations of mathematics analysis students have been taught to consider $p \geq 1$ as natural, the demands of real world signal processing made it now seem natural to consider cases which would have previously seemed pathological[17].

Combinatorial Geometry. We earlier mentioned the role that Combinatorial Geometry can play in justifying compressed sensing. Key concepts in combinatorial geometry can be learned from Branko Grunbaum's 50-year old classic *Convex Polytopes*, Grünbaum [1967] or Guenther Ziegler's more recent *Lectures on Polytopes*, Ziegler [1995]. In particular, these books define the face lattices of convex polytopes and properties of those lattices, such as k-neighborliness.

Combinatorial questions around the face lattices of random point sets (how many points are on the convex hull of a random point set? How many edges? How many faces?) were already discussed in the 1960's by Renyi and Sulanke and by Bradley Efron.

In the early 1990's, Anatoly Vershik and Pyotr Sporyshev, contemporaneous with Fernando Affentranger and Rolf Schneider, published exact formulas for the expected face numbers of various dimensions of random polytopes.

By duality, these quantities can be related to the so-called Grassmann angles defined by Branko Grunbaum, which measure the probability that linear subspaces transversely intersect a convex cone.

[15] See for example the article documenting in part my 2002 ICM Plenary address 'Multiscale Geometric Analysis' at Donoho [2002].

[16] See for example the article documenting Emmanuel Candès' 2014 ICM Plenary address 'Mathematics of Sparsity' at Candès [2014].

[17] We mention the pioneers of $p < 1$, such as Jaak Peetre and Vladimir Popov, who realized its significance in nonlinear approximation theory long ago, and in more recent times, Ron DeVore and Vladimir Temlyakov.

These diverse ingredients set the stage for the high-dimensional geometric calculations needed to determine the success of Compressed Sensing. Knowing them, and knowing that compressed sensing can be posed in terms of questions about Grassmann angles, one can calculate the probability that was mentioned in the Congressional Briefing: namely, the chance that a 900,000-dimensional random linear subspace is transverse to a certain cone with a 10,000-dimensional field of apices.

Those are merely two of the many fields that contributed to our knowledge of compressed sensing. In my lecture I mentioned four others. I can only very briefly allude to some contributions.

High-dimensional Analysis. Geometric functional analysis has played a key role in Compressed Sensing theory. Boris Kashin, in his solution of Kolmogorov's problem on the ϵ-entropy of Sobolev spaces, calculated the Gel'fand n-widths of ℓ_1^N balls and uncovered surprising structure, showing that they are controlled as $O(\sqrt{\log(N)/n})$. Today, by work in information-based complexity, we know that this implies that compressed sensing works for objects which are sparse in the sense that they belong to $\ell_{1,N}$ balls. More recently, Holger Rauhut and coauthors had pushed the calculation to encompass $\ell_{p,N}$, with $0 < p < 1$, which is really the case of interest for CS, Foucart, Pajor, Rauhut, and Ullrich [2010].

Over the last 50 years, random matrix theory has grown up into a powerful tool, and it too had its impact on compressed sensing. Concentration of measure results for eigenvalues were used by Candés and Tao to establish the important Restricted Isometry Property in their analyses of CS.

Optimization. At the center of CS is the solution of high dimensional optimization problems which are in a sense non-smooth and nonregular. And without algorithms there are *no* applications.

Pioneers in efficient high dimensional algorithms include Arkady Nemirovksii and Yuri Nesterov, whose broad influence is nearly staggering.

At the time that CS was first being developed, Ingrid Daubechies, Michel DeFrise and Christine DeMol had just written an influential paper in CPAM about provably solving ℓ_1 minimization problems in high dimension by iterative soft thresholding, Daubechies, Defrise, and De Mol [2004]. Today, the algorithm FISTA by Amir Beck and Marc Teboulle became very popular for these problems, see Beck and Teboulle [2009].

Mathematical Statistics. With Arian Maleki and Andrea Montanari, I published a paper in 2009 that exposed an unsuspected connection between Mathematical Statistics and Compressed Sensing, Donoho, Maleki, and Montanari [2009].

It showed that the phase transition boundary calculated by combinatorial geometry was identical to the minimax mean squared error in a signal denoising problem, an unrelated problem that Iain Johnstone and I had worked on 15 years earlier, and where our calculations used unrelated methods. Later, Johnstone and

Montanari and I put forward a reason for this connection and validated it empirically in many other compressed sensing-like problems, Donoho, Johnstone, and Montanari [2013]. It is simply spooky to come across an echo of an earlier result in a long-ago period of one's life – with no known reason for the echo.

Information Theory. The full story about the connection between denoising and compressed sensing has since been developed to a very fine point in modern information theory. We now understand that certain 'master conjectures' in spin glass theory would imply that connection as a special case, and important new techniques have developed to prove the conjectures in important situations. Here some of the relevant work has been done by Andrea Montanari and collaborators, other work by Florent Krzkala and Lenka Zdeborova and co-authors.

An amazing fact about the above list is: *all this important work is very recent.* One might have thought to study compressed sensing 30 years ago, but *the mathematics to study it simply did not exist*!

2 *Esprit de L'Escalier*

Denis Diderot was a renowned *encyclopediste* of the French enlightenment. Most mathematicians would know of him via E.T. Bell's *Men of Mathematics*, which recounts an amusing story of Diderot supposedly debating Euler at Catherine the Great's court[18].

Diderot recorded for posterity the notion of *Esprit de L'Escalier* - in English, literally 'staircase wit' - which refers to the experience of finding the right words to say only after leaving an important meeting or event.

Yves Meyer, the 2010 IMU Gauss Prize winner, once told me about a pilgrimage he made to Auxerre, the hometown of Fourier. Thinking about this - after the Gauss lecture - it occurred to me that I should really have mentioned some of my mathematical heroes. Here is my little attempt to make amends.

2.1 Constantin Carathéodory. Books have been written about Carathéodory's life and times, Georgiadou [2004], but still not enough of our young generation know of him. He made several lasting contributions, including what is today known as dynamic programming in optimal control. Relevant to our story, Carathéodory discovered what would today be called the *neighborliness of the moment curve* and of the trigonometric moment curve. Such neighborliness properties can be rephrased as follows (see also Donoho and Tanner [2005]: If a function $f(t)$ is a nonnegative combination of at most s complex sinusoids, $f(t) = \sum_{k=1}^{s} c_k \exp(i\omega_k t)$, it can be uniquely recovered from measurements of any $2s+1$ points $f(t_i)$, $i = 1, \ldots, 2s+1$, where $t_i \neq t_j$, $i \neq j$[19]. Also, if a function $g(t)$ is a nonnegative combination of at most s decaying exponentials,

[18] This story was engagingly related to my Honors Multivariable Calculus class many years ago, by our professor, Charles Fefferman. Unfortunately, E.T. Bell's story is now said to have been a fabrication of Augustus DeMorgan.

[19] We don't know the frequencies ω_k, nor do we know the values c_k of the coefficients.

$g(t) = \sum_{k=1}^{s} c_k \exp(-\lambda_k t)$, it can be uniquely recovered from measurements of any $2s+1$ points $g(t_i)$, $i = 1, \ldots, 2s+1$, where $t_i \neq t_j$, $i \neq j$. In our telling, such results contain some of the key features of compressed sensing: for example, a sparsity measure – in this case, s – and a sparsity-dependent sampling limit – in this case, $2s$.

2.2 Arne Beurling. As a young assistant professor, I purchased the collected works of Arne Beurling and, for a time, read them daily, see Beurling [1989b]. These works contain many beautiful results, even some important unpublished ones, but I found especially inspiring his work on interpolation, balayage, and minimal extrapolation. Here is an example most relevant to the Gauss Lecture. Beurling considered the problem: given a finite measure $\nu(dt)$ on the real line, suppose we observe the the Fourier transform $y(\omega) = \hat{\nu}(\omega) = \int \exp(i\omega t)\nu(dt)$ - but only over frequencies $|\omega| \leq \Omega$. He defined *Beurling minimal extrapolation*, Beurling [1989a], i

$$\nu^{\#} = argmin_{\eta} \int |\eta|(dt) : \text{ subject to: } \{\hat{\eta}(\omega) = \hat{\nu}(\omega), \ |\omega| \leq \Omega\}.$$

From our current perspective we can see that he is doing a form of L^1 minimization[20]. I don't recall if Beurling wrote this down or not, but a key point about his notion of minimal extrapolation is that (with the right interpretations) it perfectly recovers sparse probability measures: *if ν is a finitely supported nonnegative measure and yet the argmin runs over all nonnegative measures, then the solution perfectly recovers ν: $\nu^{\#} = \nu$.* In short, we have perfectly recovered a discrete measure from only low frequency information. This is a strengthening of Carathéodory's result above, since it gives an algorithm. Of course, Beurling did much more than just define important concepts; I learned a great deal of very specific ideas both in soft and hard analysis from his papers.

2.3 Benjamin Logan. Ben Logan worked at Bell Labs during the glory days, and teamed up with Larry Shepp for many important and well-known results, for example on tomography during the immediate explosion of computer-assisted tomography during the 1970's, and on counting of Young Tableaux. But Ben Logan had an independent career developing interesting and still very surprising results in nonlinear signal processing, long before the rest of the world understood that signal processing would soon be a ubiquitous latent presence in human life, for the rest of time[21].

Some of Logan's I think very interesting results concerned signals designed for recovery after arbitrary clipping, Logan Jr. [1984a,b]. He designed a signal processing scheme that allowed a bandlimited signal (i.e. an entire function of exponential type) to be distorted by the signum nonlinearity $sgn()$ and yet still be recovered. He even designed a practical recovery scheme and published it in the Bell System Technical Journal.

[20] AKA total variation minimization, *pace* Stanley Osher (2014 IMU Gauss Medalist).

[21] Logan had interesting pursuits outside mathematics. He was an outstanding bluegrass fiddler (violinist) called 'Tex' Logan in the bluegrass music community, and served as a studio musician for important country music stars.

A result from Ben's Ph.D. thesis Logan Jr. [1965] bears special mention. Suppose that g is a function zero outside an interval $(-T/2, T/2)$. Suppose that f is a bandlimited signal, i.e. $f(t) = \int_{-\Omega/2}^{\Omega/2} \exp\{i\omega t\}\hat{f}(\omega)d\omega$. We are interested in the L^1 distance between f and g. Ben's thesis showed that, if $T\Omega < 1/2$, and $f \neq 0$,

$$\|f - g\|_{L_1(\mathbf{R})} > \|g\|_{L_1(\mathbf{R})}.$$

The simplicity of the statement and even the proof are deceptive. No such statement could possibly be true for the L^2 norm. This expresses in a simple but profound way the nonlinearity and nonsmoothness of the L^1 norm, and the interaction with sparsity, the same phenomenon we exploit in compressed sensing. The main difference in approach is simply that in Logan's earlier result, sparsity was replaced by the more restrictive condition of narrowness of support.

3 Meta Remarks

3.1 The ICM. With the 2018 ICM, I have now attended 4 ICM's spanning 32 years (1986, 1994, 2002, 2018). During that period, this event has changed utterly. The 1986 and 1994 ICM's were held on university campuses and so had the flavor of traditional academic meetings. The opening ceremonies were straightforward affairs. The 2018 Rio ICM was held in a convention center away from any university, and the opening ceremony was polished in a way that would have been inconceivable in 1986[22]. The ceremony had extensive professional video presentations of the Fields Medalists and even entertainment with Samba Dancers from Brazil. The new persona is more fitting to today's situation. The Fields Medals are now a global brand.

What has stayed constant, in my view, is the quality of the ICM presentations. I remember very clearly some of the talks from the 1986 Congress; they were given by deservedly prominent figures and for me were era-defining. The talks I saw at the 2018 ICM met the same standard.

3.2 The Gauss Medal, and the Gauss Prize. At the 1986 ICM, I remember hallway discussions about the fact that Applied Mathematics was not traditionally recognized at events like the ICM, for example no aApplied Mathematician had ever been awarded a Fields Medal. The question young people were asking at the time was whether a young person should go into applications when there was no recognition. Apparently young people are very attuned to cues provided by who gets recognized.

This supposed situation of Applied Mathematics was partially addressed in 1994, with the award of the Fields Medal to Pierre-Louis Lions. It has now been directly addressed with the creation of the Gauss prize of the IMU, that directly recognizes impactful work in applications.

[22] The essential book Curbera [2009], has many interesting photos. They show that earlier ICMs had much simpler opening ceremonies than the 2018 iteration.

During the Rio Congress, I learned that Martin Grötschel had the idea and took the initiative to found the Gauss Prize, using the support of the German Mathematical Society, DMV. In a sense, the previous situation – of the supposed lack of importance of Applied Mathematics – was not some inherent fact about mathematics. It was simply due to the fact that no-one previously had Grotschel's organizational skills, energy, and clear vision. Take note.

The Gauss Prize medal has a story relevant to the 2018 Lecture. The artist Jan Arnold actually undersampled the image of Gauss! At the IMU website we read

> *"Dissolved into a linear pattern, the Gauss effigy is incomplete. It is the viewer's eye which completes the barcode of lines and transforms it into the portrait of Gauss."*

This really caught my attention, because the same scheme of undersampled vertical lines is *precisely* the k-space sampling pattern that is often used with compressed sensing[23]. Here art anticipates science!

3.3 My Collaborators. I have had wonderful collaborators. In the context of the ICM, note that my two close collaborators Iain Johnstone and Emmanuel Candès, have each given Plenary Addresses at ICM (2006 and 2014, respectively). They are each very serious and powerful scientists in their own right. I am very lucky to have been able to work with them.

This is only the start. At the 2018 ICM itself, my co-author Raphy Coifman (Yale) gave a plenary lecture and my co-author Andrea Montanari (Stanford) gave an invited talk, as did my former PhD student Noureddine El Karoui (UC Berkeley). It is my great good fortune to have been able to interact with, and learn from, such leading figures[24]. The broad range of work these different scientists presented was very striking.

3.4 The Long Tail. The professional writers working with the ICM to prepare profiles of the Fields, Nevanlinna, Chern and Gauss awardees are inclined to portray these otherwise fallible individuals as heroes. But in the case of the Gauss Award this idea is poorly conceived.

An award for applications, one that specifically recognizes real-world impact, ought to also recognize that applications only arise from a 'long tail' of mathematical scientists who, while not professional mathematicians themselves, are very mathematically minded and talented, while being much more applications facing than any mathematician would be. Such a long tail needs to be inspired and to be creative and driven in its own right, or no impact will result.

[23] Monajemi, H. and Donoho, D. L., 2017. Sparsity/Undersampling Tradeoffs in Anisotropic Undersampling, with Applications in MR Imaging/Spectroscopy, arXiv 1702.03062.

[24] I mentioned earlier the names of several other co-authors: Benjamin F. Logan, Jr. (deceased, ATT Labs), Michael Lustig (UC Berkeley), Philip Stark (UC Berkeley), Jared Tanner (Oxford). I should also have mentioned co-authors whose work was deeply connected to this lecture but not easy to work into the limited space here: Xiaoming Huo (Georgia Tech), Jeffrey Hoch (U. Conn), Gerard Kerkyacharyan, (Université de Paris), Brenda MacGibbon, Alan Stern (Rowland Institute), Dominique Picard (Université de Paris).

In this case, clearly Michael Lustig and John Pauly have distinguished themselves as electrical engineers with exceptional feel for mathematics and its proper deployment into MRI, and exceptional capacity for being inspired by mathematics. Lustig mentions several other MRI researcher colleagues who did major work translating Compressed Sensing into practice, including Tobias Block, Zhi-Pei Lian, Ricardo Otazo, Alexey Samsonov, Daniel Sodickson, Joshua Trzasko, and Julia Velikina. Doubtless there are many others that could also be mentioned.

In my own case, I would like to mention particularly two other mathematical scientists who have always been impressed with their ability to take mathematical inspiration and deliver practical results: Jean-Luc Starck (Centre Européen d'Astronomie) and Michael Elad (Technion). Their successful empirical results have consistently, over decades, gone far beyond what I would have dared to suggest based on mathematical analysis alone. In some sense they have always been followers of Wigner's 'unreasonable effectiveness of mathematics' while in many cases I have been too pessimistic.

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REGULARITY OF INTERFACES IN PHASE TRANSITIONS VIA OBSTACLE PROBLEMS – FIELDS MEDAL LECTURE

Alessio Figalli

Abstract

The aim of this note is to review some recent developments on the regularity theory for the stationary and parabolic obstacle problems.

After a general overview, we present some recent results on the structure of singular free boundary points. Then, we show some selected applications to the generic smoothness of the free boundary in the stationary obstacle problem (Schaeffer's conjecture), and to the smoothness of the free boundary in the one-phase Stefan problem for almost every time.

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1 Introduction

1.1 The classical Stefan problem. The classical Stefan problem aims to describe the temperature distribution in a homogeneous medium undergoing a phase change, typically the melting of a body of ice maintained at zero degrees centigrade. Given are the initial temperature distribution of the water and the energy contributed to the system through the boundary of the domain. The unknowns are the temperature distribution of the water as a function of space and time, and the ice-water interface.

This problem is named after Josef Stefan, a Slovenian physicist who introduced the general class of such problems around 1890 in relation to problems of ice formation, although this question had already been considered by Lamé and Clapeyron in 1831.

In its most classical formulation, the Stefan problem can be formulated as follows: let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and let

$$\theta = \theta(t, x)$$

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Keywords: Stefan problem, obstacle problem, free boundary, regularity.

denote the temperature of the medium at a point $x \in \Omega$ at time $t \in \mathbb{R}^+$. We assume that $\theta \geq 0$ in $\mathbb{R}^+ \times \Omega$, so that $\{\theta = 0\}$ represents the ice while $\{\theta > 0\}$ represents the water, see Figure 1.

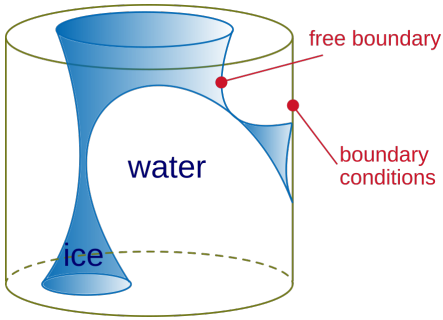

Figure 1: The Stefan problem. Because the ice-water interface is an unknown of the problem, it is called "free boundary".

We prescribe an initial condition

$$\theta(0, x) = \theta_0(x) \geq 0$$

at time $t = 0$, and a boundary condition

$$\theta(t, x) = \theta_b(t, x) \geq 0 \qquad \text{for } x \in \partial\Omega \text{ and } t \geq 0.$$

In the water, the temperature evolves in time according to the classical heat equation, that is

$$\partial_t \theta = \Delta\theta \qquad \text{inside } \{\theta > 0\}. \tag{1-1}$$

Also, the interface ice-water moves accordingly to the so-called "Stefan condition"

$$\dot{x}(t) = -\nabla\theta(t, x(t)) \qquad \forall\, x(t) \in \partial\{\theta(t) > 0\}, \tag{1-2}$$

where $\nabla\theta(t)$ denotes the spatial gradient of $\theta(t)$ computed from inside the region $\{\theta(t) > 0\}$, see Figure 2.

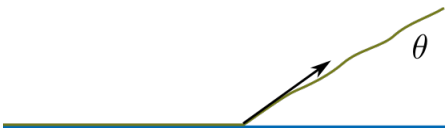

Figure 2: The gradient of $\theta(t)$ at a free boundary point is computed from inside the region $\{\theta(t) > 0\}$.

Observe that $-\nabla\theta(t, x(t))$ points always towards the region $\{\theta(t) = 0\}$, hence this set shrinks in time, see Figure 3. In other words, ice is melting.

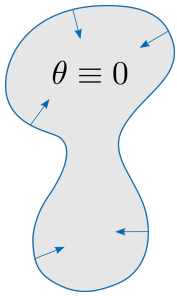

Figure 3: The vector $-\nabla\theta$ points into $\{\theta = 0\}$, which shrinks as time evolves.

This problem belongs to the general class of "free-boundary problems", since θ solves a PDE (the heat equation) inside the time-evolving domain $\{\theta > 0\}$ which depends on the solution itself, and so in particular it is an unknown of the problem. In this regard, we say that the boundary $\partial\{\theta > 0\}$ is a *free-boundary*.

1.2 From Stefan to the parabolic obstacle problem. In order to study this problem it is convenient to perform the so-called Duvaut's transformation, see Duvaut [1973, 1976]: let

$$u(t,x) := \int_0^t \theta(s,x)\,ds.$$

Then $u : \mathbb{R}^+ \times \Omega \to \mathbb{R}$ solves the so-called *parabolic obstacle problem*

$$\partial_t u = \Delta u - \chi_{\{u>0\}}, \qquad u \geq 0, \qquad \partial_t u \geq 0, \tag{1-3}$$

with Dirichlet boundary conditions

$$u(t,x) = \int_0^t \theta_b(s,x)\,ds \qquad \text{for } x \in \partial\Omega \text{ and } t \geq 0$$

(we shall explain the name "obstacle problem" in the next section).

To understand how to obtain (1-3) from the Stefan problem, we give here an informal derivation assuming that the set $\partial\{\theta > 0\}$ can be represented as the graph of a smooth function $\tau : \Omega \to \mathbb{R}^+$, that is

$$\partial\{\theta > 0\} = \{(t,x) : t = \tau(x)\} = \operatorname{graph}(\tau).$$

In other words, $\tau(x)$ represents the moment when the ice present at x melts into water.

Since

$$\theta(\tau(x),x) = 0 \qquad \forall\, x \in \Omega, \tag{1-4}$$

differentiating this relation with respect to x we obtain

$$0 = \nabla\big[\theta(\tau(x),x)\big] = \partial_t\theta(\tau(x),x)\,\nabla\tau(x) + \nabla\theta(\tau(x),x). \tag{1-5}$$

Also, since

$$\theta(t,x(t)) = 0 \qquad \text{for any curve } t \mapsto x(t) \in \partial\{\theta(t) > 0\},$$

differentiating this relation in time and using the Stefan condition (1-2), we get

$$0 = \frac{d}{dt}\theta(t, x(t)) = \partial_t \theta + \nabla\theta \cdot \dot{x}(t) = \partial_t \theta - |\nabla\theta|^2 \qquad \text{on } \partial\{\theta > 0\}. \tag{1-6}$$

Hence, combining (1-5) and (1-6), we deduce that

$$\nabla\tau(x) \cdot \nabla\theta(\tau(x), x) = -\frac{|\nabla\theta(\tau(x), x)|^2}{\partial_t \theta(\tau(x), x)} = -1. \tag{1-7}$$

Note that, because the sets $\{\theta(t) = 0\}$ shrink in time, we have

$$\theta(t, x) = 0 \qquad \text{for } t \in [0, \tau(x)].$$

In particular

$$u(t, x) = \int_0^t \theta(s, x)\, dx = 0 \qquad \text{for } t \le \tau(x),$$

which implies that

$$u(t, x) = \int_{\tau(x)}^t \theta(s, x)\, ds \qquad \forall\, t > \tau(x) \tag{1-8}$$

and that

$$\{\theta > 0\} = \{(t, x) \,:\, t > \tau(x)\} = \{u > 0\}. \tag{1-9}$$

We now want to compute the equation for u for $t > \tau(x)$.

Differentiating (1-8) with respect to x_i and recalling (1-4), we obtain

$$\partial_{x_i} u(t, x) = \int_{\tau(x)}^t \partial_{x_i}\theta(s, x)\, ds - \theta(\tau(x), x)\, \partial_{x_i}\tau(x) = \int_{\tau(x)}^t \partial_{x_i}\theta(s, x)\, ds.$$

Differentiating again with respect to x_i yields

$$\partial_{x_i x_i} u(t, x) = \int_{\tau(x)}^t \partial_{x_i x_i}\theta(s, x)\, ds - \partial_{x_i}\theta(\tau(x), x)\, \partial_{x_i}\tau(x),$$

so that summing over $i = 1, \dots, n$ gives

$$\Delta u(t, x) = \int_{\tau(x)}^t \Delta\theta(s, x)\, ds - \nabla\theta(\tau(x), x) \cdot \nabla\tau(x).$$

Hence, since $\theta = \partial_t u$, recalling (1-1), (1-4), and (1-7), we obtain

$$\Delta u(t, x) = \int_{\tau(x)}^t \partial_t\theta(s, x)\, ds + 1 = \theta(t, x) + 1 = \partial_t u(t, x) + 1$$

inside the region $\{\theta > 0\}$. Recalling (1-9), we proved that

$$\partial_t u = \Delta u - 1 \qquad \text{inside } \{u > 0\}. \tag{1-10}$$

It is important to remark that (1-10) is not equivalent to (1-3), as the latter equation has a further hidden condition on the free boundary. To understand this, assume that v solves (1-3). Then, since

$$\partial_t v - \Delta v = -\chi_{\{v>0\}} \in L^\infty,$$

it follows by parabolic regularity theory that $v(t) \in C^1(\Omega)$ for any $t > 0$. In particular, since $v(t) \geq 0$ and $v(t)|_{\partial\{v(t)>0\}} = 0$, it holds

$$\nabla v(t) = 0 \qquad \text{on } \partial\{v(t) > 0\}.$$

Hence, we need to show that the function $u(t, x)$ defined in (1-8) satisfies also this extra condition.

To prove this, we recall (see (1-4)) that

$$0 = \theta(\tau(x), x) = \partial_t u(\tau(x), x).$$

Also, since $u(\tau(x), x) = 0$, differentiating this relation with respect to x we get

$$0 = \nabla\big[u(\tau(x), x)\big] = \partial_t u(\tau(x), x)\, \nabla\tau(x) + \nabla u(\tau(x), x).$$

Combining these two relations we obtain

$$\nabla u(\tau(x), x) = 0,$$

that is

$$\nabla u(t) = 0 \qquad \text{on } \partial\{u(t) > 0\}, \tag{1-11}$$

see Figure 4.

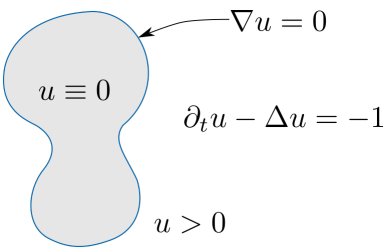

Figure 4: The parabolic obstacle problem.

It is interesting to observe that, in the equation for u, no transmission condition appears on the free boundary (cp. the Stefan condition for θ, see (1-2)). Hence, one may wonder what is determining the evolution of the free boundary. This can be explained as follows: recalling (1-11), we showed that both $u(t)$ and $\nabla u(t)$ vanish on the free boundary for any $t > 0$. One may notice that this is a very strong condition, since we are saying that $u(t)$ solves the parabolic equation

$$\partial_t u = \Delta u - 1 \qquad \text{in } \{u > 0\}$$

with *two* boundary conditions on the free boundary:

$$u = 0 \quad \text{and} \quad \nabla u = 0 \qquad \text{on } \partial\{u > 0\},$$

see Figure 4. This is an over-determined system: in classical PDE theory one can only prescribe either Dirichlet or Neumann boundary conditions, but not both! This means that the free boundary has to evolve so to ensure that both boundary conditions hold for every time. In other words, the transmission condition determining the evolution of the free boundary is now hidden inside the equation (1-3) via the condition (1-11).

Thanks to this informal discussion, since $\{\theta > 0\} = \{u > 0\}$, we have reduced the study of the free boundary in the Stefan problem to the one in the parabolic obstacle problem. Our goal now is to understand the free boundary regularity in the parabolic obstacle problem. In order to simplify the analysis, it makes sense to start first from the stationary case where u is independent of time, and then move to the general case. This will be the focus of the next sections.

2 The elliptic obstacle problem

In this section we study the free boundary regularity in the stationary obstacle problem.

Thus, given a domain $\Omega \subset \mathbb{R}^n$ and some fixed smooth boundary conditions $f : \partial\Omega \to \mathbb{R}$ with $f > 0$, we want to investigate the *elliptic obstacle problem*

$$\begin{cases} \Delta u = \chi_{\{u>0\}} & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases} \tag{2-1}$$

Note that, because $\Delta u = \chi_{\{u>0\}} \in L^\infty(\Omega)$, it follows by elliptic regularity that $u \in C^1(\Omega)$. In particular, as in the previous section, since $u \geq 0$ and $u|_{\partial\{u>0\}} = 0$, one deduces that

$$\nabla u = 0 \qquad \text{on } \partial\{u > 0\}, \tag{2-2}$$

see Figure 5.

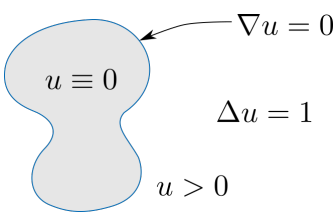

Figure 5: The stationary obstacle problem.

Before beginning our study, we first want to explain the origin of the name "obstacle problem" associated to (2-1).

Consider an elastic membrane that coincides with the graph of f on $\partial\Omega$, subject to the action of gravity, and forced to lie above the plane $\{x_{n+1} = 0\}$ (the "obstacle"). If we represents the membrane as the graph of a nonnegative function $u : \Omega \to \mathbb{R}$, this function minimizes the functional

$$\min_{v \geq 0} \left\{ \int_\Omega \left(\frac{|\nabla v|^2}{2} + g\, v \right) : v|_{\partial\Omega} = f \right\}, \tag{2-3}$$

where

$$\int_\Omega \frac{|\nabla v|^2}{2} \quad \text{represents the elastic energy of the graph of } v,$$

and

$$g \int_\Omega v \quad \text{represents the gravitational energy of the graph of } v$$

(here $g > 0$ is the gravitational constant), see Figure 6.

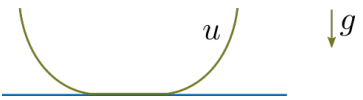

Figure 6: The obstacle problem models an elastic surface lying above the plane $\{x_{n+1} = 0\}$ and subject to gravity.

The existence and uniqueness of a minimizer for (2-3) follows by standard techniques in the calculus of variations, see for instance Figalli [2018, Section 2]. Then, computing the Euler–Lagrange equations for the minimizer u, one can prove that u satisfies the equation

$$\Delta u = g\, \chi_{\{u>0\}},$$

see for instance Figalli [ibid., Section 3]. In particular, up to replacing u by u/g we can assume that $g = 1$, which shows that (2-1) corresponds to the Euler–Lagrange equations associated to the minimization problem (2-3). This justifies the name *obstacle problem*. Also, since the set $\{u = 0\}$ corresponds to the region where u touches the obstacle, one refers to $\{u = 0\}$ as the *contact set*.

2.1 Regularity properties of u. Let u solve (2-1). Since

$$\Delta u = \chi_{\{u>0\}} \in L^\infty(\Omega),$$

standard elliptic regularity (see for instance Gilbarg and Trudinger [2001, Corollary 9.10 and Theorem 9.13]) guarantees that $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$ for any $p < \infty$. In other words

$$D^2 u \in L^p_{\mathrm{loc}}(\Omega) \qquad \forall\, p < \infty.$$

A key result in the theory of obstacle problems states that the estimate above holds even for $p = \infty$, see Frehse [1972], Brézis and Kinderlehrer [1973/74], Caffarelli [1998], and Figalli [2018]:

Theorem 2.1. *Let u solve (2-1). Then*

$$D^2u \in L^\infty_{\mathrm{loc}}(\Omega).$$

It is worth noticing that the result above is optimal. Indeed, since $\Delta u = \chi_{\{u>0\}}$ is discontinuous, the statement $u \in C^2(\Omega)$ is clearly false. Thus the boundedness of D^2u is the best one can hope for.

As a consequence of Theorem 2.1, we deduce that u grows at most quadratically away from the free boundary.

Corollary 2.2. *Let u solve (2-1), let $\Omega' \subset\subset \Omega$, let $x_0 \in \partial\{u > 0\}$, and assume that $B_r(x_0) \subset \Omega'$. Then there exists $C = C(\Omega')$ such that*

$$0 \le \sup_{B_r(x_0)} u \le C\, r^2.$$

Proof. Given $x \in B_r(x_0)$, we can write $u(x)$ using the Taylor formula centered at x_0:

$$u(x) = u(x_0) + \nabla u(x_0)\cdot(x - x_0) + \int_0^1 (1-t) D^2u\big(x_0 + t(x-x_0)\big)[x-x_0, x-x_0]\, dt.$$

Since $u(x_0) = 0$ and $\nabla u(x_0) = 0$, setting $C := \|D^2u\|_{L^\infty(\Omega')}$ we get

$$0 \le u(x) \le \frac{C}{2}|x-x_0|^2 \le \frac{C}{2} r^2,$$

as desired. □

As shown in Caffarelli [1977, 1998], see also Figalli [2018], the upper bound is always attained.

Proposition 2.3. *Let u solve (2-1), let $x_0 \in \partial\{u > 0\}$, and assume that $B_r(x_0) \subset \Omega$. Then there exists a dimensional constant $c = c(n) > 0$ such that*

$$\sup_{B_r(x_0)} u \ge c\, r^2.$$

2.2 Blow-up analysis and Caffarelli's dichotomy. Thanks to Corollary 2.2 and Proposition 2.3, we know that

$$\sup_{B_r(x_0)} u \simeq r^2 \qquad \forall\, x_0 \in \partial\{u > 0\}.$$

This suggests the following rescaling: for $x_0 \in \partial\{u > 0\}$ and $r > 0$ small, we define the family of functions

$$u_{x_0,r}(x) := \frac{u(x_0 + rx)}{r^2}. \tag{2-4}$$

In this way, recalling Theorem 2.1, we get (see Figure 7):

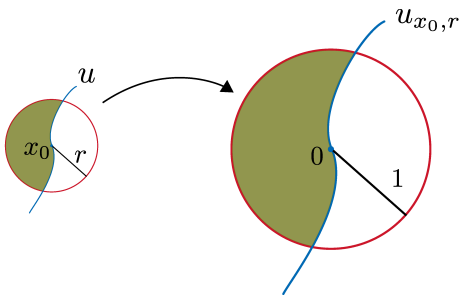

Figure 7: By scaling, we look at functions of size 1 defined inside B_1.

- $u_{x_0,r}(0) = 0, \qquad \sup_{B_1} u_{x_0,r} \simeq 1;$
- $|D^2 u_{x_0,r}|(x) = |D^2 u|(x_0 + rx) \le C.$

Thanks to these bounds, it follows by Ascoli–Arzelà Theorem that the family of functions $\{u_{x_0,r}\}_{r>0}$ is compact in C^1. So, one can consider a possible limit (up to subsequences) as $r \to 0^+$. Such a limit is called a *blow-up of u at x_0*.

The first goal is to classify the possible blow-ups, since they give us information on the infinitesimal behavior of u near x_0. We begin by considering two possible natural type of blow-ups that one may find.

2.2.1 Regular free boundary points. Let us first consider the case when the free boundary is smooth near x_0, with $u > 0$ on one side and $u = 0$ on the other side. In this case, as we rescale u around x_0 we expect in the limit to see a one dimensional "half-parabola", see Figure 8.

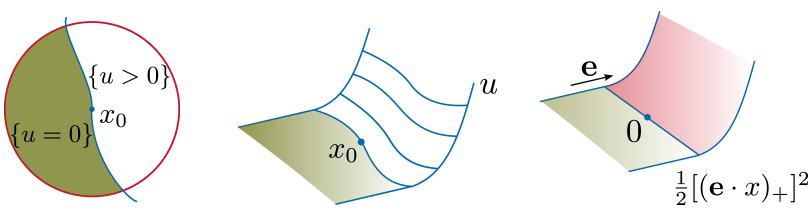

Figure 8: Performing a blow-up near a "thick" free boundary point.

This motivates the following:

Definition 2.4. A free boundary point $x_0 \in \partial\{u > 0\}$ is called a *regular point* if, up to a subsequence of radii,

$$\frac{u(x_0 + rx)}{r^2} \to \frac{1}{2}[(\mathbf{e}\cdot x)_+]^2 \qquad \text{as } r \to 0^+$$

for some unit vector $\mathbf{e} \in \mathbb{S}^{n-1}$.

2.2.2 Singular free boundary points. Now, imagine that the contact-set is very narrow near x_0. Since $\Delta u = 1$ outside of the contact set, as we rescale u around x_0 we expect to see in the limit a function that has Laplacian equal to 1 everywhere. In dimension two, a natural behavior that one may expect to observe is represented in Figure 9.

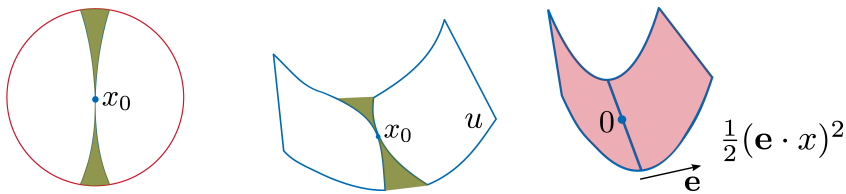

Figure 9: Performing a blow-up near a "thin" free boundary point.

More in general, since any nonnegative quadratic polynomial with Laplacian equal to 1 solves (5), one introduces the following:

Definition 2.5. A free boundary point $x_0 \in \partial\{u > 0\}$ is called a *singular point* if, up to a subsequence of radii,

$$\frac{u(x_0 + rx)}{r^2} \to p(x) := \frac{1}{2}\langle Ax, x\rangle \qquad \text{as } r \to 0^+$$

for some nonnegative definite matrix $A \in \mathbb{R}^{n\times n}$ with $\mathrm{tr}(A) = 1$.

It is worth noticing that the form of the polynomial p is strictly related to the shape of the contact set near the origin. For instance, if $n = 3$ and $p(x) = \frac{1}{2}(\mathbf{e}\cdot x)^2$ for some unit vector $\mathbf{e} \in \mathbb{S}^2$, then the contact set will be close to a 2-dimensional plane, see Figure 10.

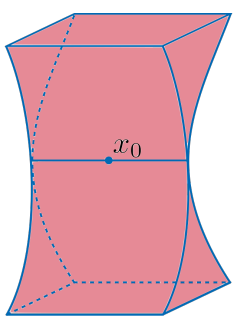

Figure 10: A possible contact set near a singular point in dimension 3.

On the other hand one could also see points where the contact set is close to a line, which may correspond for instance to a polynomial of the form $p(x) = \frac{1}{4}(x_1^2 + x_2^2)$, see Figure 11.

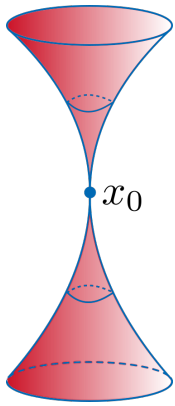

Figure 11: A possible contact set near a singular point in dimension 3.

2.2.3 Caffarelli's dichotomy theorem. Notice that, at the moment, the definitions of regular and singular points may not be mutually exclusive, since a free boundary point point could be regular along some sequence of radii and singular along a different sequence. Also, it is not clear that regular and singular points should exhaust the whole free boundary.

These highly nontrivial and deep issues have been answered in Caffarelli [1977]:

Theorem 2.6. *Let $x_0 \in \partial\{u > 0\}$. Then one of these two alternatives hold (see Figure 12):*

(i) either x_0 is regular, and then there exists a radius $r_0 > 0$ such that $\partial\{u > 0\} \cap B_{r_0}(x_0)$ is an analytic hypersurface consisting only of regular points;

(ii) or x_0 is singular, in which case for any $r > 0$ there exists a unit vector $\mathbf{e}_r \in \mathbb{S}^{n-1}$ such that

$$\partial\{u > 0\} \cap B_r(x_0) \subset \big\{x \,:\, |\mathbf{e}_r \cdot (x - x_0)| \le o(r)\big\}.$$

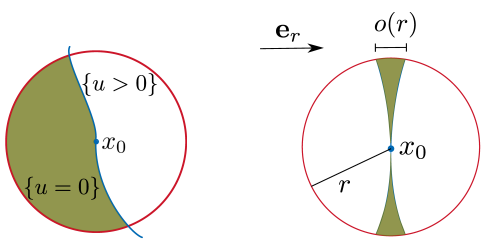

Figure 12: A regular (left) and a singular (right) free boundary point.

Theorem 2.6 states that a free boundary point can be either regular or singular. Also, if it is regular then the free boundary is smooth in a neighborhood and all points nearby are regular as well. From this one deduces that the convergence in Definition 2.4 holds without the need of taking a subsequence of radii.

While Theorem 2.6(i) gives a complete answer on the structure of regular points, Theorem 2.6(ii) is still not conclusive. Indeed, in the statement the vector $\mathbf{e}_r$ may depend on r. Also, the quantity $o(r)$ comes from a compactness argument, so it is not quantified.

Hence, from now on we shall focus on the study of singular points. To simplify the notation, we denote

$$\Sigma := \{\text{singular points}\} \subset \partial\{u > 0\}.$$

Note that, since the set of regular points is relatively open inside the free boundary (see Theorem 2.6(i)), it follows that Σ is a closed set.

2.3 Uniqueness of blow-up at singular points. As observed in the previous section, a priori the vector $\mathbf{e}_r$ appearing in the statement of Theorem 2.6(ii) may depend on r. This fact is essentially related to the question of whether the convergence in Definition 2.5 holds up to subsequences or not: indeed, if one could prove that the convergence to a polynomial p holds without passing to a subsequence, then one could easily deduce that

$$\partial\{u > 0\} \cap B_r(x_0) \subset \big\{x \,:\, \operatorname{dist}(x - x_0, \{p = 0\}) \le o(r)\big\}.$$

The complete answer to this questions has been given in Caffarelli [1998], after important previous results in the two dimensional case by Caffarelli and Rivière [1977] and Sakai [1991, 1993].

From now on, we use the notation

$$\mathcal{P} := \left\{ p(x) = \frac{1}{2}\langle Ax, x\rangle \;:\; A \in \mathbb{R}^{n\times n} \text{ symmetric nonnegative definite, } \operatorname{tr} A = 1 \right\}.$$

Theorem 2.7. *Let $x_0 \in \Sigma$. Then there exists $p_{*,x_0} \in \mathcal{P}$ such that*

$$\lim_{r\to 0} \frac{u(x_0 + rx)}{r^2} = p_{*,x_0}(x).$$

In addition, the map

$$\Sigma \ni x_0 \mapsto p_{*,x_0}$$

is locally uniformly continuous.

We present here a proof of this result given few years later in Monneau [2003]. To this aim, we first recall the so-called Monneau's monotonicity formula, whose proof relies on a previous monotonicity formula in Weiss [1999].

Lemma 2.8. *Let $0 \in \Sigma$, $p \in \mathcal{P}$, and define*

$$M(r, u, p) := \frac{1}{r^{n+3}} \int_{\partial B_r} (u - p)^2.$$

Then

$$\frac{d}{dr} M(r, u, p) \ge 0.$$

Using this lemma, the uniqueness and the continuity of the blow-up at singular points follow rather easily.

Proof of Theorem 2.7. We first prove the existence of the limit.

Assume with no loss of generality that $x_0 = 0$, and set

$$u_r(x) := r^{-2}u(rx).$$

With this notation, noticing that $r^{-2}p(rx) = p(x)$, it follows by a change of variables that

$$M(r,u,p) = \frac{1}{r^{n+3}}\int_{\partial B_r}(u-p)^2 = \int_{\partial B_1}(u_r-p)^2. \tag{2-5}$$

Now, let p_1 and p_2 be two different limits obtained along two sequences $r_{k,1}$ and $r_{k,2}$ both converging to zero. Up to taking a subsequence of $r_{k,2}$ and relabeling the indices, we can assume that $r_{k,2} \le r_{k,1}$ for all k. Thus, thanks to Lemma 2.8 and (2-5), we have

$$\int_{B_1}(u_{r_{k,1}}-p_1)^2 = M(r_{k,1},u,p_1) \ge M(r_{k,2},u,p_1) = \int_{B_1}(u_{r_{k,2}}-p_1)^2,$$

and letting $k \to \infty$ we obtain

$$0 = \lim_{k\to\infty}\int_{B_1}(u_{r_{k,1}}-p_1)^2 \ge \lim_{k\to\infty}\int_{B_1}(u_{r_{k,2}}-p_1)^2 = \int_{B_1}(p_2-p_1)^2.$$

This proves that there is a unique possible limit for the functions u_r as $r \to 0$, which implies that the limit exists. From now on, given a singular point x_0, we shall denote this limit by p_{*,x_0}.

We now prove the continuity of the map $x_0 \mapsto p_{*,x_0}$ at $0 \in \Sigma$. Fix $\varepsilon > 0$, and consider a sequence $x_k \in \Sigma$ with $x_k \to 0$. Since $u_r \to p_{*,0}$ as $r \to 0$, there exists a small radius $r_\varepsilon > 0$ such that

$$\int_{\partial B_1}\left|\frac{u(r_\varepsilon x)}{r_\varepsilon^2} - p_{*,0}(x)\right|^2 \le \varepsilon. \tag{2-6}$$

Then, applying Lemma 2.8 at x_k with $p = p_{*,0}$, we deduce that

$$\begin{aligned}\int_{\partial B_1}|p_{x_k,*}-p_{*,0}|^2 &= \lim_{r\to 0}\int_{\partial B_1}\left|\frac{u(x_k+rx)}{r^2} - p_{*,0}(x)\right|^2\\ &\le \int_{\partial B_1}\left|\frac{u(x_k+r_\varepsilon x)}{r_\varepsilon^2} - p_{*,0}(x)\right|^2.\end{aligned}$$

Hence, letting $k \to \infty$ and recalling (2-6) we obtain

$$\limsup_{k\to\infty}\int_{\partial B_1}|p_{x_k,*}-p_{*,0}|^2 \le \lim_{k\to\infty}\int_{\partial B_1}\left|\frac{u(x_k+r_\varepsilon x)}{r_\varepsilon^2} - p_{*,0}(x)\right|^2 \le \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this proves the continuity at 0.

Because Σ is locally compact (recall that Σ is closed), this actually implies that the map

$$\Sigma \ni x_0 \mapsto p_{*,x_0}$$

is locally uniformly continuous. □

2.4 Stratification and C^1 regularity of the singular set. With Theorem 2.7 at hand, we can now investigate the regularity of Σ. Note that singular points may look very different, depending on the dimension of the set $\{p_{*,x_0} = 0\}$, see Figures 10 and 11. This suggests to stratify the set of singular points according to this dimension. More precisely, given $x_0 \in \Sigma$ we set

$$k_{x_0} := \dim(\ker D^2 p_{*,x_0}) = \dim(\{p_{*,x_0} = 0\}).$$

Then, given $m \in \{0, \dots, n-1\}$, we define

$$\Sigma_m := \{x_0 \in \Sigma : k_{x_0} = m\}.$$

Note that, with this definition, the point in Figure 10 belongs to Σ_2, while the point in Figure 11 belongs to Σ_1. Hence one may expect that Σ_m should correspond to the m-dimensional part of Σ, see Figure 13.

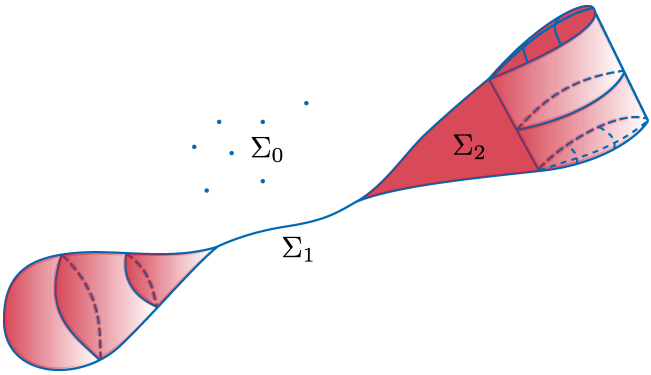

Figure 13: A possible example of contact set in 3 dimensions.

This intuition is confirmed by the following result in Caffarelli [1998]:

Theorem 2.9. *For any $m \in \{0, \dots, n-1\}$, Σ_m is locally contained in a m-dimensional manifold of class C^1.*

Idea of the proof. Recalling (2-2), we have $u|_{\Sigma_m} = \nabla u|_{\Sigma_m} \equiv 0$. Also, thanks to Theorem 2.7,

$$u(x_0 + y) = p_{*,x_0}(y) + o(|y|^2).$$

Hence, at least formally, p_{*,x_0} corresponds to the second order term in the Taylor expansion of u, namely "$p_{*,x_0}(y) = \frac{1}{2}\langle D^2 u(x_0) y, y\rangle$", or equivalently

$$\text{“}D^2 p_{*,x_0} = D^2 u(x_0)\text{”}.$$

Since the map

$$\Sigma \ni x_0 \mapsto p_{*,x_0}(y) = \frac{1}{2}\langle D^2 p_{*,x_0} y, y\rangle$$

is continuous, we deduce that

$$\Sigma \ni x_0 \mapsto D^2 p_{*,x_0} \in \mathbb{R}^{n\times n}$$

is continuous as well.

This allows us to apply Whitney's extension theorem to find a map $F : \mathbb{R}^n \to \mathbb{R}^n$ of class C^1 such that

$$F(x_0) = \nabla u(x_0) = 0 \qquad \text{and} \qquad \nabla F(x_0) = D^2 p_{*,x_0} \qquad \forall\, x_0 \in \Sigma_m.$$

Noticing that, by the definition of Σ_m,

$$\dim(\ker \nabla F(x_0)) = \dim(\ker D^2 p_{*,x_0}) = m \qquad \text{on } \Sigma_m,$$

it follows by the Implicit Function Theorem that

$$\Sigma_m = \{F = 0\} \cap \Sigma_m$$

is locally contained in a C^1 m-dimensional manifold, as desired. □

Remark 2.10. The proof above shows that the estimate

$$\|u(x_0 + \cdot) - p_{*,x_0}\|_{L^\infty(B_r)} = o(r^2),$$

with an error $o(r^2)$ independent of x_0, implies that Σ_m is locally contained in a C^1 m-dimensional manifold.

More in general, if one could prove that

$$\|u(x_0 + \cdot) - p_{*,x_0}\|_{L^\infty(B_r)} \le C\, r^{2+\alpha} \tag{2-7}$$

for some constant C independent of x_0, then by applying Whitney's extension theorem in Hölder spaces one would conclude that Σ_m is contained in a m-dimensional manifold of class $C^{1,\alpha}$.

Remark 2.11. The fact that Σ_m is only contained in a manifold (and does not necessarily coincide with it) is optimal: already for $n = 2$, one can build examples where Σ_1 coincides with a Cantor set contained in a line, see Schaeffer [1977].

2.5 Recent developments. In 1999, Weiss proved a monotonicity formula that allowed him to obtain the following result, see Weiss [1999]:

Theorem 2.12. *Let $n = 2$. Then there exist $C, \alpha > 0$ such that*

$$\|u(x_0 + \cdot) - p_{*,x_0}\|_{L^\infty(B_r)} \le C\, r^{2+\alpha} \qquad \forall\, x_0 \in \Sigma.$$

In particular Σ_1 is locally contained in a $C^{1,\alpha}$ curve.

Weiss' proof was restricted to two dimensions because of some delicate technical assumptions in some steps of the proof. Still, one could have hoped to extend his argument to higher dimensions. This was achieved by Colombo, Spolaor, and Velichkov [2018]. There, the authors introduced a quantitative argument to avoid a compactness step in Weiss' proof. However, the price to pay for working in higher dimensions was that they could only get a logarithmic improvement in the convergence of u to p_{*,x_0}:

Theorem 2.13. *Let $n \geq 3$. Then exist dimensional constants $C, \epsilon > 0$ such that*

$$\|u(x_0 + \cdot) - p_{*,x_0}\|_{L^\infty(B_r)} \leq C\, r^2 |\log(r)|^{-\epsilon} \qquad \forall\, x_0 \in \Sigma.$$

In particular, for any $m \in \{0, \dots, n-1\}$, Σ_m is locally contained in a $C^{1,\log^\epsilon}$ m-dimensional manifold.

In other words, in dimension $n \geq 3$ one can improve the C^1 regularity of Caffarelli to a quantitative one, with a logarithmic modulus of continuity. This result raises the question of whether one may hope to improve such an estimate, or if this logarithmic bound is optimal.

In a recent paper Figalli and Serra [n.d.] showed that, at most points, (2-7) holds with $\alpha = 1$. However, there exist some "anomalous" points of higher codimension where not only (2-7) does not hold with $\alpha = 1$, but actually (2-7) is false for any $\alpha > 0$.

As a consequence we deduce that, up to a small set, singular points can be covered by $C^{1,1}$ (and in some cases C^2) manifolds. As we shall discuss in Remark 2.15 below, this result provides the optimal decay estimate for the contact set.

Finally, it is important to observe that anomalous points may exist and our bound on their Hausdorff dimension is optimal.

Before stating our result we note that, as a consequence of Theorem 2.7, points in Σ_0 are isolated and u is strictly positive in a neighborhood of them. In particular u solves $\Delta u = 1$ in a neighborhood of Σ_0, hence it is analytic there. Thus, it is enough to focus on the cases $m = 1, \dots, n-1$.

Here and in the sequel, $\dim_{\mathcal{H}}(E)$ denotes the Hausdorff dimension of a set E. The main result in Figalli and Serra [ibid.] the following:

Theorem 2.14. *Let $\Sigma := \cup_{m=0}^{n-1} \Sigma_m$ denote the set of singular points. Then:*

($n = 2$) Σ_1 is locally contained in a C^2 curve.

($n \geq 3$) *(a) The higher dimensional stratum Σ_{n-1} can be written as the disjoint union of "generic points" Σ^g_{n-1} and "anomalous points" Σ^a_{n-1}, where:*
- Σ^g_{n-1} is locally contained in a $C^{1,1}$ $(n-1)$-dimensional manifold;
- Σ^a_{n-1} is a relatively open subset of Σ_{n-1} satisfying

$$\dim_{\mathcal{H}}(\Sigma^a_{n-1}) \leq n - 3$$

(actually, Σ^a_{n-1} is discrete when $n = 3$).
Furthermore, the whole stratum Σ_{n-1} can be locally covered by a $C^{1,\alpha_\circ}$ $(n-1)$-dimensional manifold, for some dimensional exponent $\alpha_\circ > 0$.

(b) For all $m = 1, \dots, n-2$ we can write $\Sigma_m = \Sigma^g_m \cup \Sigma^a_g$, where:
- Σ^g_m is locally contained in a $C^{1,1}$ m-dimensional manifold;
- Σ^a_m is a relatively open subset of Σ_m satisfying

$$\dim_{\mathcal{H}}(\Sigma^a_m) \leq m - 1$$

(actually, Σ^a_m is discrete when $m = 1$).
In addition, the whole stratum Σ_m can be locally covered by a $C^{1,\log^{\epsilon_\circ}}$ m-dimensional manifold, for some dimensional exponent $\epsilon_\circ > 0$.

This result needs several comments.

Remark 2.15. We first discuss the optimality of the theorem above, and then make some general considerations.

1. Our $C^{1,1}$ regularity provides the optimal control on the contact set in terms of the density decay. Indeed our result implies that, at all singular points up to a $(n-3)$-dimensional set, the following bound holds:
$$\frac{|\{u=0\}\cap B_r(x_0)|}{|B_r(x_0)|}\le Cr \qquad \forall\, r>0.$$
In view of the two dimensional Example 1 in Schaeffer [1977, Section 1], this estimate is optimal.

2. The possible presence of anomalous points comes from different reasons depending on the dimension of the stratum. More precisely, the following holds:

   (a) The possible presence of points in Σ^a_{n-1} comes from the potential existence, in dimension $n\ge 3$, of λ-homogeneous solutions to the so-called Signorini problem with $\lambda\in(2,3)$, see for instance Athanasopoulos and Caffarelli [2004] and Athanasopoulos, Caffarelli, and Salsa [2008]. Whether this set is empty or not is an important open problem.

   (b) The anomalous points in the strata Σ^a_m for $m\le n-2$ come from the possibility that, around a singular point x_0, the function u behaves as
$$u(x_0+rx)=r^2\,p_{*,x_0}(x)+r^2\varepsilon_r\,q(x)+o(r^2\varepsilon_r),$$
where:
- $\varepsilon_r\in\mathbb{R}^+$ is infinitesimal as $r\to 0^+$, but $\varepsilon_r\gg r^\alpha$ for any $\alpha>0$;
- q is a nontrivial second order harmonic polynomial.
This behavior may look rather strange: indeed we are saying that, after one removes from u its second order Taylor expansion p_{*,x_0}, one still sees as a reminder a second order polynomial. However, it turns out that such anomalous points may exist, and we can construct examples of solutions for which $\dim_{\mathcal{H}}(\Sigma^a_m)=m-1$.

3. Our result on the higher dimensional stratum Σ_{n-1} extends Theorem 2.12 to every dimension, and improves it in terms of the regularity.

4. The last part of the statement in the case $(n\ge 3)$-(b) corresponds to Theorem 2.13. In Figalli and Serra [n.d.] we obtain the same result as a simple byproduct of our analysis. In addition, our result on the existence of anomalous points shows that Theorem 2.13 is essentially optimal.

3 Generalizations and applications

3.1 The parabolic obstacle problem. The first natural extension of Theorem 2.14 consists in understanding the structure of the free boundary in the parabolic case

$$\partial_t u=\Delta u-\chi_{\{u>0\}},\qquad u\ge 0,\qquad \partial_t u\ge 0. \tag{3-1}$$

As shown in Caffarelli [1977] (see also Caffarelli and Friedman [1979]), solutions to this problem are C^1 in time and $C^{1,1}$ in space, namely

$$|\partial_t u| + |u| + |\nabla u| + |D^2 u| \in L^\infty_{\mathrm{loc}}, \qquad \partial_t u \in C^0.$$

Also, as in the elliptic case, points of the free boundary $\partial\{u > 0\}$ are divided into two classes: regular points and singular points. A free boundary point $z_0 = (t_0, x_0)$ is either regular or singular depending on the type of blow-up of u at that point. More precisely:

(3-2) z_0 is called *regular* point $\Leftrightarrow$

$$\frac{u(t_0 + r^2 t, x_0 + rx)}{r^2} \xrightarrow{r\downarrow 0} \frac{1}{2}[(\mathbf{e}\cdot x)_+]^2$$

for some $\boldsymbol{e} = \boldsymbol{e}_{z_0} \in \mathbb{S}^{n-1}$, and

(3-3) z_0 is called *singular* point $\Leftrightarrow$

$$\frac{u(t_0 + r^2 t, x_0 + rx)}{r^2} \xrightarrow{r\downarrow 0} p_{*,z_0}(x) := \frac{1}{2}\langle Ax, x\rangle$$

for some symmetric nonnegative definite matrix $A = A_{z_0} \in \mathbb{R}^{n\times n}$ with $\mathrm{tr}(A) = 1$. The existence of the previous limits in (3-2) and (3-3), as well as the classification of possible blow-ups are well-known results; see Caffarelli [1977], Caffarelli, Petrosyan, and Shahgholian [2004], and Blanchet [2006]. It is interesting to observe that both at regular and singular points the blow-ups are independent of time. This can be explained as follows: since solutions are C^1 in time and $C^{1,1}$ in space and $\partial_t u = \nabla u = 0$ on the free boundary, near a free boundary point $z_0 = (t_0, x_0)$ the function u satisfies

$$u(t_0 + t, x_0 + x) = o(t) + O(|x|^2).$$

Because of this, it follows immediately that the blow-ups considered above will not depend on t.

By the theory in Caffarelli [1977] and Caffarelli, Petrosyan, and Shahgholian [2004], the free boundary is an analytic hypersurface near regular points. On the other hand, near singular points the contact set $\{u = 0\}$ may form cusps and can be rather complicated.

To understand the structure of the singular points, we consider again a stratification based on the size of the zero set of the blow-up. More precisely, given a singular point $z_0 = (t_0, x_0)$, we set

$$L_{z_0} := \{p_{*,z_0} = 0\} = \ker(A_{z_0}).$$

Then, given a time $t > 0$ and $m \in \{0, 1, 2, \dots, n-1\}$, we define the m-th stratum at time t as

$$\Sigma_{m,t} := \big\{z_0 = (t_0, x_0) \ : \ \text{singular point with } \dim(L_{z_0}) = m \text{ and } t_0 = t\big\}.$$

The natural generalization of Theorem 2.9 would be to prove that, for any t and m, the set $\Sigma_{m,t}$ is contained in a m-dimensional manifold of class C^1. Actually, as proved in

Lindgren and Monneau [2015] (see also Blanchet, Dolbeault, and Monneau [2006] and Blanchet [2006] for some previous contributions)

$$\bigcup_{t>0}(\{t\}\times\Sigma_{m,t})\subset\mathbb{R}^+\times\mathbb{R}^n$$

is locally contained in a m-dimensional manifold of class C^1, where here C^1 regularity has to be intended with respect to the parabolic metric

$$d_P(z,z') := |x-x'|+|t-t'|^{1/2}, \qquad \text{where } z=(x,t),\ z'=(x',t').$$

In our paper Figalli, Ros-Oton, and Serra [2018] we take a different approach. More precisely, because the function u is mononically increasing in time, the free boundaries $\partial\{u(t)>0\}$ are nested. Exploiting this monotonicity one could use the arguments in Lindgren and Monneau [2015] to show that

$$\bigcup_{t>0}\Sigma_{m,t}\subset\mathbb{R}^n$$

is locally contained in a m-dimensional manifold of class C^1. Hence, it is natural to try to prove the analogue of Theorem 2.14 for $\cup_{t>0}\Sigma_{m,t}$. This is done in Figalli, Ros-Oton, and Serra [2018], where we obtain the following result (we state here a simplified version):

Theorem 3.1. *Let u be a solution of (3-1), let $m\in\{1,\dots,n-1\}$, and let $\Sigma_m := \cup_{t>0}\Sigma_{m,t}$. Then $\Sigma_m=\Sigma_m^g\cup\Sigma_m^a$, where:*
- *Σ_m^g can be locally covered by a $C^{1,1}$ m-dimensional manifold;*
- *Σ_m^a is a relatively open subset of Σ_m satisfying $\dim_{\mathcal{H}}(\Sigma_m^a)\le m-1$.*

The extension from the elliptic to the parabolic problem is far from trivial, as it requires relating the behavior of the solution at different times for different singular points. This involves both finer analytic arguments and a series of new covering-type theorems that allow us to take care of sets coming from different times. This result, besides considerably improving the previous knowledge on the structure of the free boundary for the Stefan problem, can be used to estimate the size of the set of singular times.

Let $\Sigma_t := \cup_{m=0}^{n-1}\Sigma_{m,t}$, and define the set of *singular times*

$$\mathcal{S} := \{t>0 : \Sigma_t\neq\varnothing\}.$$

Because singular points are a closed subset of the free boundary, if we ensure that the free boundary is contained in a bounded domain then the set Σ is compact, in which case $\mathcal{S}$ is a compact subset of $\mathbb{R}^+$. In particular, if $t_0\notin\mathcal{S}$ then there exists $\tau_0>0$ such that

$$\partial\{u>0\}\cap\big((t_0-\tau_0,t_0+\tau_0)\times\mathbb{R}^n\big) \quad \text{is an analytic hypersurface.}$$

A fundamental question is to estimate the size of $\mathcal{S}$.

Recalling that $\partial\{u>0\}$ coincides with the free boundary for the Stefan problem (recall Section 1.2), in Figalli, Ros-Oton, and Serra [ibid.] we prove the following result:

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain, and let θ solve the Stefan problem in $\mathbb{R}^+ \times \Omega$, with $\theta(t) > 0$ on $\partial\Omega$. Then*

$$\partial\{\theta > 0\} \cap \big((\mathbb{R}^+ \setminus \mathcal{S}) \times \Omega\big) \quad \textit{is analytic,}$$

where $\mathcal{S} \subset \mathbb{R}^+$ is a compact set satisfying

$$\dim_{\mathcal{H}}(\mathcal{S}) \le \frac{1}{2}.$$

At least to our knowledge, this is the first result on the size of the singular times.

3.2 Generic obstacle problems. A famous conjecture by Schaeffer on the elliptic obstacle problem states that, generically, the set of singular points in the free boundary should be empty, see Schaeffer [1977]. This result has been proved in dimension 2 by Monneau [2003].

To attack this problem, given a domain $\Omega \subset \mathbb{R}^n$, let $t \mapsto f_t \in C^0(\partial\Omega)$ be a one parameter family of nonnegative functions such that $\partial_t f_t > 0$ inside the set $\{f_t > 0\}$. For instance one may consider $f_t(x) := f(x) + t$ as in Monneau [ibid.], but many other choices are possible.

Then, for any t we can consider u_t the solution of the obstacle problem

$$\begin{cases} \Delta u_t = \chi_{\{u_t > 0\}} & \text{in } \Omega, \\ u_t \ge 0 & \text{in } \Omega, \\ u_t = f_t & \text{on } \partial\Omega. \end{cases}$$

It is interesting to observe that this one-parameter family of elliptic obstacle problems can be used to investigate the regularity of the free boundary both in the study of injection of fluid into a finite Hele-Shaw cell, see Elliott and Janovský [1981] and in the two dimensional annular electrochemical machining problem, see Elliott [1980].

Note that, since the boundary data are increasing,

$$u_t \ge u_s \qquad \text{for } t \ge s.$$

For each t we define $\Sigma_{m,t}$ as the m-th stratum of the singular points for u_t. Of course Theorem 2.14 applies to each u_t. However, exploiting the monotonicity with respect to t, as in the parabolic case we can prove that the very same theorem holds with $\Sigma_{m,t}$ replaced by $\cup_{t>0}\Sigma_{m,t}$ (actually, we can prove even a finer version of that result).

To obtain this, several new difficulties arise with respect to the parabolic case. Indeed, while on the one hand the fact that each u_t solves an elliptic problem simplifies the analysis, on the other hand much more work is needed to relate the behavior of different solutions u_t and u_s are different singular points.

As an application, define $\Sigma_t := \cup_{m=0}^{n-1}\Sigma_{m,t}$. Then we can prove the following result:

Theorem 3.3. *Let u_t be as before. Then, for a.e. t, the singular set Σ_t satisfies*

$$\dim_{\mathcal{H}}(\Sigma_t) \le n - 4.$$

In particular, for $n \le 3$ we have $\Sigma_t = \varnothing$ for a.e. t.

Notice that, by the discussion above, this result implies the validity of Schaeffer's conjecture in dimension $n \leq 3$.

Actually, as in the case of Theorem 3.2, for $n = 2, 3$ we can give an estimate on the Hausdorff dimension of the set of singular times: if we define

$$\mathcal{S} := \{t > 0 \,:\, \Sigma_t \neq \varnothing\},$$

then

$$\dim_{\mathcal{H}}(\mathcal{S}) \leq 1/4 \text{ for } n = 2, \qquad \dim_{\mathcal{H}}(\mathcal{S}) \leq 1/2 \text{ for } n = 3.$$

Recalling the connection for $n = 2$ to the Hele-Shaw flow and the electrochemical maching problem discussed above, we deduce that in these problems the free boundary is smooth outside a closed set of singular times of dimension at most 1/4.

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CRYSTAL BASES AND CATEGORIFICATIONS – CHERN MEDAL LECTURE

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Abstract

This is a survey of the theory of crystal bases, global bases and cluster algebra structure on the quantum coordinate rings.

1 Crystal bases

The notion of *quantum groups* (or *quantized universal enveloping algebras*) was introduced by Drinfeld [1985] and Jimbo [1985] around 1985 in order to explain trigonometric R-matrices in 2-dimensional solvable models in statistical mechanics. Since then, the quantum group has been one of the important tools to describe new symmetries in representation theory and other fields.

The quantum group $U_q(\mathfrak{g})$ is an algebra over $\mathbb{C}(q)$, in which q is a parameter of temperature in the 2-dimensional solvable model, and $q = 0$ corresponds to the absolute temperature zero. The notion of crystal bases was motivated by the belief that something extraordinary should happen at the absolute temperature zero. In fact, it turns out that the representations of $U_q(\mathfrak{g})$ have good bases at $q = 0$, which we call *crystal bases* (Kashiwara [1990] and Kashiwara [1991]). Here a basis at $q = 0$ of a $\mathbb{C}(q)$-vector space V means a pair (L, B) consisting of a free module L of V over the ring $A_0 := \{f \in \mathbb{C}(q) \mid f \text{ is regular at } q = 0\}$ and a basis B of the $\mathbb{C}$-vector space L/qL together with an isomorphism $\mathbb{C}(q) \otimes_{A_0} L \simeq V$.

The crystal bases have nice properties such as uniqueness, stability by tensor products, etc. Moreover the modified actions of the simple root vectors induce a combinatorial structure on the crystal basis, called *crystal graph*. This permits us to reduce many problems in the representation theory to problems of the combinatorics.

For example, the combinatorics describing the representation theory of $\mathfrak{gl}_n$ can be well understood with crystal bases. The irreducible representation over $\mathfrak{gl}_n$ is parameterized by the highest weight, which corresponds to a Young diagram in the combinatorial language. Then Young tableaux with the given Young diagram label the crystal basis of

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the corresponding irreducible representation. The Littlewood–Richardson rule, describing the decomposition of the tensor product of a pair of representations into irreducible components, may be also clearly explained by means of crystal bases.

The combinatorial description of crystal bases by Young tableaux is generalized to other types of simple Lie algebras in Kashiwara and Nakashima [1994]. Littelmann gave a completely different combinatorial description of crystal bases (path model) in Littelmann [1995]. Lustzig gave also bases at $q = 0$ from the PBW bases in the ADE case (Lusztig [1990]).

See Kashiwara [1995, 2002] for surveys of crystal bases.

2 Global bases

A crystal basis is a basis at $q = 0$. However, we can extend this basis to the whole q-space in a unique way to obtain a true basis of the representation, which we call the *lower global basis* (Kashiwara [1991]). More precisely, the representation V is a $\mathbb{C}(q)$-vector space and it is equipped a bar-involution $c\colon V \xrightarrow{\sim} V$, i.e., a $\mathbb{C}$-linear involution of V satisfying $c(qv) = q^{-1}c(v)$ for any $v \in V$. Moreover V has a certain $\mathbb{C}[q^{\pm 1}]$-form, i.e., a free $\mathbb{C}[q^{\pm 1}]$-submodule $V_{\mathbb{C}[q^{\pm 1}]}$ of V such that $\mathbb{C}(q) \otimes_{\mathbb{C}[q^{\pm 1}]} V_{\mathbb{C}[q^{\pm 1}]} \xrightarrow{\sim} V$. Then we can prove that the map $L \cap c(L) \cap V_{\mathbb{C}[q^{\pm 1}]} \to L/qL$ is an isomorphism and the inverse image $\mathbf{B} \subset L \cap c(L) \cap V_{\mathbb{Q}}$ of B is a basis of the $\mathbb{C}(q)$-vector space V and also a basis of the A_0-module L. We call $\mathbf{B}$ the *lower global basis* of V.

The negative half $U_q^-(\mathfrak{g})$ of $U_q(\mathfrak{g})$ has also a crystal basis $B(U_q^-(\mathfrak{g}))$, and it lifts to the lower global basis $\mathbf{B}^{\mathrm{low}}$.

It is first introduced by Lusztig under the name of canonical basis in the ADE cases inspired by the work of Ringel [1990] describing $U_q^-(\mathfrak{g})$ as the Hall algebras associated with quivers (see Lusztig [1990] and Lusztig [1993]). It is shown that the canonical basis and the global basis coincide (Grojnowski and Lusztig [1993]).

The dual basis of the lower global basis is called the *upper global basis*. The lower global basis $\mathbf{B}^{\mathrm{low}}$ of $U_q^-(\mathfrak{g})$ is a basis of the integral form $U^-_{\mathbb{Z}[q^{\pm 1}]}(\mathfrak{g})$ of the negative half of $U_q(\mathfrak{g})$. The upper global basis $\mathbf{B}^{\mathrm{up}}$ is the basis of $A_q(\mathfrak{n})$, the dual form of the integral form $U^-_{\mathbb{Z}[q^{\pm 1}]}(\mathfrak{g})$ (Kashiwara [1993]). With a canonical symmetric bilinear form on $U_q^-(\mathfrak{g})$, the dual form $A_q(\mathfrak{n})$ may be regarded as a subalgebra of $U_q^-(\mathfrak{g})$. However, at $q = 1$, the integral form $U^-_{\mathbb{Z}[q^{\pm 1}]}(\mathfrak{g})$ becomes the universal enveloping algebra $U(\mathfrak{n})$ of the negative half $\mathfrak{n}$ of $\mathfrak{g}$, while $A_q(\mathfrak{n})$ becomes the coordinate ring $\mathbb{C}[\mathfrak{n}]$ of $\mathfrak{n}$.

3 Quiver Hecke algebras

The notion of *quiver Hecke algebras* (sometimes called the *KLR algebras*) was introduced independently by Rouquier [2008, 2012] and Khovanov and Lauda [2009, 2011].

It is a family of $\mathbb{Z}$-graded algebras which *categorifies* the negative half $U_q^-(\mathfrak{g})$ of a quantum group. More precisely, there exists a family of algebras $\{R(n)\}_{n\in\mathbb{Z}_{\geqslant 0}}$ (quiver Hecke algebras) such that the (split) Grothendieck group $K(R\text{-gproj})$ of the direct sum $R\text{-gproj} := \bigoplus_{n\in\mathbb{Z}_{\geqslant 0}} R(n)\text{-gproj}$ of the categories of finitely generated projective graded

$R(n)$-modules is isomorphic to the integral form $U^-_{\mathbb{Z}[q^{\pm 1}]}(\mathfrak{g})$ of the negative half of $U_q(\mathfrak{g})$ as a $\mathbb{Z}[q^{\pm 1}]$-algebra. Note that the Grothendieck group $K(R\text{-gproj})$ has a natural structure of $\mathbb{Z}[q^{\pm 1}]$-algebra, where the multiplication is induced by the monoidal category structure of R-gproj given by the convolution product and the action of q is induced by the grading shift functor.

Remark. (i) The quiver Hecke algebra $\{R(n)\}_{n\in\mathbb{Z}_{\geqslant 0}}$ depends on a base ring $\mathbf{k}$, an index set I (the index of simple roots) and a family of polynomials $Q_{ij}(u,v) \in \mathbf{k}[u,v]$ $(i \neq j \in I)$ such that $Q_{ij}(u,v) = Q_{ji}(v,u)$. It is related with $U_q(\mathfrak{g})$ as follows. Let $A = \{a_{ij}\}_{i,j\in I}$ be a generalized Cartan matrix for $\mathfrak{g}$. Then $Q_{ij}(u,v)$ is a polynomial in u (with coefficients in $\mathbf{k}[v]$) of degree $-a_{ij}$ with an element of $\mathbf{k}^\times$ as the top coefficient. (For the precise definition, see e.g., Rouquier [2008] and Kang, Kashiwara, Kim, and Oh [2018].)

(ii) For a given generalized Cartan matrix A, the representation theory of R depends on a choice of $\{Q_{ij}\}$ (see Kashiwara [2012]).

(iii) The quiver Hecke algebra is called *symmetric* if $Q_{ij}(u,v) = (u-v)^{-a_{ij}}$ up to a constant multiple.

The Grothendieck group $K(R\text{-gproj})$ has the basis consisting of the isomorphism classes of indecomposable project modules. If R is symmetric (and the base field is characteristic 0), then the lower global basis corresponds to this basis of $K(R\text{-gproj})$ (Varagnolo and Vasserot [2011]).

There is a dual statement. The Grothendieck group $K(R\text{-gmod})$ of the direct sum $R\text{-gmod} := \bigoplus_{n\in\mathbb{Z}_{\geqslant 0}} R(n)\text{-gmod}$ of the categories of finite-dimensional graded $R(n)$-modules is isomorphic to the dual of $K(R\text{-gproj})$. Hence $K(R\text{-gmod})$ is isomorphic to $A_q(\mathfrak{n})$, the dual form of the integral form $U^-_{\mathbb{Z}[q^{\pm 1}]}(\mathfrak{g})$. Moreover, if R is symmetric (and the base field is characteristic 0), then the upper global basis corresponds to the set of the isomorphism classes of simple modules.

As a consequence, the coefficients $C^{\,b}_{b_1,b_2}$ appearing in the multiplication

$$b_1 \cdot b_2 = \sum_b C^{\,b}_{b_1,b_2}\, b \quad \text{where } b_1, b_2, b \in \mathbf{B}^{\mathrm{low}} \text{ or } \mathbf{B}^{\mathrm{up}}$$

belong to $\mathbb{Z}_{\geqslant 0}[q,q^{-1}]$ when the generalized Cartan matrix is symmetric. Here, $\mathbb{Z}_{\geqslant 0}[q,q^{-1}]$ is the set of Laurent polynomials in q with non-negative integers as coefficients. In general, the positivity fails (first observed in the G_2 case by Yamane [1994]).

Note that Lauda and Vazirani [2011] proved that the set of the isomorphism classes of simple modules in R-gmod is canonically isomorphic to $B(U_q^-(\mathfrak{g}))$ for an arbitrary quiver Hecke algebra.

The cyclotomic quotient $R^\Lambda(n)$ of $R(n)$ provides a categorification of the integrable highest weight module $V(\Lambda)$ of $U_q(\mathfrak{g})$ (Kang and Kashiwara [2012]) with a dominant integral weight Λ as a highest weight.

One of the motivations of these categorification theorems originated from the so-called *LLT-Ariki theory*. In 1996, Lascoux, Leclerc, and Thibon [1996] conjectured that

the irreducible representations of Hecke algebras of type A are controlled by the upper global basis of the basic representation of the quantum affine algebra $U_q(A^{(1)}_{N-1})$. Soon after, Ariki [1996] proved this conjecture by showing that the cyclotomic quotients of affine Hecke algebras categorify the irreducible highest weight modules over $U(A^{(1)}_{N-1})$, the universal enveloping algebra of affine Kac–Moody algebra of type $A^{(1)}_{N-1}$. In Brundan and Kleshchev [2009] and Rouquier [2008], Brundan and Kleshchev [2009] and Rouquier [2012] showed that the affine Hecke algebra of type A is isomorphic to the quiver Hecke algebra of type $A^{(1)}_{N-1}$ or of type A_∞ up to a specialization and a localization. Thus the quiver Hecke algebras can be understood as a graded version of the affine Hecke algebras of type A, and Kang–Kashiwara's cyclotomic categorification theorem (Kang and Kashiwara [2012]) is a generalization of Ariki's theorem on type $A^{(1)}_{N-1}$ and A_∞ to all symmetrizable Cartan datum.

4 Cluster algebras

As one can imagine from the fact that $A_q(\mathfrak{n})$ is a commutative algebra at $q = 1$, the upper global basis of $A_q(\mathfrak{n})$ has an interesting multiplicative property.

Berenstein and Zelevinsky [1993] conjectured that, when the the generalized Cartan matrix is of finite type, there exists a family $\mathfrak{F}$ of finite subsets of the upper global basis $\mathbf{B}^{\mathrm{up}}$ of $A_q(\mathfrak{n})$ satisfying the following properties:

(a) Any pair (x, y) of elements of $C \in \mathfrak{F}$ is q-commutative (i.e., $xy = q^n yx$ for some $n \in \mathbb{Z}$),

(b) For any $C \in \mathfrak{F}$, any C-monomial, i.e., an element of the form $x_1 \cdots x_\ell$ with $x_1, \dots, x_\ell \in C$, belongs to $q^{\mathbb{Z}}\mathbf{B}^{\mathrm{up}} := \{q^n b \mid n \in \mathbb{Z}, b \in \mathbf{B}^{\mathrm{up}}\}$.

(c) $\mathbf{B}^{\mathrm{up}}$ is the union of C-monomials (up to a constant multiple) where C ranges over $\mathfrak{F}$.

Then Leclerc [2003] gave a counterexample to this conjecture. He called an element $b \in \mathbf{B}^{\mathrm{up}}$ a *real* vector if $b^2 \in q^{\mathbb{Z}}\mathbf{B}^{\mathrm{up}}$. Otherwise, b is called *imaginary*. If Conjecture were true, then any $b \in \mathbf{B}^{\mathrm{up}}$ would be a real vector. He gave examples of imaginary vectors for types A_n ($n \geqslant 5$), B_n ($n \geqslant 3$), C_n ($n \geqslant 3$), D_n ($n \geqslant 4$), and all exceptional types.

Although their conjecture failed, their idea survives and it was one of the motivations of the introduction of cluster algebras by Fomin and Zelevinsky [2002]. They replaced condition (c) with a weaker condition

(c)′ $U_q^-(\mathfrak{g})$ is generated by $\bigcup_{C\in\mathfrak{F}} C$ as a $\mathbb{C}(q)$-algebra.

They call $C \in \mathfrak{F}$ a cluster, and reformulated the conjecture in the language of cluster algebras.

A *cluster algebra* is a $\mathbb{Z}$-subalgebra of a rational function field given by a set of generators, called the *cluster variables*. These generators are grouped into overlapping subsets, called *clusters*. The clusters are defined inductively by a procedure called *mutation* from the *initial cluster* $\{X_i\}_{1\leqslant i\leqslant r}$. The mutation is controlled by an exchange

matrix $\widetilde{B} = (b_{ij})_{ij}$ as follows. By the mutation at k ($1 \leqslant k \leqslant r$), a new cluster is created from the old cluster by replacing the k-th variable X_k with

$$X'_k = \frac{\prod_{i;\, b_{ik}>0} X_i^{b_{ik}} + \prod_{i;\, b_{ik}<0} X_i^{-b_{ik}}}{X_k} \tag{1}$$

The exchange matrix $\widetilde{B} = (b_{ij})_{ij}$ is also mutated to the new exchange matrix $\mu_k(\widetilde{B}) = (b'_{ij})_{ij}$ given by

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k, \\ b_{ij} + (-1)^{\delta(b_{ik}<0)} \max(b_{ik}b_{kj}, 0) & \text{otherwise.} \end{cases}$$

A *cluster monomial* is a monomial of cluster variables in one cluster. In the conjecture above, every $C \in \mathfrak{F}$ is a cluster. Moreover, any two members of $\mathfrak{F}$ are connected by successive mutations. We do not assume (c) but we assume that the algebra is generated (as an algebra) by the cluster monomials.

Fomin and Zelevinsky proved that every cluster variable is a Laurent polynomial of the initial cluster $\{X_i\}_{1\leqslant i\leqslant r}$. They conjectured that this Laurent polynomial has positive coefficients (Fomin and Zelevinsky [ibid.]). This *positivity conjecture* was proved by Lee and Schiffler in the *skew-symmetric* cluster algebra case in Lee and Schiffler [2015]. The *linearly independence conjecture* on cluster monomials was proved in the skew-symmetric cluster algebra case in Cerulli Irelli, Keller, Labardini-Fragoso, and Plamondon [2013].

The notion of quantum cluster algebras, introduced by Berenstein and Zelevinsky [2005], can be considered as a q-analogue of cluster algebras. It is an algebra over $\mathbb{Z}[q^{\pm 1/2}]$. The cluster variables in a cluster q-commute with each other. As in the cluster algebra case, every cluster variable belongs to $\mathbb{Z}[q^{\pm 1/2}][X_i^{\pm 1}]_{1\leqslant i\leqslant r}$ for the initial cluster $\{X_i\}_{1\leqslant i\leqslant r}$ (Berenstein and Zelevinsky [ibid.]), and is expected to be an element of $\mathbb{Z}_{\geqslant 0}[q^{\pm 1/2}][X_i^{\pm 1}]_{1\leqslant i\leqslant r}$, which is referred to as the *quantum positivity conjecture* (cf. Davison, Maulik, Schürmann, and Szendrői [2015, Conjecture 4.7]). In Kimura and Qin [2014], they proved the quantum positivity conjecture for quantum cluster algebras containing *acyclic* seed and specific coefficients.

Assume that the generalized Cartan matrix is symmetric. In a series of papers Geiß, Leclerc, and Schröer [2011, 2005] and Geiß, Leclerc, and Schröer [2013], Geiß, Leclerc and Schröer showed that the quantum unipotent coordinate algebra $A_q(\mathfrak{n}(w))$ has a skew-symmetric quantum cluster algebra structure whose initial cluster consists of *quantum minors*. Here, $A_q(\mathfrak{n}(w))$ is a $\mathbb{Z}[q^{\pm 1}]$-subalgebra of $A_q(\mathfrak{n})$ associated with a Weyl group element w. In Kimura [2012], he proved that $A_q(\mathfrak{n}(w))$ is *compatible* with the upper global basis $\mathbf{B}^{\mathrm{up}}$ of $A_q(\mathfrak{n})$; i.e., the set $\mathbf{B}^{\mathrm{up}}(w) := A_q(\mathfrak{n}(w)) \cap \mathbf{B}^{\mathrm{up}}$ is a basis of $A_q(\mathfrak{n}(w))$. Thus, one can expect that every cluster monomial of $A_q(\mathfrak{n}(w))$ is contained in the upper global basis $\mathbf{B}^{\mathrm{up}}(w)$, which is named *the quantization conjecture* by Kimura [ibid.].

5 Monoidal categorification

This conjecture (in the symmetric generalized Cartan matrix case) is proved affirmatively by Kang, Kashiwara, Kim, and Oh [2018], using the monoidal categorification of $A_q(\mathfrak{n}(w))$ by a subcategory of the module category R-gmod over the quiver Hecke algebras.

In Hernandez and Leclerc [2010], Hernandez and Leclerc introduced the notion of *a monoidal categorification of a cluster algebra*. Let $(\mathcal{C}, \otimes)$ be a monoidal category. We say that a simple object S of $\mathcal{C}$ is *real* if $S \otimes S$ is simple. We say that a simple object S is *prime* if there exists no non-trivial factorization $S \simeq S_1 \otimes S_2$. They say that $\mathcal{C}$ is a monoidal categorification of a cluster algebra A if the Grothendieck ring of $\mathcal{C}$ is isomorphic to A and if

(M1) any cluster monomial of A corresponds to the class of a real simple object of $\mathcal{C}$,

(M2) any cluster variable of A corresponds to the class of a real simple prime object of $\mathcal{C}$.

(Note that the above version is weaker than the original definition of the monoidal categorification in Hernandez and Leclerc [ibid.].) They proved that certain categories of modules over symmetric quantum affine algebras $U_q'(\mathfrak{g})$ give monoidal categorifications of cluster algebras. Nakajima extended it to the cases of the cluster algebras of type A, D, E (Nakajima [2011]) (see also Hernandez and Leclerc [2013]).

Once a cluster algebra A has a monoidal categorification, the positivity of cluster variables of A and the linear independence of cluster monomials of A follow.

In order to give a monoidal categorification of $A_q(\mathfrak{n}(w))$, we use a monoidal cluster, a categorification of a cluster. It is a finite set of real simple objects $\{M_i\}_{1\leqslant i\leqslant r}$ in the monoidal category $(R\text{-gmod}, \otimes)$ of the finite-dimensional graded modules over the quiver Hecke algebras, which satisfies the condition: $M_i \otimes M_j \simeq M_j \otimes M_i$ up to a grading shift. Then the mutation at k $(1 \leqslant k \leqslant r)$ creates a new monoidal cluster from the old monoidal cluster by replacing the k-th object M_k with a real simple object M_k'. Here, the mutated object M_k' is related with the original monoidal cluster by the following exact sequences (up to grading shifts) instead of the relation (1):

$$
\begin{aligned}
&0 \longrightarrow \bigotimes_{i;\, b_{ik}<0} M_i^{\otimes(-b_{ik})} \longrightarrow M_k \otimes M_k' \longrightarrow \bigotimes_{i;\, b_{ik}>0} M_i^{\otimes b_{ik}} \longrightarrow 0,\\
&0 \longrightarrow \bigotimes_{i;\, b_{ik}>0} M_i^{\otimes b_{ik}} \longrightarrow M_k' \otimes M_k \longrightarrow \bigotimes_{i;\, b_{ik}<0} M_i^{\otimes(-b_{ik})} \longrightarrow 0.
\end{aligned}
\tag{2}
$$

Note that if one passes to the Grothendieck group level, then (1) is a consequence of (2).

In Kang, Kashiwara, Kim, and Oh [2018], it is proved that if the first step mutations starting from the initial monoidal cluster are possible, then every successive mutations are possible. As its consequence, the quantum unipotent coordinate algebra $A_q(\mathfrak{n}(w))$ has a monoidal categorification, and the quantization conjecture follows. Note that F.

Qin also provided a proof of the conjecture for a large class with a condition on the Weyl group element w in a completely different method (Qin [2017]).

Note that the converse of (M1) is conjectured by several experts (e.g., see Geiß, Leclerc, and Schröer [2011] and Kimura [2012]):

(M1)′ any real element of $B\big(A_q(\mathfrak{n}(w))\big)$ is a cluster monomial.

It is still open.

6 Real objects

By the monoidal categorification, a real element of the upper global basis corresponds to a real object of the monoidal category R-gmod of finite-dimensional graded modules over the quiver Hecke algebra. Real simple objects have remarkable properties. Let $(\mathcal{C}, \otimes)$ be a monoidal category which is either the monoidal category R-gmod of finite-dimensional graded modules over the *symmetric* quiver Hecke algebra, or the monoidal category of finite-dimensional modules over an affine quantum group. Let $\mathbf{k}$ be the base field of $\mathcal{C}$. Then we have the following propositions.

Proposition 1 (Kang, Kashiwara, Kim, and Oh [2015, 2018]). *Let M and N be simple objects of $\mathcal{C}$. We assume that either M or N is real. Then we have (forgetting grading shifts)*

(i) $\operatorname{Hom}(M \otimes N, M \otimes N) = \mathbf{k}\,\mathrm{id}_{M \otimes N}$,

(ii) *there exists a non-zero morphism* $\mathbf{r}\colon M \otimes N \to N \otimes M$ *such that*

$$\operatorname{Hom}(M \otimes N, N \otimes M) = \mathbf{k}\mathbf{r},$$

(iii) $\operatorname{Im}(\mathbf{r})$ *is simple and it coincides with the head of $M \otimes N$ and also with the socle of $N \otimes M$.*

Conversely, a simple object M is real as soon as $\operatorname{End}(M \otimes M) = \mathbf{k}\,\mathrm{id}_{M \otimes M}$.
We denote by $M \nabla N$ the simple head of $M \otimes N$.

Proposition 2 (Kang, Kashiwara, Kim, and Oh [2015]). *Let M be a real simple object of $\mathcal{C}$ and let N, N' be simple objects. If $M \nabla N \simeq M \nabla N'$, then $N \simeq N'$.*

Remark. (i) Note that the Grothendieck ring $K(\mathcal{C})$ is commutative if $\mathcal{C}$ is the module category over an affine quantum group. Similarly, the Grothendieck ring $K(R\text{-gmod})$ is commutative if we forget the grading shifts.

(ii) In (2), we have

$$M_k \nabla M_k' \simeq \bigotimes_{i;\, b_{ik}>0} M_i^{\otimes b_{ik}} \text{ and } M_k' \nabla M_k \simeq \bigotimes_{i;\, b_{ik}<0} M_i^{\otimes -b_{ik}}.$$

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SIMPLE MEMORIES FROM A SIMPLE MATHEMATICS LECTURER – LEELAVATI PRIZE LECTURE

Ali Nesin

I was told I needed to write this about 6 months ago. Since then I've been wondering what to write. Along the years I've said or written almost everything there is to say about the maths village. Saying the same things twice, three times bores me to death. I can give the same lecture numerous times, even making it better and more fun to give each time. But for ordinary life which isn't maths? Never mind twice, I don't even want to tell it once.

I decided to write a few memories from the Maths Village - moments which surprised me, made me think, shook me or made me happy. They are fairly ordinary memories, they might not include any original ideas but these are moments that have affected me. Let's say they are what have given my life some spice.

A man's voice. He introduces himself on the phone.

"I am Colonel X…"

My colleague on the phone is afraid of course, we are then in a period in Turkey when the military has all the power and when ordinary citizens are afraid of soldiers. With a trembling voice, she (?) says

"How can I help you?"

"What have you done to my son!"

Not a question, an exclamation! Our colleague is even more frightened.

"What did we do my Colonel? What happened to your son?"

Clearly the colonel is a bit of a prankster and knows the effect that he has.

"How did you do in fifteen days what I couldn't do in fifteen years?!" he continues.

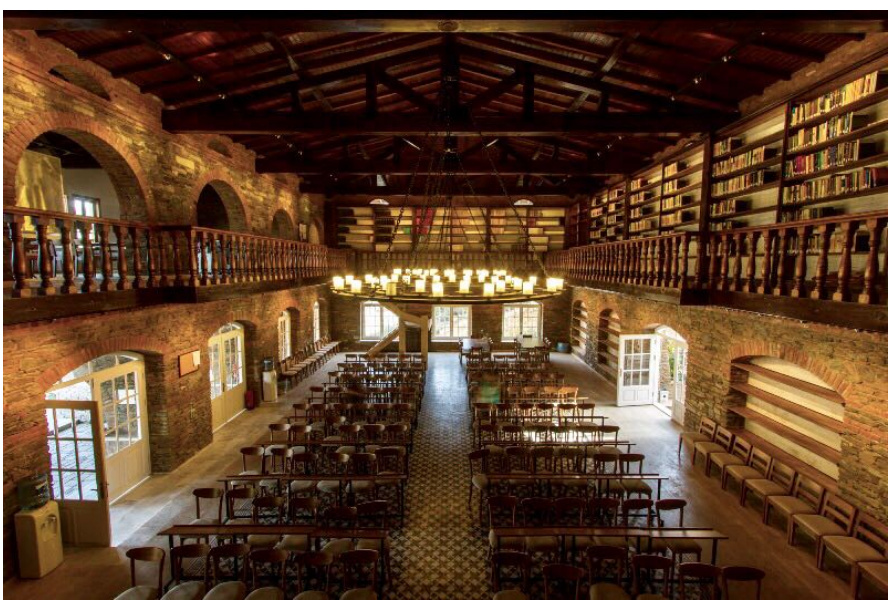
Our colleague relaxes, it seems the change has been positive. Apparently the kid, who used to be a complete recluse, never did his chores and was constantly glued to a screen, came home a completely different person…

When they told me about this phone call, I started thinking about what we had actually done that could have caused such a change and I think I found the answer: we hadn't done anything! We specifically did nothing. A relaxed and free environment, lecturers that trusted and respected the students, architecture that was modest and unpretentious, nature as far as the eye can see… And of course hundreds of other young people sharing the space day and night, cooperating in the daily work of the village, from cleaning toilets to chopping carrots…

This next story is really unbelievable, it seems straight out of a film. A mother called a colleague and thanked her a thousand times for the changes the Village brought about in her daughter. Our colleague kept wondering what we had done to deserve such a shower of praise. Then she eventually found out. Apparently the woman's daughter had previously been cripplingly shy; unable to make eye contact with anyone; she always

had her head bowed, mumbled. Her mother assumed we knew all this, as it would have been impossible to miss. This girl was also completely transformed when she came home from the Village. She was confident, stood up straight and was able to speak aloud. This is why the mother was thanking us.

We often have kids come to the Village who are different, or who have behavioural issues. Some are autistic, some are dyslexic, some are hyperactive, some, like the girl in this story, are passive and shy. A problem we encounter often is diction issues. Interestingly, very often these particular students have a great talent for mathematics. We determine their needs and act accordingly. Their classmates are understanding of this. But in the programme this girl participated in, no-one in particular had jumped out at us. They were the most ordinary group in the world! According to the mother, we should have been able to see the daughter's issues immediately - so why didn't we? The girl must have changed the day she set foot in the Village. I really have no idea what happened to this girl that could cause such a sudden change.



One day I am supposed to teach about a hundred high school students. I enter the classroom and start observing the students, all the while chatting about this and that. My purpose is to see how curious, how motivated they are, and to deduce from how they sit and how they communicate with each other what kind of a lecture I should give. I always do this. In the first two or three minutes I observe them while I am talking. I decide whether I should be strict or soft, how much I should talk, how I will present the concepts - all in those first few minutes. I must say that even though I have been wrong a few times, my rate of successful guesses is quite high. In this case, I decided: I can't lecture this kind of students, and this many of them, for two hours. They have to be active, think, find things out on their own. If I talk at them for two hours, they won't listen to me, they'll get bored. I decided to pose a question. My questions are

generally hard enough to need at least one hour of thought, but not so hard that the student can't make at least a bit of headway. Usually for the first 10-15 minutes I make sure that everybody has understood the question, and then I retreat. I position myself as a psychologist does to his patients: the students can only see me out of the corner of their eyes, and then only if they want to. But this time the students are too excited. They are constantly raising their hands and asking questions. I'm running around, unable to keep up. Clearly they love the question. I am happy as can be, drunk on the victory of having vanquished a hundred young monsters. One kid in particular is very excited. He not only raises his hand but jumps up and down to make sure I can see him! The little devil is very clever in his thinking, too. Anyway, the lecture was drawing to an end. I solved the question and gave its explanation on the board. They already knew more or less what I was going to write, after all they had thought about the question for more than an hour. They couldn't express it in the same way but they had the intuition, they nodded at everything I said impatiently as if to say "we know that, move on". The lecture was very successful. On my way out I met one of their teachers. She (?) had been worried about one of the kids who was hyperactive. Apparently he would never sit still, he would walk around, climb up the walls. She had been worried he would annoy me by doing this. But her fears had been in vain, the kid had thought for two hours. She couldn't believe it!

Oh! I was like that at school too. I can't begin to describe how bored I got in primary school and in middle school. But I loved when the teacher had to leave us alone for a while and would leave us with a problem. I'd go into a trance. This kid was not actually hyperactive, he was just bored of hyper-passivity.

One day, I had a very small class. I was teaching very well, if I say so myself. It's not always like this - I can usually do quite well but this time my success was extraordinary. Thirty to forty students were hypnotized. I again asked a question and sat off to one side while they thought, and I wondered what made that lecture so successful. I got it! I had loved those children! It might be a cliche that you need to be able to love in order to teach, but I had never seen this fact illustrated so concretely.

Some youngsters hate thinking. This doesn't happen with children, children are always curious, but it can happen after puberty. At first I would argue with these young people. "Think!", I'd say, "You don't have to find the answer, but think!". Of course it was no use. If you tell someone to raise their arm, they will. If you tell them to nod, they'll nod. But what are they to do when one tells them to think? Thinking is not an activity like running, there is no recipe for it. After a while I stopped telling these youngsters to think. Let me tell you an episode related to this that surprised me, even though it is not surprising at all.

I gave a class a problem and pulled back, as usual. The class were not bad - they all understood the question quite quickly and started working on it. Except for a young man in the first row. He was looking outside, staring at the ceiling, making the boy on his right laugh, playing with the hair of the girl on his right… I kept watching him. It took all I had not to go and say something. I kept telling myself "It won't be any use, this is how he is, nothing I can say will change that". About 15 minutes later he

picks up his pencil. I thought "I think he's taken the bait!". After all, how long can one sit without doing anything? It's almost always the case - these distracted people eventually start thinking. It was the case with this one. He started scribbling something as he slouched on his bench. Then he sat up and wrote some more. He sat back and thought. He leaned against his friend and thought. He started writing again, turned his paper over, wrote some more, asked his friend for more paper… That young man thought for more than an hour. I had won another victory. By doing what? Specifically by doing nothing!

When I pose a question in class, I tell the ones who have partially or completely solved it to go sit in the back of the class, so that they don't tell the others, to give the others a chance to think. Eventually being sent to the back of the class becomes a reward. One day I'm in front of a very good class - the back of the class is getting more crowded every second. The back of the class gets fuller than the front; there is no more room back there. Of course, then the ones in front feel like they are being punished. I feel sorry for them and guide them towards a partial answer at least. One of these students comes up with an answer that merits being sent to the back for, but the back is completely packed. "Sir, am I not going to be sent to the back?" he asked. I said "But there's no room…". "Sir pleeeease send me to the back". I did of course!

One day I posed a problem in class again. I intended to sit back as usual, but they weren't letting me. The class is in turmoil, everyone's giving a different answer. And they're all wrong! Even though I told them "no answer you will give before half an hour has passed can possibly be correct!", nothing can stop them. I say "Look guys, we didn't make this Village so that we could ask easy questions and you could answer them right away… Think a little!". It's no use. I decree that no-one is allowed to give an answer before half an hour has passed. Still they persist. I'm despairing, I don't know what else to do. Every one of the 40 students comes up with an answer every two seconds. At last, I said "Ok, I'm leaving. I'll be in the courtyard right outside. You can see me out the window. Talk, argue among yourselves, and decide on a single answer. And I don't want only the answer, I also want the justification. Convince each other, and when you are all convinced, come and get me." And that's what I did. Fifteen minutes later I peek through the window. What an amazing scene! The students are huddled in little groups and are arguing excitedly. Some have gone up to the chalkboard and are explaining things to others. I forget now whether they actually found the answer. They probably got quite close. I think it shocked them that I would trust them enough to leave the classroom. That had never happened to them. Nor had I ever done such a thing myself. It is such a difficult problem to know what kind of attitude it is best to take towards which kind of student…

One day in class there is a kid who refuses to think. He doesn't bother anyone, he just doesn't care about the question. I don't think he even understands it. I go up to him and ask "Have you understood the question?". He looks at me impassively and just says "No." It's clear that he is rebelling, against me and everything around him. That "no" isn't one of helplessness, it's a "no" of defiance. I ignore the challenge. I touch his

shoulder softly and bend towards him. “Shall I explain?” I ask. “All right”, he answers. It was the magic touch!

I get a group of 20 primary school students one day. It’s clear that they are very smart. I need to give them a hard problem, or they’ll solve it in 15 minutes. So I ask a tough one. I must have overshot; no-one can find the answer. Grumbles start rising in the classroom. They demand a hint. I say “Not a chance!”, because they are working really well - they are sure to get it eventually. They are insisting on the hint, but some of them don’t want one. The class splits into two factions, those who want a hint and those who don’t. I sit back, pretend to read my book. All of a sudden half of the class erupts in protest. They’ve prepared a sign “WE WANT A HINT!”. They start marching around the classroom, chanting “HINT, HINT!!”. They even climbed to the top floor of the library and shouted from there. I laughed so much but I never did give them that hint.

I’m teaching a class of about a hundred 12- or 13-year-olds. I decide to ask them little combinatorics questions. I ask my first one. I tell them I don’t want them to tell me the answer - I want them to write it down and show me, so that the others can’t hear and get a chance to think. A girl sitting in the front row writes the correct answer, and explains it very clearly and in perfect Turkish when the time comes. This kind of student is very rare. Clearly she is very talented. My second question is a bit harder. After a couple of minutes she again gives the right answer, and again explains it very well. The third question is even harder. After 5 to 10 minutes she says “360”. I am surprised she got it wrong - she must have made a calculation error somewhere even if her reasoning is correct.

> “You must have made a calculation mistake somewhere, check again, I tell her.”
>
> “Ok.”

and she bows over her calculations once more. Meanwhile, the other students are showering me with answers. All of them are wrong. The girl says:

> “Sir, I haven’t made a mistake.”
>
> “There has got to be one, I am sure you can find it.”

Eventually another student says 720, and I tell them they are correct. In no time the answer 720 has spread throughout the classroom. They must have told one another. The girl insists,

> “Sir, I still get 360.”
>
> “Ok, I’ll explain it on the board now.”

The moment I get to the chalkboard I realize the answer is 360!

Tell me, in which other discipline but mathematics can a 13-year-old girl challenge a wizened 60-year-old man who has devoted his life to the subject? What a contrast to the others, who happily copied the wrong answer from one another.

I have more memories of course - how could I not? I teach more than a thousand hours per year, to thousands of students. For example, the slight smile that glows on a student's face when they understand something. Every teacher has experienced this, enjoyed it. Are those memories? Definitely. Are they worth telling? They are so common… But so important to me. Just one of those smiles makes my day.

Take the girl that was in our last programme. She's a triplet - two girls, one boy, and they are all at the Village. On the first day, she tells me "Sir, please don't ask me any questions, my math is terrible". I said "But you've gone and sat in the front row?". "Yes," she said "I came to the Village as a last shot, to see if I can finally understand…". She became one of the best in the class almost immediately. On the third day, she solved a problem that no-one else in the class could solve. When she arrived at the Village she had been determined to study medicine; 15 days later she had changed her mind to mathematics…

Her brother, on the contrary, was already getting good marks in maths before he came to the Village. He really was very intelligent. On his last day he told me "The thing that surprised me the most was that my questions got answered completely and thoroughly".

I almost forgot! One day I am teaching Analysis to high school students. Not calculus, analysis with all its epsilons and deltas. It's a great class. I witnessed such scenes: while I was writing on the board, one of them goes

> "Wait a minute… What on earth is this, I don't understand a thing…"

Another goes,

> "I know, dude… Sir!"
>
> "Sir wait a sec!"

The kids were rebelling against not understanding, protesting until they did. The three other lecturers on this programme were flabbergasted, like me. It was an amazing two weeks.

I am sorry to have taken your time with my simple memories. But I had to write this note, and this is what I wanted to write about. I have shared my most precious memories with you. Let this be the last one:

It was a year before my father's death. My cartoonist friends from years ago had come to visit him, we were sitting on his balcony. My father introduced me to his friends like this:

> "Some people are born with a mission. Some are born painters, some are born musicians, this boy of mine was born a teacher."

At that time I was doing research. I guess my father knew me so well that he could see this side of me. It's the best compliment I've gotten in my life.

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COHOMOLOGY OF ARITHMETIC GROUPS – FIELDS MEDAL LECTURE

Akshay Venkatesh

Abstract

The topology of "arithmetic manifolds", such as the space of lattices in $\mathbf{R}^n$ up to rotation, encodes subtle features of the arithmetic of algebraic varieties. In some cases, this can be explained because the arithmetic manifold itself carries the structure of an algebraic variety.

I will talk about some of the phenomena one encounters in the other, "non-algebraic," cases.

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1 Introduction

I have tried to write this paper, an expanded version of my ICM talk, for a broad audience. The style is very informal: I have suppressed many technicalities, at times I have been imprecise and used idiosyncratic terminology, and the final sections are rather

speculative. I apologize for any confusion this causes. I will try in future to write down a more detailed account of the novel parts.

Informally speaking, an *arithmetic group* is obtained by taking the integral points of a matrix group. For example, $\mathrm{SL}_n(\mathbf{Z})$ is the group of $n \times n$ integer matrices with determinant 1. We will be interested in the cohomology of such groups. More geometrically: writing Y_n for the space of lattices in $\mathbf{R}^n$ up to rotation, we have

$$H^*(\mathrm{SL}_n(\mathbf{Z}), \mathbf{Q}) \simeq H^*(Y_n, \mathbf{Q}),$$

see §2 for details. The space Y_n is what one might call the "arithmetic manifold"[1] associated to $\mathrm{SL}_n(\mathbf{Z})$; every arithmetic group Γ has such an arithmetic manifold Y_Γ.

Now, for some arithmetic groups, for example $\Gamma = \mathrm{Sp}_{2g}(\mathbf{Z})$, the associated arithmetic manifolds Y_Γ live in the world of algebraic geometry: they can be identified with the complex points of an algebraic variety. We will call these cases the "Hermitian cases," because the universal cover of Y_Γ is a Hermitian symmetric domain. Over the past 50 years these situations have been deeply studied; this examination has motivated many fundamental ideas in the theory of automorphic forms, and the field remains extremely active.

Based on this study and other evidence, Langlands and others proposed that there is an association

(1) cohomology of arithmetic groups $\rightsquigarrow$ arithmetic local systems

This association is part of Langlands' reciprocity conjectures. The phrase "arithmetic local system" is meant to capture several related concepts (Galois representations and motives) in a way that is not too threatening. The reader should think of it as a local system on the spectrum Spec $\mathbb{Z}$ of the integers although it is not simple to give a precise definition. See §5 for details.

1.1 Outline of the paper. Non-Hermitian cases – e.g. $\mathrm{SL}_n(\mathbf{Z})$ when $n > 2$ – have been (relatively speaking) neglected. However, despite their absence of direct algebraic structure, they have other new features that are not seen in the Hermitian case. I want to give some flavour of these features in this paper.

(i) In the first part of the paper (sections §2 - §4) we will try to describe some of the structures that exist on the cohomology of arithmetic groups, but emphasizing the phenomena (torsion, derived Hecke operators) that are only prominent in the non-Hermitian case.

(ii) In the middle part, §5 and §6, we try to explain a little bit about Langlands' reciprocity conjecture (1).

(iii) In the final part of the paper, §7 – §9, we examine how various features on one side of (1) manifest themselves on the other side. The discussion of §8 and §9 is quite speculative.

[1] This is not a standard term. It might be more usually described as "the locally symmetric space associated to Γ."

For example, as we will discuss in §8, there is a natural notion of complexity (or "height") on both sides of (1), and we propose that these notions should match up with each other. Most of our discussion is not specific to the non-Hermitian case, but is by far the most interesting there.

In an ideal world, one would understand (1) by exhibiting a machine that takes input from the left-hand side and produces output on the right-hand side. In this world, the kind of exercise we do in (iii) is pointless. After all, we could simply stick an invariant of one side of (1) into our machine, wind the crank, and see what it produces on the other side.

But for the moment, *in the non-Hermitian case*, the mechanism for (1) remains a complete mystery (despite substantial progress recently in verifying some of its implications). Perhaps the conjectures formulated in §7 – §9 might, in fact, teach us something about this still unknown mechanism.

1.2 Acknowledgements. In some form or the other, the cohomology of arithmetic groups has been my primary object of study for the past few years.

The ideas presented here draw specifically on my joint work Bergeron and Venkatesh [2013] and Bergeron, Şengün, and Venkatesh [2016]with Haluk Sengun and Nicolas Bergeron, and also Prasanna and Venkatesh [2016] with Kartik Prasanna. My overall picture of the situation has been fundamentally shaped by my work Calegari and Venkatesh [2012] with Frank Calegari, and by Calegari's many patient explanations about basic properties of Galois representations. I learned a lot from discussions with Soren Galatius, and these discussions led to the paper Galatius and Venkatesh [2018]. My understanding of derived Hecke algebra has been much advanced through my collaboration Harris and Venkatesh [2018] and Darmon, Harris, Rotger, and Venkatesh [n.d.] with Michael Harris and with Henri Darmon, Harris, and Victor Rotger. Finally, Aravind Asok and Nicolas Bergeron gave me helpful feedback on this text.

I thank all of them, and all the other mathematicians, past and present, who discovered and developed this particular intellectual paradise. (I'm sorry that this brief text can't do justice to their contributions.)

2 Arithmetic groups and their cohomology

Warning: what we call "arithmetic groups," for short, would be usually called "congruence arithmetic group."

This introduction is partly borrowed from my similar introduction to Venkatesh [2017].

2.1 Arithmetic groups. An *arithmetic group* is a group such as

$$\mathrm{SL}_n(\mathbf{Z}), \mathrm{Sp}_{2n}(\mathbf{Z}), \mathrm{SL}_2(\mathbb{Z}[\sqrt{-1}]), \cdots$$

obtained, roughly speaking, by taking the "$\mathbb{Z}$-points" of a classical group of matrices.

Now $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-1}])$ looks different to the others at first, but it can be presented similarly: one can regard 2×2 matrices over $\mathbb{Z}[i]$ as 4×4 matrices over $\mathbb{Z}$:

$$\begin{pmatrix} a+bi & c+di \\ e+fi & g+hi \end{pmatrix} \mapsto \begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ e & f & g & h \\ -f & e & -h & g \end{pmatrix}$$

A precise definition, adequate for our purposes, is given as follows: A "congruence arithmetic group" (which we shall abbreviate simply to "arithmetic group" for the purpose of this document) is a group Γ obtained by taking a semisimple $\mathbf{Q}$-group $\mathbf{G} \subset \mathrm{SL}_N$, an integer M, and taking

$$\Gamma = \{g \in \mathbf{G}(\mathbf{Q}) : g \text{ has integral entries, } g \equiv \mathrm{Id}_N \text{ modulo } M\}$$

or, more generally, allowing any Γ that contains such a subgroup with finite index. Each such group Γ is contained in an ambient Lie group, namely the real points of $\mathbf{G}$:

$$\Gamma \leqslant G = \mathbf{G}(\mathbf{R}).$$

In fact, Γ is a "lattice" in G: this just means that the quotient G/Γ has finite volume.

In the examples above, the ambient Lie groups are as follows:

$$\mathrm{SL}_n(\mathbf{Z}) \leqslant \mathrm{SL}_n(\mathbf{R}), \mathrm{Sp}_{2g}(\mathbf{Z}) \leqslant \mathrm{Sp}_{2g}(\mathbf{R}), \mathrm{SL}_2(\mathbb{Z}[\sqrt{-1}]) \leqslant \mathrm{SL}_2(\mathbf{C}).$$

2.2 Symmetric spaces. Each such group Γ acts on a canonically associated Riemannian manifold S (the "symmetric space" for the ambient Lie group G). In general S is, as a manifold, the quotient of G by a maximal compact subgroup $K \subset G$; it is known that all such K are conjugate inside G. It's easy to verify that G preserves a Riemannian metric on S.

For example, if $\Gamma = \mathrm{SL}_2(\mathbf{Z})$, we have $G = \mathrm{SL}_2(\mathbf{R})$ and can take $K = \mathrm{SO}_2$; the associated geometry $S = G/K$ can be identified with the Poincaré upper-half plane

$$S = \{z \in \mathbf{C} : \mathrm{Im}(z) > 0\}$$

and the action of G is by fractional linear transformations; it preserves the standard hyperbolic metric $|dz|^2/\mathrm{Im}(z)^2$.

If $\Gamma = \mathrm{SL}_2(\mathbb{Z}[i])$ we have $G = \mathrm{SL}_2(\mathbf{C})$ and can take $K = \mathrm{SU}_2$; the associated geometry S can be identified with the three-dimensional hyperbolic space $\mathbb{H}^3$.

Finally, for $\Gamma = \mathrm{SL}_n(\mathbf{Z})$ the situation is less familiar: $G = \mathrm{SL}_n(\mathbf{R})$, we can take $K = \mathrm{SO}_n$, and the geometry S can be identified with the space of positive definite, symmetric, real-valued $n \times n$ matrices A with $\det(A) = 1$. It is an enjoyable exercise to determine the invariant notion of distance in this case.

2.3 Motivation via reduction theory. Although not important for our later purposes, we now describe, by way of motivation, a problem that naturally leads to considering the cohomology of arithmetic groups.

For $\Gamma = \mathrm{SL}_n(\mathbf{Z})$, it is a classical problem – the reduction theory of quadratic forms – to explicitly describe a fundamental domain for Γ acting on S. The case of $n = 2$ was very well understood by Gauss, and Minkowski wrote down explicit fundamental domains for $n \leq 6$; for example, for a 3×3 positive definite, symmetric matrix

$$S = \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{13} & S_{23} & S_{33} \end{pmatrix},$$

one can define a fundamental domain:

$$S_{11} \leq S_{22} \leq S_{33}, 0 \leq S_{12} \leq S_{11}/2, 0 \leq S_{23} \leq S_{22}/2, |S|_{13} \leq S_{11}/2, \text{ eight more.}$$

Such examples suggest that the complexity of such a fundamental domain (e.g. the number of inequalities needed to define it) increases very rapidly as one increases n.

Must a fundamental domain be complicated? One reason that would force it to be so is if the quotient

$$Y_\Gamma = S/\Gamma = \text{“arithmetic manifold” associated to } \Gamma$$

is topologically complicated, since one can make a model for Y_Γ by gluing sides in the fundamental domain.[2] This leads us naturally to the study of the Betti numbers, and thus the cohomology, of Y_Γ.

The cohomology of Y_Γ as a topological space coincides with the group cohomology of Γ as a discrete group:

$$H^*(\Gamma, \mathbf{C}) \simeq H^*(Y_\Gamma, \mathbf{C}).$$

Thus we can study the topology of Y_Γ without explicit recourse to the geometry of S, if so desired.

Two technical warnings:

- What we call “arithmetic manifold” would be, in more standard terminology, the “locally symmetric space” associated to Γ.
- It often happens that it is convenient and important to group together Y_Γ for several very closely related arithmetic groups Γ. This process is achieved in number theory by considering adèle groups, that is to say, by replacing the role of $\mathbb{Z} \subset \mathbb{R}$ by $\mathbb{Q} \subset \mathbf{A}$. For simplicity we suppress this in this document, but the statements in the later section should be understood with the enlarged Y_Γ. In the case when $\mathbf{G}$ is simply connected, there is no difference.

3 The defect δ of an arithmetic group

Let Γ be an arithmetic group. As we have discussed (§2) it lies as a lattice $\Gamma \leqslant G$ inside some ambient Lie group G. The behavior of its cohomology depends very strongly on

[2] If Γ is torsion-free, S/Γ has the structure of manifold; even if not, it still has the structure of an orbifold. We will, however, continue to use the world “manifold” to describe S/Γ, even though it may have finite quotient singularities.

G. For example, numerical computations suggest that $H^*(\Gamma, \mathbf{Q})$ is "typically" much larger when $G = \mathrm{SL}_2$ than when $G = \mathrm{SL}_3$. There is an important numerical invariant that accounts for this discrepancy.

Let $K \subset G$ be a maximal compact subgroup, and let

$$\delta = \mathrm{rank}(G) - \mathrm{rank}(K).$$

For example, when $G = \mathrm{SL}_n(\mathbf{R})$, we have $K = \mathrm{SO}_n$, and

$$\delta = \mathrm{rank}(\mathrm{SL}_n(\mathbf{R})) - \mathrm{rank}(\mathrm{SO}_n) = (n-1) - [n/2] = \lceil \frac{n-1}{2} \rceil,$$

so that $\delta = 0$ for SL_2 but $\delta = 1$ for SL_3. We call δ the *defect*.

Now, all Hermitian cases have $\delta = 0$. By contrast, in this paper, many of the statements and situations we consider will primarily be of interest in the case when $\delta > 0$.

The invariant δ controls much of the behavior of the cohomology (for reasons we cannot get into here). We will discuss just one example, namely, it controls growth of cohomology as Γ shrinks. There are different ways of quantifying Γ "shrinking." The weakest possible notion is asking that the volume of G/Γ get larger. For ease of statement we impose simplifying conditions (see Abert, Bergeron, Biringer, Gelander, Nikolov, Raimbault, and Samet [2017] for discussion of the general case).

We consider a sequence

$$\Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_N \supset \ldots$$

of congruence arithmetic subgroups of $\Gamma_1 = \Gamma$, such that the common intersection of all Γ_N is the trivial subgroup. We suppose also that S/Γ is compact.

Theorem 1. *(de George and Wallach, de George and Wallach [1978]; see also Lück Lück [1994]): The limit*

$$\lim_{N\to\infty} \frac{\dim H^j(\Gamma_N, \mathbf{C})}{\mathrm{vol}(S/\Gamma_N)}$$

exists. It equals zero unless $\delta = 0$ and $j = \frac{1}{2}\dim(S)$, in which case it is a nonzero quantity depending only on S (and volume normalization).

Thus $\delta = 0$ is distinguished by the rapid growth of Betti numbers. The following counterpart, suggesting that $\delta = 1$ is distinguished by rapid growth of torsion, was proposed in Bergeron and Venkatesh [2013]. This paper also contains significant evidence for the conjecture (e.g., replacing $\mathbb{Z}$ by other Γ-modules) that we don't describe.

Conjecture 1. *(Bergeron–Venkatesh, Bergeron and Venkatesh [ibid.]): The limit*

$$\lim_{N\to\infty} \frac{\log \# H^j(\Gamma_N, \mathbf{Z})_{\mathrm{tors}}}{\mathrm{vol}(S/\Gamma_N)}$$

exists. It equals zero unless $\delta = 1$ and $j = \frac{1}{2}\dim(S) + 1$, in which case it is a nonzero quantity depending only on S (and volume normalization).

The picture suggested by these statements is as follows:

- when the defect δ vanishes, lattices Γ tend to have a lot of cohomology in characteristic zero; this is concentrated in the"middle degree" $\frac{\dim(S)}{2}$.
- When the defect δ equals 1, lattices Γ tend to have a lot of torsion in their cohomology; this is concentrated in the degree immediately above middle.
- When the defect is larger than 1, both torsion and characteristic zero cohomology are scarcer.[3]

As we have mentioned, until fairly recently, there was relatively little study of cases with $\delta > 0$ (but it was certainly appreciated that there are interesting new phenomena there – see, for example, Mennicke and Grunewald [1980] and Ash and McConnell [1992]). Our emphasis in this paper is on this case of positive defect.

4 Operators

The defining feature of the cohomology of arithmetic groups is the existence of a large number of extra symmetries, i.e. extra operations

$$T : H^j(\Gamma) \longrightarrow H^j(\Gamma). \tag{2}$$

In a slightly different setting, the existence of such "Hecke" operators goes back at least to Mordell Mordell [1917].

When $\delta > 0$ there also exist operators that shift cohomological degree, i.e. maps like this:

$$H^j(\Gamma) \to H^{j+1}(\Gamma).$$

These were introduced in my recent paper Venkatesh [2016], where they are called derived Hecke operators. However, in a purely algebraic context – i.e. separate from the application to arithmetic groups – the derived Hecke algebra was studied previously by Schneider Schneider [2018].

We now describe these in a little more detail.

4.1 Hecke operators. Operations as in (2) comes from "almost automorphisms" of the group.

Namely, call an "almost automorphism" of Γ a triple $(\Gamma_1, \Gamma_2, \varphi)$ of two finite index subgroups Γ_1 and Γ_2, and an isomorphism $\varphi : \Gamma_2 \to \Gamma_1$ between them. Although not quite an automorphism of Γ itself, this data nonetheless induces an endomorphism of cohomology, as follows:

$$H^*(\Gamma) \to H^*(\Gamma_1) \xrightarrow{\varphi^*} H^*(\Gamma_2) \dashrightarrow H^*(\Gamma). \tag{3}$$

[3] These results should not be interpreted as saying that there is very little cohomology for $\delta > 1$. An heuristic proposed in Bergeron and Venkatesh [2013], phrased informally, states that if we choose Γ "at random," the chance that $H^*(\Gamma, \mathbb{Z}/p\mathbf{Z})$ is nonzero is about $p^{-\delta}$. This has to be applied with great care, but should give the rough picture: if we fix p, there is a positive, though small, probability that one sees at least a mod p cohomology class. Characteristic zero cohomology is rarer, but it is again reasonable to think if one chooses K different choices of Γ, a positive power $K^?$ will have interesting cohomology classes in $H^*(\Gamma, \mathbf{C})$.

The first two maps here come from functoriality. The last map is the transfer: a finite index inclusion of groups also induces a map on cohomology in the "wrong" direction.

Arithmetic groups, such as $\mathrm{SL}_n(\mathbf{Z})$, admit a rich supply of almost-automorphisms. Namely, if $\alpha \in \mathrm{SL}_n(\mathbf{Q})$, then conjugation by α induces

$$\Gamma_1 := \Gamma \cap \alpha\Gamma\alpha^{-1} \simeq \alpha^{-1}\Gamma\alpha \cap \Gamma =: \Gamma_2.$$

The resulting endomorphisms of homology or cohomology are called Hecke operators.

What is surprising, and not obvious, is that these operators commute, at least if we mildly restrict α. For example, if $\Gamma \leqslant \mathrm{SL}_n(\mathbf{Z})$, there exists an integer

$$N = N(\Gamma) \tag{4}$$

such that this is true if we restrict to α whose denominators are relatively prime to N. In what follows we suppose that we always restrict to Hecke operators arising from such α. We will call any choice of such an integer N the *level* of Γ; we can harmlessly replace N by any multiple of itself. A similar notion of level may be defined for any arithmetic group.

The resulting commutative algebra of Hecke operators can be used to decompose cohomology. For example, if K is a field, we may try to decompose $H^*(\Gamma, K)$ into common eigenspaces (or generalized eigenspaces) of the Hecke operators. An element of such a common eigenspace is often called a *Hecke eigenclass.* The corresponding "eigenvalue" is a homomorphism to K, from the algebra generated by all Hecke operators.

This commutative algebra generated by all Hecke operators is usually called the Hecke algebra $\mathbb{T}$ (for Γ). For simplicity we will always regard $\mathbb{T}$ as what we get by considering endomorphisms of integral cohomology; thus, $\mathbb{T}$ is a commutative subalgebra

$$\mathbb{T} \leqslant \text{endomorphisms of } H^*(\Gamma, \mathbf{Z}). \tag{5}$$

(To handle torsion and other issues it is better to make a more careful definition, see Khare and J. A. Thorne [2017, §6.5].)

A key idea is that:

> much of the information in $H^*(\Gamma)$ is actually encoded by the ring-theoretic structure of $\mathbb{T}$.

For example, rather than deal directly with the generalized eigenspace decomposition of $H^*(\Gamma, K)$, we will prefer to directly deal with homomorphisms $\pi : \mathbb{T} \to K$. It is helpful to think of such a π as indexing a Hecke eigenclass. Indeed, for such π, there exists $\alpha \in H^*(\Gamma, \mathbf{Z}) \otimes K$ which is scaled by $\mathbb{T}$ according to the character π:

$$T \cdot \alpha = \pi(T)\alpha$$

for all $T \in \mathbb{T}$.

4.2 Derived Hecke operators, when $\delta > 0$. When $\delta > 0$, one often observes in a closer study of $H^*(\Gamma, \mathbf{Q})$ that many phenomena are not confined to a single cohomological degree: if they happen in one degree, they happen in a range of several contiguous degrees. For an attempted introduction to this phenomenon, see Venkatesh [2017].

On the basis of this, it is reasonable to expect the existence of operators that can shift degree in cohomology by 1. In order to make them, one wants to cup with some classes (like the Lefschetz operators of algebraic geometry). Unfortunately, in most cases of interest, $H^1(\Gamma, \mathbf{Q}) = 0$, so one cannot imitate the idea of Lefschetz operators too directly.

The key observation is that:

> There is (for any fixed m) a large supply of subgroups $\Gamma_1 \leqslant \Gamma$ with $H^1(\Gamma, \mathbb{Z}/m\mathbf{Z})$ nonvanishing.

In fact, we can even find such a Γ_1 which arises as in (3). Given this, and a class $\beta \in H^1(\Gamma_1, \mathbb{Z}/m\mathbf{Z})$, we can "insert" cupping with β into the sequence (3), where we now take cohomology with $\mathbb{Z}/m\mathbb{Z}$ coefficients:

$$H^*(\Gamma) \longrightarrow H^*(\Gamma_1) \xrightarrow{\cup\beta} H^*(\Gamma_1) \xrightarrow{\varphi^*} H^*(\Gamma_2) \longrightarrow H^*(\Gamma).$$

Call this composite $T_{\varphi,\beta}$. Finally, one can find a sequence of φ, β and m, and patch the resulting operators $T_{\varphi,\beta}$ together into a characteristic zero limit. This gives rise to an algebra

$$\widetilde{\mathbb{T}} \leqslant \text{endomorphisms of } H^*(\Gamma, \mathbb{Q}_p),$$

which is called in Venkatesh [2016] the derived Hecke algebra. The real content of Venkatesh [ibid.] is providing evidence that $\widetilde{\mathbb{T}}$ is bigger than $\mathbb{T}$, i.e. really does contain interesting degree-shifting operators.

This derived Hecke algebra is a feature of life when $\delta > 0$. It explains why many phenomena occur in several cohomological degrees. However, it has the following failing, compared to the usual Hecke algebra: because it is constructed from operations that exist only for torsion coefficients, it does not produce endomorphisms of *rational* or *integral* cohomology. We return to this point in §7.

5 Local systems in topology and arithmetic

Let K be a field. Let Γ be an arithmetic group, and N the level of Γ, as in (4). In practice, the prime divisors of N are primes where something complicated happens. In this document we will ignore them, but really one needs to instead study them rather carefully to obtain a good understanding of the situation.

As we will see later, Langlands' reciprocity conjecture predicts that there is a mapping

(6)

$$\text{elements of a certain basis for } H^*(\Gamma, K) \rightsquigarrow \begin{array}{c}\dim(G)\text{-dimensional "local system"}\\ \text{of } K\text{-vector spaces on } \operatorname{Spec}\mathbb{Z}[1/N]\end{array}$$

But we need to make sense of the right hand side; note that Spec $\mathbb{Z}[1/N]$ is the prime spectrum of $\mathbb{Z}[1/N]$ – i.e., it has a point for every prime ideal of $\mathbb{Z}$, excepting divisors

of N – and is thus far from a nice topological space like a manifold. It is easy to make a good definition for the right hand side in some cases – for example, for K finite. But for $K = \mathbf{Q}$, say, it is quite difficult to make a definition that is large enough to accommodate (6). So we digress to discuss this issue, which is part of the reason why the Langlands correspondence is so difficult to formulate correctly.

To try to make our discussion a bit more comprehensible we use the paradigm of "arithmetic topology," namely, the analogy

$$\operatorname{Spec} \mathbb{Z}[1/N] \leftrightarrow \text{(open, nonorientable) 3-manifold.}$$

I do not have a strong philosophical commitment to this viewpoint, but it certainly makes exposition easier.

Accordingly, we first review local systems over a manifold, and then try to describe local systems on $\mathbb{Z}[1/N]$ in a parallel way. Whatever the exact definition, in what follows, we will use the phrase *arithmetic local system* to mean a local system of K-vector spaces on $\operatorname{Spec} \mathbb{Z}[1/N]$.

5.1 Local systems in topology. For a manifold M, a local system of K-vector spaces M is a locally constant sheaf of K-modules on M. In plainer language: it is an assignment $m \rightsquigarrow L_m$ of a K-vector space to each point $m \in M$, together with compatible identifications $L_m \simeq L_{m'}$ for nearby points $m \approx m'$.

Assuming M connected, the local system L is entirely specified by its monodromy representation

$$\pi_1(M, m_0) \longrightarrow \operatorname{Aut}_K(L_{m_0})$$

for any basepoint $m_0 \in M$. Conversely, any such representation gives a local system on M, with fiber L_{m_0} at m_0.

Such local systems have the following important properties:

(i) A fiber bundle $E \to M$ and a choice of integer $j \geq 0$ gives rise to a local system: to $m \in M$ we assign the cohomology $H^j(E_m; K)$,

(ii) We can take cohomology with coefficients in a local system. These groups $H^*(M, L)$ are K-vector spaces. (If L arose from the construction of (i), then these K-vector spaces are related to the cohomology of E.)

5.2 Arithmetic local systems. How does this discussion adapt to the space $\operatorname{Spec} \mathbb{Z}[1/N]$? The answer depends on the nature of the coefficient field K.

5.2.1 Finite or p-adic fields. If K is finite, there is a reasonable way of adapting this discussion to the space $\operatorname{Spec} \mathbb{Z}[1/N]$, simply by using the étale topology. Namely, a local system of K-vector spaces should be a locally constant sheaf of K-vector spaces for the étale topology. In order that this behave well, we must always require that the characteristic of K divide N; we can arrange this by increasing N.

The situation is very similar if K is a p-adic field, i.e. a finite extension of the p-adic numbers $\mathbb{Q}_p$. Again, if we are using this notion, we will always require that p divide N.

In both cases, then, a "local system with K-coefficients" is specified by giving a homomorphism

$$\pi_1^{\mathrm{et}}(\operatorname{Spec} \mathbb{Z}[1/N]) \longrightarrow \mathrm{GL}_n(K). \tag{7}$$

In these settings, one obtains analogues of (i) and (ii) just from formal properties of the étale topology.

In number theory one usually formulates things in a different (equivalent) way. Namely, the étale π_1 of $\mathbb{Z}[1/N]$ is a quotient of the absolute Galois group of $\overline{\mathbb{Q}}/\mathbf{Q}$, and thus a local system of K-vector spaces is specified by giving a representation

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbf{Q}) \longrightarrow \mathrm{GL}_n(K). \tag{8}$$

This is the perspective that is often adopted in number theory: local systems are encoded by means of such "Galois representations".

5.2.2 Number fields. What if K is $\mathbf{Q}$, or a finite extension of $\mathbf{Q}$? The reader without familiarity with the theory of motives should skip this section and just take on faith that:

- one can make a working definition of "local system with K coefficients" for K a finite extension of $\mathbf{Q}$, but
- it is much more subtle than the case of K finite, being closely related to the theory of motives and motivic cohomology.

Suppose, then, that $K = \mathbf{Q}$. Of course we can still try to use (7) or (8) as a definition, but the resulting class of objects is far too small. There are simply not enough homomorphisms from the profinite group π_1^{et} to the group $\mathrm{GL}_n(\mathbf{Q})$ to accommodate the left-hand side of (6). The problem is that $\mathrm{GL}_n(\mathbf{Q})$ has too few finite subgroups.

We can make a definition that captures more natural phenomena by replacing étale sheaves with their motivic analogues. Informally, this corresponds to trying to turn point (i) of §5.1 into an actual definition, declaring that all local systems arise from "factors" of this construction.

Recent advances Ayoub [2007a,b] and Cisinski and Déglise [2009] permit one to construct, at least, a derived category of motivic sheaves over $\mathbb{Z}[1/N]$, and it may well be possible to make a good working definition from this theory. Historically, however, the theory of motivic sheaves was not available, so number theory resorted to various cruder substitutes. Since my own familiarity with motivic sheaves is minimal, we will use such a crude substitute in this document, with the hope that the sophisticated reader can construct a better version.

Working definition:

- For K a number field, we declare for this document that a "local system over $\mathbb{Z}[1/N]$ with coefficients in K" means a Chow motive M over $\mathbf{Q}$ and with coefficients in K, pure of some fixed weight w, and whose associated p-adic Galois representation has good reduction outside Np for every prime p.

- For a local system arising from M as in (i), we declare the "cohomology with coefficients in the local system"

$$H^i(\mathbb{Z}[1/N], M) \tag{9}$$

to be the $\mathbf{Q}$-vector space obtained by applying the construction of Scholl [2000], informally, "classes in the motivic cohomology $H^{i+w}(M, \mathbf{Q})$ of M that extend to a projective, flat, regular model over $\mathbb{Z}$."

At the very least, as best as we can see from automorphic forms, such a provisional definition does a pretty good job at producing a large enough class of local systems for the needs of the Langlands program (cf. conjectures in Clozel [2016]). But the given definition certainly has some rather undesirable properties – simple results like finite dimensionality of cohomology are in most cases unknown. (See e.g. Prasanna and Venkatesh [2016, §2.1.8, 2.19] for some other problems posed by this definition that are relevant to our later analysis.) For our purposes it is in any case just a working definition.

To summarize, here are our various definitions of "arithmetic local systems of K-vector spaces":

K	local system of K vector spaces on $\mathbb{Z}[1/N]$
$\mathbf{F}_p$	étale sheaf of n-dimensional $\mathbf{F}_p$-vector spaces on $\mathbb{Z}[1/N]$
$\mathbf{Q}_p$	étale sheaf of n-dimensional $\mathbb{Q}_p$-vector spaces on $\mathbb{Z}[1/N]$
$\mathbf{Q}$	n-dimensional Chow motive + extra properties
$\mathbf{C}$	I have no idea

We added $K = \mathbf{C}$ in this table to draw attention to its plight. The Langlands program (which is much broader than the cohomology of arithmetic groups) suggests that there should be a notion of arithmetic local system with $\mathbf{C}$ coefficients, substantially larger than what one gets by extending from $\mathbf{Q}$ or any number field. However, at present, there is no suggestion how to define it (cf. Arthur [2002]). This seems to me a fundamental problem of number theory.

These definitions all have the analogue of properties (i) and (ii) for manifolds; in particular, a local system ρ of K-vector spaces gives rise to a K-vector space $H^i(\mathbb{Z}[1/N], \rho)$ (which we will call $H^i(\rho)$ for short, that is, the cohomology with coefficients in the local system ρ), given by étale cohomology in the first two cases, and as described in (9) when K is a number field. Also a local system ρ of $\mathbf{Q}$-vector spaces gives, in a natural way, a local system $\rho \otimes \mathbb{Q}_p$ of $\mathbb{Q}_p$-vector spaces.

For each of the field K above there is a canonical one-dimensional local system called $K(1)$, the "Tate twist." This local system plays a distinguished role. When K is p-adic or finite, for example, $K(1)$ may be regarded (in the metaphor of arithmetic topology) as the orientation sheaf of $\mathbb{Z}[1/N]$.

5.3 Trace of monodromy. We will want to extract certain numerical invariants from local systems. For a rank m local system L of K-vector spaces on a manifold M, a map

$$\gamma : S^1 \longrightarrow M$$

defines a conjugacy class of $m \times m$ matrices, the monodromy around the loop; in particular, by taking the trace, we get a well defined invariant in K. Thus:

$$\text{trace of monodromy around } \gamma \in K.$$

There is a natural class of "loops" on Spec $\mathbb{Z}[1/N]$. In algebraic geometry, the spectrum of $\mathbf{F}_\ell$ looks like a circle; its étale fundamental group is a (pro-)cyclic group with a distinguished generator. The inclusion

$$\text{Spec } \mathbf{F}_\ell \hookrightarrow \text{Spec } \mathbb{Z}[1/N]$$

behaves similarly to the inclusion $S^1 \to M$ of a circle into a manifold. If ρ is a local system of K-vector spaces, and ℓ is a prime number not dividing N we may then similarly define

$$\text{trace of monodromy around Spec } \mathbf{F}_\ell \in K. \tag{10}$$

(Strictly, we can only do this for K finite or p-adic using the étale topology, and for K a number field we must make a choice of p and tensor with $\mathbb{Q}_p$; it is expected, although not proved in general, that the result is in K and independent of choice, see Illusie [2006]. We will suppress this issue, which causes no problem for our purposes.)

5.4 Boundary conditions. We warn that with our definitions, the association $\rho \rightsquigarrow H^i(\rho)$ does *not* commute with passings from $\mathbf{Q}$ to $\mathbb{Q}_p$. There is a map

$$H^i(\rho) \otimes \mathbb{Q}_p \longrightarrow H^i(\rho \otimes \mathbb{Q}_p), \tag{11}$$

but it need not be an isomorphism.

There are multiple reasons for this failure. One serious reason is that étale cohomology of $\mathbb{Z}[1/N]$ has a good duality theory, and motivic cohomology does not seem to.

Another reason has to do with boundary conditions. The space Spec $\mathbb{Z}[1/N]$ fails to be compact, in the arithmetic topology paradigm, because it is missing points related to the prime divisors of N, and it is also reasonable to try to further compactify it by adjoining Spec $\mathbb{R}$. Correspondingly it admits different types of cohomology according to what conditions we impose at the boundary. For example, we can consider an analog of "compactly supported cohomology," or, if we fix a subspace $L \subset H^i(\mathbb{Q}_p, \rho)$, we may consider the subspace

$$H^i_L(\rho) = \{\alpha \in H^i(\rho) : \text{image of } \alpha \text{ in } H^i(\mathbb{Q}_p, \rho) \text{ lies in } L.\},$$

or perhaps carry out a chain-level version of this condition. One can think of this as akin to considering classes in the cohomology of a 3-manifold M which lie in some specified subspace of $H^i(\partial M)$.

By imposing suitable boundary conditions on the definition of $H^i(\rho)$, for ρ a p-adic local system, we can (at least conjecturally) force (11) to be an isomorphism again for $i = 0, 1$. How to get these conditions right is a very subtle manner, addressed in generality by Bloch and Kato in their famous paper Bloch and Kato [1990].

5.5 Determinants of cohomology (topology) . (This section §5.5 is motivational. Both it and and §5.6 are only used in the final section, §9).

Local systems on a manifold can be assigned, in some cases, an interesting numerical invariant: Reidemeister torsion. We describe the algebra of it. In discussing this we will introduce some space-saving notation: if V is a vector space over the field k, we write $[V]$ for the determinant of V, i.e., the top exterior power of V. We write $[V][W]$ instead of $[V] \otimes [W]$, $\frac{[V]}{[W]}$ instead of $[V] \otimes [W]^*$, and so on. For a graded vector space V^i we define $[V^*] = \prod_i [V^i]^{(-1)^i}$. These are all k-lines, i.e. one-dimensional k vector spaces. [4]

Suppose, in what follows, that M is a compact odd-dimensional manifold. and that L is a local system of K-vector spaces on M. One can form the determinant of cohomology

$$[H^*(M, L)] = \prod_i [H^i(M, L)]^{(-1)^i},$$

which is a K-line.

Now fix a smooth triangulation T of M. This identifies the above determinant with the determinant of a certain chain complex, the complex of cochains for T valued in L:

$$[H^*(M, L)] \simeq \prod_i [C^i(L)]^{(-1)^i} \tag{12}$$

For L trivial there is a preferred basis for the right hand space, at least up to sign, coming from the basis of C^i arising from the dual basis of cells; and even for L nontrivial one can obtain a preferred basis after fixing some extra data (an "Euler structure," for this and independence of choice of triangulation see Turaev [1989]). Thus, for a smooth manifold (+ a little extra data) we obtain a preferred trivialization of the determinant of cohomology of any local system.

This algebraic data gives rise to a numerical invariant. Suppose, for example, that L is acyclic, i.e. $H^j(M, L) = 0$ for every j, then both sides of the above isomorphism (12) have distinguished bases. However, these bases do not match up under (12); they differ by an element of K^*, well defined at least up to sign. We will ignore the sign ambiguity in what follows. This element of K^* is the Reidemeister torsion:

$$L \text{ local system} \rightsquigarrow \mathrm{RT}(M, L) \in K \tag{13}$$

(let us agree this invariant is zero if L is not acyclic). If, for example, $K = \mathbf{Q}$ and L has an integral structure $L_{\mathbb{Z}}$, this invariant computes an alternating product of the sizes of $H^i(M, L_{\mathbb{Z}})$.

5.6 Determinant of cohomology (arithmetic). (This section §5.6 is only used in the final section, §9).

The theory of L-functions in number theory, in favorable circumstances, attaches an invariant valued in $(K \otimes \mathbf{C})$ to a local system of K-vector spaces ρ:

$$\rho \rightsquigarrow L(\rho) \in (K \otimes \mathbf{C}). \tag{14}$$

[4] Since we do not need to be careful about signs that occur in this theory, and so we will not attempt to think of $[V]$ as a graded k-line, just as a k-line.

(Read this only if you want details: according to our definition, a local system of K-vector spaces is a Chow motive M over $\mathbf{Q}$ with coefficients in K, and we define $L(\rho)$ so that its image under an embedding $\sigma : K \hookrightarrow \mathbb{C}$ gives the value at $s = 0$ of the L-function $L(s, M^\sigma)$, see Deligne [1979b, §2]. The phrase "in favorable circumstances" above refers to the fact that this L-function is not known in general to extend holomorphically to a region containing $s = 0$.)

Now, with respect to the analogy between Spec $\mathbb{Z}[1/N]$ and a 3-manifold, the Reidemeister torsion has strong formal analogies with L-functions in number theory. (Unfortunately, I do not know how to attribute this analogy). Besides various straightforward formal analogies, there is an interesting parallel between Fried's formula Fried [1986] and the product formula for an L-function.

By analogy with the situation with manifolds, then, one might hope that there is an extra algebraic structure at play: a trivialization of cohomology determinant lines. This viewpoint has been proposed in papers of Fontaine and Perrin-Riou [1994] and by Kato [1993]. For example Kato [ibid., §3.2] formulates a precise version of the following

> Conjecture: If ρ is a local system over Spec $\mathbb{Z}[1/N]$ with finite or profinite coefficients, then there is a distinguished trivialization of the determinant line $[H_c^*(\rho)]$ (see below for definition), characterized in terms of L-functions.

This conjecture, an algebraic incarnation of the theory of L-functions, seems to be difficult to approach directly, at least in any generality.

As we have mentioned in §5.4 one needs to be careful, when discussing cohomology of arithmetic local systems, to get the boundary conditions right. This is certainly the case in the above conjecture: it uses not usual cohomology but a modification H_c^*. Here Kato [1993] and Fontaine and Perrin-Riou [1994] use different normalizations and we have followed Fontaine and Perrin-Riou [1994]. For detailed formulations of the following discussion see Fontaine and Perrin-Riou [ibid., §4] and also the survey article Flach [2004].

- For ρ a local system of $\mathbf{F}_p$ or $\mathbb{Q}_p$-vector spaces, we define $[H_c^*(\rho)]$ to be $[H^*(\rho)]$ modified by a boundary term:

  $$[H_c^*(\rho)] = [H^*(\rho)] \otimes [\partial\rho]^{-1}$$

  where (see Fontaine and Perrin-Riou [1994, §4.1]) $[\partial\rho]$ is the K-line defined by

  $$[\partial\rho] = \prod_{v \in V} \text{determinant of étale cohomology of } \rho \text{ restricted to Spec } \mathbb{Q}_v$$

  where V is the set of primes dividing N, together with ∞; we regard $\mathbb{Q}_\infty = \mathbb{R}$.

- If ρ is a $\mathbf{Q}$-local system, there is an analogous but more *ad hoc* definition of $[H_c^*(\rho)]$.

  First of all, we define $H^j(\mathbb{Z}[1/N], \rho)$ by means of motivic cohomology as in (9) for $j \leq 1$, and extend this definition to degrees 2 and 3 by forcing it to satisfy an analogous duality to étale cohomology.

Having done this, we again define $[H_c^*(\rho)]$ to be $[H^*(\rho)]$ modified by a boundary term:

$$[H_c^*(\rho)] = [H^*(\rho)] \otimes [\partial\rho]^{-1} \tag{15}$$

where $[\partial\rho]$ is now the $\mathbf{Q}$-vector space defined by

$$[\partial\rho] = [H_B^+] \otimes [H_{\mathrm{dR}}/F^0 H_{\mathrm{dR}}]^{-1}. \tag{16}$$

Here, since ρ corresponds to a Chow motive, one can speak of its Betti cohomology H_B and its de Rham cohomology H_{dR}, both taken with $\mathbf{Q}$-coefficients; H_B^+ is the fixed part under complex conjugation. For details see Fontaine and Perrin-Riou [1994, p. 4.4.1]; this $[\partial\rho]$ is an algebraic incarnation of the Deligne period introduced in Deligne [1979b].

With the above definitions, the two definitions given above are conjecturally compatible with one another: under certain conjectures about comparison between motivic and étale cohomology, there is a canonical isomorphism

$$[H_c^*(\rho)] \simeq [H_c^*(\rho \otimes \mathbb{Q}_p)] \tag{17}$$

if ρ is an arithmetic local system with $\mathbf{Q}$ coefficients. See again Fontaine and Perrin-Riou [1994, §3.2] for discussion of the precise conjectures under which (17) is valid.

6 Langlands' vision

Having introduced a working definition of an "arithmetic local system," i.e. a local system of K-modules on $\mathbb{Z}[1/N]$, for various classes of fields K, we are in a position to formulate the special case of Langlands' reciprocity conjectures[5] that applies to the cohomology of arithmetic groups.

We formulate the conjectures in a non-traditional way. It is also imprecise, because we do not explicate the compatibility condition with Hecke operators. For a better formulation of this conjecture, in standard language, see Clozel [2016].

6.1 The reciprocity conjecture. The basic content is that *cohomology of arithmetic groups provide arithmetic local systems.*

To be a little more specific we have to use the Hecke algebra: as in §4.1, $H^*(\Gamma, \mathbf{Z})$ admits a commutative algebra $\mathbb{T}$ of endomorphisms.

Langlands reciprocity proposes that[6], attached to this situation, there should exist an arithmetic local system of locally free $\mathbb{T}$-modules on $\mathbb{Z}[1/N]$, with rank equal to

[5] The word "reciprocity" here, deriving from quadratic reciprocity, is traditional but confusing. Namely quadratic reciprocity permits us to understand how the number of solutions to $x^2 = N$ modulo p varies with the prime p. When unwound, the conjecture below yields similar (if far more complicated) statements, with $x^2 = N$ replaced by a more complicated system of equations.

[6] Note that the usual formulation of this conjecture involves the dual group to G, but we are describing a simplified version using the adjoint representation of that dual group –see §6.2 below.

$\dim(G)$, which comes along with a very explicit formula for the monodromy around each loop Spec $\mathbf{F}_\ell \hookrightarrow$ Spec $\mathbb{Z}[1/N]$ – see (10).

Unfortunately, we cannot give a precise formulation of this version, because we have not defined local systems of $\mathbb{T}$-modules for rings as general as $\mathbb{T}$. So to make a more precise statement, we use homomorphisms $\pi : \mathbb{T} \to K$, where K is one of the fields discussed earlier. Also, to be careful, we must allow a finite extension of K.

Conjecture 2. *(Langlands' reciprocity, simplified version):*

Let K be a finite or p-adic or number field. Let $\mathbb{T}$ be the commutative subalgebra of endomorphisms of $H^*(\Gamma, \mathbf{Z})$ defined in §4.1.

For each character $\pi : \mathbb{T} \to K$, there exists an arithmetic local system ρ_π of dimension equal to $(\dim G)$, with coefficients in a finite extension of K, and such that

$$\text{monodromy trace of } \rho_\pi \text{ around } \operatorname{Spec} \mathbf{F}_\ell = \pi(T_\ell), \tag{18}$$

where $T_\ell \in \mathbb{T}$ is described by a completely explicit recipe.

This perhaps seems underwhelming or incomprehensible at first sight. The important features for us are, firstly, that there are $(\dim G)$-dimensional arithmetic local systems that can be extracted from the cohomology of Γ; and, secondly, that they control via (18) how Hecke operators act on that cohomology.

For simplicity in the rest of the text we suppose that this arithmetic local system in fact has K coefficients (as should be the case in most cases), rather than a finite extension of K. Recalling our prior discussion after (5) we can then informally think of the conjecture as yielding an association

(19)

Hecke eigenclasses in $H^*(\Gamma, K) \rightsquigarrow$ arithmetic local systems ρ of K-vector spaces.

6.2 Translation to usual terminology. As we now explain for readers familiar with the usual formulations, our "arithmetic local system" is, in effect, the adjoint Galois representation attached to π. To fix ideas take $K = \overline{\mathbb{Q}_p}$ and assume that the semisimple $\mathbf{Q}$-group $\mathbf{G}$ associated to Γ is simply connected. It has an L-group, ${}^L\widehat{G}$. Usual formulations propose that to π is attached a Galois representation

$$\rho : \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbf{Q}) \longrightarrow {}^L\widehat{G}(K).$$

Write $d = \dim(\mathbf{G}) = \dim(G)$, so that one has an adjoint representation ${}^L\widehat{G} \to \mathrm{GL}_d$; composing ρ with this we get

$$\mathrm{Ad} \circ \rho : \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbf{Q}) \longrightarrow \mathrm{GL}_d(K).$$

This is the local system proposed in Conjecture 2. Of course, we are throwing away information by only considering the adjoint of ρ, but this extra information isn't needed for our current purposes.

6.3 The role of the defect. Recall from §3 that attached to Γ there is a numerical invariant, the "defect" δ, that controls many aspects of the cohomology of Γ.

There is an enormous amount of evidence for Conjecture 2 in cases of vanishing defect. See for example Shin [2011]. But until rather recently there were very few *general statements* supporting it in the case of positive defect – although progress in the broader Langlands program nonetheless provided a great deal of evidence for it. The difficulty lies in the fact that the space Y_Γ never has the structure of complex algebraic variety when the defect is positive.

Recent work has improved the situation, particularly with torsion coefficients. Works of Harris, Lan, Taylor, and J. Thorne [2016] and Scholze [2015] have provided many examples where one can construct the association $\pi \rightsquigarrow \rho_\pi$ for K a finite or p-adic field (under some technical conditions, which we omit). The importance of this work demands more than a paragraph, but for lack of space I cannot go further into it, beyond saying that:

- the techniques involves *reducing* to a case with vanishing defect. Indeed, some of the deepest parts of Harris, Lan, Taylor, and J. Thorne [2016] and Scholze [2015] are new theorems in the case $\delta = 0$.
- the techniques do not seem, in their current form, to really give anything when K is a number field. They are essentially based on the extra flexibility of torsion coefficients.

There is a more fundamental reason, beyond technique alone, that cases with $\delta > 0$ are more complicated. The deformation theory of the arithmetic local systems that appear in Langlands' conjecture are controlled by δ. (Here, to make sense of deformation theory, we must work with finite or profinite coefficients, using the étale topology.) But when $\delta > 0$, the deformation theory is always obstructed, and to fully capture it, we need to consider deformations over differential graded rings or a similar concept.

For this reason, when we study cases of positive defect, one needs some form of derived geometry to accurately describe the situation. Because of the technical nature of this we will not discuss it here. See Galatius and Venkatesh [2018], which is partly a reformulation in more intrinsic terms of ideas in the fundamental paper Calegari and Geraghty [2018a] of Calegari and Geraghty.

7 Rational structures and motivic cohomology

Our task in this and the remaining sections is to examine how various features (on one or the other side) interact with Langlands' reciprocity, (19):

$$\text{Hecke eigenclasses in } H^*(\Gamma, K) \rightsquigarrow \text{arithmetic local systems } \rho \text{ of } K\text{-vector spaces.}$$

We have already described, in §4.2, the existence of operators on $H^*(\Gamma, K)$ that shift cohomological degree. However, these operators exist only with torsion coefficients, or (by passing to a limit) p-adic coefficients. The construction does not work with rational coefficients. It is thus reasonable to ask to:

> Construct a family of endomorphisms of $H^*(\Gamma, \mathbf{Q})$ which, upon extension of scalars to $\mathbb{Q}_p$, recovers the derived Hecke algebra of §4.2.

The answer to this seems to be quite subtle, and at present exists only at a conjectural level. It involves cohomology with coefficients in the arithmetic local system ρ; that is why we could not formulate it without first formulating the reciprocity conjecture.

7.1 Rough version. We begin with an informal version before we get to details. Recall the first, informal formulation of Langlands reciprocity from §6.1: there is an arithmetic local system of locally free $\mathbb{T}$-modules associated to $H^*(\Gamma, \mathbf{Q})$. The cohomology of $\mathbb{Z}[1/N]$ with coefficients in that local system gives a $\mathbb{T}$-module Λ. Then

> Rough conjecture: The $\mathbb{T}$-dual of Λ, i.e. $\mathrm{Hom}_{\mathbb{T}}(\Lambda, \mathbb{T})$, acts on $H^*(\Gamma, \mathbf{Q})$, compatibly with $\mathbb{T}$, and increasing cohomological degree by 1.

As before we do not really have the language to formulate this properly, so we localize everything by means of a character $\pi : \mathbb{T} \to \mathbf{Q}$. In order to make statements that are reasonably precise we shall need to impose some genericity conditions on π, and we turn to this in §7.2. The reader can probably skip this and refer back to it as needed.

7.2 Standing assumptions for the remainder of this document. In §7, 8, 9, we will work with a Hecke eigenclass

$$\alpha \in H^*(\Gamma, \mathbf{Q}), \text{ with eigenvalue given by } \pi : \mathbb{T} \to \mathbb{Q}.$$

As mentioned earlier, it is most convenient to express everything in terms of π, and we will do so, but the reader can think of everything as being indexed by α if they prefer. (We may also replace $\mathbf{Q}$ by a finite extension of $\mathbf{Q}$, but then some of the later statements must be altered.)

We impose some genericity conditions on π, referring to Prasanna and Venkatesh [2016] for details. Namely, as in Prasanna and Venkatesh [ibid., §1.1] we assume that π is *cuspidal* and *tempered*. What we mean by this is that every automorphic representation for $\mathbf{G}$ that has Hecke eigenvalues given by π is cuspidal and tempered.[7]

We will also suppose that Y_Γ is a genuine manifold, as opposed to an orbifold, to avoid some mild issues with torsion.

Let $H^*(\Gamma, \mathbf{Q})_\pi$ be the π-eigenspace on cohomology: those classes which transform, under the Hecke algebra, according to π:

$$H^*(\Gamma, \mathbf{Q})_\pi = \{\alpha' \in H^*(\Gamma, \mathbf{Q}) : T\alpha' = \pi(T)\alpha' \text{ for all } T \in \mathbb{T}.\}. \tag{20}$$

We can make the same definition with coefficients in $\mathbb{R}$ or $\mathbb{C}$. Our assumptions on π imply that this is the same as the generalized π eigenspace, and in particular is a direct summand of cohomology.

Hodge theory – that is to say, representing cohomology classes for Y_Γ by means of harmonic differential forms on Y_Γ – allows one to compute various features of this. For example, we know the range of dimensions in which $H^*(\Gamma, \mathbf{Q})_\pi$ is supported: it turns out that

$$H^j(\Gamma, \mathbf{Q})_\pi \neq 0 \iff j \in [q, \dim(Y_\Gamma) - q], \tag{21}$$

[7] This setting is really the "generic" one. It is still desirable to generalize beyond it; then one must, at least, take account of the Arthur SL_2 parameter.

where q is chosen so that $\dim(Y_\Gamma) - 2q = \delta$, with δ the defect of §3. One can also compute the dimensions of these groups: they turn out to be a constant multiple of $\binom{\delta}{(j-q)}$.

We now supply a more precise version of the discussion of §7.1, by specializing to the situation described in §7.2. Associated to $\pi : \mathbb{T} \to \mathbf{Q}$ there is, as in §6.1, a $\dim(G)$-dimensional arithmetic local system ρ_π with $\mathbf{Q}$ coefficients. We may twist ρ_π by $\mathbb{Q}(1)$ (see page 278) and then, as for any local system, we can form the cohomology with coefficients in it:

$$\Lambda := H^1 \text{ with coefficients in } \rho_\pi \otimes \mathbb{Q}(1)$$

which is a $\mathbf{Q}$-vector space. Conjecturally (see Prasanna and Venkatesh [2016] for discussion of which conjectures are involved!) $\dim_{\mathbb{Q}} \Lambda = \delta$.

The key conjecture is that the dual vector space Λ^* acts on $H^*(\Gamma, \mathbf{Q})_\pi$, shifting degrees by 1, and when we extend scalars to $\mathbb{Q}_p$ this recovers the derived Hecke action:

Conjecture 3. *(An action of a motivic cohomology group on the cohomology of an arithmetic group:) There is a (graded, free) action of $\wedge^*\Lambda^*$ on $H^*(\Gamma, \mathbf{Q})_\pi$, specified by either of the following properties:*

- *the corresponding action of $\wedge^*(\Lambda \otimes \mathbf{C})^*$ is as described in Prasanna and Venkatesh [ibid.] (explicitly described in terms of operations on differential forms).*
- *the corresponding action of $\wedge^*(\Lambda \otimes \mathbb{Q}_p)^*$ is as described in Venkatesh [2016] (explicitly described in terms of the derived Hecke algebra).*

This conjecture makes predictions that are numerically testable, and these have been tested in different ways in Harris and Venkatesh [2018], Darmon, Harris, Rotger, and Venkatesh [n.d.], and Prasanna and Venkatesh [2016].

The conjecture says that there are are extra, degree-shifting, operations on $H^*(\Gamma, \mathbf{Q})$, indexed by the dual of $H^1(\rho_\pi \otimes \mathbb{Q}(1))$. Now, according to our definitions, $H^1(\rho_\pi \otimes \mathbb{Q}(1))$ is really a certain motivic cohomology group; our statement above says that there are *extra operations on $H^*(\Gamma, \mathbf{Q})$ indexed by classes in the motivic cohomology of certain algebraic varieties.*

At present I don't know of any framework, even heuristic, into which this phenomenon should fit. One hopes that there is a way to formulate Langlands' reciprocity conjecture in such a way that this should "obviously" be true, but I don't know how to do this at the moment.

8 Heights on cohomology and heights of motives

This section is speculative. As with §7, our goal is to match up various invariants on either side of the correspondence (see §6):

$$\text{Hecke eigenclasses in } H^*(\Gamma, K) \rightsquigarrow \text{arithmetic local systems } \rho \text{ of } K\text{-vector spaces.}$$

In this section, we explain how there are natural notions of "complexity" on either side, which we will call "height" in line with arithmetic tradition, and propose that they

should match up. Although we use the same word in both cases, the nature of the definitions is quite different on either side.

Up to the exact formulation, our proposal is well-known in the case of the classical modular curve. The main point is to suggest that it *always* works. This becomes much more surprising in cases when $\delta > 0$.

As in §7.2, starting with an eigenvector for Hecke operators on $H^*(\Gamma, \mathbf{Q})$ we get a character $\pi : \mathbb{T} \to \mathbf{Q}$ of the Hecke algebra. We impose the same assumptions on this π as in §7.2. We want to assign to π two notions of height:

- An automorphic height
$$\text{height}(\pi) \in \mathbb{R}_{>0},$$
which measures, roughly speaking, how complicated it is to exhibit cycles on Y_Γ representing elements of $H^*(\Gamma, \mathbf{Q})_\pi \simeq H^*(Y_\Gamma, \mathbf{Q})_\pi$.

- An arithmetic height
$$\text{height}(\rho_\pi) \in \mathbb{R}_{>0}$$
for the associated arithmetic local system ρ_π.

  Recalling that ρ arises, in effect, from a Chow motive. Although we did not define this concept, a motive is an object constructed out of a $\mathbf{Q}$-variety V. Then height(ρ) is intended to give a measure of how large the coefficients of a defining equation for V must be. Of course, a motive is not the same as a variety, but fortunately Kato [2018] has proposed a way to assign a height to a motive, rather than just to a variety, and we will make use of his idea.

After introducing these two different notions of height, we discuss the conjectured relationship between them in §8.4.

8.1 The height of π: motivation. We will give some motivation for the definition that follows in §8.2.

In number theory, if X is a (preferably projective) variety over $\mathbb{Z}$ and x an integral point of X, the "height of x" measures the size of the coordinates of x. Of course, this requires some choices to make sense of "size of coordinates." A nice way to do it is to pick

(i) a line bundle $\mathfrak{L}$ on X

(ii) for each complex point x a norm $|\cdot|_x$ on the complex line $\mathfrak{L}_x$, varying smoothly.[8]

Given these data, we get a height function:

$$\text{height} : \text{integral points of } X \longrightarrow \mathbf{R}_{>0}$$

as follows: pulling back $\mathfrak{L}$ to an integral point gives a free, rank one, $\mathbb{Z}$-module; choose a generator s and compute $|s|^{-1}$. (This definition is hard to wrap your mind around

[8]For us, a norm on the one-dimensional complex vector space W is just a function $W \to \mathbf{R}_{\geq 0}$, not identically zero, with $|zw| = |z||w|$ ($z \in \mathbf{C}, w \in W$).

when you first see it, but it is short and has good formal properties. It can be motivated by analogy with computing the degree of a line bundle on a curve.)

Now, because the Hecke algebra $\mathbb{T}$ is finite over $\mathbb{Z}$, our homomorphism π actually has image in the integers,

$$\pi : \mathbb{T} \longrightarrow \mathbb{Z},$$

so we can think of π as an integral point of Spec $\mathbb{T}$. This scheme has only finitely many complex points, so it is not geometrically very interesting, but as a scheme over $\mathbb{Z}$ it can be quite complicated. Anyway, according to our discussion above, to define a "height" in this setting, we should specify a line bundle over Spec $\mathbb{T}$, and a hermitian metric on this line bundle.

Now $\mathbb{T}$ does come with a certain class of modules, namely, the cohomology $H^*(\Gamma, \mathbf{Z})$. Let us suppose that, for some r, $H^r(\Gamma, \mathbf{Z})$ is actually *free* over $\mathbb{T}$; thus it gives rise to a vector bundle over Spec $\mathbb{T}$. We can always make a line bundle out of a vector bundle (take its determinant). Moreover, identifying cohomology classes for Γ with harmonic differential forms on Y_Γ gives rise to a norm on this line bundle. This extra data gives rise to a height function on Spec $\mathbb{T}$, as desired.

In practice, it is hard to check the freeness result mentioned above, but one can just write down a formula inspired by this reasoning, and it makes sense even without freeness. This is what we do next.

8.2 Height of π: definition. As in §7.2 let q be the minimum degree for which $H^q(\Gamma, \mathbf{Q})_\pi \neq 0$. The identification

$$H^q(\Gamma, \mathbf{R})_\pi \simeq \text{harmonic } q\text{-forms on } Y_\Gamma, \text{ transforming according to } \pi \tag{22}$$

together with the Riemannian structure on Y_Γ, endows $H^q(\Gamma, \mathbf{R})_\pi$ with the structure of a finite-dimensional inner product space (since we can take inner product of two harmonic forms, using the Riemannian metric).

Put

$$H^q(\Gamma, \mathbf{Z})_\pi := \text{integral classes inside } H^q(\Gamma, \mathbf{R})_\pi.$$

i.e. the intersection of $H^q(\Gamma, \mathbf{R})_\pi$ with the image of $H^q(\Gamma, \mathbf{Z}) \to H^q(\Gamma, \mathbf{R})$, or, said differently, classes in $H^q(\Gamma, \mathbf{R})_\pi$ all of whose periods over cycles are integral. Now define height(π) in such a way that

$$\begin{aligned} \text{height}(\pi)^{h_q} &= \text{volume}\left(H^q(Y, \mathbf{Z})_\pi\right)^2, \\ &= (\det\langle e_i, e_j\rangle) \end{aligned}$$

where $e_1, \ldots, e_{h_q}$ is a basis for the free $\mathbb{Z}$ module $H^q(\Gamma, \mathbf{Z})_\pi$. For example:

- When $h_q = 1$, we can choose a harmonic differential q-form ω generating the one-dimensional real vector space $H^q(\Gamma, \mathbf{R})_\pi$. It has a lattice of periods: the collection of integrals of ω over q-cycles is of the form $\mathbb{Z}.m$ for some $m \in \mathbb{R}$. Scale ω so that $m = \pm 1$. Then the height of π is given by $\langle \omega, \omega \rangle$.

- In Bergeron, Şengün, and Venkatesh [2016] it is shown that, when Y_Γ is a 3-manifold, height(π) is related to the minimal Thurston norm of an element of $H_2(Y_\Gamma, \mathbb{Q})$ that transforms according to π. In that case, then, this can truly be seen as a measure of the minimum geometric complexity of a cycle whose eigenvalue is given by π.

8.3 Kato's height. We are now going to define the arithmetic height of an arithmetic local system ρ of $\mathbf{Q}$-vector spaces, or, rather, we are going to summarize Kato's definition. See Kato [2018, 2017] and Koshikawa [2015]. Any errors here are due to to me, of course.

Stepping back, one would like to define the *height* of an algebraic variety V over $\mathbf{Q}$ as a measure of the amount of information needed to describe V. One way of doing this is to measure the largest coefficient in a system of defining equations. Since V can be presented in many ways, one takes the minimum over all such presentations. This is however a rather inconvenient definition. Faltings found (see Faltings [1983]) a much more elegant definition for abelian varieties, and Kato found a generalization to all motives. We briefly sketch some of the key ideas:

Think of V, for a moment, as a point on some moduli space $\mathcal{V}$ of varieties, which we suppose is a projective $\mathbf{Q}$-variety. According to the discussion of §8.1, to attach a height to V, we need a line bundle $\mathcal{L}$ on this moduli space, normed at each complex point, and an extension of $(\mathcal{V}, \mathcal{L})$ from $\mathbf{Q}$ to $\mathbb{Z}$. Faltings observed that, in the case of abelian varieties of dimension g,

- we can take $\mathcal{L}$ to be the line bundle that attaches to an abelian variety A the line $H^0(A, \Omega^g)$ of algebraic volume forms, the squared norm being given by integrating $\Omega \wedge \overline{\Omega}$ over $A(\mathbf{C})$, for $\Omega \in H^0(A, \Omega^g)$, and

- One can get by without extending $(\mathcal{V}, \mathcal{L})$ to $\mathbb{Z}$. Rather, it is enough to specify an integral structure on $H^0(A, \Omega^g)$ for each abelian variety over $\mathbf{Q}$, and this can be done using the miracle of the Néron model, which provides for each such abelian variety a "best" model over $\mathbb{Z}$.

Said slightly differently, up to constants,

$$\text{Faltings height of } A = \left|\langle \Omega, \overline{\Omega} \rangle\right|^{-1/2} \tag{23}$$

where $\Omega \in H^0(A, \Omega^g) \subset H^g(A, \mathbf{C})$ is normalized up to sign using the Néron model, and the pairing is the duality pairing on $H^g(A, \mathbf{C})$.

In this way, the resulting definition makes no mention of the moduli space $\mathcal{V}$; admittedly, to prove that the resulting height has reasonable properties requires some analysis on $\mathcal{V}$ (see Faltings [ibid.]).

Kato adapts this definition to a general motive; for simplicity let us talk only about general (projective, smooth) $\mathbf{Q}$-varieties V. In the discussion above, we replace $H^0(A, \Omega^g)$ by Griffiths' "canonical line" constructed out of various pieces of the Hodge filtration on $H^*(V)$ (cf. Griffiths [1970, (7.13)]), and one uses elementary properties of Hodge structures to make a norm on it. Now there is no "best" model for V over $\mathbb{Z}$;

but by using integral p-adic Hodge theory one can still make a good candidate for an integral structure on the Hodge filtration, and this method is so flexible that it works for motives too.

Having given this rough motivation, we now explain the version of it that is relevant to our situation.

Recall that the "arithmetic local system" ρ_π is, by definition, a Chow motive M over $\mathbf{Q}$ (with some extra properties). In fact, although we did not specify it, this Chow motive is actually self-dual, and we suppose it comes equipped with a preferred pairing (for example, associated to the Killing form, see Prasanna and Venkatesh [2016, §4] for discussion). It has a de Rham realization $H_{\mathrm{dR}}(M)$, which is a filtered $\mathbf{Q}$-vector space by means of the Hodge filtration, and inherits a $\mathbf{Q}$-valued pairing. Let $F^1 H_{\mathrm{dR}}(M)$ be the degree 1 step of this filtration. Then $F^1 H_{\mathrm{dR}}(M)$ is equipped with a Hermitian form, arising from pairing ω and $\overline{\omega}$. Correspondingly, the determinant of $F^1 H_{\mathrm{dR}}(M)$ also inherits a norm, given by

$$\|\omega_1 \wedge \cdots \wedge \omega_k\|^2 = \left|\det\langle \omega_i, \overline{\omega_j}\rangle\right|.$$

where $\langle -, -\rangle$ denotes the duality on $H_{\mathrm{dR}}(M)$. (This is related to Kato's $H_\clubsuit$, in the case when M is of the form $N \otimes N^*$.)

Now suppose that the $\mathbf{Q}$-vector space $H_{\mathrm{dR}}(M)$ comes with an integral lattice. Kato and Koshikawa describe a specific way to produce such a lattice, which we will mention below; call a lattice that arises by their construction *admissible.* Writing $\omega_1, \dots, \omega_k$ for a basis for the integral lattice induced on $F^1 H_{\mathrm{dR}}(M)$, we may now define

$$\begin{aligned} \text{arithmetic height of } \pi &= \|\omega_1 \wedge \cdots \wedge \omega_k\|^{-1} \\ &= \left|\det\langle \omega_i, \overline{\omega_j}\rangle\right|^{-1/2.} \end{aligned}$$

compare with (23)!

This is not a real definition, because it depends on the choice of an admissible integral lattice. However, we will permit ourselves to use it with this ambiguity, bearing in mind:

(i) Kato and Koshikawa describe (in slightly different ways) how to construct the integral lattice in $H_{\mathrm{dR}}(M)$ starting from a Galois-invariant lattice in the Betti cohomology $H_B(M)$ and using p-adic Hodge theory. An integral lattice in $H_{\mathrm{dR}}(M)$ thus constructed will be called *admissible*.

While there can be infinitely many possible Galois-stable lattices in $H_B(M)$, and correspondingly infinitely many possible admissible integral lattices in $H_{\mathrm{dR}}(M)$, the height does not change too much. In particular as one changes the integral lattice, only finitely many possibilities for the height occur, at least in Koshikawa's definition: Koshikawa [2015, Theorem 9.8].

(ii) For many primes p, in our situation, there is a unique admissible lattice at p (at least up to rescaling, which doesn't affect height). Namely, if π is not Eisenstein at p in the rather strong sense that the p-adic local system $\rho_\pi \otimes \mathbb{Q}_p$ remains irreducible modulo p, then there will be a unique admissible lattice at p, up to rescaling.

8.4 Automorphic versus arithmetic height. I would (tentatively) conjecture that there is an identity

$$\frac{\text{automorphic height}}{\text{arithmetic height}} \stackrel{?}{=} \text{fudge factor} \cdot L(\rho_\pi \otimes \mathbb{Q}(1)). \tag{24}$$

valid for all Γ, π.

In more conventional notation, $L(\rho_\pi \otimes \mathbb{Q}(1))$ is the value of the adjoint L-function at the edge of its critical strip. From the analytic point of view L-functions at the edge of their critical strip behave in a very mild fashion – not too big, not too small – and so we should think of (24) as saying that the sizes of automorphic and arithmetic heights are comparable.

Recall that, in our current state of affairs, the arithmetic height requires a choice of admissible lattice – see discussion in §8.3 – so we interpret (24) as asserting that there is a choice of admissible lattice for which the above equality is valid.

I apologize for the existence of the "fudge factor" – I have not yet worked out enough examples in the detail needed to make a precise formulation, and without some limitations on this fudge factor, the equation (24) is contentless. But it seems reasonable to expect that this factor is a product of purely local contributions:

$$c_\infty \prod_{p|N} c_p \prod_{p \text{ Eis}} c_p :$$

- c_∞ is a constant depending only on the ambient group G and the choice of invariant metric on S;
- c_p depends only on the local behavior of Γ and π at primes p dividing the level of Γ.
- The final factor tracks contributions arising from "Eisenstein" primes p. These are primes at which there are multiple choices for admissible lattice up to scaling. Optimistically, this term could be eliminated by making the correct choice of admissible lattice.

8.5 Discussion. Despite the common use of the word height, such a relationship

$$\text{heights of automorphic forms } \pi \longleftrightarrow \text{height of local systems (or motives) } \rho_\pi$$

seems to me surprising. To see why, let us interpret both sides as the height of an integral point on a variety, in the sense of §8.1.

- π defines a $\mathbb{Z}$-point of the spectrum of $\mathbb{T}$, and the left hand side measures the height of this point.
- If ρ_π arises from an algebraic variety V over $\mathbf{Q}$, we should think of the right-hand side as measuring the height of V considered as a point of some moduli space $\mathcal{V}$ of algebraic varieties.

Thus, both sides have to do with heights on some algebraic varieties – but the two varieties have nothing to do with one another! $\mathcal{V}$ is often some positive-dimensional moduli space; Spec $\mathbb{T}$ is (over $\mathbb{C}$) zero-dimensional. Moreover, Spec $\mathbb{T}$ is morally speaking determined by just the "topological" structure of Spec $\mathbb{Z}$, whereas $\mathcal{V}$ really uses its algebraic structure.

Also, different aspects are clear on the two sides. On the automorphic side, it is easy – given Γ – to give a (terrible) *a priori* upper bound on the height of π, whereas on the arithmetic side this is a very subtle question related to effective bounds for Diophantine equations. In the reverse direction, if we start from a given variety V (explicitly presented in terms of some polynomial equations) it is clear that the height is bounded by a polynomial in the coefficients of those equations, but that bound is not at all clear on the automorphic side! See Frey [1989] and Murty and Pasten [2013] for examples where this was exploited.

A few words about the origin of the conjecture: The motivation to study height(π) arose from studying the growth of torsion cohomology, i.e. Conjecture 1. It turns out that Conjecture 1 is intimately connected to getting good upper bounds on height(π), see my paper Bergeron, Şengün, and Venkatesh [2016] with N. Bergeron and H. Sengun. This paper along with Calegari and Venkatesh [2012] gave some evidence that the conjecture might be true for hyperbolic 3-manifolds. That it might be true more generally was suggested by a striking coincidence between some of the expressions that appear in my work Prasanna and Venkatesh [2016] with Kartik Prasanna, and some of the formulas in Kato's paper Kato [1993].

9 Special cycles and L-functions

This section is speculative. We continue to examine how various features that appear on one side of

$$\text{Hecke eigenclasses in } H^*(\Gamma, K) \rightsquigarrow \text{arithmetic local systems } \rho \text{ of } K\text{-vector spaces.}$$

(see §6) manifest themselves on the other side.

Recall that we have discussed, in §5.6, how – by analogy with the case of manifolds – we might like to produce trivializations of cohomology determinants for arithmetic local systems. In this section we explain how we can use special cycles on Y_Γ to produce such trivializations – at least for certain very special *virtual* arithmetic local systems.

The ideas here are in some sense a reworking of computations contained in my joint work Prasanna and Venkatesh [2016] with Kartik Prasanna. The main point of carrying out this reworking is that everything is set up in a way that it could also work with torsion coefficients (although we do not examine this here).

9.1 Cohomology determinants. We continue with the setting of §7.2, and the discussion following it.

Recall that we have defined a $\mathbf{Q}$-vector space Λ there:

$$\Lambda = H^1(\rho_\pi \otimes \mathbb{Q}(1)).$$

Let q and q^* be (respectively) the minimum and maximum dimensions where $H^*(\Gamma, \mathbf{Q})_\pi$ is nonzero. Conjecture 3 implies that the highest exterior power $[\Lambda^*]$ of the $\mathbf{Q}$-dual Λ^* indexes an isomorphism between them. In symbols, $H^q(\Gamma, \mathbf{Q})_\pi \otimes [\Lambda^*] \cong H^{q^*}(\Gamma, \mathbf{Q})_\pi$ and now, taking determinants, we get

$$[H^q(\Gamma, \mathbf{Q})_\pi] \cdot [\Lambda^*]^{h_q} \simeq [H^{q^*}(\Gamma, \mathbf{Q})_\pi],$$

where $h^q = \dim H^q(\Gamma, \mathbf{Q})_\pi$. Taking account of Poincaré duality (our assumptions in §7.2 mean that the π-eigenspace is self-dual) we arrive at an isomorphism:

$$[\Lambda]^{h_q} \simeq [H^q(\Gamma, \mathbf{Q})_\pi]^2. \tag{25}$$

In the situations we are in, the cohomology with coefficients in the arithmetic local system $\rho_\pi \otimes \mathbb{Q}(1)$ vanishes in degrees other than 1. Therefore $[\Lambda]$ is identified with $[H^*(\rho_\pi \otimes \mathbb{Q}(1))]^{-1}$ (the negation because the contribution comes from cohomological degree 1). Thus (25) can be reformulated as an identification

$$[H^*(\rho_\pi \otimes \mathbb{Q}(1))]^{-h_q} \simeq [H^q(\Gamma, \mathbf{Q})_\pi]^2. \tag{26}$$

To emphasize, (26) says:

> the determinant of $H^q(\Gamma, \mathbf{Q})_\pi$ can be (up to a certain power) canonically identified with the determinant of cohomology of an arithmetic local system.

This conclusion is just a restatement of a small piece of Conjecture 3, and is pretty much useless taken in isolation (although it is very much related to (24), relating arithmetic and automorphic heights).

The next step is to use certain distinguished classes in $H^q(\Gamma, \mathbf{Q})$ to produce classes in the cohomology determinant which are meaningful from the point of view of L-functions (see §5.6).

We can only do this in some very special cases. The details are a real mess, so we give only a brief sketch below.

9.2 Special cycles. The spaces Y_Γ often admit natural submanifolds: an inclusion of arithmetic groups, such as $\mathrm{SL}_{n-1}(\mathbf{Z}) \to \mathrm{SL}_n(\mathbf{Z})$, induces an inclusion $Z \hookrightarrow Y$ of associated arithmetic manifolds. These "special cycles" play an important role in the theory of automorphic forms and are related to L-functions in a way sketched below.

In Prasanna and Venkatesh [ibid., §6.2] we examine four cases that fit into this paradigm. Assume for simplicity that we are in a case with $h_q = 1$ (in the other cases the discussion is slightly more complicated, see Prasanna and Venkatesh [ibid.]). In the cases examined in Prasanna and Venkatesh [ibid.], Y_Γ admits a sub-arithmetic manifold Z of dimension q (as in (21)), and one can either prove, or expects Gan, Gross, and Prasad [2012] the validity of, a result of the following nature:

$$\langle [Z]_\pi, [Z]_\pi \rangle \sim \frac{L(\rho')}{L(\rho_\pi \otimes \mathbb{Q}(1))}. \tag{27}$$

Here:

- $[Z]_\pi$ is the projection of the fundamental class of Z to $H^*(Y, \mathbf{Q})_\pi$;
- $\langle -, - \rangle$ denotes the inner product on cohomology induced by an identification with harmonic forms, as in (22);
- ρ' is another arithmetic local system, functorially associated with ρ_π (in terms of the discussion of §6.2, it is obtained from yet another representation of ${}^L\hat{G}$).
- L denotes L-function, as in (14).

The notation $\sim$ means that the equality holds up to certain factors of π and algebraic numbers, at least for suitable choice of metric.

We will now indicate how to use the cycle Z to construct a

distinguished basepoint for the $\mathbf{Q}$-line $\frac{\det H_c^*(\rho')}{\det H_c^*(\rho_\pi \otimes \mathbb{Q}(1))}$,

in the case when $L(\rho') \neq 0$. This might be seen as an algebraic incarnation of formula (27). As we have mentioned, the work Fontaine and Perrin-Riou [1994] and Kato [1993] proposes that there should be such a basepoint for $[H_c^*(\rho')]$ and $[H_c^*(\rho_\pi \otimes \mathbb{Q}(1))]$ individually, and presumably the basepoint we construct should be closely related to the ratio of these basepoints.

Recall that $[H_c^*]$ is not the determinant of cohomology, but rather a modification of it (see §5.6 especially (15)) by a "boundary" term that we have called $[\partial\rho]$. The key point is that there is an identification

$$[\partial(\rho_\pi \otimes \mathbb{Q}(1))] \sim [\partial\rho'] \tag{28}$$

(the $\sim$ means that there is an identification up to twisting by factors related to cyclotomic motives, which we will ignore here). The equality (28) is a puzzling piece of numerology, which is verified implicitly by the computations in Prasanna and Venkatesh [2016].

Accordingly,

$$\frac{[H_c^*(\rho')]}{[H_c^*(\rho_\pi \otimes \mathbb{Q}(1)]} = \frac{[H^*(\rho')]}{[H^*(\rho_\pi \otimes \mathbb{Q}(1))]} \left(\frac{[\partial\rho']}{[\partial(\rho_\pi \otimes \mathbb{Q}(1))]} \right)^{-1}.$$

where $=$ really means there is a canonical identification. But we can now perceive a distinguished element on the right-hand side, up to sign and possibly after extending scalar from $\mathbf{Q}$ to a quadratic extension:

- Using the identification (26), in the case $h_q = 1$, the fundamental class $[Z]_\pi$ give rise to an element $z_\pi \in [H^*(\rho_\pi \otimes \mathbb{Q}(1))]^{-2}$.

  We may take the square root of this element to get a class (defined up to sign) in $[H^*(\rho_\pi \otimes \mathbb{Q}(1))] \otimes_{\mathbb{Q}} E$ for some quadratic extension $E/\mathbf{Q}$, well defined up to sign. (Hopefully we can take $E = \mathbf{Q}$).
- The second fraction on the right is trivialized using (28).
- Conjecturally, the fact that $Z_\pi \neq 0$ implies that $H^*(\rho')$ is vanishing. Thus there is an identification $[H^*(\rho')] \simeq \mathbf{Q}$.

In this way (up to possibly extending scalars to a quadratic field) we have constructed[9] a class in the left-hand side, i.e. a class in

determinant of cohomology $[H_c^*]$ for the *virtual* representation
$\rho' - \rho_\pi \otimes \mathbb{Q}(1)$.

I believe this construction is interesting since it gives some (slightly) more explicit understanding, in some special cases, of the trivializations of cohomology determinants predicted by Fontaine and Perrin-Riou [1994] and Kato [1993]. To emphasize, the key non-formal point in the construction is that the "boundary term" from (16) *vanishes* for this virtual representation – at least up to simple factors related to Tate twists.

To me, the most interesting direction is this: in the whole discussion above we could replace $\mathbf{Q}$ coefficients by $\mathbf{F}_q$ coefficients. The case of $\mathbf{F}_q$ coefficients is particularly interesting because it cannot be accessed through the theory of L-functions.

10 Conclusion

We have been discussing

Hecke eigenclasses in $H^*(\Gamma, K) \rightsquigarrow$ arithmetic local systems ρ of K-vector spaces.

It important to note that the arrow is not surjective. When $K = \mathbf{Q}$, say, one sees only a small fraction of all the possible arithmetic local systems this way. Said more plainly, only very particular algebraic varieties are related to the cohomology of arithmetic groups.

As a result, while the conjectures that we have presented are important in the context of the above correspondence, they are of relatively small significance in the greater scope of arithmetic algebraic geometry. For example, nothing in this document has any relevance whatsoever to a smooth projective curve C over $\mathbf{Q}$ of genus 3.

The Langlands program, however, is larger than the study of $H^*(\Gamma, K)$ and it does not forget about C. It relates C to an eigenfunction of the Hecke operators, but no longer in cohomology – rather, to an eigenfunction inside the space of all L^2-functions on a suitable Y_Γ. This eigenfunction and its ambient L^2-space do not possess the rich algebraic structures of cohomology.

Although it seems far-fetched at the moment, any extension of our discussion to this more transcendental setting would be of great importance.

In the unlikely event of a reader having gotten this far, I thank them for their patience!

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[9] This construction as stated is conditional on various conjectures, but it should be possible to make unconditional shadows of it.

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CALCULUS, HEAT FLOW AND CURVATURE-DIMENSION BOUNDS IN METRIC MEASURE SPACES

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Abstract

The theory of curvature-dimension bounds for nonsmooth spaces has several motivations: the study of functional and geometric inequalities in structures which are very far from being Euclidean, therefore with new non-Riemannian tools, the description of the "closure" of classes of Riemannian manifolds under suitable geometric constraints, the stability of analytic and geometric properties of spaces (e.g. to prove rigidity results). Even though these goals may occasionally be in conflict, in the last few years we have seen spectacular developments in all these directions, and my text is meant both as a survey and as an introduction to this quickly developing research field.

1 Introduction

I will mostly focus on *metric measure spaces* (m.m.s. in brief), namely triples $(X, \mathsf{d}, \mathfrak{m})$, where (X, d) is a complete and separable metric space and $\mathfrak{m}$ is a non-negative Borel measure, finite on bounded sets, typically with $\operatorname{supp} \mathfrak{m} = X$. The model case that should always be kept in mind is a *weighted* Riemannian manifold $(\mathbb{M}, \mathrm{g}, \mathfrak{m})$, with $\mathfrak{m}$ given by

$$\mathfrak{m} := e^{-V} \mathrm{vol}_{\mathrm{g}} \tag{1-1}$$

for a suitable weight function $V : \mathbb{M} \to \mathbb{R}$. It can be viewed as a metric measure space by taking as $\mathsf{d} = \mathsf{d}_{\mathrm{g}}$ the Riemannian distance induced by g.

In order to achieve the goals I mentioned before, it is often necessary to extend many basic calculus tools from smooth to nonsmooth structures. Because of this I have organized the text by starting with a presentation of these tools: even though some new developments of calculus in m.m.s. have been motivated by the theory of curvature-dimension bounds, the validity of many basic results does not depend on curvature and it is surely of more general interest. In this regard, particularly relevant are results which provide a bridge between the so-called "Eulerian" point of view (when dealing with gradients, Laplacians, Hessians, etc.) and the so-called "Lagrangian" point of view (when dealing with curves in the ambient space). In the theory of curvature-dimension bounds,

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these bridges are crucial to connect the Lott–Villani and Sturm theory, based on Optimal Transport (therefore Lagrangian) to the Bakry-Émery theory, based on Γ-calculus (therefore Eulerian), in many cases of interest.

The limitation on the length of this text forced me to make difficult and subjective choices, concerning both references and topics; for this reason and not for their lack of importance I will not mention closely related areas of investigation, such as the many variants and regularizations of optimal transport distances, curvature-dimension bounds in sub-Riemannian structures, rigidity results, time-dependent metric measure structures, and others.

2 Calculus tools in metric spaces

Let us start with some basic tools and terminology, at the metric level. Recall that a curve $\gamma : [0,T] \to X$ is said to be *absolutely continuous* if there exists $g \in L^1(0,T)$ satisfying

$$\mathsf{d}(\gamma_s, \gamma_t) \le \int_s^t g(r)\, dr \qquad \forall 0 \le s \le t \le T.$$

Among absolutely continuous curves, Lipschitz curves play a special role. Among them, we shall denote by $\mathrm{Geo}(X)$ the class of *constant speed geodesics* $\gamma : [0,1] \to X$, characterized by

$$\mathsf{d}(\gamma_s, \gamma_t) = |s-t|\mathsf{d}(\gamma_1, \gamma_0) \qquad \forall s,\, t \in [0,1].$$

A metric space (X, d) is said to be *geodesic* if any pair of points can be connected by at least one $\gamma \in \mathrm{Geo}(X)$.

In this survey, K-*convex* functions, with $K \in \mathbb{R}$, play an important role. In the smooth setting, K-convexity corresponds to the lower bound $\mathrm{Hess}\, f \ge K\,\mathrm{Id}$ on the Hessian of f, but the definition is immediately adapted to the metric setting, by requiring that f is K-convex (i.e for all $x,\ y \in X$ there exists $\gamma \in \mathrm{Geo}(X)$ such that $\gamma_0 = x$, $\gamma_1 = y$ and $t \mapsto f(\gamma_t) - \frac{1}{2}Kt^2\mathsf{d}^2(\gamma_0, \gamma_1)$ is convex in $[0,1]$).

Definition 2.1 (Metric derivative)**.** Let $\gamma : [0,T] \to X$ be absolutely continuous. Then, it can be proved that for $\mathcal{L}^1$-a.e. $t \in (0,T)$ the limit

$$|\gamma'|(t) := \lim_{h\to 0} \frac{\mathsf{d}(\gamma_{t+h}, \gamma_t)}{|h|}$$

exists. We call this limit metric derivative: it is indeed the minimal function $g \in L^1(0,T)$, up to $\mathcal{L}^1$-negligible sets, such that the inequality $\mathsf{d}(\gamma_s, \gamma_t) \le \int_s^t g(r)\, dr$ holds for all $0 \le s \le t \le T$.

Building on this definition, one can define the space of curves $\mathrm{AC}^p([0,T]; X)$, $1 \le p \le \infty$, by requiring p-integrability of the metric derivative. Also, as in the smooth setting, the metric derivative provides an integral representation to the curvilinear integrals

$$\int_\gamma g\, d\sigma := \int_0^T g(\gamma_s)|\gamma'|(s)\, ds = \int g\, dJ\gamma \quad \text{with } J\gamma := \gamma_\#(|\gamma'|\mathcal{L}^1) \tag{2-1}$$

which otherwise should be defined using integration on $\gamma([0,T])$ with respect to the 1-dimensional Hausdorff measure $\mathcal{H}^1$ (counting multiplicities if γ is not 1-1). In turn, the inequality

$$|f(\gamma_1) - f(\gamma_0)| \le \int_\gamma g \, d\sigma, \tag{2-2}$$

valid with $g = |\nabla f|$ in a smooth setting, leads to the notion of *upper gradient* Koskela and MacManus [1998].

Definition 2.2 (Upper gradient)**.** We say that a Borel function $g : X \to [0,\infty]$ is an upper gradient of $f : X \to \mathbb{R}$ if the inequality (2-2) holds for any $\gamma \in \mathrm{AC}([0,1]; X)$.

Clearly the upper gradient should be thought of as an upper bound for the modulus of the gradient of f[1]. Without appealing to curves, the "dual" notion of *slope* (also called local Lipschitz constant) simply deals with difference quotients:

Definition 2.3 (Slope)**.** For $f : X \to \mathbb{R}$ the slope $|\nabla f|(x)$ of f at a non-isolated point $x \in X$ is defined by

$$|\nabla f|(x) := \limsup_{y \to x} \frac{|f(y) - f(x)|}{\mathsf{d}(y,x)}.$$

It is simple to check that the slope is an upper gradient for Lipschitz functions. In the theory of metric gradient flows a key role is also played by the *descending* slope, a one-sided counterpart of the slope:

$$|\nabla^- f|(x) := \limsup_{y \to x} \frac{\max\{f(x) - f(y), 0\}}{\mathsf{d}(x,y)}. \tag{2-3}$$

The notion of gradient flow, closely linked to the theory of semigroups, also plays an important role. If we are given a K-convex and lower semicontinuous function $F : X \to (-\infty,\infty]$, with X Hilbert space, the theory of evolution problems for maximal monotone operators (see for instance Brézis [1973]) provides for any $\bar{x} \in \overline{\{F < \infty\}}$ a locally absolutely continuous map $x_t : (0,\infty) \to X$ satisfying

$$\frac{d}{dt} x_t \in -\partial_K F(x_t) \ \text{ for } \mathcal{L}^1\text{-a.e. } t \in (0,\infty), \qquad \lim_{t \to 0} x_t = \bar{x}, \tag{2-4}$$

where $\partial_K F$ stands for the K-subdifferential of F, namely

$$\partial_K F(x) := \left\{ \xi \in X : \ F(y) \ge F(x) + \langle \xi, y - x \rangle + \frac{K}{2}|y - x|^2 \ \ \forall y \in X \right\}.$$

Besides uniqueness, a remarkable property of the gradient flow x_t is a selection principle, which turns the differential inclusion into an equation: for $\mathcal{L}^1$-a.e. $t \in (0,\infty)$ one

[1] Strictly speaking it should be the modulus of the differential, the natural object in duality with curves, but the "gradient" terminology is by now too established to be changed. However, as emphasized in Gigli [2015, Sec 3], this distinction is crucial for the development of a good theory.

has that $-\frac{d}{dt}x_t$ is the element with minimal norm in $\partial_K F(x_t)$. Moreover, differentiating the square of the Hilbert norm one can write (2-4) in an equivalent form, called *Evolution Variational Inequality* (in short EVI_K)

(2-5)
$$\frac{d}{dt}\frac{1}{2}|x_t - y|^2 \le F(y) - F(x_t) - \frac{K}{2}|y - x_t|^2 \qquad \mathcal{L}^1\text{-a.e. in } (0,\infty), \text{ for all } y \in X.$$

This way, the scalar product does not appear anymore and this formulation, involving energy and distance only, makes sense even in metric spaces.

We conclude this section by recalling the *metric* notion of gradient flow, based on a deep intuition of E. De Giorgi, see Ambrosio, Gigli, and Savaré [2008] for much more on this subject. Assume for the moment that we are in a smooth setting (say F of class C^1 on a Hilbert space X). Then, we can encode the *system* of ODE's $\gamma' = -\nabla F(\gamma)$ into a *single* differential inequality: $-2(F \circ \gamma)' \ge |\gamma'|^2 + |\nabla F|^2(\gamma)$. Indeed, for any $\gamma \in C^1$ one has

$$-2(F \circ \gamma)' = -2\langle \nabla F(\gamma), \gamma' \rangle \le 2|\nabla F|(\gamma)|\gamma'| \le |\gamma'|^2 + |\nabla F|^2(\gamma).$$

Now, the first inequality is an equality iff $\nabla F(\gamma)$ is parallel to $-\gamma'$, while the second one is an equality iff $|\nabla F|(\gamma) = |\gamma'|$, so that by requiring the validity of the converse inequalities we are encoding the full ODE. In the metric setting, using metric derivatives and the descending slope (2-3) and moving to an integral formulation of the differential inequality, this leads to the following definition:

Definition 2.4 (Metric gradient flow)**.** Let $F : X \to (-\infty, \infty]$ and $\bar{x} \in \{F < \infty\}$. We say that a locally absolutely continuous curve $\gamma : [0,\infty) \to X$ is a metric gradient flow of F starting from $\bar{x}$ if

$$F(\gamma_t) + \int_0^t \frac{1}{2}|\gamma'|^2(r) + \frac{1}{2}|\nabla^- F|^2(\gamma_r)\, dr \le F(\bar{x}) \qquad \forall t \ge 0. \tag{2-6}$$

Under the assumption that $|\nabla^- F|$ is an upper gradient of F (this happens for Lipschitz functions or, in geodesic spaces, for K-convex functions) one obtains that equality holds in (2-6), that $t \mapsto F(\gamma_t)$ is absolutely continuous in $[0,\infty)$, and that $|\gamma'| = |\nabla^- F|(\gamma)$ $\mathcal{L}^1$-a.e. in $(0,\infty)$. Reasoning along these lines one can prove that, for K-convex and lower semicontinuous functions in Hilbert spaces, the metric and differential notions of gradient flow coincide. However, in general metric spaces the existence of an EVI_K-flow is a much stronger requirement than the simple energy-dissipation identity (2-6): it encodes not only the K-convexity of Φ (this has been rigorously proved in Daneri and Savaré [2008]) but also, heuristically, some infinitesimally Hilbertian behaviour of d.

3 Three basic equivalence results

Curvature conditions deal with second-order derivatives, even though often - as happens for convexity - their synthetic formulation at least initially involves difference quotients or first-order derivatives. Before coming to the discussion of synthetic curvature conditions, in this section I wish to describe three basic equivalence results at the level of

"first order differential calculus" (weakly differentiable functions, flow of vector fields, metric versus energy structures), which illustrate well the Eulerian–Lagrangian duality I mentioned in the introduction.

3.1 Cheeger energy and weakly differentiable functions. The theory of weakly differentiable functions, before reaching its modern form developed along different paths, with seminal contributions by B. Levi, J. Leray, L. Tonelli, C. B. Morrey, G. C. Evans, S. L. Sobolev (see Naumann [2010] for a nice historical account). In Euclidean spaces, we now recognize that three approaches are essentially equivalent: approximation by smooth functions, distributional derivatives and of good behaviour along almost all lines. More surprisingly, this equivalence persists even in general metric measure structures. In what follows, I will restrict my discussion to the case of p-integrable derivatives with $1 < p < \infty$; in the limiting case $p = 1$ the results are weaker, while for BV functions the full equivalence still persists, see Ambrosio and Di Marino [2014], Marino [2014], and Ambrosio, Pinamonti, and Speight [2015].

To illustrate this equivalence, let me start from the approximation with smooth functions, now replaced by Lipschitz functions in the m.m.s. category. The following definition is inspired by Cheeger's Cheeger [1999], who dealt with a larger class of approximating functions (the functions with p-integrable upper gradient), see also Cheeger and Colding [1997, Appendix 2].

Definition 3.1 ($H^{1,p}$ Sobolev space)**.** We say that $f \in L^p(X, \mathfrak{m})$ belongs to $H^{1,p}(X, \mathsf{d}, \mathfrak{m})$ if there exist a sequence $(f_i) \subset \mathrm{Lip}_b(X, \mathsf{d})$ with $f_i \to f$ in L^p and $\sup_i \||\nabla f_i|\|_p < \infty$.

This definition is also closely related to the so-called *Cheeger energy* $\mathrm{Ch}_p : L^p(X, \mathfrak{m}) \to [0, \infty]$, namely

(3-1)
$$\mathrm{Ch}_p(f) := \inf\left\{\liminf_{i\to\infty} \int_X |\nabla f_i|^p \, d\mathfrak{m} : \ f_i \to f \text{ in } L^p(X, \mathfrak{m}),\ f_i \in \mathrm{Lip}_b(X, \mathsf{d})\right\},$$

which turns out to be a convex and $L^p(X, \mathfrak{m})$-lower semicontinuous functional, whose finiteness domain coincides with $H^{1,p}$ and is dense in L^p. Then, by looking for the optimal approximation in (3-1), J. Cheeger identified a distinguished object, the *minimal relaxed slope*, denoted $|\nabla f|_*$: it provides the integral representation

$$\mathrm{Ch}_p(f) = \int_X |\nabla f|_*^p \, d\mathfrak{m} \qquad \forall f \in H^{1,p}(X, \mathsf{d}, \mathfrak{m})$$

which corresponds, in the smooth setting and for $p = 2$, to the *weighted Dirichlet energy* $\int_M |\nabla f|^2 e^{-V} \, d\mathrm{vol}_g$.

Even at this high level of generality one can then estabilish basic calculus rules, such as the chain rule. In addition, $f \mapsto |\nabla f|_*$ has strong locality properties, which pave the way to connections with the theory of Dirichlet forms, when $p = 2$ and Ch_p is a quadratic form.

The convexity and lower semicontinuity of Ch_p allow us, when $p = 2$, to apply the well-established theory of gradient flows in Hilbert spaces to provide, for all $\bar{f} \in$

$L^2(X,\mathfrak{m})$ the unique gradient flow of $\frac{1}{2}\,\mathrm{Ch}_2$ starting from $\bar{f}$. In addition, the selection principle of the Hilbertian theory of gradient flows motivates the following definition and terminology, consistent with the classical setting.

Definition 3.2 (Laplacian Δ and Heat flow P_t)**.** Let $g \in L^2(X,\mathfrak{m})$ be such that $\partial_0\,\mathrm{Ch}_2(g)$ is not empty. We call Laplacian of g, and denote Δg, the element with minimal norm in $-\frac{1}{2}\partial_0\,\mathrm{Ch}_2(g)$. With this notation, for all $f \in L^2(X,\mathfrak{m})$ we denote by $\mathsf{P}_t f$ the unique solution to (2-4) with $F = \frac{1}{2}\,\mathrm{Ch}_2$, thus solving the equation

$$\frac{d}{dt}\mathsf{P}_t f = \Delta \mathsf{P}_t f \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in (0,\infty).$$

Notice that Δ, also called weighted Laplacian or Witten Laplacian in the smooth context, depends *both* on d and $\mathfrak{m}$: this can be immediately understood in the setting of weighted Riemannian manifolds, since ∇f depends on d (i.e. the Riemannian metric g) but the divergence, viewed as adjoint of the gradient operator in $L^2(X,\mathfrak{m})$, depends on $\mathfrak{m}$, so that

$$\Delta f = \Delta_{\mathrm{g}} f - \langle \nabla V, \nabla f\rangle \tag{3-2}$$

and Δ reduces to the Laplace–Beltrami operator Δ_{g} when $V \equiv 0$ in (1-1).

By the specific properties of Ch_2, under the global assumption (5-11) the semigroup P_t can also be extended to a semigroup of contractions in every $L^p(X,\mathfrak{m})$, $1 \le p \le \infty$, preserving positivity, mass, and constants. We retain the same notation P_t for this extension.

As simple examples illustrate (see Section 3.2), Ch_2 need not be a quadratic form, so that in general neither the operator Δ nor the semigroup P_t are linear; nevertheless basic calculus rules and differential inequalities still apply, see Ambrosio, Gigli, and Savaré [2014a] and Gigli [2015].

Coming back to our discussions about Sobolev spaces, one can try to define or characterize weakly differentiable functions by appealing to the behaviour of functions along lines (and, in a nonsmooth setting, curves). Actually, this is the very first approach to the theory of weakly differentiable functions, pioneered by Levi [1906] in his efforts to put the Dirichlet principle on firm grounds. Later on, it was revisited and, at the same time, made frame-indifferent by Fuglede [1957] in this form: $f : \Omega \subset \mathbb{R}^N \to \mathbb{R}$ belongs to the Beppo Levi space if, for some vector $\nabla f \in L^p(\Omega;\mathbb{R}^N)$, one has

$$f(\gamma_1) - f(\gamma_0) = \int_\gamma \nabla f \qquad \text{for } \mathrm{Mod}_p\text{-almost every } \gamma.$$

Here Fuglede used a potential-theoretic notion, the so-called p-Modulus: for a family Γ of (non parametric) curves, in $\mathbb{R}^N$, one defines

$$\mathrm{Mod}_p(\Gamma) := \inf\left\{\int_{\mathbb{R}^N} \rho^p\, d\mathfrak{m} : \int_\gamma \rho\, d\sigma \ge 1 \ \ \forall \gamma \in \Gamma\right\}. \tag{3-3}$$

In more recent times, Shanmugalingam [2000] adapted this concept to the metric measure setting, with the introduction of the *Newtonian space* $N^{1,p}(X,\mathsf{d},\mathfrak{m})$; notice that the notion of p-Modulus immediately extends to the metric measure setting, understanding curvilinear integrals as in (2-1).

Definition 3.3 ($N^{1,p}$ Sobolev space)**.** We say that $f \in L^p(X, \mathfrak{m})$ belongs to $N^{1,p}(X, \mathsf{d}, \mathfrak{m})$ if there exist $\tilde{f} : X \to \mathbb{R}$ and $g \in L^p(X, \mathfrak{m})$ non-negative such that $\tilde{f} = f$ $\mathfrak{m}$-a.e. in X and $|\tilde{f}(\gamma_1) - \tilde{f}(\gamma_0)| \le \int_\gamma g \, d\sigma$ holds for Mod_p-almost every curve γ.

Even in this case one can identify a distinguished object playing the role of the modulus of the gradient, namely the g with smallest L^p-norm among those satisfying $|\tilde{f}(\gamma_1) - \tilde{f}(\gamma_0)| \le \int_\gamma g \, d\sigma$ Mod_p-a.e.: it is called *minimal p-weak upper gradient*, and denoted by $|\nabla f|_w$. This point of view has been deeply investigated by the Finnish school, covering also vector-valued functions and the relation with the original H Sobolev spaces of Cheeger [1999], see the recent monographs A. Björn and J. Björn [2011], Heinonen, Koskela, Shanmugalingam, and Tyson [2015].

Having in mind the theory in Euclidean spaces, we might look for analogues in the metric measure setting of the classical point of view of weak derivatives, within the theory of distributions. I will describe this last point of view, even though for the moment it does not play a substantial role in the theory of curvature-dimension bounds for m.m.s. On a Riemannian manifold $(\mathbb{M}, \mathrm{g})$, with $\mathfrak{m} = \mathrm{Vol}_{\mathbb{M}}$, it is natural to define the weak gradient ∇f by the integration by parts formula

$$\int \mathrm{g}(\nabla f, \boldsymbol{b}) \, d\mathfrak{m} = -\int f \operatorname{div} \boldsymbol{b} \, d\mathfrak{m} \tag{3-4}$$

against smooth (say compactly supported) vector fields $\boldsymbol{b}$. In the abstract m.m.s. setting, the role of vector fields is played by *derivations*, first studied in detail by in Weaver [2000]. Here we adopt a definition close to the one adopted in Weaver [ibid.], but using Gigli [2014b] to measure of the size of a derivation.

Definition 3.4 (Derivations and their size)**.** An L^p-derivation is a linear map from $\mathrm{Lip}_b(X)$ to $L^p(X, \mathfrak{m})$ satisfying:

(a) (Leibniz rule) $\boldsymbol{b}(f_1 f_2) = f_1 \boldsymbol{b}(f_2) + f_2 \boldsymbol{b}(f_1)$;

(b) for some $g \in L^p(X, \mathfrak{m})$, one has $|b(f)| \le g|\nabla f|_*$ $\mathfrak{m}$-a.e. in X for all $f \in \mathrm{Lip}_b(X)$;

(c) (Continuity) $b(f_n)$ weakly converge to $b(f)$ in $L^p(X, \mathfrak{m})$ whenever $f_n \to f$ pointwise, with $\sup_X |f_n| + \mathrm{Lip}(f_n) \le C < \infty$.

The smallest function g in (b) is denoted by $|\boldsymbol{b}|$.

Now, the definition of divergence $\operatorname{div} \boldsymbol{b}$ of a derivation is based on (3-4), simply replacing $\mathrm{g}(\nabla f, \boldsymbol{b})$ with $\boldsymbol{b}(f)$, and we define

$$\mathrm{Div}^q(X, \mathsf{d}, \mathfrak{m}) := \left\{ \boldsymbol{b} : \ |\boldsymbol{b}| \in L^q(X, \mathfrak{m}), \operatorname{div} \boldsymbol{b} \in L^1 \cap L^\infty(X, \mathfrak{m}) \right\}.$$

According to the next definition, bounded Lipschitz functions f in $L^p(X, \mathfrak{m})$ belong to the $W^{1,p}$ Sobolev space (with $L(\boldsymbol{b}) = \boldsymbol{b}(f)$) introduced in Marino [2014] (in the Euclidean setting, but already with general reference measures, a closely related definition appeared also in Bouchitte, Buttazzo, and Seppecher [1997]):

Definition 3.5 ($W^{1,p}$ Sobolev space)**.** We say that $f \in L^p(X, \mathfrak{m})$ belongs to the space $W^{1,p}(X, \mathsf{d}, \mathfrak{m})$ if there exists a Lip_b-linear functional $L_f : \mathrm{Div}^q(X, \mathsf{d}, \mathfrak{m}) \to L^1(X, \mathfrak{m})$ satisfying

$$\int L_f(\boldsymbol{b})\, d\mathfrak{m} = -\int f \operatorname{div} \boldsymbol{b}\, d\mathfrak{m} \qquad \forall \boldsymbol{b} \in \mathrm{Div}^q(X, \mathsf{d}, \mathfrak{m}),$$

where $q = p/(p-1)$ is the dual exponent of p.

The following result has been established for the $H^{1,p}$ and $N^{1,p}$ spaces first in the case $p = 2$ in Ambrosio, Gigli, and Savaré [2014a], then in Ambrosio, Gigli, and Savaré [2013] for general p. In Marino [2014] the equivalence has been completed with the $W^{1,p}$ spaces.

Theorem 3.6. *For all $p \in (1, \infty)$ the spaces $H^{1,p}$, $N^{1,p}$, $W^{1,p}$ coincide. In addition the minimal relaxed slope coincides $\mathfrak{m}$-a.e. with the minimal p-weak upper gradient.*

Finally, let me conclude this "calculus" section with a (necessarily) brief mention of other important technical aspects and research directions.

Test plans. In connection with Theorem 3.6, the inclusion $H^{1,p} \subset N^{1,p}$ is not hard to prove, while the converse requires the construction of a "good" approximation of f by Lipschitz functions, knowing only the behaviour of f along Mod_p-almost all curves. To achieve this goal, in Ambrosio, Gigli, and Savaré [2014a, 2013] besides nontrivial tools (Hopf–Lax semigroup (4-3), superposition principle, etc, described in the next sections) we also use a new and more "probabilistic" way to describe the exceptional curves. This is encoded in the concept of *test plan*.

Definition 3.7 (Test plan)**.** We say that $\boldsymbol{\eta} \in \mathcal{P}(C([0,T]; X))$ is a p-test plan if it is concentrated on $\mathrm{AC}^q([0,1]; X)$, with $q = p/(p-1)$, and there exists $C = C(\boldsymbol{\eta}) \geq 0$ such that

$$(e_t)_\# \boldsymbol{\eta} \leq C\mathfrak{m} \quad \forall t \in [0,1], \quad \text{where } e_t : C([0,1]; X) \to X,\ e_t(\gamma) := \gamma_t.$$

Then, we say that a Borel set $\Gamma \subset C([0,T]; X)$ is p-negligible if $\boldsymbol{\eta}(\Gamma) = 0$ for any p-test plan $\boldsymbol{\eta}$. Now, we can say that a function $f \in L^p(X, \mathfrak{m})$ belongs to the Beppo Levi space $BL^{1,p}(X, \mathsf{d}, \mathfrak{m})$ if, for some $g \in L^p(X, \mathfrak{m})$, the upper gradient property (2-2) holds for p-almost every curve. Since Mod_p-negligible sets are easily seen to be p-negligible one has the inclusion $N^{1,p} \subset BL^{1,p}$, and with the proof of the equality $BL^{1,p} = H^{1,p}$ we have closed the circle of equivalences.

Test plans are useful not only to describe null sets of curves. They are natural objects in the development of calculus in metric measure spaces (for instance the proof of Theorem 3.9 below deeply relies on this concept), since they induce vector fields, i.e. derivations, via the formula

$$\int_X \boldsymbol{b}_{\boldsymbol{\eta}}(f)\phi\, d\mathfrak{m} := \int \int_0^1 \phi \circ \gamma \frac{d}{dt} f \circ \gamma\, dt\, d\boldsymbol{\eta}(\gamma) \qquad \forall \phi \in L^1(X, \mathfrak{m}),$$

which implicitly defines $\boldsymbol{b}_{\boldsymbol{\eta}}$. These connections are further investigated in Gigli [2015], where the notion of test plan representing the gradient of a function is introduced, see

also Schioppa [2016], where an analogous analysis is done with the so-called *Alberti representations* $\int J\gamma\, d\boldsymbol{\eta}(\gamma)$ of $\mathfrak{m}$, with J the measure-valued operator on $\mathrm{AC}([0,1];X)$ defined in (2-1). In addition, along these lines one obtains Ambrosio, Di Marino, and Savaré [2015] also a useful "dual" representation of the p-Modulus:

$$\big(\mathrm{Mod}_p(\Gamma)\big)^{1/p} = \sup\left\{\frac{1}{\|\mathrm{bar}\,\boldsymbol{\eta}\|_q} : \boldsymbol{\eta}(\Gamma)=1,\ \boldsymbol{\eta}\in T_q\right\}, \tag{3-5}$$

where $q = p/(p-1)$ and $T_q \subset \mathcal{P}(C([0,1];X))$ is defined by the property $\int J\gamma\, d\boldsymbol{\eta}(\gamma) \ll \mathfrak{m}$, with density, the barycenter bar $\boldsymbol{\eta}$, in $L^q(X,\mathfrak{m})$.

Differentiable structures on metric measure spaces. One of the main motivation of the seminal paper Cheeger [1999] has been the statement and proof of a suitable version of Rademacher's theorem in m.m.s. Roughly speaking, J. Cheeger proved that in doubling m.m.s. satisfying the Poincaré inequality (the so-called PI spaces, see Heinonen, Koskela, Shanmugalingam, and Tyson [2015] for much more on this subject) one has a countable Borel atlas X_i and Lipschitz maps $F^i : X \to \mathbb{R}^{N_i}$ with the property that $\sup_i N_i < \infty$ and, for any i and $f \in \mathrm{Lip}_b(X)$, there exist $\alpha_i = (\alpha_{i,1},\dots,\alpha_{i,N_i}) : X_i \to \mathbb{R}^{N_i}$ with

$$\Big|\nabla\Big(f(\cdot) - \sum_{j\le N_i}\alpha_{i,j}(x)F^i_j(\cdot)\Big)\Big|(x) = 0 \qquad \text{for } \mathfrak{m}\text{-a.e. } x\in X_i.$$

By letting F_i play the role of local coordinates, this fact can be used to develop a good first order (nonsmooth) differential geometry and to prove, among other things, reflexivity of the spaces $H^{1,p}(X,\mathsf{d},\mathfrak{m})$. An abstraction of this property has led to the concept of *Lipschitz differentiability space*, now studied and characterized by many authors (see e.g. Cheeger, Kleiner, and Schioppa [2016] and the references therein). A somehow parallel philosophy, not based on a blow-up analysis and initiated by N. Weaver [2000], aims instead at a kind of implicit description of the tangent bundle via the collection of its sections, i.e. the family of derivations. It is proved in Gigli [2014b] that the L^∞-module generated by gradient derivations is dense in the class of L^2 derivations and that, within the class of PI spaces, the two points of view are equivalent. Finally, notice that the point of view of Γ-calculus seems to be closer to Weaver's one, since as we will see that several objects (carré du champ, Hessian, etc.) are defined by their action against gradient derivations.

3.2 Metric versus energy structures. In this section I want to emphasize key connections between metric and energy structures, using for the latter point of view the well-established theory of Dirichlet forms Fukushima, Oshima, and Takeda [2011] and Ma and Röckner [1992]. In this text, by *Dirichlet form* we mean a $L^2(X,\mathfrak{m})$-lower semicontinuous quadratic form $\mathcal{E} : L^2(X,\mathfrak{m}) \to [0,\infty]$ with the Markov property ($\mathcal{E}(\eta(f)) \le \mathcal{E}(f)$ for any 1-Lipschitz function with $\eta(0)=0$) and with a dense finiteness domain $\mathbb{V}$. The domain $\mathbb{V}$ is endowed with the Hilbert norm $\|\cdot\|^2_{\mathbb{V}} = \mathcal{E} + \|\cdot\|^2_2$. Even if this is not strictly needed, to simplify the discussion I assume that (X,τ) is a Hausdorff topological space and that $\mathfrak{m}$ is a non-negative and finite Borel measure in X.

Still denoting by $\mathcal{E}$ the associated symmetric bilinear form, we also assume the following properties:

(a) (strong locality) $\mathcal{E}(u, v) = 0$ if $u(c + v) = 0$ $\mathfrak{m}$-a.e. in X for some $c \in \mathbb{R}$;

(b) (carré du champ) there exists a continuous bilinear form $\Gamma : \mathbb{V} \times \mathbb{V} \to L^1(X, \mathfrak{m})$ such that

$$\int_X g\Gamma(f, f)\, d\mathfrak{m} = \mathcal{E}(fg, f) - \frac{1}{2}\mathcal{E}(f^2, g) \qquad \forall f, g \in \mathbb{V} \cap L^\infty(X, \mathfrak{m}).$$

If we apply these constructions to the Dirichlet energy on a Riemannian manifold, we see that $\Gamma(f, g)$ corresponds to the scalar product between ∇f and ∇g; in this sense we may think that Lipschitz functions provide a differentiable structure (with global sections of the tangent bundle provided by gradient vector fields) and Dirichlet forms provides a metric structure (via the operator Γ).

We may move from the metric to the "energy" structure in a canonical way, setting $\mathcal{E} = \mathrm{Ch}_2$ if Cheeger's energy Ch_2 is a quadratic form. However, this property of being a quadratic form is far from being true in general: for instance, if we apply Cheeger's construction to the metric measure structure $(\mathbb{R}^N, \mathsf{d}, \mathcal{L}^N)$, where $\mathsf{d}(x, y) = \|x - y\|$ is the distance induced by a norm, we find that $|\nabla f|_* = \|\nabla f\|_*$ for all $f \in H^{1,p}$, where ∇f is the weak (distributional) derivative and $\|\cdot\|_*$ is the dual norm. Hence, the norm is Hilbertian iff Ch_2 is a quadratic form. This motivates the following terminology introduced in Gigli [2015].

Definition 3.8 (Infinitesimally Hilbertian m.m.s.)**.** We say that a m.m.s. $(X, \mathsf{d}, \mathfrak{m})$ is infinitesimally Hilbertian if Cheeger's energy Ch_2 in (3-1) is a quadratic form in $L^2(X, \mathfrak{m})$.

For infinitesimally Hilbertian m.m.s., the following consistency result has been proved in Ambrosio, Gigli, and Savaré [2014b, Thm. 4.18], see also Gigli [2015].

Theorem 3.9. *If $(X, \mathsf{d}, \mathfrak{m})$ is infinitesimally Hilbertian, then Ch_2 is a strongly local Dirichlet form and its carré du champ $\Gamma(f)$ coincides with $|\nabla f|_*^2$.*

In order to move in the opposite direction, we need to build a distance out of $\mathcal{E}$. The canonical construction starts from the class

$$\mathcal{C} := \{f \in \mathbb{V} \cap C_b(X, \tau) : \Gamma(f) \le 1 \ \mathfrak{m}\text{-a.e. in } X\}$$

and defines the *intrinsic distance* by

$$\mathsf{d}_{\mathcal{E}}(x, y) := \sup\{|f(x) - f(y)| : f \in \mathcal{C}\}. \tag{3-6}$$

Under the assumption that $\mathcal{C}$ generates a *finite* distance (this is not always the case, as for the Dirichlet form associated to the Wiener space, leading to *extended* metric measure structures Ambrosio, Erbar, and Savaré [2016]) and assuming also that the topology induced by $\mathsf{d}_{\mathcal{E}}$ coincides with τ, we have indeed obtained a metric measure structure.

This happens for instance in the classical case of quadratic forms in $L^2(\mathbb{R}^N)$ induced by symmetric, uniformly elliptic and bounded matrices A:

$$\mathcal{E}_A(f) := \begin{cases} \int_{\mathbb{R}^N} \langle A(x)\nabla f(x), \nabla f(x)\rangle \, dx & \text{if } f \in H^1(\mathbb{R}^N); \\ +\infty & \text{otherwise.} \end{cases} \tag{3-7}$$

Given that, with some limitations, one can move back and forth from metric to energy structures, one may wonder what happens when we iterate these procedures, namely from $\mathcal{E}$ one builds $\mathsf{d}_{\mathcal{E}}$ and then the Cheeger energy $\mathrm{Ch}_{2,\mathsf{d}_{\mathcal{E}}}$ induced by $\mathsf{d}_{\mathcal{E}}$ (or, conversely, one first moves from the metric to the energy structure and then again to the metric structure). To realize that this is a nontrivial issue, I recall what happens at the level of the relation between energy and distance in the case of the quadratic forms $\mathcal{E}_A$ in (3-7): first, even though $\mathcal{E}_A$ is in 1-1 correspondence with A (and this observation is at the basis of the theories of G-convergence for diffusion operators A, and of Γ-convergence), we also know from Sturm [1997] that $\mathcal{E}_A$ need not be uniquely determined by $\mathsf{d}_{\mathcal{E}_A}$. Moreover, the analysis of the construction in Sturm [ibid.] reveals that, given an intrinsic distance d induced by some $\mathcal{E}_B$, there is no "minimal" $\mathcal{E}_A$ whose intrinsic distance is d. Second, an example in Koskela, Shanmugalingam, and Zhou [2014] shows that, for some $\mathcal{E} = \mathcal{E}_A$, the Cheeger energy $\mathrm{Ch}_{2,\mathsf{d}_{\mathcal{E}}}$ need not be a quadratic form as in (3-7).

In order to clarify the relations between these objects in the general setting, the following property plays an important role.

Definition 3.10 (τ-upper regularity)**.** We say that $\mathcal{E}$ is τ-upper regular if for all $f \in \mathbb{V}$ there exist $f_i \in \mathrm{Lip}_b(X, \mathsf{d}_{\mathcal{E}})$ and upper semicontinuous functions $g_i \geq \Gamma(f_i)$ $\mathfrak{m}$-a.e. in X with $f_i \to f$ in $L^2(X, \mathfrak{m})$ and

$$\limsup_{i\to\infty} \int_X g_i \, d\mathfrak{m} \leq \mathcal{E}(f).$$

The definition can also be adapted to the metric measure setting, replacing $\Gamma(f)$ with $|\nabla f|_*^2$. It has been proved in Ambrosio, Gigli, and Savaré [2013] that Ch_2 is always τ-upper regular, with τ given by the metric topology.

The following result, taken from Ambrosio, Gigli, and Savaré [2015] and Ambrosio, Erbar, and Savaré [2016] (see also Koskela, Shanmugalingam, and Zhou [2014], under additional curvature assumptions), deals with the iteration of the two operations I described above and provides a "maximality" property of Cheeger's energy.

Theorem 3.11. *Assume that $\mathcal{E}$ is a Dirichlet form satisfying (a) and (b), and that $\mathsf{d}_{\mathcal{E}}$ induces the topology τ. Then $\mathcal{E} \leq \mathrm{Ch}_{2,\mathsf{d}_{\mathcal{E}}}$, with equality if and only $\mathcal{E}$ is τ-upper regular. Conversely, if we start from an infinitesimally Hilbertian m.m.s. $(X, \mathsf{d}, \mathfrak{m})$, and if we set $\mathcal{E} = \mathrm{Ch}_2$, then $\mathrm{Ch}_{2,\mathsf{d}_{\mathcal{E}}} = \mathcal{E}$ and $\mathsf{d}_{\mathcal{E}} \geq \mathsf{d}$. Equality holds iff one has the "Sobolev-to-Lipschitz" property: any $f \in H^{1,2}(X, \mathsf{d}, \mathfrak{m}) \cap C_b(X)$ with $|\nabla f|_* \leq 1$ is 1-Lipschitz.*

3.3 Flow of vector fields and the superposition principle. Let $\boldsymbol{b}_t$, $t \in (0, T)$, be a time-dependent family of vector fields. In a nice (say Euclidean or Riemannian) framework, it is a classical fact that the ordinary differential equation

$$\text{(ODE)} \qquad \begin{cases} \gamma_t' = \boldsymbol{b}_t(\gamma_t) \\ \gamma_0 = x \end{cases}$$

is closely related to the continuity equation

$$\text{(CE)} \qquad \frac{d}{dt}\varrho_t + \operatorname{div}(\boldsymbol{b}_t \varrho_t) = 0, \qquad \varrho_0 = \bar{\varrho}.$$

Indeed, denoting by

$$\boldsymbol{X}(t, x) : [0, T] \times X \to X$$

the flow map of the ODE, under appropriate assumptions, the push-forward measures $\mu_t := \boldsymbol{X}(t, \cdot)_\#(\bar{\rho}\mathfrak{m})$ are shown to be absolutely continuous w.r.t. $\mathfrak{m}$ and their densities ϱ_t solve the weak formulation of (CE), namely

$$\text{(3-8)} \qquad \frac{d}{dt}\int_X \phi \varrho_t \, d\mathfrak{m} = \int_X \langle \boldsymbol{b}_t, \nabla\phi \rangle \varrho_t \, d\mathfrak{m}$$

for any test function ϕ (notice that the operator div in (CE), according to (3-4), does depend on the reference measure $\mathfrak{m}$). Under appropriate regularity assumptions (for instance within the Cauchy–Lipschitz theory) one can then prove that this is the unique solution of (CE). Starting from the seminal paper DiPerna and Lions [1989], these connections have been extended to classes of nonsmooth (e.g. Sobolev, or even BV Ambrosio [2004]) vector fields, with applications to fluid mechanics and to the theory of conservation laws, see the lecture notes Ambrosio and Crippa [2014] for much more information on this topic. One of the basic principles of the theory is that, as I illustrate below, well-posedness can be transferred from the (ODE) to (CE), and conversely.

More recently it has been understood in Ambrosio and Trevisan [2014] that not only can one deal with nonsmooth vector fields, but even with general (nonsmooth) metric measure structures. Therefore from now on I come back to this high level of generality. We have already seen that in the m.m.s. setting the role of vector fields is played by derivations, and that the divergence operator can be defined; on the other hand, the definition of solution to the ODE is more subtle. If we forget about the measure structure, looking only at the metric one, there is by now a well-established theory for ODE's $\boldsymbol{b} = -\nabla\mathcal{E}$ of gradient type Ambrosio, Gigli, and Savaré [2008]: in this setting, as we have seen in Section 2, one can characterize the gradient flow by looking at the maximal rate of dissipation of $\mathcal{E}$. In general, for vector fields which are not gradients, one can use *all* Lipschitz functions as "entropies"; taking also into account the role of the measure $\mathfrak{m}$, this leads to the following definition of *regular Lagrangian flow*, an adaptation to the nonsmooth setting of the notion introduced in Ambrosio [2004].

Definition 3.12 (Regular Lagrangian Flow)**.** Let $\boldsymbol{b}_t$ be derivations. We say that $\boldsymbol{X}(t, x)$ is a regular Lagrangian flow relative to $\boldsymbol{b}_t$ (in short RLF) if the following three properties hold:

(a) $\boldsymbol{X}(\cdot, x) \in \mathrm{AC}([0,T]; X)$ for $\mathfrak{m}$-a.e. $x \in X$;

(b) for all $f \in \mathrm{Lip}_b(X)$ and $\mathfrak{m}$-a.e. $x \in X$, one has $\frac{d}{dt} f(\boldsymbol{X}(t,x)) = \boldsymbol{b}_t(f)(\boldsymbol{X}(t,x))$ for $\mathcal{L}^1$-a.e. $t \in (0,T)$;

(c) for some $C \geq 0$, one has $\boldsymbol{X}(t,\cdot)_\# \mathfrak{m} \leq C\mathfrak{m}$ for all $t \in [0,T]$.

The basic principle of the theory is the following result, reminiscent of the uniqueness in law/pathwise uniqueness results typical of the theory of stochastic processes.

Theorem 3.13. *Assume that $|\boldsymbol{b}_t| \in L^1((0,T); L^2(\mathfrak{m}))$. Then (CE) is well-posed in the class*

(3-9)
$$\mathcal{L} := \left\{\varrho \in L^\infty((0,T); L^1 \cap L^\infty(X,\mathfrak{m})) : \varrho_t \geq 0,\ \varrho_t\ w^* - L^\infty(X,\mathfrak{m}) \text{ continuous}\right\}$$

if and only if there exists a unique Regular Lagrangian Flow $\boldsymbol{X}$.

It is clear, by the simple transfer mechanism I described at the beginning of this section, that distinct RLF's lead to different solutions to (CE). The description of the path from existence of solutions to (CE) to existence of the RLF deserves instead more explanation, and requires a basic result about moving from Eulerian to Lagrangian representations, the *superposition principle*. Its origins go back to the work of L. C. Young (see Bernard [2008]), but in its modern form it can be more conveniently stated in the language of the theory of currents, following Smirnov [1993]: any normal 1-dimensional current in $\mathbb{R}^N$ can be written as the superposition of elementary 1-dimensional currents associated to curves (see also Paolini and Stepanov [2012] for versions of this result within the metric theory of currents I developed with B. Kirchheim Ambrosio and Kirchheim [2000]). The version of this principle I state below, taken from Ambrosio and Trevisan [2014], is adapted to the space-time current $J = (\varrho_t \mathfrak{m}, \boldsymbol{b}_t \varrho_t \mathfrak{m})$ associated to (CE), see also Ambrosio, Gigli, and Savaré [2008, Thm. 8.2.1] for stronger formulations in Euclidean spaces:

Theorem 3.14 (Superposition principle)**.** *Let $\boldsymbol{b}_t$, $\varrho_t \in \mathcal{L}$ be as in Theorem 3.13. If ϱ_t solves (CE), then there exists $\boldsymbol{\eta} \in \mathcal{P}(C([0,T]; X))$ concentrated on absolutely continuous solutions to (ODE), such that*

(3-10) $$\varrho_t \mathfrak{m} = (e_t)_\# \boldsymbol{\eta} \quad \forall t \in [0,T], \qquad \text{where } e_t : C([0,T]; X) \to X,\ e_t(\gamma) := \gamma_t.$$

Using this principle, as soon as we have w^*-L^∞ continuous solutions to (CE) starting from $\varrho_0 \equiv 1$ we can lift them to probabilities $\boldsymbol{\eta}$ in $C([0,T]; X)$ concentrated on solutions to (ODE), thus providing a kind of generalized solution to the (ODE). The uniqueness of (CE) now comes into play, in the proof that the conditional measures $\boldsymbol{\eta}_x$ associated to the map $e_0(\gamma) = \gamma_0$ should be Dirac masses, so that writing $\boldsymbol{\eta}_x = \delta_{\boldsymbol{X}(\cdot,x)}$ we recover our RLF $\boldsymbol{X}$.

As the theory in Euclidean spaces shows (see Ambrosio and Crippa [2014]), some regularity of the vector field is necessary to obtain uniqueness of solutions to (CE), even within the class $\mathcal{L}$ in (3-9). Assuming until the end of the section that $(X, \mathsf{d}, \mathfrak{m})$ is infinitesimally Hilbertian, we introduce the following regularity property for derivations; for gradient derivations $\boldsymbol{b}_h(f) = \Gamma(f,h)$ it corresponds to the integral form of Bakry's definition of Hessian (see (5-10) and Bakry [1997]).

Definition 3.15 (Derivations with deformation in L^2)**.** Let $\boldsymbol{b}$ be a derivation in L^2. We write $D^{sym}\boldsymbol{b} \in L^2(X, \mathfrak{m})$ if there exists $c \geq 0$ satisfying

$$\left| \int D^{sym}\boldsymbol{b}(f,g)\, d\mathfrak{m} \right| \leq c\|\Gamma(f)\|_2^{1/2}\|\Gamma(g)\|_2^{1/2}, \tag{3-11}$$

for all $f,\ g \in H^{1,2}(X, \mathsf{d}, \mathfrak{m})$ with $\Delta f,\ \Delta g \in L^4(X, \mathfrak{m})$, where

$$\int D^{sym}\boldsymbol{b}(f,g)\, d\mathfrak{m} := -\frac{1}{2}\int \big[\boldsymbol{b}(f)\Delta g + \boldsymbol{b}(g)\Delta f - (\operatorname{div}\boldsymbol{b})\Gamma(f,g)\big]\, d\mathfrak{m}. \tag{3-12}$$

We denote by $\|D^{sym}\boldsymbol{b}\|_2$ be the smallest constant c in (3-11).

Under a mild regularizing property of the semigroup P_t, satisfied for instance in all $\mathrm{RCD}(K,\infty)$ spaces (see Ambrosio and Trevisan [2014, Thm. 5.4] for the precise statement), the following result provides well posedness of (CE), and then existence and uniqueness of the RLF $\boldsymbol{X}$, in a quite general setting.

Theorem 3.16. *If*

- $|\boldsymbol{b}_t| \in L^1((0,T); L^2(X, \mathfrak{m}))$,
- $\|D^{sym}\boldsymbol{b}_t\|_2 \in L^2(0,T)$ *and*
- $|\operatorname{div}\boldsymbol{b}_t| \in L^1((0,T); L^\infty(X, \mathfrak{m}))$

then (CE) is well posed in the class $\mathfrak{L}$ *in* (3-9).

4 Background on optimal transport

Building on the metric structure (X, d), optimal transport provides a natural way to introduce a geometric distance between probability measures, which reflects well the metric properties of the base space. We call $\mathcal{P}_2(X)$ the space of Borel probability measures with finite quadratic moment, namely μ belongs to $\mathcal{P}_2(X)$ if $\int_X \mathsf{d}^2(x,\bar{x})\, d\mu(x) < \infty$ for some (and thus any) $\bar{x} \in X$. Given $\mu_0,\ \mu_1 \in \mathcal{P}_2(X)$ we consider the collection $\mathrm{Plan}(\mu_0,\mu_1)$ of all *transport plans* (or *couplings*) between μ_0 and μ_1, i.e. measures $\boldsymbol{\mu} \in \mathcal{P}(X \times X)$ with marginals $\mu_0,\ \mu_1$, i.e. $\mu_0(A) = \boldsymbol{\mu}(A \times X)$, $\mu_1(A) = \boldsymbol{\mu}(X \times A)$. The squared Kantorovich–Rubinstein–Wasserstein distance $\mathrm{W}_2(\mu_0,\mu_1)$ (Wasserstein distance, in short) is then defined as

$$\mathrm{W}_2^2(\mu_0,\mu_1) := \min_{\boldsymbol{\mu}\in\mathrm{Plan}(\mu_0,\mu_1)} \int_{X\times X} \mathsf{d}^2(x_0,x_1)\, d\boldsymbol{\mu}(x_0,x_1). \tag{4-1}$$

The duality formula

$$\frac{1}{2}\mathrm{W}_2^2(\mu_0,\mu_1) = \sup_{f\in\mathrm{Lip}_b(X)} \int_X \mathsf{Q}_1 f\, d\mu_1 - \int_X f\, d\mu_0 \tag{4-2}$$

where Q_t is the *Hopf–Lax* semigroup

$$\mathsf{Q}_t f(x) := \inf_{y\in Y} f(y) + \frac{1}{2t}\mathsf{d}^2(x,y),\ t > 0, \qquad \mathsf{Q}_0 f(x) = f(x) \tag{4-3}$$

plays an important role in the proof of many estimates (e.g. contractivity properties) involving W_2.

The distance W_2 induces on $\mathcal{P}_2(X)$ the topology of weak convergence with quadratic moments, i.e. continuity of all the integrals $\mu \mapsto \int_X \phi \, d\mu$ with $\phi : X \to \mathbb{R}$ continuous and with at most quadratic growth. The metric space $(\mathcal{P}_2(X), \mathrm{W}_2)$ is complete and separable and it inherits other useful properties from (X, d) such as compactness, completeness, existence of geodesics, nonnegative sectional curvature (see e.g. Ambrosio, Gigli, and Savaré [2008], Santambrogio [2015], and Villani [2009]). Particularly relevant for our discussion are the geodesic properties. In the same spirit of the concepts illustrated in Section 3 (regular Lagrangian flows, test plans, superposition principle, etc.) there is a close connection between $\mathrm{Geo}(X)$ and $\mathrm{Geo}(\mathcal{P}_2(X))$: very informally we can say that "a geodesic in the space of random variables is always induced by a random variable in the space of geodesics".

Proposition 4.1. *Any $\boldsymbol{\eta} \in \mathcal{P}(\mathrm{Geo}(X))$ with $(e_0, e_1)_\# \boldsymbol{\eta}$ optimal transport plan induces $\mu_t := (e_t)_\# \boldsymbol{\eta} \in \mathrm{Geo}(\mathcal{P}_2(X))$. Conversely, any $\mu_t \in \mathrm{Geo}(\mathcal{P}_2(X))$ is representable (in general not uniquely) in this way.*

In the sequel we shall denote by $\mathrm{OptGeo}(X)$ the *optimal geodesic plans*, namely the distinguished class of probabilities $\boldsymbol{\eta}$ in $\mathrm{Geo}(X)$ which induce optimal plans between their marginals at time 0, 1. By the previous proposition, these probability measures canonically induce geodesics in $\mathcal{P}_2(X)$ by taking the time marginals, and it will be very useful to lift all geodesics in $\mathcal{P}_2(X)$ to elements of $\mathrm{OptGeo}(X)$.

If the ambient space X is Euclidean or Riemannian, it was understood at the end of the 90s that even the Riemannian structure could be lifted from X to $\mathcal{P}_2(X)$, see Benamou and Brenier [2000], Otto [2001], and Jordan, Kinderlehrer, and Otto [1998]. In this direction the key facts are the Benamou–Brenier formula

$$\text{(BB)} \qquad \mathrm{W}_2^2(\mu_0, \mu_1) = \inf \left\{ \int_0^1 \int_X |\boldsymbol{b}_t|^2 \, d\mu_t \, dt : \frac{d}{dt}\mu_t + \mathrm{div}\,(\boldsymbol{b}_t \mu_t) = 0 \right\}$$

and the Jordan–Kinderlehrer-Otto interpretation of the heat flow $\mathsf{P}_t f$ as the gradient flow of the *Entropy functional*

$$\text{(4-4)} \qquad \mathrm{Ent}(\mu) := \begin{cases} \int_X \varrho \log \varrho \, d\mathfrak{m} & \text{if } \mu = \varrho \mathfrak{m}, \\ +\infty & \text{otherwise} \end{cases}$$

w.r.t. W_2. In particular, according to Otto's calculus Otto [2001] we may at least formally endow $\mathcal{P}_2(X)$ with the metric tensor[2]

$$\mathrm{g}_\mu(s_1, s_2) := \int_X \langle \nabla \phi_1, \nabla \phi_2 \rangle \, d\mu \qquad \text{where } -\mathrm{div}\,(\nabla \phi_i \mu) = s_i, \, i = 1, 2,$$

so that, after recognizing that gradient velocity fields $\boldsymbol{b}_t = \nabla \phi_t$ are the optimal ones in (BB) (see also (4-6) below), we may interpret the (BB) formula by saying that W_2

[2]Compare with Fisher–Rao's metric tensor, used in Statistics, formally given by $\mathrm{g}_\mu(s_1, s_2) = \int_X s_1 s_2 \, d\mu$, with $s_i \, \mu$ tangent vectors.

is the Riemannian distance associated to the metric g (see also Lott [2008], with calculations of curvature tensors in $\mathcal{P}_2(X)$ along these lines). Similarly, according to this calculus, the heat equation can be interpreted as the gradient flow with respect to this "Riemannian structure"; as illustrated in Otto [2001] and Otto and Villani [2000] and many subsequent papers (see e.g. Carrillo, McCann, and Villani [2003] and other references in Ambrosio, Gigli, and Savaré [2008]), this provides a very powerful heuristic principle which applies to many more PDE's (Fokker–Planck equation, porous medium equation, etc.) and to the proof of functional/geometric inequalities. Particularly relevant for the subsequent developments is the formula

$$\int_{\{\varrho>0\}} \frac{|\nabla\varrho|^2}{\varrho}\, d\mathfrak{m} = |\nabla^-\mathrm{Ent}|^2(\varrho\mathfrak{m}) \tag{4-5}$$

which corresponds to the energy dissipation rate of Ent along the heat equation, when seen from the classical, "Eulerian", point of view (the left hand side) and from the new, "Lagrangian", point of view (the right hand side). The left hand side, also called *Fisher information*, can be written in the form $4\int |\nabla\sqrt{\varrho}|^2\, d\mathfrak{m}$.

After these discoveries, many attempts have been made to develop a systematic theory based on Otto's calculus, even though no approach based on local coordinates seems to be possible. In this direction, the building block in Ambrosio, Gigli, and Savaré [ibid.] is the identification of absolutely continuous curves in $(\mathcal{P}_2(X), \mathrm{W}_2)$ (a purely metric notion) with solutions to the continuity equation (a notion that appeals also to the differentiable structure).

Theorem 4.2. *Assume that either $X = \mathbb{R}^N$, or X is a compact Riemannian manifold. Then, for any $\mu_t \in \mathrm{AC}^2([0,1];(\mathcal{P}_2(X),\mathrm{W}_2))$ there exists a velocity field $\boldsymbol{b}_t$ such that the continuity equation $\frac{d}{dt}\mu_t + \mathrm{div}\,(\boldsymbol{b}_t\mu_t) = 0$ holds, and $\int_X |\boldsymbol{b}_t|^2\, d\mu_t \le |\mu_t'|^2$ for $\mathcal{L}^1$-a.e. $t \in (0,1)$. Conversely, for any solution $(\mu_t, \tilde{\boldsymbol{b}}_t)$ to the continuity equation with $\int_0^1\int_X |\tilde{\boldsymbol{b}}_t|^2\, d\mu_t dt < \infty$ one has that $\mu_t \in \mathrm{AC}^2([0,1];(\mathcal{P}_2(X),\mathrm{W}_2))$ with $|\mu_t'|^2 \le \int_X |\tilde{\boldsymbol{b}}_t|^2\, d\mu_t$ for $\mathcal{L}^1$-a.e. $t \in (0,1)$. Finally, the minimal velocity field $\boldsymbol{b}_t$ is characterized by*

$$\boldsymbol{b}_t \in \overline{\{\nabla\phi : \phi \in C_c^\infty(X)\}}^{L^2(\mu_t)} \qquad \text{for } \mathcal{L}^1\text{-a.e. } t \in (0,1). \tag{4-6}$$

While for applications to PDE's it is very useful to transfer differential information from X to $\mathcal{P}_2(X)$, it has been realized only more recently that also the converse path can be useful, namely we may try to use information at the level of $\mathcal{P}_2(X)$ to get information on the energy/differentiable structure of X, or its curvature, that seem to be difficult to obtain, or to state, with different means. Besides the Lott–Villani and Sturm theory, one of the first applications of this viewpoint and of the identification (4-5) has been the following result from Ambrosio, Savaré, and Zambotti [2009] (the full strength of the analogous identification (5-15) in $\mathrm{CD}(K,\infty)$ spaces will also play an important role in Section 6).

Theorem 4.3. *Let X be an Hilbert space and let $\mathfrak{m} \in \mathcal{P}(X)$ be log-concave, i.e.*

$$\log\mathfrak{m}\big((1-t)A + tB\big) \ge (1-t)\log\mathfrak{m}(A) + t\log\mathfrak{m}(B) \qquad \forall t \in [0,1]$$

for any pair of open sets A, B in X. Then the quadratic form

$$\mathcal{E}(f) := \int_X |\nabla f|^2 \, d\mathfrak{m} \qquad f \text{ smooth, cylindrical}$$

is closable in $L^2(X, \gamma)$, and its closure is a Dirichlet form.

While traditional proofs of closability use quasi-invariant directions (whose existence is an open problem for general log-concave measures), here the proof is based on (4-5): lower semicontinuity of $|\nabla^- \mathrm{Ent}|$, granted by the convexity of Ent along W_2-geodesics, provides lower semicontinuity in $L^1_+(X, \mathfrak{m})$ of Fisher information, and then closability of $\mathcal{E}$.

5 Curvature-dimension conditions

In this section I will illustrate two successful theories dealing with synthetic notions of Ricci bounds from below and dimension bounds from above. The first one, the Bakry-Émery theory, can be formulated at different levels of smoothness; I have chosen to describe it at the level of Dirichlet forms and Γ-calculus (see Section 3.2), since at this level the comparison with the Lott–Villani and Sturm theory (or, better, the Riemannian part of it) is by now well understood.

In the Bakry-Émery theory the starting point is Bochner–Lichnerowicz's formula

$$\frac{1}{2}\Delta_{\mathrm{g}}\big(|\nabla f|^2\big) - \langle \nabla f, \nabla \Delta_{\mathrm{g}} f\rangle = |\mathrm{Hess}\, f|^2 + \mathrm{Ric}(\nabla f, \nabla f), \tag{5-1}$$

valid in Riemannian manifolds, and its modification, accounting for the weight, on weighted Riemannian manifolds. We have already seen in (3-2) that the natural operator in the weighted setting is $\Delta f = \Delta_{\mathrm{g}} f - \langle \nabla V, \nabla f\rangle$, and the replacement of Δ_{g} with Δ in the left hand side of (5-1) gives

$$\frac{1}{2}\Delta\big(|\nabla f|^2\big) - \langle \nabla f, \nabla \Delta f\rangle = |\mathrm{Hess}\, f|^2 + \mathrm{Ric}_{\mathfrak{m}}(\nabla f, \nabla f) \tag{5-2}$$

where now $\mathrm{Ric}_{\mathfrak{m}}$ is the "weighted" Ricci tensor

$$\mathrm{Ric}_{\mathfrak{m}} := \mathrm{Ric} + \mathrm{Hess}\, V.$$

Still in the smooth setting, the starting point of the CD theory, instead, is a concavity inequality satisfied by the Jacobian function

$$\mathfrak{J}(s, x) := \det\big[\nabla_x \exp(s\nabla\phi(x))\big],$$

in N-dimensional manifolds with $\mathrm{Ric} \geq K\,\mathrm{g}$, as long as $\mathfrak{J}(s, x) > -\infty$. Namely, a careful ODE analysis (see Villani [2009, Thm. 14.12]) shows that

$$\mathfrak{J}^{1/N}(s, x) \geq \tau^{(s)}_{K,N}(\theta)\mathfrak{J}^{1/N}(1, x) + \tau^{(1-s)}_{K,N}(\theta)\mathfrak{J}^{1/N}(0, x) \qquad \forall s \in [0, 1] \tag{5-3}$$

where $\theta = \mathsf{d}(x, \exp(\nabla\phi(x)))$, $\tau^{(s)}_{K,N}(\theta) = s^{1/N}\sigma^{(s)}_{K/(N-1)}(\theta)^{1-1/N}$ and, for $s \in [0,1]$,

$$\sigma^{(s)}_\kappa(\theta) := \begin{cases} \frac{\mathfrak{s}_\kappa(s\theta)}{\mathfrak{s}_\kappa(\theta)} & \text{if } \kappa\theta^2 \neq 0 \text{ and } \kappa\theta^2 < \pi^2, \\ s & \text{if } \kappa\theta^2 = 0, \\ +\infty & \text{if } \kappa\theta^2 \geq \pi^2, \end{cases} \tag{5-4}$$

with

$$\mathfrak{s}_\kappa(r) := \begin{cases} \frac{\sin(\sqrt{\kappa}r)}{\sqrt{\kappa}} & \text{if } \kappa > 0, \\ r & \text{if } \kappa = 0, \\ \frac{\sinh(\sqrt{-\kappa}r)}{\sqrt{-\kappa}} & \text{if } \kappa < 0. \end{cases} \tag{5-5}$$

For $\kappa\theta < \pi^2$, the coefficients $\sigma^{(s)}_\kappa(\theta)$ solve the ODE $\sigma'' + \kappa\theta^2\sigma = 0$ on $[0,1]$, with $\sigma(0) = 0$, $\sigma(1) = 1$. In the limit as $N \to \infty$ the inequality (5-3) becomes

$$\log \mathfrak{J}(s,x) \geq s \log \mathfrak{J}(1,x) + (1-s)\log \mathfrak{J}(0,x) + K\frac{s(1-s)}{2}\theta^2. \tag{5-6}$$

5.1 The BE **theory.** In the framework of Dirichlet forms and Γ-calculus (see Section 3.2) there is still the possibility to write (5-2) in the weak form of an inequality. Let us start from the observation that, because of the locality assumption, one has $\Gamma(f) = \frac{1}{2}\Delta f^2 - f\Delta f$. Now, we may write (5-2) in terms of the *iterated* Γ operator $\Gamma_2(f) = \frac{1}{2}\Delta\Gamma(f) - \Gamma(f, \Delta f)$, to get the formula

$$\Gamma_2(f) = |\mathrm{Hess}\, f|^2 + \mathrm{Ric}_{\mathfrak{m}}(\nabla f, \nabla f). \tag{5-7}$$

Still, using *only* Γ and Δ the left hand side in (5-7) can be given a meaning, if one has an algebra $\mathcal{A}$ of "nice" functions dense in $\mathbb{V}$, where nice means stable under the actions of the operators Γ and Δ (such as C^∞_c in Riemannian manifolds, smooth cylindrical functions in Gaussian spaces, etc.). By estimating from below the right hand side in (5-7) with objects which makes sense in the abstract setting, this leads to the the following definition:

Definition 5.1 (BE(K,N) condition)**.** Let $K \in \mathbb{R}$ and $N \in [1,\infty)$. We say that the Bakry-Émery condition BE(K,N) holds if

$$\Gamma_2(f) \geq \frac{(\Delta f)^2}{N} + K\Gamma(f) \qquad \forall f \in \mathcal{A}. \tag{5-8}$$

It is not hard to see that this is a *strongly consistent* definition of upper bound on dimension and lower bound on Ricci tensor, in the smooth setting of weighted n-dimensional Riemannian manifolds: more precisely when V is constant BE(K,n) holds if and only if $\mathrm{Ric} \geq K\mathrm{g}$ and, when $N > n$, BE(K,N) holds if and only if

$$\mathrm{Ric}_{\mathfrak{m}} \geq K\mathrm{g} + \frac{1}{N-n}\nabla V \otimes \nabla V. \tag{5-9}$$

The expression $\mathrm{Ric}_{\mathfrak{m}} - (N-n)^{-1}\nabla V \otimes \nabla V$ appearing in (5-9) is also called Bakry-Émery N-dimensional tensor and denoted $\mathrm{Ric}_{N,\mathfrak{m}}$, so that the formula reads $\mathrm{Ric}_{N,\mathfrak{m}} \geq K\mathsf{g}$. The possibility to introduce an "effective" dimension, possibly larger than the topological one is a richness of the BE and the CD theories. This separation of dimensions is very useful to include warped products, collapsing phenomena (i.e. changes of dimension under measured Gromov–Hausdorff limits) and it reveals to be a crucial ingredient also in the localization technique (see Section 5.3 below), where N-dimensional isoperimetric problems are factored into a family on N-dimensional isoperimetric problems on segments endowed with a weighted Lebesgue measure.

Last but not least, it is remarkable Bakry [1997] that iterated Γ operators can also provide a consistent notion of Hessian in this abstract setting, via the formula

$$\text{Hess}\, f(\nabla g, \nabla h) := \frac{1}{2}\big[\Gamma(g, \Gamma(f,h)) + \Gamma(h, \Gamma(f,g)) - \Gamma(f, \Gamma(g,h))\big]. \tag{5-10}$$

As illustrated in the recent monograph Bakry, Gentil, and Ledoux [2015], curvature-dimension bounds in the synthetic form (5-8), when combined with clever interpolation arguments originating from Bakry and Émery [1985], lead to elegant and general proofs of many functional inequalities (Poincaré, Sobolev and Logarithmic Sobolev, Nash inequalities, Gaussian isoperimetric inequalities, etc..), often with sharp constants.

5.2 The CD **theory.** This theory is formulated in terms of suitable convexity properties, along geodesics in $\mathcal{P}_2(X)$, of integral functionals. A good analogy that should be kept in mind is with the purely metric theory of Alexandrov spaces (see e.g. D. Burago, Y. Burago, and Ivanov [2001]), where lower bounds on sectional curvature depend on concavity properties of $\mathsf{d}^2(\cdot, y)$, $y \in X$. The main new ingredient in the CD theory is the role played by the reference measure $\mathfrak{m}$.

Throughout this section we assume that the reference measure $\mathfrak{m}$ satisfies the growth condition

$$\mathfrak{m}(B_r(\bar{x})) \leq a\, \mathrm{e}^{b r^2} \quad \forall r > 0, \text{ for some } \bar{x} \in X \text{ and } a,\, b \geq 0. \tag{5-11}$$

In his pioneering paper McCann [1997], McCann pointed out the interest of convexity along constant speed geodesics of $\mathcal{P}_2(X)$ of integral functionals in Euclidean spaces such as the logarithmic entropy in (4-4), introducing the notion of *displacement convexity*, i.e. convexity along $\mathrm{Geo}(\mathcal{P}_2(X))$. More generally, by considering the dimensional counterparts of Ent, Rényi's entropies

$$\mathcal{E}_N(\mu) := -\int_X \varrho^{1/N}\, d\mathfrak{m} \quad \text{if } \mu = \varrho\mathfrak{m} + \mu^{\perp}, \tag{5-12}$$

(here $\rho\mathfrak{m} + \mu^{\perp}$ denotes the Radon–Nikodým decomposition of μ w.r.t. $\mathfrak{m}$) he provided an elegant proof of the Brunn–Minkowksi inequality in $\mathbb{R}^N$ based on displacement convexity.

Moving from Euclidean spaces to Riemannian manifolds it was soon understood in Cordero-Erausquin, McCann, and Schmuckenschläger [2006] and Otto and Villani

[2000] on the basis of (5-6) or Otto's calculus that the lower bound $\mathrm{Ric} \geq K$ implies the K-convexity inequality

$$\mathrm{Ent}(\mu_s) \leq (1-s)\mathrm{Ent}(\mu_0) + s\mathrm{Ent}(\mu_1) - \frac{K}{2}s(1-s)\mathrm{W}_2^2(\mu_0,\mu_1). \tag{5-13}$$

The key idea is to average the distorsion of volume along geodesics, using the inequality (5-6). Therefore (5-13) provides a *consistent* definition of Ricci lower bounds of Riemannian manifolds, later on proved to be strongly consistent in von Renesse and Sturm [2005] (namely on Riemannian manifolds, (5-13) implies $\mathrm{Ric} \geq K\mathrm{g}$). This motivated the definition of $\mathrm{CD}(K,\infty)$, given independently by Sturm [2006a,b] and Lott and Villani [2009].

Definition 5.2 ($\mathrm{CD}(K,\infty)$ condition)**.** A m.m.s. $(X,\mathsf{d},\mathfrak{m})$ satisfies the $\mathrm{CD}(K,\infty)$ condition if Ent is geodesically K-convex in $(\mathcal{P}_2(X),\mathrm{W}_2)$: every couple $\mu_0,\,\mu_1 \in D(\mathrm{Ent})$ can be connected by $\mu_s \in \mathrm{Geo}(\mathcal{P}_2(X))$ along which (5-13) holds.

For this and the many variants of the CD condition we will add the adjective *strong* to mean that the convexity property holds for *all* geodesics, and the suffix *loc* to mean that the property is only satisfied locally (i.e. for measures with localized support).

A crucial advantage of the CD theory is a clear separation of the roles of the distance d and the reference measure $\mathfrak{m}$: the former enters only in W_2, the latter enters only in Ent; in the theories based on energy structures, instead, the measure $\mathfrak{m}$ and the "metric" Γ both enter in the construction of a single object, namely $\mathcal{E}$. The following result, obtained in Ambrosio, Gigli, and Savaré [2014a] (see also Gigli, Kuwada, and S.-I. Ohta [2013], dealing with Alexandrov spaces) extends the key identity (4-5) and the representation of P_t as metric gradient flow of Ent w.r.t. W_2 to the whole class of $\mathrm{CD}(K,\infty)$ m.m.s. Its proof motivated some of the development of calculus in m.m.s. (particularly the notion of test plan) I illustrated in Section 3: in particular it involves a metric version of the superposition principle Lisini [2007] and the validity of the Hamilton–Jacobi equation

$$\frac{d}{dt}\mathsf{Q}_t f(x) + \frac{1}{2}|\nabla \mathsf{Q}_t f|^2(x) = 0 \tag{5-14}$$

even in the metric setting (with a few exceptional points in space-time). From (5-14) one can obtain Kuwada [2010] another key connection between the Lagrangian and Eulerian points of view: the estimate of metric derivative with Fisher information:

$$|\mu_t'|^2 \leq \int_{\{\mathsf{P}_t\varrho>0\}} \frac{|\nabla \mathsf{P}_t\varrho|_*^2}{\mathsf{P}_t\varrho}\,d\mathfrak{m} \quad \text{for } \mathcal{L}^1\text{-a.e. } t>0, \text{ with } \mu_t := \mathsf{P}_t\varrho\,\mathfrak{m}.$$

Theorem 5.3. *Let $(X,\mathsf{d},\mathfrak{m})$ be a $\mathrm{CD}(K,\infty)$ m.m.s. and let $\varrho \in L^1(X,\mathfrak{m})$ be nonnegative with $\varrho\mathfrak{m} \in \mathcal{P}_2(X)$. Then:*

(a) the curve of measures $\mu_t := \mathsf{P}_t\varrho\,\mathfrak{m}$ is the unique W_2-gradient flow of Ent *starting from $\varrho\mathfrak{m}$;*

(b) $|\nabla^-\mathrm{Ent}|(\varrho\mathfrak{m})$ is finite if and only if $\mathrm{Ch}_2(\sqrt{\varrho}) < \infty$ and

$$|\nabla^-\mathrm{Ent}|^2(\varrho\mathfrak{m}) = \int_{\{\varrho>0\}} \frac{|\nabla\varrho|_*^2}{\varrho}\,d\mathfrak{m} = 4\,\mathrm{Ch}_2(\sqrt{\varrho}). \tag{5-15}$$

Even though the two notions of heat flow can be identified, they are conceptually different and their natural domains differ (in particular when, thanks to contractivity, the W_2-gradient flow of Ent can be extended to the whole of $\mathcal{P}_2(X)$). For this reason we will use the distinguished notation

$$\mathcal{H}_t\mu := \mathsf{P}_t\varrho\,\mathfrak{m} \qquad \text{whenever } \mu = \varrho\mathfrak{m} \in \mathcal{P}_2(X). \tag{5-16}$$

Let us move now to the "dimensional" theory, i.e. when we want to give an upper bound $N < \infty$ on the dimension, with $N > 1$. In this case the convexity conditions should take into account also the parameter N, and the distorsion coefficients $\tau^{(s)}_{K,N}(\theta) = s^{1/N}\sigma^{(s)}_{K/(N-1)}(\theta)^{1-1/N}$ are those of (5-3), (5-4). In this text I follow more closely Sturm's axiomatization Sturm [2006a,b] (J. Lott and C. Villani's one Lott and Villani [2009] uses a more general classes of entropies, not necessarily power-like, singled out by R. McCann).

Definition 5.4 (CD(K,N) spaces)**.** We say that $(X,\mathsf{d},\mathfrak{m})$ satisfies the curvature dimension condition CD(K,N) if the functionals $\mathcal{E}_M$ in (5-12) satisfy: for all $\mu_0 = \varrho_0\mathfrak{m}$, $\mu_1 = \varrho_1\mathfrak{m} \in \mathcal{P}_2(X)$ with bounded support there exists $\boldsymbol{\eta} \in \mathrm{OptGeo}(\mu_0,\mu_1)$ with

$$\mathcal{E}_{N'}(\mu_s) \le -\int \big[\tau^{(1-s)}_{K,N'}(\mathsf{d}(\gamma_0,\gamma_1))\varrho_0^{-1/N'}(\gamma_0) + \tau^{(s)}_{K,N'}(\mathsf{d}(\gamma_0,\gamma_1))\varrho_1^{-1/N'}(\gamma_1)\big]\,d\boldsymbol{\eta}(\gamma) \tag{5-17}$$

for all $N' \ge N$ and $s \in [0,1]$, where $\mu_s := (e_s)_\sharp\boldsymbol{\eta}$.

Besides N-dimensional Riemannian manifolds with Ricci $\ge K\,\mathrm{Id}$ and Finsler manifolds S.-i. Ohta [2009], it has been proved by A. Petrunin in Petrunin [2011] that the class CD$(0,N)$ includes also positively curved N-dimensional spaces, in the sense of Alexandrov. The definition is built in such a way that the curvature dimension condition becomes weaker as N increases, and it implies (by taking $N' \to \infty$ in (5-17) and using that $N' + N'\mathcal{E}_{N'} \to \mathrm{Ent}$) the CD$(K,\infty)$ condition. These curvature dimension conditions, besides being stable w.r.t. m-GH convergence, can be used to establish functional and geometric inequalities, often with sharp constants, see Section 5.3. However, except in the cases $K = 0$ or $N = \infty$ (and under the non-branching assumption) it is not clear why the CD condition holds globally, when it holds locally, and T. Rajala built indeed in Rajala [2016] a highly branching $\mathrm{CD}_{\mathrm{loc}}(0,4)$ space which, for no value of K and N, is CD(K,N). This *globalization* problem is a fundamental issue, since only the global condition, without artificial scale factors, can be proved to be stable w.r.t. m-GH convergence. Recently, in the class of essentially non-branching m.m.s. (see Definition 5.6 below), the globalization problem has been brilliantly solved by Cavalletti and Milman [n.d.], building on a very refined analysis of the metric Hamilton–Jacobi equation (5-14) and the regularity of Q_t. The globalization problem led K. Bacher and K. T. Sturm to the introduction in Bacher and Sturm [2010] of a weaker curvature-dimension condition CD*, involving the smaller coefficients $\sigma^{(s)}_\kappa(\theta)$:

Definition 5.5 (CD$^*(K,N)$ spaces)**.** We say that $(X,\mathsf{d},\mathfrak{m})$ satisfies the reduced curvature dimension condition CD$^*(K,N)$ if the functionals $\mathcal{E}_M$ in (5-12) satisfy: for all

$\mu_0 = \varrho_0 \mathfrak{m}$, $\mu_1 = \varrho_1 \mathfrak{m} \in \mathcal{P}_2(X)$ with bounded support there exists an optimal geodesic plan $\boldsymbol{\eta} \in \mathrm{OptGeo}(\mu_0, \mu_1)$ with

(5-18)
$$\mathcal{E}_{N'}(\mu_s) \le -\int \big[\sigma^{(1-s)}_{K/N'}(\mathsf{d}(\gamma_0,\gamma_1))\varrho_0^{-1/N'}(\gamma_0) + \sigma^{(s)}_{K/N'}(\mathsf{d}(\gamma_0,\gamma_1))\varrho_1^{-1/N'}(\gamma_1)\big]\, d\boldsymbol{\eta}(\gamma)$$

for all $N' \ge N$ and $s \in [0,1]$, where $\mu_s := (e_s)_\sharp \boldsymbol{\eta}$.

At the local level the two classes of spaces coincide, more precisely

$$\bigcap_{K'<K} \mathrm{CD}^*_{\mathrm{loc}}(K', N) \sim \bigcap_{K'<K} \mathrm{CD}_{\mathrm{loc}}(K', N).$$

In addition, the inclusion $\mathrm{CD}(K,N) \subset \mathrm{CD}^*(K,N)$ can be reversed at the price of replacing, in $\mathrm{CD}(K,N)$, K with $K^* = K(N-1)/N$ (in particular one can still obtain from $\mathrm{CD}^*(K,N)$ functional inequalities, but sometimes with non-optimal constants). More results can be established in the class of the *essentially non-branching* m.m.s., first singled out in Rajala and Sturm [2014]. Recall that a metric space (X,d) is said to be non-branching if the map $(e_0, e_t) : \mathrm{Geo}(X) \to X^2$ is injective for all $t \in (0,1]$ (for instance Riemannian manifolds and Alexandrov spaces are non-branching). Analogously we can define the non-branching property of a subset E of $\mathrm{Geo}(X)$.

Definition 5.6 (Essential non-branching)**.** We say that $(X,\mathsf{d},\mathfrak{m})$ is essentially non-branching if any $\boldsymbol{\eta} \in \mathrm{OptGeo}(\mu_0,\mu_1)$ with $\mu_i \in \mathcal{P}_2(X)$ and $\mu_i \ll \mathfrak{m}$ is concentrated on a Borel set of non-branching geodesics.

It has been proved in Rajala and Sturm [ibid.] that *strong* $\mathrm{CD}(K,\infty)$ spaces are essentially non-branching. In the class of essentially non-branching m.m.s. the CD^* condition gains the local-to-global property, namely $\mathrm{CD}^*_{\mathrm{loc}}(K,N) \sim \mathrm{CD}^*(K,N)$.

Finally, we can complete the list of CD spaces with the *entropic* CD^e spaces, introduced in Erbar, Kuwada, and Sturm [2015]. Their definition involves the new notion of (K,N)-*convexity*. In a geodesic space X, a function S is said to be (K,N)-convex if for any pair of points γ_0, $\gamma_1 \in X$ there exists $\gamma \in \mathrm{Geo}(X)$ connecting these two points such that $(S\circ\gamma)'' \ge K\mathsf{d}^2(\gamma_1,\gamma_0) + |(S\circ\gamma)'|^2/N$ in $(0,1)$, in the sense of distributions. In the smooth setting, this is equivalent to either the inequalities

$$\mathrm{Hess}\, S \ge K\,\mathrm{Id} + \frac{1}{N}(\nabla S \otimes \nabla S), \qquad \mathrm{Hess}\, S_N \le -\frac{K}{N} S_N \tag{5-19}$$

for $S_N := \exp(-S/N)$, while in the metric setting this property can be formulated in terms of the inequality

$$S_N(\gamma_t) \ge \sigma^{(1-t)}_{K/N}(\mathsf{d}(\gamma_0,\gamma_1))S_N(\gamma_0) + \sigma^{(t)}_{K/N}(\mathsf{d}(\gamma_0,\gamma_1))S_N(\gamma_1)$$

for all $t \in [0,1]$ and $\gamma \in \mathrm{Geo}(X)$. These facts, and the differential inequality $\ell''(s) \ge (\ell'(s))^2/n + \mathrm{Ric}\,(\gamma'(s),\gamma'(s))$, valid in the smooth setting with $\ell(s) = -\log \mathfrak{J}(s,x)$ and $\gamma(s) = \exp(s\nabla\phi(x))$, motivate the following definition.

Definition 5.7 ($\mathrm{CD}^e(K,N)$ spaces)**.** We say that $(X,\mathsf{d},\mathfrak{m})$ satisfies the entropic curvature dimension condition $\mathrm{CD}^e(K,N)$ if the functional Ent is (K,N)-convex in $\mathcal{P}_2(X)$.

The following result (due for the first part to Erbar, Kuwada, and Sturm [ibid.], for the second part to Cavalletti and Milman [n.d.]) provides, under the essential non-branching assumption, a basic equivalence between all these definitions. In addition, Cavalletti and Milman [ibid.] provides also equivalence with another definition based on disintegrations of $\mathfrak{m}$ (as in Alberti's representations mentioned in Section 3.1) induced by transport rays of the optimal transport problem with cost=distance.

Theorem 5.8 (Equivalence under essential non-branching)**.** *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an essentially non-branching m.m.s. with* $\mathfrak{m}(X) < \infty$. *Then* $(X, \mathsf{d}, \mathfrak{m})$ *is* $\mathrm{CD}^e(K, N)$ *iff it is* $\mathrm{CD}^*(K, N)$ *iff it is* $\mathrm{CD}(K, N)$.

Finally, inspired by the calculations done in the smooth setting in Villani [2009] (see (29.2) therein) we also proved in Ambrosio, Mondino, and Savaré [2016] that, for essentially non-branching m.m.s., the $\mathrm{CD}^*(K, N)$ condition is equivalent to a distorted convexity inequality for Rényi's entropy

$$\mathcal{E}_N(\mu_s) \le (1-s)\mathcal{E}_N(\mu_0) + s\mathcal{E}_N(\mu_1) - K\mathfrak{A}_N^{(s)}(\mu) \qquad \forall s \in [0,1],$$

where the dimensional distorsion is present also in the action term

$$\mathfrak{A}_N^{(s)}(\mu) := \int_0^1 \int_X G(t,s)\varrho_t^{1-1/N}|v_t|^2\, d\mathfrak{m}\, dt \qquad \mu_t = \varrho_t\mathfrak{m} + \mu_t^\perp.$$

Here $|v_s|$ is the minimal velocity field of μ_s (which still makes sense in the metric setting, by an adaptation of Theorem 4.2) and G is a suitable Green function. In the limit as $N \to \infty$, along geodesics $\mu_s \ll \mathfrak{m}$, the action term converges to

$$\tfrac{1}{2}s(1-s)\mathrm{W}_2^2(\mu_0, \mu_1).$$

5.3 Geometric and functional inequalities. We recall here some of the most important geometric and functional inequalities by now available in the setting of $\mathrm{CD}(K, N)$ spaces.

Bishop–Gromov inequality and Bonnet–Myers diameter estimate: Villani [2009] The map

$$r \mapsto \frac{\mathfrak{m}(B_r(x_0))}{\int_0^r s^{K,N}(t)\,\mathrm{d}t} \quad \text{is nonincreasing for all } x_0 \in X.$$

When N is an integer $s^{K,N}$ can be interpreted as the functions providing the measure of the spheres in the model space of Ricci curvature K and dimension N. If $K > 0$ the diameter of X is bounded by $\pi\sqrt{(N-1)/K}$.

Upper bounds on $\Delta\mathsf{d}^2$: Gigli [2014b] Under a suitable strict convexity assumption of Ch_2, in $\mathrm{CD}^*(K, N)$ spaces one has the upper bound $\Delta\mathsf{d}^2 \le \gamma_{K,N}(\mathsf{d})\mathfrak{m}$ in the weak sense (with $\gamma_{0,N} \equiv 2N$).

Spectral gap and Poincaré inequality: If $K > 0$ then

$$\int_X (f - \bar{f})^2\, d\mathfrak{m} \le \frac{N-1}{NK}\int_X |\nabla f|_*^2\, d\mathfrak{m}, \quad \text{with} \quad \bar{f} = \int_X f\, d\mathfrak{m}.$$

In more recent times, B. Klartag used L^1 optimal transportation methods and the *localization* technique (going back to the work of Payne–Weinberger Payne and Weinberger [1960] and then further developed in the context of convex geometry by Gromov–Milman and Kannan–Lovász–Simonovitz) to provide in Klartag [2017] a new proof of the Levy–Gromov isoperimetric inequality in Riemannian manifolds, one of the few inequalities not available with Γ-calculus tools. Shortly afterwards, F. Cavalletti and A. Mondino have been able to extend in Cavalletti and Mondino [2017a,b] the localization technique to obtain in the class of essentially non-branching $\mathrm{CD}(K, N)$ m.m.s. this and many other inequalities with sharp constants.

Levy–Gromov inequality: Cavalletti and Mondino [2017a] If $\mathfrak{m}(X) = 1$ and $K > 0$, then for any Borel set $E \subset X$ one has

$$\mathfrak{m}^+(E) \geq \frac{|\partial B|}{|S|}$$

where $\mathfrak{m}^+(E) = \liminf_{r\downarrow 0}(\mathfrak{m}(E_r) - \mathfrak{m}(E))/r$ is the Minkowski content of E (coinciding with the perimeter of the boundary, for sufficiently nice sets E) and B is a spherical cap in the N-dimensional sphere S with Ricci curvature equal to K such that $|B|/|S| = \mathfrak{m}(E)$. This is part of a more general isoperimetric statement proved in Cavalletti and Mondino [ibid.] involving isoperimetric profiles and model spaces for manifolds with dimension smaller than N, Ricci curvature larger than K and diameter smaller than D discovered in Milman [2015]. In $\mathrm{RCD}(K, \infty)$ spaces see also Bakry, Gentil, and Ledoux [2015, Cor. 8.5.5], Ambrosio and Mondino [2016].

Log-Sobolev and Talagrand inequalities: If $K > 0$ and $\mathfrak{m}(X) = 1$ then

$$\frac{KN}{2(N-1)}\mathrm{W}_2^2(\varrho\mathfrak{m}, \mathfrak{m}) \leq \mathrm{Ent}(\varrho\mathfrak{m}) \leq \frac{N-1}{2KN}\int_{\{\varrho>0\}} \frac{|\nabla\varrho|_*^2}{\varrho}\, d\mathfrak{m}.$$

Sobolev inequalities: If $K > 0$, $N > 2$, $2 < p \leq 2N/(N-2)$, then (see also Bakry, Gentil, and Ledoux [2015, Thm. 6.8.3])

$$\|f\|_{L^p}^2 \leq \|f\|_{L^2}^2 + \frac{(p-2)(N-1)}{KN}\int_X |\nabla f|_*^2\, d\mathfrak{m}.$$

6 Stability of curvature-dimension bounds and heat flows

In this section we deal with *pointed* m.m.s. $(X, \mathsf{d}, \mathfrak{m}, \bar{x})$, a concept particularly useful when (X, d) has infinite diameter and blow-up procedures are performed. Pointed metric measure structures are identified by measure-preserving isometries of the supports which preserve the base points. Remarkably, Gromov's *reconstruction theorem* Gromov [2007] (extended in Gigli, Mondino, and Savaré [2015] to spaces with infinite mass), characterizes the equivalence classes by the family of functionals

$$\varphi^\star\big[(X, \mathsf{d}, \mathfrak{m}, \bar{x})\big] := \int_{X^N} \varphi\big(\mathsf{d}(x_i, x_j)_{i,j=1}^N\big)\, d\delta_{\bar{x}}(x_1)\, d\mathfrak{m}^{\otimes N-1}(x_2, \dots, x_N), \tag{6-1}$$

where $N \geq 2$ and $\varphi : \mathbb{R}^{N^2} \to \mathbb{R}$ is continuous with bounded support.

A fundamental property of the CD condition is the stability w.r.t. (pointed) measured Gromov–Hausdorff convergence, established (in slightly different settings) in Lott and Villani [2009] and Sturm [2006a,b]. Building on Gromov's seminal work Gromov [2007] on convergence for metric structures, this notion of convergence for (pointed) metric measure structures was introduced by K. Fukaya in connection with spectral stability properties, and then it has been a crucial ingredient in the remarkable program developed in the 90's by Cheeger and Colding [1996, 1997, 2000a,b], dealing with the fine structure of Ricci limit spaces (particularly in the so-called non-collapsed case).

According to local and global assumptions on the sequence of metric measure structures, several definitions of convergence are possible. For the sake of illustration, I follow here the definition of *pointed measured Gromov convergence* in Gigli, Mondino, and Savaré [2015] and Greven, Pfaffelhuber, and Winter [2009], based on the reconstruction theorem. As for Sturm's $\mathbb{D}$-convergence Sturm [2006a], this notion of convergence, while avoiding at the same time finiteness of the measure and local compactness, is consistent with pointed mGH-convergence when the pointed m.m.s. have more structure (e.g. under a uniform doubling condition, ensured in the $\mathrm{CD}(K,N)$ case, $N<\infty$, by the Bishop–Gromov inequality). Within this approach, not relying on doubling and local compactness, general $\mathrm{CD}(K,\infty)$ spaces can also be treated (see also Sturm [2006a] and Shioya [2016] for a comparison with Gromov's notions Gromov [2007] of box and concentration convergence).

Definition 6.1 (pmG-convergence)**.** We say that $(X^h,\mathsf{d}^h,\mathfrak{m}^h,\bar{x}^h)$ converge to $(X,\mathsf{d},\mathfrak{m},\bar{x})$ if for every functional $\varphi^\star$ as in (6-1) one has

$$\lim_{h\to\infty}\varphi^\star\big[(X^h,\mathsf{d}^h,\mathfrak{m}^h,\bar{x}^h)\big]=\varphi^\star\big[(X,\mathsf{d},\mathfrak{m},\bar{x})\big].$$

The following result from Gigli, Mondino, and Savaré [2015], which includes as a particular case those proved in Cheeger and Colding [2000b] for Ricci limit spaces and those proved in Shen [1998] for Finsler manifolds, provides not only stability of the $\mathrm{CD}(K,\infty)$ condition, but also convergence of Cheeger's energies and heat flows; for Cheeger's energies, the right notion of convergence is Mosco convergence (see Mosco [1969]), a notion of variational convergence particularly useful in connection to stability of variational inequalities, that can be adapted also to the case when sequences of metric measure structures are considered.

Theorem 6.2. *Assume that* $(X^h,\mathsf{d}^h,\mathfrak{m}^h,\bar{x}^h)$ *are* $\mathrm{CD}(K,\infty)$ *pointed m.m.s., pmG-convergent to* $(X,\mathsf{d},\mathfrak{m},\bar{x})$. *Then:*

(a) $(X,\mathsf{d},\mathfrak{m})$ *is* $\mathrm{CD}(K,\infty)$*;*

(b) the Cheeger energies $\mathrm{Ch}_{2,h}$ *relative to* $(X^h,\mathsf{d}^h,\mathfrak{m}^h)$ *Mosco converge to the Cheeger energy* Ch_2 *relative to* $(X,\mathsf{d},\mathfrak{m})$*;*

(c) the heat flows P^h_t *relative to* $(X^h,\mathsf{d}^h,\mathfrak{m}^h)$ *converge to the heat flow* P_t *relative to* $(X,\mathsf{d},\mathfrak{m})$.

In order to give a mathematically rigorous and specific meaning to (b) and (c) one has to use the so-called *extrinsic* approach, embedding isometrically all spaces into a single complete and separable metric space (Z, d_Z); within this realization of the convergence, which is always possible, pmG-convergence corresponds to weak convergence of $\mathfrak{m}^h$ to $\mathfrak{m}$. The proof of parts (b) and (c) of Theorem 6.2 relies once more on Theorem 5.3 and particularly on the key identification (5-15), to transfer information from the Lagrangian level (the one encoded in the definition of convergence) to the Eulerian level.

7 Adding the Riemannian assumption

One of the advantages of the CD theory, when compared to the BE theory dealing essential with quadratic energy structures, is its generality: it provides a synthetic language to state and prove functional and geometric inequalities in structures which are far, even on small scales, from being Euclidean. On the other hand, as advocated in Gromov [1991] and Cheeger and Colding [1997, Appendix 2], the description of the closure with respect to m-GH convergence of Riemannian manifolds requires a finer axiomatization, possibly based on the linearity of the heat flow. Within the CD theory, a good step forward in this direction has been achieved in Ambrosio, Gigli, and Savaré [2014b], see also Ambrosio, Gigli, Mondino, and Rajala [2015]:

Definition 7.1 (RCD(K,∞) condition)**.** A $(X, \mathsf{d}, \mathfrak{m})$ m.m.s. satisfies the RCD(K,∞) condition if it is CD(K,∞) and Ch_2 is a quadratic form, i.e. if $(X, \mathsf{d}, \mathfrak{m})$ is infinitesimally Hilbertian according to Definition 3.8.

This new definition is useful (for instance in the proof of rigidity results by compactness arguments) only if the additional "Riemannian" axiom, equivalent to the linearity of the semigroup $\mathsf{P}_\dagger$, is stable with respect to the measured Gromov–Hausdorff convergence and its variants. Simple examples show that, by itself, it is not. However, the remarkable fact is that the extra axiom is stable, when combined with the CD(K,∞) condition. This stability property could be seen as a consequence of Mosco convergence (see Theorem 6.2), since quadraticity is stable under Mosco convergence. However, the original proof of stability of the RCD(K,∞) condition given in Ambrosio, Gigli, and Savaré [2014b] uses the full strength of the Riemannian assumption and relies on the characterization of RCD(K,∞) in terms of the EVI_K-property of the heat flow $\mathcal{H}_t$.

Theorem 7.2 (Ambrosio, Gigli, and Savaré [ibid.], Ambrosio, Gigli, Mondino, and Rajala [2015])**.** $(X, \mathsf{d}, \mathfrak{m})$ *is* RCD(K,∞) *if and only if the heat semigroup* $\mathcal{H}_t$ *in* (5-16) *satisfies the* EVI_K *property*

$$\frac{d}{dt}\frac{1}{2}\mathrm{W}_2^2(\mathcal{H}_t\mu, \nu) \le \mathrm{Ent}(\nu) - \mathrm{Ent}(\mathcal{H}_t\mu) - \frac{K}{2}\mathrm{W}_2^2(\mathcal{H}_t\mu, \nu) \tag{7-1}$$

for all initial datum $\mu = \varrho\mathfrak{m} \in \mathcal{P}_2(X)$*, and all* $\nu \in \mathcal{P}_2(X)$*.*

Since EVI_K solutions are metric gradient flows, the previous theorem could also have been stated in terms of a semigroup satisfying the EVI_K property (this formulation, only apparently weaker, is useful for instance in connection with the stability of heat flows in

the RCD setting). EVI_K solutions are a crucial technical tool for more than one reason: first, as we have seen in Theorem 7.2, they encode in a single condition both the CD and the Riemannian assumption; even more (see Ambrosio, Gigli, and Savaré [2008] and Daneri and Savaré [2014] for a more complete account of the EVI theory), they enjoy strong stability and contractivity properties that allow at once the extension of $\mathcal{H}_t$ to the whole of $\mathcal{P}_2(X)$, with $\mathrm{W}_2(\mathcal{H}_t\mu, \mathcal{H}_t\nu) \le e^{-Kt}\mathrm{W}_2(\mu,\nu)$. Finally, S. Daneri and G. Savaré discovered in Daneri and Savaré [2008] that the existence of EVI_K solutions, for a given function S in a geodesic space, encodes also the strong convexity (i.e. convexity along all constant speed geodesics). As a consequence, $\mathrm{RCD}(K,\infty)$ spaces are strong $\mathrm{CD}(K,\infty)$ spaces and we obtain from Rajala and Sturm [2014] also the essential non-branching property of this new class of spaces.

In Ambrosio, Gigli, and Savaré [2014b] we proved several properties of RCD spaces, and many more have been proved in subsequent papers (see the next section). To conclude this section, I will describe results which establish an essential equivalence between the RCD and the BE theories, both in the dimensional and adimensional case. The connection can be established in one direction using Cheeger's energy Ch_2, in the other direction using the intrinsic distance $\mathsf{d}_{\mathcal{E}}$. A precursor of these results is Kuwada [2010], which first provided the equivalence in the Riemannian setting of gradient contractivity $|\nabla \mathsf{P}_t f|^2 \le e^{-2Kt}\mathsf{P}_t|\nabla f|^2$ (namely the integrated form of $\mathrm{BE}(K,\infty)$) and contractivity of W_2 under the heat flow. The advantages of this "unification" of the theories are evident: at the RCD level one can use (with the few limitations I already mentioned) all power of Γ-calculus, having at the same time all stability and geometric properties granted by the metric point of view.

For the sake of simplicity, I will state the next results for the case when $\mathfrak{m}$ is finite measure, but most results have been proved also in the more general setting, under suitable global assumptions analogous to (5-11).

Theorem 7.3 (Ambrosio, Gigli, and Savaré [2014b], Ambrosio, Gigli, and Savaré [2015]). *If $(X,\mathsf{d},\mathfrak{m})$ is a $\mathrm{RCD}(K,\infty)$ m.m.s., then the Dirichlet form $\mathcal{E} = \mathrm{Ch}_2$ in $L^2(X,\mathfrak{m})$ satisfies the $\mathrm{BE}(K,\infty)$ condition. Conversely, assume that (X,τ) is a topological space, that $\mathcal{E} : L^2(X,\mathfrak{m})$ is a strongly local Dirichlet form with a carré du champ and that*

(a) the intrinsic distance $\mathsf{d}_{\mathcal{E}}$ induces the topology τ and is complete;

(b) any $f \in \mathbb{V}$ with $\Gamma(f) \le 1$ has a τ-continuous representative;

(c) the condition $\mathrm{BE}(K,\infty)$ holds.

Then the m.m.s. $(X,\mathsf{d}_{\mathcal{E}},\mathfrak{m})$ is a $\mathrm{RCD}(K,\infty)$ space.

From RCD **to** BE**.** This implication, in Theorem 7.3, requires strictly speaking a weaker formulation of $\mathrm{BE}(K,\infty)$, since in the metric measure setting no algebra stable under the action both of Δ and Γ is known. Nevertheless, not only can $\mathrm{BE}(K,\infty)$ be written in weak form, but it can be even proved (adapting to this setting Bakry's estimates in the frame of Γ-calculus, see Bakry, Gentil, and Ledoux [2014]) that

$$\mathrm{TestF}(X,\mathsf{d},\mathfrak{m}) := \left\{ f \in \mathrm{Lip}_b(X) \cap H^{1,2}(X,\mathsf{d},\mathfrak{m}) : \ \Delta f \in H^{1,2}(X,\mathsf{d},\mathfrak{m}) \right\} \tag{7-2}$$

is an algebra, and that the restriction of the iterated Γ operator Γ_2 to $\mathrm{TestF}(X, \mathsf{d}, \mathfrak{m})$ is measure-valued, with absolutely continuous negative part, and density bounded from below as in (5-8), see Savaré [2014].

From BE **to** RCD**.** The proof is based on the verification of the EVI_K-property using the (BB) representation of W_2^2 and suitable action estimates, an approach discovered for the purpose of contractivity in Otto and Westdickenberg [2005], and then improved and adapted to the metric setting in Daneri and Savaré [2014]. A different strategy, illustrated in Bakry, Gentil, and Ledoux [2015] (and then used also in Ambrosio, Erbar, and Savaré [2016], in the class of *extended* m.m.s.) involves instead the dual representation (4-2) of W_2^2.

Moving now to the dimensional case, the following definition (first proposed in Gigli [2015]) is natural.

Definition 7.4 ($\mathrm{RCD}^*(K, N)$ condition)**.** For $N \geq 1$, a $\mathrm{CD}^*(K, N)$ m.m.s. $(X, \mathsf{d}, \mathfrak{m})$ satisfies the $\mathrm{RCD}^*(K, N)$ condition if Ch_2 is a quadratic form, i.e. if $(X, \mathsf{d}, \mathfrak{m})$ is infinitesimally Hilbertian according to Definition 3.8.

In light of the recent equivalence result Cavalletti and Milman [n.d.] between the CD^* and CD conditions in essential non-branching m.m.s. (since, as we have seen, $\mathrm{RCD}(K, \infty)$ spaces are essentially non-branching), we now know that $\mathrm{RCD}^*(K, N)$ is equivalent to $\mathrm{RCD}(K, N)$, i.e. $\mathrm{CD}(K, N)$ plus infinitesimally Hilbertian.

Building on Theorem 7.2, the equivalence between the $\mathrm{BE}(K, N)$ and $\mathrm{RCD}^*(K, N)$ with $N < \infty$ has been proved, independently, in Erbar, Kuwada, and Sturm [2015] and Ambrosio, Mondino, and Savaré [2016]. The "distorsion" of the EVI_K property due to the dimension has been treated quite differently in the two papers: in Erbar, Kuwada, and Sturm [2015], instead of Rényi's entropies, a suitable dimensional modification of Ent, the so-called power entropy functional

$$\mathrm{Ent}_N(\mu) := \exp\Big(-\frac{1}{N}\mathrm{Ent}(\mu)\Big) \tag{7-3}$$

has been used. We have already seen in (5-19) that, in the smooth setting, the (K, N)-convexity condition for S can also be reformulated in terms of $S_N = \exp(-S/N)$. It turns out that, still in a Riemannian setting, the (K, N)-convexity condition can be formulated in terms of a $\mathrm{EVI}_{K,N}$ condition satisfied by the gradient flow γ_t of S: more precisely

$$\frac{d}{dt}\mathfrak{s}^2_{K/N}\left(\frac{1}{2}\mathsf{d}(\gamma_t, z)\right) + K\mathfrak{s}^2_{K/N}\left(\frac{1}{2}\mathsf{d}(\gamma_t, z)\right) \leq \frac{N}{2}\left(1 - \frac{S_N(z)}{S_N(\gamma_t)}\right)$$

for all $z \in X$, where $\mathfrak{s}_\kappa$ are defined in (5-5).

These facts are at the basis of the following result from Erbar, Kuwada, and Sturm [ibid.].

Theorem 7.5. *$(X, \mathsf{d}, \mathfrak{m})$ is a $\mathrm{RCD}^*(K, N)$ m.m.s. if and only if (X, d) is a length space and the heat semigroup $\mathcal{H}_t$ starting from any $\mu \in \mathcal{P}_2(X)$ satisfies the $\mathrm{EVI}_{K,N}$ property:*

$$\frac{d}{dt}\mathfrak{s}^2_{K/N}\left(\frac{1}{2}\mathrm{W}_2(\mathcal{H}_t\mu, \nu)\right) + K\mathfrak{s}^2_{K/N}\left(\frac{1}{2}\mathrm{W}_2(\mathcal{H}_t\mu, \nu)\right) \leq \frac{N}{2}\left(1 - \frac{\mathrm{Ent}_N(\nu)}{\mathrm{Ent}_N(\mathcal{H}_t\mu)}\right) \tag{7-4}$$

for all $\nu \in \mathcal{P}_2(X)$.

The characterization of $\mathrm{RCD}^*(K,N)$ provided in Ambrosio, Mondino, and Savaré [2016], involves, instead, a distorted EVI property of McCann's N-displacement convex entropies $\int_X U(\varrho)\,d\mathfrak{m}$ and their gradient flow, which is a *nonlinear diffusion equation*

$$\frac{d}{dt}\varrho_t = \Delta P(\varrho_t) \qquad \text{with} \qquad P(z) := zU'(z) - U(z).$$

This is very much in the spirit of Otto's seminal paper Otto [2001], motivated precisely by the long term behaviour, in Euclidean spaces, of solutions to these equations.

As we will see in Section 8, distorted Evolution Variational Inequalities lead also to new contractivity estimates, besides those which already characterize the curvature-dimension condition Wang [2011] and those that can be obtained by adapting Γ-calculus techniques to the RCD setting.

8 Properties of RCD spaces

Heat kernel and contractivity. In $\mathrm{RCD}(K,\infty)$ spaces, the EVI_K-property of the heat flow immediately leads to $\mathrm{W}_2^2(\mathcal{H}_t\mu,\mathcal{H}_t\nu) \le e^{-2Kt}\mathrm{W}_2^2(\mu,\nu)$ and then, by duality to the gradient contractivity property $|\nabla \mathsf{P}_t f|_*^2 \le e^{-2Kt}\mathsf{P}_t|\nabla f|_*^2$ and to the Feller property, namely $\mathsf{P}_t : L^\infty(X,\mathfrak{m}) \to C_b(X)$, $t>0$. Wang's log-Harnack inequality Wang [ibid.] also implies the regularization of $\mathcal{H}_t$, $t>0$, from $\mathcal{P}_2(X)$ to absolutely continuous probability measures with density in LlogL. These inequalities can be improved, taking the dimension into account, in various ways, see Wang [ibid.] and the more recent papers Bolley, Gentil, and Guillin [2014] and Erbar, Kuwada, and Sturm [2015]. On the Lagrangian side, from (7-4) one obtains

$$\begin{aligned}(8\text{-}1)\quad & \mathfrak{s}_{K/N}^2\left(\frac{1}{2}\mathrm{W}_2(\mathcal{H}_t\mu,\mathcal{H}_s\nu)\right) \le \\ & e^{-K(s+t)}\mathfrak{s}_{K/N}^2\left(\frac{1}{2}\mathrm{W}_2(\mathcal{H}_t\mu,\mathcal{H}_s\nu)\right) + \frac{N}{K}\left(1-e^{-K(s+t)}\right)\frac{(\sqrt{s}-\sqrt{t})^2}{2(s+t)},\end{aligned}$$

while on the Eulerian side one can recover in the RCD setting the inequality

$$|\nabla \mathsf{P}_t f|_*^2 + \frac{4Kt^2}{N(e^{2Kt}-1)}|\Delta \mathsf{P}_t f|^2 \le e^{-2Kt}\mathsf{P}_t|\nabla f|_*^2 \qquad \mathfrak{m}\text{-a.e. on } X$$

proved by Γ-calculus techniques in Bakry and Ledoux [2006]. In connection with nearly optimal heat kernel bounds, see Jiang, Li, and Zhang [2016].

Li–Yau and Harnack inequalities: If $K \ge 0$, $N < \infty$, $f > 0$ then the Γ-techniques (see for instance Bakry, Gentil, and Ledoux [2015, Cor. 6.7.6]) have been adapted in Garofalo and Mondino [2014] to the RCD setting to obtain the Li–Yau and Harnack inequalities:

$$\Delta(\log \mathsf{P}_t f) \ge -\frac{N}{2t} \quad t>0, \qquad \mathsf{P}_t f(x) \le \mathsf{P}_{t+s} f(y)\left(\frac{t+s}{t}\right)^{N/2} e^{\mathsf{d}^2(x,y)/(2s)}.$$

Tensorization: Tensorization is the persistence of geometric/analytic properties when we we consider two factors $(X_1, \mathsf{d}_1, \mathfrak{m}_1)$, $(X_2, \mathsf{d}_2, \mathfrak{m}_2)$ having both these properties, and their product

$$\big(X_1 \times X_2, \mathsf{d}, \mathfrak{m}_1 \times \mathfrak{m}_2\big) \qquad \text{with } \mathsf{d}^2((x_1', x_2'), (x_1, x_2)) := \mathsf{d}_1^2(x_1, x_1') + \mathsf{d}_2^2(x_2, x_2').$$

For instance, it is easily seen that the completeness and geodesic properties tensorize. At the level of CD spaces, we know from Sturm [2006b], Bacher and Sturm [2010], and Villani [2009] that essentially non-branching $\mathrm{CD}(0, N)$, $\mathrm{CD}(K, \infty)$ and $\mathrm{CD}^*(K, N)$ spaces all have the tensorization property. When we add the Riemannian assumption we get the strong $\mathrm{CD}(K, \infty)$ property and then the essential non-branching property. Therefore, taking also into account the tensorization of the infinitesimally Hilbertian property Ambrosio, Gigli, and Savaré [2014b] and Ambrosio, Pinamonti, and Speight [2015], we obtain that all spaces $\mathrm{RCD}^*(K, N)$ tensorize. Alternatively, one can use the equivalence results of Theorem 7.2 and Theorem 7.5 to obtain the tensorization from the BE theory.

Improved stability results: Thanks to the more refined calculus tools available in RCD spaces, and to the gradient contractivity available in the RCD setting, in Ambrosio and Honda [2017] the convergence result of Theorem 6.2 has been extended to the whole class of p-th Cheeger energies Ch_p, including also the total variation norm. This gives, among other things, also the stability of isoperimetric profiles and Cheeger's constant.

Splitting theorem: In Gigli [2014a], N. Gigli extended to the RCD setting the Cheeger–Gromoll splitting theorem: If $K \geq 0$, $N \in [2, \infty)$ and X contains a line, i.e. there exists $\gamma : \mathbb{R} \to X$ such that $\mathsf{d}(\gamma(s), \gamma(t)) = |t - s|$ for every $s, t \in \mathbb{R}$, then $(X, \mathsf{d}, \mathfrak{m})$ is isomorphic to the product of $\mathbb{R}$ and a $\mathrm{RCD}(0, N-1)$ space.

Universal cover: Mondino and Wei [2016] $\mathrm{RCD}^*(K, N)$ have a universal cover, this is the first purely topological result available on this class of spaces.

Maximal diameter theorem: Ketterer [2015] If $(X, \mathsf{d}, \mathfrak{m})$ is a $\mathrm{RCD}(N, N+1)$ space with $N > 0$ and there exist $x, y \in X$ with $\mathsf{d}(x, y) = \pi$, then $(X, \mathsf{d}, \mathfrak{m})$ is isomorphic to the spherical suspension of $[0, \pi]$ and a $\mathrm{RCD}(N-1, N)$ space with diameter less than π.

Volume-to-metric cones: De Philippis and Gigli [2016] If $K = 0$, there exists $\bar{x} \in X$ such that $\mathfrak{m}(B_R(\bar{x})) = (R/r)^N \mathfrak{m}(B_r(\bar{x}))$ for some $R > r > 0$ and $\partial B_{R/2}(\bar{x})$ contains at least 3 points, then $B_R(\bar{x})$ is locally isometric to the ball $B_R(0)$ of the cone Y built over a $\mathrm{RCD}(N-2, N-1)$ space. This extends the Riemannian result of Cheeger and Colding [1996].

Local structure: The k-dimensional regular set $\mathcal{R}_k$ of a $\mathrm{RCD}^*(K, N)$-space $(X, \mathsf{d}, \mathfrak{m})$ is the set of points $x \in \operatorname{supp} \mathfrak{m}$ such that

$$(X, r^{-1}\mathsf{d}, s_{x,r}\mathfrak{m}, x) \overset{m-GH}{\to} (\mathbb{R}^k, \mathsf{d}_{\mathbb{R}^k}, c_k \mathcal{H}^k, 0) \qquad \text{as } r \to 0^+,$$

where $c_k^{-1} = \int_{B_1(0)} (1 - |x|)\, d\mathcal{H}^k(x)$, and $s_{x,r}^{-1} = \int_{B_r(x)} (1 - \mathsf{d}(x, \cdot)/r)\, d\mathfrak{m}$. For $k \geq 1$ integer, we say that a set $S \subset X$ is $(\mathfrak{m}, k)$-rectifiable if $\mathfrak{m}$-almost all of S can be covered by Lipschitz images of subsets of $\mathbb{R}^k$. The following theorem provides some information on the local structure of $\mathrm{RCD}^*(K, N)$ spaces, analogous to those obtained for Ricci limit spaces in Cheeger and Colding [1997, 2000a,b]; see Mondino and Naber

[2014] for the proof of the first two statements (more precisely, it has been proved the stronger property that $\mathfrak{m}$-almost all of $\mathcal{R}_k$ can be covered by bi-Lipschitz charts with bi-Lipschitz constant arbitrarily close to 1) and Kell and Mondino [2016], De Philippis, Marchese, and Rindler [2017], and Gigli and Pasqualetto [2016] for the proof of the absolute continuity statement.

Theorem 8.1. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a $\mathrm{RCD}^*(K, N)$ space with $N \in (1, \infty)$. For all $k \in [1, N]$ the set $\mathcal{R}_k$ is $(\mathfrak{m}, k)$-rectifiable and*

$$\mathfrak{m}\Big(X \setminus \bigcup_{1 \le k \le N} \mathcal{R}_k\Big) = 0.$$

In addition, the restriction $\mathfrak{m} \llcorner \mathcal{R}_k$ of $\mathfrak{m}$ to $\mathcal{R}_k$ is absolutely continuous w.r.t. $\mathcal{H}^k$.

Second order calculus: Building on Bakry's definition of Hessian (5-10), N. Gigli has been able to develop in Gigli [2014b] a full second-order calculus in $\mathrm{RCD}(K, \infty)$ spaces, including covariant derivatives for vector fields, connection Laplacian, Sobolev differential forms of any order and Hodge Laplacian. The starting points are, besides the formalism of L^p-normed modules inspired by Weaver [2000], the Riemannian formulas

$$\langle \nabla_{\nabla g} X, \nabla h \rangle = \big\langle \nabla \langle X, \nabla g \rangle, \nabla h \big\rangle - \mathrm{Hess}\,(h)(X, \nabla g),$$

$$d\omega(X_1, X_2) = \langle X_1, \nabla \omega(X_2) \rangle - \langle X_2, \nabla \omega(X_1) \rangle - \omega(\nabla_X Y - \nabla_Y X)$$

which grant the possibility, as soon as one has a good definition of Hessian, to define first the covariant derivative of X and then the exterior differential of ω. The RCD assumption enters to provide good integrability estimates and non-triviality of the objects involved (for instance the existence of a rich set of $H^{2,2}(X, \mathsf{d}, \mathfrak{m})$ functions). Remarkably, at the end of this process also the Hessian term in the right hand side of (5-7) is well defined, so that one can define a measure-valued Ricci tensor by $\Gamma_2(f) - \mathrm{Hess}\,(f)$ and the lower bounds on Ricci tensor can be localized.

9 Open problems

Finally, I wish to conclude this survey by stating a few open questions, on which I expect to see new developments in the near future.

• As we have seen, many equivalence and structural results of the CD theory hold under the essential non-branching assumption. At this moment, the only stable class of spaces satisfying this condition is the one of $\mathrm{RCD}(K, \infty)$ spaces. Is there a larger "non-Riemannian" stable class satisfying this condition, or how should the notion of essential non-branching be adapted to this purpose?

• Presently, as we have seen, the BE and CD theories can be closely related only in the class of infinitesimally Hilbertian m.m.s. Is there a "nonlinear" BE theory corresponding to the CD theory, without assuming Ch_2 to be quadratic? In the setting of Finsler manifolds some important steps in this direction have already been achieved, see the survey paper S.-i. Ohta [2017].

• In connection with Theorem 8.1, in a remarkable paper Colding and Naber [2012] proved that, for Ricci limit spaces, only one of the sets $\mathcal{R}_k$ has positive $\mathfrak{m}$-measure (so that the dimension is constant). Is this property true also for $\mathrm{RCD}^*(K, N)$ spaces?
• Even though many properties of Ricci limit spaces (i.e. limits of Riemannian manifolds) are being proved for RCD spaces, the characterization of limit spaces within RCD ones is a challenging question. Using the fact that 3-dimensional non-collapsed limits are topological manifolds Simon [2012] as well as the existence of $\mathrm{RCD}^*(0, 3)$ spaces which are not topological manifolds[3], a gap between Ricci limits and RCD spaces surely exists, at least if one looks at non-collapsed limits.
• The definition of Laplacian in the metric measure setting corresponds, in the smooth setting, to the (weighted) Laplacian with homogeneous Neumann boundary conditions. For this reason the "boundary" is somehow hidden and it is not clear, not even in the RCD setting, how a reasonable definition of boundary can be given at this level of generality. A definition based on the n-dimensional Hausdorff measure of small balls, thus using only the metric structure, is proposed in Kapovitch, Lytchak, and Petrunin [2017], dealing with geodesic flow in n-dimensional Alexandrov spaces.

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DELOCALIZATION OF SCHRÖDINGER EIGENFUNCTIONS

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Abstract

A hundred years ago, Einstein wondered about quantization conditions for classically ergodic systems. Although a mathematical description of the spectrum of Schrödinger operators associated to ergodic classical dynamics is still completely missing, a lot of progress has been made on the delocalization of the associated eigenfunctions.

1 Some history

One can date the birth of quantum mechanics back to Planck's 1900 paper (Planck [1900]), when he realized that the statistical model leading to the spectrum of the "black body" had to be discrete, not continuous. To that effect, he introduced the "Planck constant" h, but this was for him a mathematical artifact, without physical foundation. It was Einstein [1905] who gave this notion a physical meaning, introducing the idea of quantum of energy in the exchange of energy between electromagnetic field and matter (later called photon). The amount of energy that can be exchanged between light of frequency ν and matter is discrete, "quantized", it must be an integer multiple of $h\nu$.

This idea was applied to the planetary model of the atom by Bohr [1913]. Trying to explain the discrete emission/absorption spectrum of the hydrogen, he used the Rutherford model where the electron gravitates around the nucleus submitted to Coulomb attraction, and postulated the quantization of the kinetic momentum: it must be an integer multiple of h. This in turn implied that the energy can only take a discrete set of values, that fitted perfectly well with the experimental spectrum. However, setting up quantization rules for larger atoms turned out to be an inextricable task.

In 1917, Einstein wrote a theoretical paper with an aim to extend the quantization rules to systems with higher degrees of freedom (Einstein [1917]). He modified some rules given earlier by Epstein and Sommerfeld, and he noted that his new rules only made sense if (using modern vocabulary) the system is completely integrable: that is, if there exist some action/angle canonical coordinates, such that the actions are invariants of motion (Einstein's quantization rule is that the values taken by the action variables have to be integer multiples of h). At the end of Einstein's paper, there is a sentence that looks incidental, but may be considered to be the starting point of a whole field of research: *"on the other hand, classical statistical mechanics is essentially only concerned with Type b) [i.e. non integrable systems], for in this case the microcanonical average*

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is the same as the time average". The equivalence of time average with the average over phase space is the property called "ergodicity". Einstein's point is the following: if a classical dynamical system is ergodic, the quantization rules do not apply, so how can we describe its spectrum ?

Facing the failure to find quantization rules even for an atom as simple as the helium, entirely new rules of mechanics were set up by Heisenberg [1925]. One should work only with observable quantities such as the position or the momentum (but for instance, the trajectory of an electron is not observable); and these "observables" are modelled by matrices (operators), subject to certain commutation rules. The momentum observable p and the position observable q must satisfy $qp - pq = i\hbar I$, where $\hbar$ is the reduced Planck constant $h/2\pi$. Time evolution is governed by the energy observable H; Heisenberg gives a recipe to build the operator H starting from the classical expression of energy. Any other observable A evolves according to the linear equation $i\hbar\frac{dA}{dt} = [A, H]$, where $[\cdot,\cdot]$ stands for the commutator of two operators. The physical spectrum of the system (emitted or absorbed energies) is given by the differences $E_n - E_m$, where (E_n) are the eigenvalues of H.

At the same time, a concurrent theory emerged. In 1923, De Broglie had formulated the idea of *wave mechanics*: in the same way as light, considered to be a wave, was discovered to have a discrete behaviour embodied by the photons, one could do the reverse operation with the particles composing matter, and consider them to be waves as well. In 1926 an evolution equation for a wave/particle of mass m evolving in a force field coming from a potential V was proposed by Schrödinger [1926a,b]:

$$i\hbar\frac{\partial\psi}{\partial t} = \left(-\frac{\hbar^2}{2m}\Delta + V\right)\psi \tag{1}$$

where Δ is the Laplacian, and where $\psi = \psi(t,x)$ is a function of time t and of the position $x \in \mathbb{R}^3$ of the particle, called the *wave function*.

The linear partial differential equation (1) can be solved by diagonalizing the differential operator $H = -\frac{\hbar^2}{2m}\Delta + V$. Assume, for instance, we can find an orthonormal basis of the Hilbert space $L^2(\mathbb{R}^3)$ consisting of functions ϕ_n satisfying $H\phi_n = E_n\phi_n$ with $E_n \in \mathbb{R}$. Then the general solution of (1) is

$$\psi(t,x) = \sum_n c_n\phi_n(x)e^{-itE_n/\hbar}$$

where the coefficients $c_n \in \mathbb{C}$ are given by the initial condition at $t = 0$. The physical spectrum is again given by the differences $E_n - E_m$.

Both the Heisenberg and the Schrödinger theories yielded exact results for the hydrogen atoms, but also for larger ones. In fact, they can be shown to be mathematically equivalent. But, as Schrödinger wrote it, mathematical equivalence is not the same as physical equivalence (Schrödinger [1926c]). The wave function ψ is absent from Heisenberg's theory. Soon afterwards, Born gave a probabilistic interpretation of the function ψ: $|\psi(x,t)|^2$ represents the probability, in a measurement, to find a particle at position x, at time t. This was in complete disagreement with Schrödinger's intuition, but this is the interpretation that has been retained.

After 1925, Einstein's question may be reformulated as follows: if a classical system is ergodic, and if H is the energy operator governing the system from the point of view of quantum mechanics, what are the patterns exhibited by the eigenvalues of the operator H ? How is classical ergodicity transferred to the quantum system ?

One may broaden the question by asking about the properties of the wave functions, that is, the eigenfunctions of H (solutions of $H\phi = E\phi$, $E \in \mathbb{R}$), or more generally the solutions $\psi(x,t)$ of the time-dependent solutions of (1). How are the probability densities $|\psi|^2$ localized in space ?

In the mid-fifties, Wigner introduced Random Matrix Theory to deal with the scattering spectrum of heavy nuclei. Although there is no doubt about the validity of the Schrödinger equation, it seems impossible to effectively work with it, in view of the high number of degrees of freedom of such systems. Wigner's hypothesis was that the spectrum of heavy nuclei resembles, statistically, that of certain *ensembles* of large random matrices (the Gaussian Orthogonal Ensemble or the Gaussian Unitary Ensemble). This turns out to fit the experimental data extraordinarily well (pictures may be found in Bohigas [1991]).

Unexpectedly, the spectral statistics of Random Matrix Theory were discovered to also fit extremely well with the spectra of certain Schrödinger operators with very few degrees of freedom: the hydrogen atom in a strong magnetic field, as well as some 2-dimensional billiards (in the latter case, the Schrödinger operator is just the Laplacian in a bounded open set of $\mathbb{R}^2$, with Dirichlet boundary condition). See Delande [1991] for illustrations. The common point of all these examples is that the underlying classical dynamical system is ergodic, or even *chaotic*, meaning a very strong sensitivity to initial conditions. So, it seems that the answer to Einstein's question could be that: if the classical dynamics is ergodic, or sufficiently chaotic, then the spectrum of the corresponding Schrödinger operator looks like that of a large random matrix. This is what is conjectured by Bohigas, Giannoni, and Schmit [1984]. However,:

- there is to this day no mathematical proof of this fact; the question may be considered fully open, except for the heuristic arguments given by Sieber and Richter [2001], that seem impossible to make mathematically rigorous;
- there are some counter-examples to this assertion, given by Luo and Sarnak [1994]; and they come from very strongly chaotic classical dynamics, so the source of the problem does not lie there.

The counter-examples are Laplacians on *arithmetic* hyperbolic surfaces (such as the modular surface and finite covers thereof); they are believed to be "non-generic" in some vague sense, and thus one may conjecture that the assertions above hold for "generic" systems. But even in such a weakened form, the question is fully open.

On the other hand, the question of localization of wave functions, although very difficult, has known steady progress in the last decade. In this paper, we will

- report on recent progress on delocalization of wave functions for chaotic systems, in the semiclassical limit (limit of small wavelengths), Section 2;
- discuss delocalization of eigenfunctions on large finite systems, such as large finite graphs, or Riemann surfaces of high genus, Sections 3 and 4;

- find a link between spatial delocalization and spectral delocalization on infinite systems, Section 3.4;

- note that delocalization of eigenvectors of large random matrices has also undergone intensive study lately. Although these eigenvectors have no direct physical interpretation, they are directly related to the Green function and were studied in relation with the question of universality of the spectrum. The spectacular recent progress on Wigner matrices and large random graphs will be mentioned in Section 4.1 – in a largely non exhaustive manner, as we will focus on results pertaining to delocalization of eigenvectors.

2 High frequency delocalization

In this section, we let (M, g) be a compact smooth Riemannian manifold of dimension d, and Δ be the Laplace–Beltrami operator on M. It is a self-adjoint operator on the Hilbert space $L^2(M, \mathrm{Vol})$, where Vol is the Riemannian volume measure. We diagonalize Δ: it is known that there is a non-decreasing sequence $\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \longrightarrow +\infty$, and an orthonormal basis $(\phi_k)_{k\in\mathbb{N}}$ of $L^2(M, \mathrm{Vol})$, such that

$$\Delta\phi_k = -\lambda_k\phi_k.$$

If M has a boundary, we impose a boundary condition, for instance the Dirichlet condition (i.e. we ask that ϕ_k vanishes on ∂M). The case when M is a billiard table, that is, a bounded domain in $\mathbb{R}^2$ with piecewise smooth boundary, already contains all the difficulties of the subject: actually, the presence of a boundary induces additional technical difficulties, and all the theorems given below have been proven for boundariless manifolds first.

In this part of the paper, we are interested in notions of delocalization defined in the high-frequency limit $\lambda_k \longrightarrow +\infty$. This is the same as the small wavelength limit, and it is also known as a *semiclassical limit*, meaning that classical dynamics emerges from quantum mechanics in this limit.

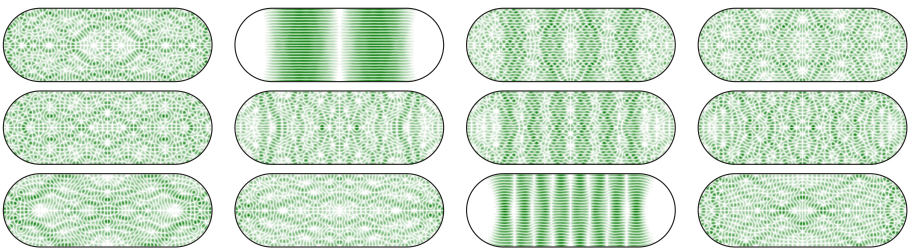

Figure 1: Plot of $|\phi_n(x, y)|^2$ for the stadium billiard with odd-odd symmetry, for consecutive states starting from $n = 319$. Darker shades correspond to large values of the eigenfunctions. Courtesy A. Bäcker

2.1 The role of the geodesic flow . The eigenfunction equation $\Delta\phi_k = -\lambda_k\phi_k$ may be rewritten as $-\hbar^2\Delta\phi = E\phi$ (with $\lambda_k = E\hbar^{-2}$) to make a connection with the Schrödinger operators from (1) (so the external potential V vanishes here). If we impose that E stays away from 0, the limit $\lambda_k \longrightarrow +\infty$ is equivalent to $\hbar \longrightarrow 0$; in this régime, quantum mechanics should "converge to classical mechanics". This was actually a requirement of Schrödinger when he introduced his equation.

The Schrödinger operator $-\hbar^2\Delta$ corresponds to a particle moving on M in absence of any external force. In classical mechanics, this corresponds to the motion along geodesics, in other words, the motion with zero acceleration. When M is a billiard, the motion is in straight line, with reflection on the boundary. We denote by T^*M the *cotangent bundle* of M; this is the classical phase space. An element $(x,\xi) \in T^*M$ has a component $x \in M$ (the "position" of the particle) and $\xi \in T_x^*M$ (the "momentum"). For $(x,\xi) \in T^*M$, and $t \in \mathbb{R}$, we denote by $g^t(x,\xi) \in T^*M$ the position and momentum of the particle, after it has moved during time t along the geodesic starting at x with initial momentum ξ. The family $(g^t)_{t\in\mathbb{R}} : T^*M \longrightarrow T^*M$ is a flow of diffeomorphisms, meaning that $g^{t+s} = g^t \circ g^s$ and g^0 is the identity. This dynamical system is called the geodesic flow. The motion along geodesics has constant speed, and thus, the *unit cotangent bundle* $S^*M = \{(x,\xi) \in T^*M, \|\xi\|_x = 1\}$ is preserved by g^t.

In the limit of small wavelengths, ($\lambda_k \longrightarrow +\infty$), the Schrödinger equation

$$\frac{\partial\psi}{\partial t} = i\Delta\psi$$

moves the *wavefronts* along geodesics. What we mean is that, if we start with an initial condition of the special form

$$\psi^\epsilon(x) = \chi(x)e^{i\frac{S(x)}{\epsilon}}$$

with χ and S smooth, and apply the unitary group $e^{i\epsilon t\Delta}$ (note the rescaling of time), then $e^{i\epsilon t\Delta}\psi^\epsilon$ is of the form

$$e^{i\epsilon t\Delta}\psi^\epsilon(x) = \chi_t(x)e^{i\frac{S(t,x)}{\epsilon}} + O_t(\epsilon) \tag{2}$$

where

- $S(t,x)$ satisfies the Hamilton–Jacobi equation $\frac{\partial S}{\partial t} = \|d_xS\|^2$; which means that at time t the "wavefront set" $\{(x, d_xS(t,x))\} \subset T^*M$ is the image under g^t of the initial wavefront set $\{(x, d_xS(x))\}$;

- denoting $G^t : x \mapsto \pi g^t(x, d_xS)$, where $\pi : T^*M \longrightarrow M$ is the projection to the position coordinate, we have

$$\chi_t(y) = \chi(G^{-t}y)|G^{-t}y|^{1/2}$$

  where $|G^{-t}y|$ is the Jacobian of the map G^{-t} at y. This normalization factor is related to the fact that the L^2 norm must be constant in time.

Formula (2) is an approximate one, it has a remainder term $O_t(\epsilon)$. It is called a BKW approximation, as it was first obtained by Kramers [1926], Wentzel [1926], and Brillouin [1926]. Note that this description is usually only valid for small times; for larger times, it might no longer be possible to write the set $g^t\{(x, d_xS(x))\}$ in the form $\{(x, d_xS(t,x))\}$ for some smooth function $S(t,\cdot)$, as *caustics* may appear.

Using (2), and the fact that eigenfunctions satisfy $e^{i\epsilon t\Delta}\phi_k = e^{-i\epsilon t\lambda_k}\phi_k$, we can hope to establish a relation between the behaviour of the eigenfunctions and the large-time properties of the geodesic flow, however there are two main difficulties:

– a general function ϕ is not of the form (2), but can be written as a linear superposition of such functions (with $\epsilon \sim \lambda_k^{-1/2}$ in the case of eigenfunctions). This may be seen using the Fourier transform in local coordinates. It is extremely difficult to control how the different terms will add up and interfere after applying $e^{i\epsilon t\Delta}$, for large t;

– the error term in (2) grows (usually exponentially) with t; so it is extremely delicate to use the approximation (2) for large times.

2.2 L^p-norms as measures of delocalization ? One of the first question that comes to mind at the sight of Figure 1 is: how large can the eigenfunctions be, how strongly can they be peaked, and at what points ? In this section we denote by ϕ_λ any solution of $-\Delta\phi_\lambda = \lambda\phi_\lambda$, normalized so that $\|\phi_\lambda\|_{L^2} = 1$. A general bound on the L^∞-norm is the following:

Theorem 1. *(known as Hörmander's bound)*

$$\|\phi_k\|_\infty = O(\lambda_k^{(d-1)/4}).$$

In a celebrated paper, C. Sogge gave a bound for all L^p norms, $2 \le p \le +\infty$:

Theorem 2 (Sogge [1988])**.**

$$\|\phi_\lambda\|_{L^p} = O(\lambda^{\frac{\mu(p)}{2}})$$

where

- $\mu(p) = d\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{1}{2}$ *for* $\frac{2(d+1)}{(d-1)} \le p \le +\infty$;
- $\mu(p) = \frac{d-1}{2}\left(\frac{1}{2} - \frac{1}{p}\right)$ *for* $2 \le p \le \frac{2(d+1)}{(d-1)}$.

These bounds hold for any compact manifold M. Recall that d is the dimension of M. Note the role of the critical value $p_c = \frac{2(d+1)}{(d-1)}$. The upper bounds are achieved on the sphere $\mathbb{S}^d$: with zonal spherical harmonics for $p \ge p_c$ (these spherical harmonics are strongly peaked at 2 poles), and with highest weight spherical harmonics for $p \le p_c$ (these spherical harmonics are peaked in the vicinity of a circle). So, in all the L^p-norms, the sphere is a case where eigenfunctions are most strongly peaked. For $p > p_c$, several results give a partial converse, showing that manifolds where the L^p-bound is saturated must have a "pole", that is, a point where many geodesics loops go through:

If $x \in M$, let $\mathfrak{L}_x \subset S_x^*M$ be the set of directions that loop back to x, i.e.

$$\mathfrak{L}_x = \{v \in S_x^*M, \exists t > 0, g^t(x,v) \in S_x^*M\}.$$

We denote by σ_x the Lebesgue measure on the sphere S_x^*M.

Theorem 3 (Sogge and Zelditch [2002, 2016a,b]).

- *Assume there exists a subsequence $\lambda_{n_k} \longrightarrow +\infty$ and $C > 0$ such that $\|\phi_{\lambda_{n_k}}\|_\infty \geq C\lambda^{\frac{\mu(\infty)}{2}}$. Then there exists x such that $\sigma_x(\mathfrak{L}_x) > 0$;*
- *If M is real analytic, the existence of such subsequence $\phi_{\lambda_{n_k}}$ is* equivalent *to the existence of x such that $\mathfrak{L}_x = S_x^*M$, and the first return map $\eta_x : S_x^*M \longrightarrow S_x^*M$ possesses an absolutely continuous invariant probability measure. (Moreover, in that case, there exists $t_0 > 0$ such that $g^{t_0}(x, v) \in S_x^*M$ for all $v \in S_x^*M$, that is, there is a common return time).*
- *If M is real analytic and* $\dim M = 2$, *the existence of such subsequence $\phi_{\lambda_{n_k}}$ is* equivalent *to the existence of $x \in M$ and $t_0 > 0$ such that $g^{t_0}(x, v) = (x, v)$ for all $v \in S_x^*M$.*

What about our original question ? is it true that, if the geodesic flow is chaotic, eigenfunctions will be much less peaked ? To be more specific, we shall mostly be interested in manifolds with negative sectional curvatures. It is then known that the geodesic flow has the *Anosov property*, which is a very strong and very well understood form of chaos: the geodesic flow is not only ergodic, it has strong mixing properties, is measurably isomorphic to a Bernoulli system, exhibits exponential sensitivity to initial conditions,... On a negatively curved manifold, there are only countably many closed geodesic, and what's more, through a point x there pass at most countably many geodesic loops. Thus, Theorem 3 implies that $\|\phi_\lambda\|_{L^\infty} = o\left(\lambda^{\frac{\mu(\infty)}{2}}\right)$ (the big O if Theorem 1 becomes a little o).

One can in fact go further:

Theorem 4. *(i)* (Bérard [1977]) *If $d = 2$ and M has no conjugate points, or if $d \geq 2$ and M has non-positive sectional curvature, for $p = +\infty$,*

$$\|\phi_\lambda\|_{L^p} = O\left(\frac{\lambda^{\frac{\mu(p)}{2}}}{\sqrt{\log\lambda}}\right).$$

(i') (Bonthonneau [2017]) *Statement (i) actually holds if M has no conjugate points, for all $d \geq 2$.*

(ii) (Hassell and Tacy [2015]) *(i) holds for all $p > p_c$.*

(iii) (Blair and Sogge [2017b, 2015]) *If M has non-positive sectional curvature, for $p < p_c$, there exists $\sigma(p, d) > 0$ such that*

$$\|\phi_\lambda\|_{L^p} = O\left(\frac{\lambda^{\frac{\mu(p)}{2}}}{(\log\lambda)^{\sigma(p,d)}}\right)$$

(iv) (Blair and Sogge [2017a]) *Statement (iii) still holds for $p = p_c$.*

(iii) and (iv) were previously proven by Hezari and Rivière [2016] for negatively curved manifolds and for a density 1 sequence of eigenfunctions.

Although this logarithmic improvement constitutes a great progress, it is far from reaching our goal of saying that eigenfunctions are "spread around" if the geodesic flow is chaotic. In fact, after all it is only assumed that the curvature is non-positive, so the results hold already for flat tori (where the geodesic flow is completely integrable), and has not much to do with the long-term chaotic behaviour of the geodesic flow.

2.3 The Shnirelman theorem and the Quantum Unique ergodicity conjecture . As another indicator of delocalization, we can study the probability measure $|\phi_k(x)|^2 d\,\mathrm{Vol}(x)$. Ideally, the aim is to show that it is close to the uniform measure (say, asymptotically as $\lambda_k \longrightarrow +\infty$); or, maybe less ambitiously we could ask whether the measure $|\phi_k(x)|^2 d\,\mathrm{Vol}(x)$ can be large on "small" sets (sets of small dimension for instance). The Quantum Ergodicity theorem gives a first and almost complete answer in case the geodesic flow is ergodic, with respect to the Liouville measure.

Recall, this means that for any L^1-function $a : S^*M \longrightarrow \mathbb{R}$, for Lebesgue almost-every $(x_0, \xi_0) \in S^*M$, the time average $\frac{1}{T}\int_0^T a \circ g^t(x_0, \xi_0)dt$ converges as $T \longrightarrow +\infty$ to the phase-space average $\int_{S^*M} a\, dL$ where L is the normalized Liouville measure on S^*M (i.e. the Lebesgue measure, the uniform measure), arising naturally from the symplectic structure on T^*M.

Quantum Ergodicity Theorem (Shnirelman theorem).

Theorem 5 (Šnirel′man [1974], Colin de Verdière [1985], and Zelditch [1987])**.** *Let (M, g) be a compact Riemannian manifold, with the metric normalized so that* $\mathrm{Vol}(M) = 1$. *Call Δ the Laplace–Beltrami operator on M. Assume that the geodesic flow of M is* ergodic *with respect to the Liouville measure. Let $(\phi_k)_{k\in\mathbb{N}}$ be an orthonormal basis of $L^2(M, g)$ made of eigenfunctions of the Laplacian*

$$\Delta\phi_k = -\lambda_k\phi_k, \qquad \lambda_k \le \lambda_{k+1} \longrightarrow +\infty.$$

Let a be a continuous function on M. Then

$$\frac{1}{N(\lambda)} \sum_{k,\lambda_k\le\lambda} \left| \langle \phi_k, a\phi_k\rangle_{L^2(M)} - \int_M a(x)d\,\mathrm{Vol}(x) \right|^2 \underset{\lambda\longrightarrow+\infty}{\longrightarrow} 0 \tag{3}$$

where the normalizing factor is $N(\lambda) = |\{k, \lambda_k \le \lambda\}|$.

Note that $\langle \phi_k, a\phi_k\rangle_{L^2(M)} = \int_M a(x)|\phi_k(x)|^2 d\,\mathrm{Vol}(x)$.

Remark 6. *The Cesàro limit (3) implies that there exists a subset $S \subset \mathbb{N}$ of density 1 such that*

$$\langle \phi_k, a\phi_k\rangle \underset{n\longrightarrow+\infty, n\in S}{\longrightarrow} \int_M a(x)d\,\mathrm{Vol}(x). \tag{4}$$

In addition, using the fact that the space of continuous functions is separable, one can actually find $S \subset \mathbb{N}$ of density 1 such that (4) holds for all *$a \in C^0(M)$. In other words, the sequence of measures $(|\phi_k(x)|^2 d\,\mathrm{Vol}(x))_{n\in S}$ converges weakly to the uniform measure $d\,\mathrm{Vol}(x)$.*

Actually, the full statement of the theorem says that there exists a subset $S \subset \mathbb{N}$ of density 1 *such that*

$$\langle \phi_k, A\phi_k \rangle \underset{n \longrightarrow +\infty, n \in S}{\longrightarrow} \int_{S^*M} \sigma^0(A) dL \tag{5}$$

for every pseudodifferential operator A of order 0 *on M. On the right-hand side, $\sigma^0(A)$ is the principal symbol of A, that is a function on the unit cotangent bundle S^*M. Equation 4 corresponds to the case where A is the operator of multiplication by the function a.*

The theorem has subsequently been extended to manifolds with boundary by Gérard and Leichtnam [1993] and Zelditch and Zworski [1996]. It applies, in particular, to the stadium billiard in Figure 1, where the billiard flow has been proven by Bunimovich to be ergodic. The observation of large samples of eigenfunctions reveals that, indeed, most eigenfunctions are uniformly distributed over the stadium, but some of them look very localized inside the rectangle, and some of them also exhibit some mild enhancement in the neighbourhood of unstable periodic orbits, a phenomenon called "scarring" by physicists (Heller [1991]).

The theorem was also extended to general Schrödinger operators (or even pseudodifferential operators) in the limit $\hbar \longrightarrow 0$ by Helffer, Martinez, and Robert [1987]; more recently, to systems of differential operators acting on sections of vector bundles – such as Dirac operators, Dolbeault Laplacians,... (Bolte and Glaser [2004], Jakobson and Strohmaier [2007], and Jakobson, Strohmaier, and Zelditch [2008]). The case of metrics with jump-like discontinuities has been elucidated (Jakobson, Safarov, and Strohmaier [2015]), as well as the case of pseudo-riemannian Laplacians on 3-dimensional contact manifolds (Colin de Verdière, Hillairet, and Trélat [2015]). "Small scale quantum ergodicity", that is, the possibility to use in (3) a test function a whose support shrinks as $\lambda_k \longrightarrow +\infty$, has been explored in Hezari and Rivière [2016] and Han [2015] on negatively curved manifolds, and on flat tori in Han [2017] and Lester and Rudnick [2017].

Quantum Unique Ergodicity conjecture. One may wonder whether the full sequence converges in (5), without having to extract the subsequence S. Figure 1 (or larger samples of eigenfunctions) suggests that this is not the case for the billiard stadium, where we see a sparse sequence of eigenfunctions that are not at all equidistributed.

This was proven by Hassell [2010] (for "almost all" stadium billiards, meaning, for Lebesgue-almost-all lengths of the stadium).

On the other hand, Rudnick and Sarnak's Quantum Unique Ergodicity (QUE) conjecture (Rudnick and Sarnak [1994]) predicts that if M is a compact boundaryless manifold with *negative sectional curvatures*, then one has convergence of the full sequence in (5), in other words the whole sequence of eigenfunctions becomes equidistributed as $\lambda \longrightarrow +\infty$. The conjecture has been proved by Lindenstrauss in the setting of "Arithmetic Quantum Unique Ergodicity", where M is an "arithmetic" hyperbolic surface, and where the ϕ_λ are assumed to be eigenfunctions, not only of the Laplacian, but also of the Hecke operators (Lindenstrauss [2006], Bourgain and Lindenstrauss [2003], and Brooks and Lindenstrauss [2014]).

Arithmetic Quantum Unique Ergodicity will not be discussed with enough detail in this text, but the results have been presented at previous ICM's; we refer to Sarnak [2003], Einsiedler and Lindenstrauss [2006], Lindenstrauss [2010], and Soundararajan [2010a] for a more adequate overview.

For general negatively curved manifolds, the conjecture is open, but in the last 20 years significant progress has been made:

2.4 Entropy and support of semiclassical measures. In this section, M is assumed to have negative sectional curvature, and dimension d.

Let us come back to the diagonal matrix elements $\langle \phi_n, A\phi_n \rangle$ appearing in (5), where A is a pseudodifferential operator of order 0. By a general compactness argument, one may always extract subsequences so that $\langle \phi_{n_k}, A\phi_{n_k} \rangle$ converge for all A. The limit is of the form $\int_{S^*M} \sigma^0(A) d\mu$, where μ is a probability measure on S^*M. A measure obtained this way is obtained, according to sources, "microlocal defect measure", "semiclassical measure", or "microlocal lift" associated with the sequence ϕ_{n_k}. The Quantum Unique Ergodicity conjecture described above is equivalent to proving that μ has to be the Liouville measure, for every subsequence (ϕ_{n_k}). But without aiming that far, we can try to characterize specific properties of the measure μ. A priori, we only know that μ has to be invariant under the geodesic flow: that is, $g^t_\sharp \mu = \mu$ for all $t \in \mathbb{R}$. This is a consequence of the eigenfunction property and of the classical/quantum correspondence as $\lambda \longrightarrow +\infty$, as seen in Section 2.1.

Theorem 7. *Anantharaman [2008] Assume M is a compact Riemannian manifold with negative sectional curvature. Assume $\langle \phi_{n_k}, A\phi_{n_k} \rangle$ converges to $\int_{S^*M} \sigma^0(A) d\mu$ for all A. Then μ has positive entropy.*

This is the Kolmogorov–Sinai entropy of dynamical systems. We do not give its definition here, but state a few facts to help understand the implications of the theorem. To each invariant probability measure ν of a dynamical system (here the geodesic flow), one can associate a non-negative number $h_{KS}(\nu)$, having the following properties:

- if ν is carried by a periodic trajectory, then $h_{KS}(\nu) = 0$;
- $\nu \mapsto h_{KS}(\nu)$ is affine: $h_{KS}(\alpha\nu_1 + (1-\alpha)\nu_2) = \alpha h_{KS}(\nu_1) + (1-\alpha) h_{KS}(\nu_2)$, for any invariant measures ν_1, ν_2, for $\alpha \in [0,1]$;
- (Pesin–Margulis–Ruelle inequality) if the dynamical system is sufficiently smooth,

$$h_{KS}(\nu) \leq \int \left(\sum_{j=1}^{r} \chi_j^+(\rho) \right) d\nu(\rho) \tag{6}$$

  where the numbers $\chi_j^+(\rho)$ are the positive Lyapunov exponents of a point ρ – defined ν-almost everywhere, by the Oseledets theorem – that give the rate of exponential instability of the trajectory of ρ.
- in the case of the geodesic flow on a negatively curved manifold, there is equality in (6) if and only if ν is the Liouville measure L (Ledrappier and Young [1985a,b]);

- in the case of the geodesic flow on a negatively curved manifold, of constant curvature -1 and dimension d (so that S^*M has dimension $2d-1$), there are $d-1$ positive Lyapunov exponents, they do not depend on ρ and have the value 1. Thus, (6) can be written as

$$h_{KS}(\nu) \leq d-1, \tag{7}$$

with equality if and only if ν is the Liouville measure L.

Let us give two more transparent corollaries to Theorem 7:

Corollary 1. *Let $\Gamma \subset S^*M$ be the union of all points lying on a periodic trajectory on the geodesic flow (recall, if M has negative curvature, there are countably many periodic geodesics). Let μ be as in Theorem 7. Then $\mu(\Gamma) < 1$.*

Otherwise, μ would have zero entropy. In the physics literature, an eigenfunction that is enhanced near an unstable periodic classical trajectory is said to have a *scar* (Heller [1991]). In the mathematics literature, a sequence of eigenfunctions is said to be *strongly scarred* if the corresponding semiclassical measure μ is supported on some periodic trajectory. Our theorem thus shows that this is not possible on a negatively curved manifold (however, it does not rule out a *partial scar*, that is to say that $\mu(\Gamma) > 0$).

From the definition of entropy, one can also prove:

Corollary 2. *The support of μ has Hausdorff dimension > 1.*

Note that the fact that the dimension is ≥ 1 is trivial since μ is invariant under the geodesic flow.

With Nonnenmacher, we later obtained a more quantitative version if the curvature is constant.

Theorem 8 (Anantharaman and Nonnenmacher [2007b]). *Assume M is a compact Riemannian manifold of dimension d, with constant sectional curvature -1. Then μ has entropy greater than $\frac{d-1}{2}$.*

By the aforementioned properties of entropy, the QUE conjecture in constant negative curvature is equivalent to proving that μ has entropy $d-1$, so we fall short of a factor $1/2$. There are toy models of quantum chaos where it is known that the lower bound $\frac{d-1}{2}$ is sharp, i.e. there are sequences of eigenfunctions that are not equistributed and have exactly half the maximal entropy: see the quantum cat map and the quantum baker's map (Faure, Nonnenmacher, and De Bièvre [2003] and Anantharaman and Nonnenmacher [2007a]).

Corollary 3. *The support of μ has Hausdorff dimension $\geq d$.*

As a comparison, the dimension of the full phase space T^*M is $2d$, and of the energy layer S^*M is $2d-1$.

Corollary 4. *Let $\Gamma \subset S^*M$ be the union of all points lying on a closed trajectory on the geodesic flow. Let μ be as in Theorem 7, with M of constant negative curvature. Then $\mu(\Gamma) \leq 1/2$.*

Indeed, let us decompost μ as $\mu = \alpha\mu_1 + (1-\alpha)\mu_2$, where μ_1 is carried by Γ, so that $h_{KS}(\mu_1) = 0$. So $h_{KS}(\mu) = (1-\alpha)h_{KS}(\mu_2)$. The result says that this has to be $\geq \frac{d-1}{2}$; but the entropy of μ_2 is smaller than the maximal entropy $d-1$, so necessarily $\alpha \leq 1/2$. For the toy model of the quantum cat map, Corollary 4 had been proven by Faure and Nonnenmacher [2004] without using entropy.

In variable curvature, the generalization of Theorem 8 should be that the entropy of μ is greater than $\frac{1}{2}\int_{S^*M}\sum_{j=1}^{d-1}\chi_j^+ d\mu$, where χ_j^+ are the Lyapunov exponents. However our method in Anantharaman and Nonnenmacher [2007b] gives a slightly less good bound; the predicted lower bound in variable curvature has only been obtained for $d = 2$, by Rivière [2010]. Again, by Ledrappier and Young [1985a], proving QUE is equivalent to getting rid of the factor $1/2$ in Rivière's result.

Theorem 9 (Dyatlov and Jin [2017])**.** *μ has full support, that is, $\mu(\Omega) > 0$ for any non-empty open set $\Omega \subset S^*M$.*

Note that Theorems 7 and 9 are somehow independent. There are measures with positive entropy and not full support (for instance, measures supported by geodesics avoiding an open set Ω may have a large entropy). And there are measures having full support but zero entropy (for instance, a measure putting positive weight on each periodic geodesic). Both results leave open the question whether μ can be a convex combination of the Liouville measure and a measure carried on a closed geodesic. Such limit measures appeared in the aforementioned toy models of quantum chaos, see the work by Faure, Nonnenmacher, and De Bièvre [2003].

2.5 Some questions on non-compact manifolds. We have chosen to limit the scope of this text to compact manifolds (and thus, "delocalization" is understood in the limit of small wavelength, but does not deal with what happens at infinity). There are of course many interesting questions related to delocalization phenomena on non-compact manifolds, that we briefly review in this paragraph.

In keeping with the rest of this paper, let us consider the Laplacian on a non-compact riemannian manifold M (most questions also make sense for general Schrödinger operators).

2.5.1 Absolutely continuous spectrum . In the context of infinite systems, the word "delocalization" is often used to mean that the Laplacian has no pure-point spectrum (this means that eigenfunctions are not square-integrable), or even stronger, purely absolutely continuous spectrum, in some region of the spectrum.

As we will see in Section 4.3, one can sometimes prove that this implies a form of "quantum ergodicity" for eigenfunctions on large compact manifolds approximating M (see Theorem 22 for a precise statement). For the moment, this theorem is restricted to the case where M is the hyperbolic disc, where the spectrum of the Laplacian is explicit and can be seen (by direct computation) to be purely absolutely continuous. In general, it turns out to be very difficult to find examples of M having purely absolutely continuous in some interval of the L^2-spectrum, outside of the world of locally symmetric spaces. For instance, starting from X a compact riemannian manifold with *variable* negative

sectional curvature, and taking $M = \widetilde{X}$ to be its universal cover, it seems that nothing is known about the nature of the spectrum of the Laplacian on M, although one would be naturally inclined to guess that is has absolutely continuous spectrum.

2.5.2 Large frequency delocalization on non-compact manifolds. If ϕ_λ is a solution of $-\Delta\phi_\lambda = \lambda\phi_\lambda$ on M (with $\lambda \in \mathbb{R}$) one could ask the question of the behaviour of the measures $|\phi_\lambda(x)|^2 d\,\mathrm{Vol}(x)$, in the limit $\lambda \longrightarrow +\infty$, even if M is non-compact. More precisely, it seems reasonable to restrict these measures to a compact set before studying this limit.

When M is a *finite volume* hyperbolic surface – so that the ends of M are hyperbolic cusps – the question was studied by Zelditch [1991], for ϕ_λ generalized eigenfunctions corresponding to the absolutely continuous spectrum: the so-called Eisenstein series. A "quantum ergodicity" theorem was proven. It was strengthened to a "quantum unique ergodicity" result by Jakobson [1994], when M is the modular surface. When M has variable curvature in a compact subset, but still has hyperbolic cusps, quantum ergodicity was proven more recently in Bonthonneau and Zelditch [2016]. Note that for *infinite volume*, convex-cocompact manifolds, quantum (unique) ergodicity for the Eisenstein series has been studied in Dyatlov and Guillarmou [2014], Guillarmou and Naud [2014], and Ingremeau [2017] but the phenomena are quite different, as it is not the Liouville measure that appears at the semiclassical limit, but a family of measures indexed by the boundary at infinity.

Back to the case of finite volume hyperbolic surface, an example of special interest in number theory is the modular surface and its congruence covers. In this case, M has an infinite sequence of discrete eigenvalues embedded in the continuous spectrum (see the survey papers by Sarnak [1995, 2003] for more details and references). "Arithmetic quantum ergodicity" is the study of the joint L^2-eigenfunctions of the Laplacian and of the so-called "Hecke operators". In this context, Arithmetic Quantum Unique ergodicity, that is, the convergence of the full sequence of probability measures $|\phi_\lambda(x)|^2 d\,\mathrm{Vol}(x)$ to a multiple of the uniform measure, was proven by Lindenstrauss [2006]. Since the modular surface is not compact, there can be escape of mass to infinity, and thus it is not clear that the limit of the measures $|\phi_\lambda(x)|^2 d\,\mathrm{Vol}(x)$ is still a probability measure. Escape of mass was ruled out by Soundararajan [2010b].

Having discrete spectrum embedded in the continuous spectrum is non-generic. For general hyperbolic surfaces, the discrete spectrum is turned into the "resonance spectrum"; resonances are poles of the analytic continuation on the resolvent restricted to $C_c^\infty(M)$ (Selberg [1990] and Borthwick [2007]). Generically, resonances are not real. Naturally attached to resonances, there are non-L^2-eigenstates called "resonant states". The question of quantum ergodicity for resonant states is to this date fully open, and seems extremely difficult.

3 Large scale delocalization

In the mathematical physics literature, it is believed that the spectrum of the Laplacian, as well as its eigenfunctions, should exhibit universal features that depend only on qualitative geometric properties of the space. Localization/delocalization of eigenfunctions is believed to bear close relation with the nature of *spectral statistics*: localization is supposedly associated with Poissonian spectral statistics, whereas delocalization should be associated with Random Matrix statistics (GOE/GUE). In the field of quantum chaos, the former notion is often associated with *integrable dynamics* and the latter with *chaotic dynamics* (Berry and Tabor [1977] and Bohigas, Giannoni, and Schmit [1984]). However, specific examples show that the relation is not so straightforward: see Luo and Sarnak [1994], Sarnak [1995, 1997], and Marklof [2003]. Understanding how far one can push these ideas is one amongst many reasons for studying models of large graphs as toy models, as was done for instance in Keating [2008], Kottos and Smilansky [1997, 1999], and Smilansky [2007, 2010].

It seems that "quantum graphs" have been studied before discrete graphs in the context of quantum chaos. By "quantum graphs", we mean 1-dimensional CW-complexes with $\Delta = \frac{d^2}{dx^2}$ on the edges and suitable matching conditions on the vertices; the most natural ones being the "Kirchhoff" matching condition where it is asked that the functions are continuous at the vertices, and that the sum of their derivatives at a vertex vanish. On a fixed quantum graph, it is known that the analogue of Shnirelman's theorem never holds in the large frequency limit $\lambda \longrightarrow +\infty$, thanks to the work of Colin de Verdière [2013]. See also Berkolaiko, Keating, and Winn [2004], Keating, Marklof, and Winn [2003], Gnutzmann, Keating, and Piotet [2010], and Berkolaiko, Keating, and Smilansky [2007] for other results pertaining to eigenvalue or eigenfunction statistics on compact quantum graphs.

In what follows, instead of the high-frequency limit, we consider the limit where the size of the graph goes to infinity ("large scale limit"). We focus on discrete graphs and the eigenfunctions of their adjacency operators – although similar questions for large quantum graphs should also be explored in the future. We mostly focus on discrete *regular* graphs, but in Section 3.4 also report on recent progress concerning non-regular graphs.

3.1 Overview of the problem. Consider a very large graph $G = (V, E)$. Are the eigenfunctions of its adjacency matrix *localized*, or *delocalized* ? These words are used in a variety of contexts, with several different meanings.

For discrete Schrödinger operators on infinite graphs (e.g. for the celebrated Anderson model describing the metal-insulator transition), localization can be understood in a spectral, spatial or dynamical sense. Given an interval $I \subset \mathbb{R}$, one can consider

- *spectral localization :* pure point spectrum in I,
- *exponential localization :* the corresponding eigenfunctions decay exponentially,
- *dynamical localization :* an initial state with energy in I which is localized in a bounded domain essentially stays in this domain as time goes on.

On the opposite, delocalization may be understood at different levels :

- *spectral delocalization :* purely absolutely continuous spectrum in I,
- *ballistic transport :* wave packets with energies in I spread on the lattice at a specific (ideally, linear) rate as time goes on.

Here we want to discuss notions of *spatial delocalization*. Since the wavefunctions corresponding to absolutely continuous spectrum are not square-summable, a natural interpretation of spatial delocalization is to consider a sequence of growing "boxes" or finite graphs (G_N) approximating the infinite system in some sense, and ask if the eigenfunctions on (G_N) become delocalized as $N \to \infty$. Can they concentrate on small regions, or, on the opposite, are they uniformly distributed over (G_N) ? Large, finite graphs are also a subject of interest on their own. Actually, an infinite system is often an idealized version of a large finite one.

Recently, the question of delocalization of eigenfunctions of large matrices or large graphs has been a subject of intense activity. Let us mention several ways of testing delocalization that have been used. Let M_N be a large symmetric matrix of size $N \times N$, and let $(\phi_j)_{j=1}^N$ be an orthonormal basis of eigenfunctions. The eigenfunction ϕ_j defines a probability measure $\sum_{x=1}^N |\phi_j(x)|^2 \delta_x$. The goal is to compare this probability measure with the uniform measure, which puts mass $1/N$ on each point.

- ℓ^∞ *norms:* Can we have a pointwise upper bound on $|\phi_j(x)|$, in other words, is $\|\phi_j\|_\infty$ small, and how small compared with $1/\sqrt{N}$?
- ℓ^p *norms:* Can we compare $\|\phi_j\|_p$ with $N^{1/p-1/2}$? In De Luca, Altshuler, Kravtsov, and Scardicchio [2014], a state ϕ_j is called *non-ergodic* (and *multi-fractal*) if $\|\phi_j\|_p$ behaves like $N^{f(p)}$ with $f(p) \neq 1/p - 1/2$.
- *Scarring:* Can we have full concentration ($\sum_{x\in\Lambda} |\phi_j(x)|^2 \geq 1-\epsilon$) or partial concentration ($\sum_{x\in\Lambda} |\phi_j(x)|^2 \geq \epsilon$) with Λ a set of "small" cardinality ? We borrow the term "scarring" from the term coined in the theory of quantum chaos by Heller [1991].
- *Quantum ergodicity:* Given a function $a : \{1, \dots, N\} \longrightarrow \mathbb{C}$, can we compare $\sum_x a(x)|\phi_j(x)|^2$ with $\frac{1}{N}\sum_x a(x)$? This criterion is borrowed again from quantum chaos, it is inspired from the Shnirelman theorem 5. It was applied to discrete regular graphs in Anantharaman and Le Masson [2015] and Anantharaman [2017]. Quantum ergodicity means that the two averages are close for *most* j. If they are close for *all* j, one speaks of *quantum unique ergodicity*.

As was demonstrated in a recent series of papers by Yau, Erdős, Schlein, Knowles, Bourgade, Bauerschmidt, Yin, Huang... adding some randomness may allow to settle the problem completely, proving *almost sure* optimal ℓ^∞-bounds and quantum unique ergodicity for various models of *random* matrices and *random* graphs, such as Wigner matrices, sparse Erdős–Rényi graphs, random regular graphs of slowly increasing or bounded degrees: see Erdős, Schlein, and Yau [2009a,b], Bourgade and Yau [2013],

Erdős, Knowles, Yau, and Yin [2013], Bauerschmidt, Knowles, and Yau [2017], Bauerschmidt, Huang, Knowles, and Yau [2015], and Bauerschmidt, Huang, and Yau [2016]: see Section 4.1. The completely different point of view adopted in Brooks and Lindenstrauss [2013] and Anantharaman and Le Masson [2015] is to consider deterministic graphs and to prove delocalization as resulting directly from the geometry of the graphs.

3.2 Entropy. The paper by Brooks and Lindenstrauss [2013] has pioneered the study of the spatial distribution of eigenfunctions of the Laplacian on large deterministic $(q+1)$-regular graphs (that is, such that each vertex has the same number of neighbours, denoted by $q+1$).

Consider a sequence of $(q+1)$-regular connected graphs $(G_N)_{N\in\mathbb{N}} = (V_N, E_N)$. Consider the adjacency operator defined on functions on V_N by

$$\mathcal{A}_N f(x) = \sum_{x\sim y} f(y) \tag{8}$$

where $x \sim y$ means x and y are related by an edge. The discrete Laplacian is

$$\Delta_N f(x) = \sum_{x\sim y} (f(y) - f(x)). \tag{9}$$

For regular graphs these two operators are essentially the same:

$$\mathcal{A}_N - (q+1)I = \Delta_N. \tag{10}$$

Theorem 10 (Brooks and Lindenstrauss [ibid.]). *Let (G_N) be a sequence of $(q+1)$-regular graphs (with q fixed), $G_N = (V_N, E_N)$ with $V_N = \{1, \dots, N\}$. Assume that*[1] *there exists $c > 0, \delta > 0$ such that, for any $k \le c \ln N$, for any pair of vertices $x, y \in V_N$,*

$$|\{\text{paths of length } k \text{ in } G_N \text{ from } x \text{ to } y\}| \le q^{k\left(\frac{1-\delta}{2}\right)}. \tag{11}$$

Fix $\epsilon > 0$. Then, if ϕ is an eigenfunction of the discrete Laplacian on G_N and if $\Lambda \subset V_N$ is a set such that

$$\sum_{x\in\Lambda} |\phi(x)|^2 \ge \epsilon \sum_{x\in V_N} |\phi(x)|^2,$$

then $|\Lambda| \ge N^\alpha$ — where $\alpha > 0$ is given as an explicit function of ϵ, δ and c.

This theorem is reminiscent of Theorems 7 and 8 about the entropy of eigenfunctions in the large frequency limit. It is stronger than saying that the entropy

$$H_N(\phi) = -\frac{1}{\log N} \sum_x |\phi(x)|^2 \ln |\phi(x)|^2$$

is bounded from below by a positive constant.

A careful reading also reveals that the proof shows some logarithmic upper bound on the L^∞-norm of eigenfunctions: $\|\phi\|_\infty = O((\log N)^{-1/4})$. Very recently, Brooks and Le Masson have announced an improvement of the power $1/4$ under a stronger assumption than (11): see Brooks and Le Masson [2017].

[1] This assumption holds in particular if the injectivity radius is $\ge c \ln N$. The interest of the weaker assumption is that it holds for typical random regular graphs, see McKay, Wormald, and Wysocka [2004].

3.3 QE on regular graphs. In Anantharaman and Le Masson [2015], a general statement of "quantum ergodicity" was obtained for the first time in the large scale limit, namely for the discrete Laplacian on large regular graphs. We consider a sequence $G_N = (V_N, E_N)$ of $(q+1)$-regular graphs, and now assume the following:

(EXP) The sequence of graphs is a family of expanders. More precisely, there exists $\beta > 0$ such that the spectrum of $(q+1)^{-1}\mathcal{A}_N$ on $\ell^2(V_N)$ is contained in $\{1\} \cup [-1+\beta, 1-\beta]$ for all N.

Note that 1 is always an eigenvalue, corresponding to constant functions. Our assumption implies in particular that each G_N is connected and non-bipartite. It is well-known that a uniform spectral gap for $\mathcal{A}_N$ is equivalent to a Cheeger constant bounded away from 0, which means that the graph is very connected (see for instance Diaconis and Stroock [1991], Section 3).

(BST) For all R,

$$\frac{|\{x \in V_N, \rho(x) < R\}|}{N} \underset{N\longrightarrow\infty}{\longrightarrow} 0$$

where $\rho(x)$ is the "injectivity radius" of x, that is to say, the largest integer r such that the ball $B(x,r)$ is a tree.

(BST) can be rephrased by saying that our sequence of graphs converges, in the sense of Benjamini and Schramm [2001], to the $(q+1)$-regular tree. In particular, this condition is satisfied if the girth goes to infinity. In what follows we denote by $\mathfrak{X}$ the $(q+1)$-regular tree. Condition **(BST)** implies the convergence of the spectral measure, according to the Kesten–McKay law (Kesten [1959] and McKay [1981]). Call $(\lambda_1^{(N)}, \dots, \lambda_N^{(N)})$ the eigenvalues of $\mathcal{A}_N$ on G_N; then, for any interval $I \subset \mathbb{R}$,

$$\frac{1}{N}|\{j, \lambda_j^{(N)} \in I\}| \underset{N\longrightarrow+\infty}{\longrightarrow} \int_I \mathrm{m}(\lambda)d\lambda$$

where $\mathrm{m}(\lambda)$ is a probability density corresponding to the spectral measure of a Dirac mass δ_o for the operator $\mathcal{A}$ on $\ell^2(\mathfrak{X})$. This measure can be characterized by its moments,

$$\int \lambda^k \mathrm{m}(\lambda)d\lambda = \langle \delta_o, \mathcal{A}_{\mathfrak{X}}^k \delta_o\rangle_{\ell^2(\mathfrak{X})} \tag{12}$$

where $\mathcal{A}_{\mathfrak{X}}$ is the adjacency operator on $\mathfrak{X}$; this is also the number of paths $\mathfrak{X}$, starting at o and returning to o after k steps. We won't need the explicit expression of m here, but let us mention that it is smooth and positive on $(-2\sqrt{q}, 2\sqrt{q})$ and vanishes elsewhere. This implies that most of the eigenvalues $\lambda_j^{(N)}$ are in $(-2\sqrt{q}, 2\sqrt{q})$, an interval strictly smaller than $[-(q+1), q+1]$.

The main result of Anantharaman and Le Masson [2015] and Anantharaman [2017] is stated below as Theorem 11.

Theorem 11 (Anantharaman and Le Masson [2015]). *Let $(G_N) = (V_N, E_N)$ be a sequence of $(q+1)$-regular graphs with $|V_N| = N$. Assume that (G_N) satisfies* **(BST)** *and* **(EXP)**.

Let $(\phi_1^{(N)}, \dots, \phi_N^{(N)})$ *be an orthonormal basis of eigenfunctions of* $\mathcal{A}_N$ *in* $\ell^2(V_N)$.

Let $a_N : V_N \longrightarrow \mathbb{C}$ *be a sequence of functions such that* $\sup_N \sup_{x \in V_N} |a_N(x)| \leq 1$. *Define* $\langle a_N \rangle = \frac{1}{N} \sum_{x \in V_N} a_N(x)$.

Then

$$\frac{1}{N} \sum_{j=1}^{N} \left| \langle \phi_j^{(N)}, a_N \phi_j^{(N)} \rangle_{\ell^2(V_N)} - \langle a_N \rangle \right|^2 \underset{N \longrightarrow +\infty}{\longrightarrow} 0.$$

Equivalently, for any $\delta > 0$,

$$\frac{1}{N} \left| \left\{ j \in [1, N], \left| \langle \phi_j^{(N)}, a_N \phi_j^{(N)} \rangle_{\ell^2(V_N)} - \langle a_N \rangle \right| > \delta \right\} \right| \underset{N \longrightarrow +\infty}{\longrightarrow} 0. \tag{13}$$

Note that $\langle \phi_j^{(N)}, a_N \phi_j^{(N)} \rangle_{\ell^2(V_N)}$ is the scalar product between $\phi_j^{(N)}$ and $a_N \phi_j^{(N)}$, its explicit expression is $\sum_{x \in V_N} a_N(x) |\phi_j^{(N)}(x)|^2$. The interpretation of Theorem 11 is that we are trying to measure the distance between the two probability measures on V_N,

$$\sum_{x \in V_N} |\phi_j^{(N)}(x)|^2 \delta_x \quad \text{and} \quad \frac{1}{N} \sum_{x \in V_N} \delta_x \qquad \text{(uniform measure)}$$

in a rather weak sense (just by testing the function a_N against both). What (13) tells us is that for large N and for most indices j, this distance is small.

3.4 Non-regular graphs: from spectral to spatial delocalization. The results described up to now only deal with *regular* graphs. The proofs always use, in some way or the other, the explicit Fourier analysis infinite regular trees. The aim of the paper by Anantharaman and Sabri [2017a] was to extend the *quantum ergodicity theorem* to eigenfunctions of discrete Schrödinger operators on quite general large graphs. A particularly interesting point of the result below is that it gives a direct relation between *spectral delocalization* of infinite systems and *spatial delocalization* of large finite system. The result may be summarized as follows (with proper additional assumptions to be described later):

"If a large finite system is close (in the Benjamini–Schramm topology) to an infinite system having purely absolutely continuous spectrum in an interval I*, then the eigenfunctions (with eigenvalues lying in* I*) of the finite system satisfy quantum ergodicity."*

We consider a sequence of connected graphs without self-loops and multiple edges $(G_N)_{N \in \mathbb{N}}$. We assume each vertex has at least 3 and at most d neighbours.

We denote by V_N and E_N the vertices and edges of G_N, respectively. We assume $|V_N| = N$ and work in the limit $N \longrightarrow \infty$. Define the adjacency operator $\mathcal{A}_N : \mathbb{C}^{V_N} \to \mathbb{C}^{V_N}$ by

$$(\mathcal{A}_N f)(v) = \sum_{w \sim v} f(w),$$

where $v \sim w$ means v and w are nearest neighbours. The central object of our study are the eigenfunctions of $\mathcal{A}_N$, and their behaviour (localized/delocalized) as $N \longrightarrow +\infty$.

We shall assume the following conditions on our sequence of graphs:

(EXP) The sequence (G_N) forms an expander family.

More precisely, for a non-regular graph, let us define the Laplacian (generator of the simple random walk) $P_N : \mathbb{C}^{V_N} \to \mathbb{C}^{V_N}$ by

$$(P_N f)(x) = \frac{1}{d_N(x)} \sum_{y \sim x} f(y)\,, \tag{14}$$

where $d_N(x)$ stands for the number of neighbours of x. **(EXP)** means that the Laplacian P_N has a uniform spectral gap, that is, the eigenvalue 1 of P_N is simple, and the spectrum of P_N is contained in $[-1+\beta, 1-\beta] \cup \{1\}$, where $\beta > 0$ is independent of N.

Note that 1 is always an eigenvalue, corresponding to constant functions. Our assumption implies in particular that each G_N is connected and non-bipartite. It is well-known that a uniform spectral gap for P_N is equivalent to a Cheeger constant bounded away from 0 (see for instance Diaconis and Stroock [1991], Section 3).

Our second assumption is that (G_N) has few short loops:

(BST) For all $r > 0$,

$$\lim_{N\to\infty} \frac{|\{x \in V_N : \rho_{G_N}(x) < r\}|}{N} = 0\,,$$

where $\rho_{G_N}(x)$ is the *injectivity radius* at x, i.e. the largest ρ such that the ball $B_{G_N}(x,\rho)$ is a tree.

The general theory of Benjamini–Schramm convergence (or local weak convergence Benjamini and Schramm [2001]), allows us to assign a limit object to the sequence (G_N), which is a probability distribution on the set of *rooted graphs* (modulo isomorphism). More precisely, up to passing to a subsequence, assumption **(BST)** above is equivalent to the following assumption.

(BSCT) (G_N) converges in the local weak sense to a random of rooted *tree* $[\mathcal{T}, o]$.

Let us denote $\mathbb{P}$ the law of $\{[\mathcal{T}, o]\}$; thus $\mathbb{P}$ is a probability measure on the space of rooted trees.

Call $(\lambda_j^{(N)})_{j=1}^N$ the eigenvalues of $\mathcal{A}_N$ on $\ell^2(V_N)$. Assumption **(BSCT)** implies the convergence of the empirical law of eigenvalues: for any continuous $\chi : \mathbb{R} \longrightarrow \mathbb{R}$, we have

$$\frac{1}{N}\sum_{j=1}^{N} \chi(\lambda_j^{(N)}) \underset{N\longrightarrow+\infty}{\longrightarrow} \mathbb{E}\left(\langle \delta_o, \chi(\mathcal{A}_{\mathcal{T}})\delta_o\rangle\right) =: \int \chi(t)\,d\mathrm{m}(t)\,, \tag{15}$$

where $\mathcal{A}_{\mathcal{T}}$ is the adjacency matrix of $\mathcal{T}$, it is a self-adjoint operator on $\ell^2(\mathcal{T})$. Here $\mathbb{E}$ is the expectation with respect to $\mathbb{P}$. The measure m is called the *integrated density of states* in the theory of random Schrödinger operators.

The forthcoming assumption is rather technical to state; it says - in a strengthened manner - that there is an interval I in which the spectrum of $\mathcal{A}_{\mathcal{T}}$ is absolutely continuous

(for $\mathbb{P}$-almost every $[\mathcal{T}, o]$). Let $[\mathcal{T}, o]$ be a rooted tree (chosen randomly according to the law $\mathbb{P}$). Given $x, y \in \mathcal{T}$, and $\gamma \in \mathbb{C} \setminus \mathbb{R}$, we introduce the Green function

$$\mathcal{G}^{\gamma}(x, y) = \langle \delta_x, (\mathcal{A}_{\mathcal{T}} - \gamma)^{-1} \delta_y \rangle_{\ell^2(\mathcal{T})} .$$

Given $v, w \in \mathcal{T}$ with $v \sim w$, we denote by $\mathcal{T}^{(v|w)}$ the tree obtained by removing from the tree $\mathcal{T}$ the branch emanating from v that passes through w. We denote by $\mathcal{A}_{\mathcal{T}^{(v|w)}}$ the corresponding adjacency matrix, and by $\mathcal{G}^{(v|w)}(\cdot, \cdot; \gamma)$ the corresponding Green function. We then put $\zeta_w^{\gamma}(v) := -\mathcal{G}^{(v|w)}(v, v; \gamma)$.

(Green) There is a non-empty open set I, such that for all $s > 0$ we have

$$\sup_{\lambda \in I, \eta_0 \in (0,1)} \mathbb{E}\left(\sum_{y : y \sim o} |\operatorname{Im} \zeta_o^{\lambda + i\eta_0}(y)|^{-s} \right) < \infty .$$

To understand the implications of **(Green)**, define the (rooted) spectral measure of $[\mathcal{T}, o]$ by

$$\mu_o(J) = \langle \delta_o, \mathbb{1}_J(\mathcal{A}_{\mathcal{T}}) \delta_o \rangle \qquad \text{for Borel } J \subseteq \mathbb{R} . \tag{16}$$

It can be shown that Assumption **(Green)** implies that $\sup_{\lambda \in I, \eta_0 > 0} \mathbb{E}(|\mathcal{G}^{\gamma}(o, o)|^2) < \infty$. As shown by Klein [1998], this implies that for $\mathbb{P}$-a.e. $[\mathcal{T}, o]$, the spectral measure μ_o is absolutely continuous in I, with density $\frac{1}{\pi} \operatorname{Im} \mathcal{G}^{\lambda + i0}(o, o)$. Hence, **(Green)** implies that $\mathbb{P}$-a.e. operator $\mathcal{A}_{\mathcal{T}}$ has purely absolutely continuous spectrum in I. This is a natural assumption since we wish to interpret Theorem 12 as a delocalization property of eigenfunctions. Negative moments such as **(Green)**, with $s < 0$, were used in the work by Aizenman and Warzel [2012] to show ballistic transport for the Anderson model on the regular tree, that is, a form of delocalization for the time-dependent Schrödinger equation.

Let us state the main abstract result.

Let I be the open set of Assumption **(Green)**, and let us fix an interval I_1 (or finite union of intervals) such that $\bar{I}_1 \subset I$. We write G_N as a quotient $\Gamma_N \backslash \widetilde{G_N}$ where $\widetilde{G_N}$ is a tree (the universal cover of G_N). For $\tilde{x}, \tilde{y}$ vertices of $\widetilde{G_N}$, and $\gamma \in \mathbb{C} \setminus \mathbb{R}$, we introduce the Green function of the adjacency matrix $\widetilde{\mathcal{A}}_N$ of $\widetilde{G_N}$

$$\tilde{g}_N^{\gamma}(\tilde{x}, \tilde{y}) = \langle \delta_{\tilde{x}}, (\widetilde{\mathcal{A}}_N - \gamma)^{-1} \delta_{\tilde{y}} \rangle_{\ell^2(\widetilde{G_N})} . \tag{17}$$

Theorem 12 (Anantharaman and Sabri [2016]). *Assume that (G_N, W_N) satisfies **(BSCT)**, **(EXP)** and **(Green)**.*

Call $(\lambda_j^{(N)})_{j=1}^N$ the eigenvalues of $\mathcal{A}_N$ on $\ell^2(V_N)$, and let $(\phi_j^{(N)})_{j=1}^N$ be a corresponding orthonormal eigenbasis.

For each N, let $a = a_N$ be a function on V_N with $\sup_N \sup_{x \in V_N} |a_N(x)| \leq 1$.

Then

$$\lim_{\eta_0 \downarrow 0} \lim_{N \to +\infty} \frac{1}{N} \sum_{\lambda_j^{(N)} \in I_1} \left| \sum_{x \in V_N} a(x) |\phi_j^{(N)}(x)|^2 - \sum_{x \in V_N} a(x) \mu^N_{\lambda_j^{(N)} + i\eta_0}(x) \right| = 0 ,$$

for some family of probability measures μ_γ^N on V_N, indexed by a parameter $\gamma \in \mathbb{C} \setminus \mathbb{R}$, defined as follows:

$$\mu_\gamma^N(x) = \frac{\operatorname{Im} \tilde{g}_N^\gamma(\tilde{x}, \tilde{x})}{\sum_{y \in V_N} \operatorname{Im} \tilde{g}_N^\gamma(\tilde{y}, \tilde{y})}.$$

Here, $\tilde{x} \in \widetilde{G_N}$ is a lift of $x \in V_N$.

Equivalently, for any $\epsilon > 0$, we have

(18)
$$\frac{1}{N} \left| \left\{ \lambda_j^{(N)} \in I_1 : \left| \sum_{x \in V_N} a(x) |\phi_j^{(N)}(x)|^2 - \sum_{x \in V_N} a(x) \mu^N_{\lambda_j^{(N)} + i\eta_0}(x) \right| > \epsilon \right\} \right| \underset{N \to +\infty,\, \eta_0 \downarrow 0}{\longrightarrow} 0 .$$

Theorem 12 is not relevant unless we can compare the probability measures μ_γ^N with the uniform measure. A good test is to choose $a_N = \mathbb{1}_{\Lambda_N}$, the characteristic function of a set $\Lambda_N \subset V_N$ of size $\approx \alpha N$ for some $\alpha \in (0,1)$. In the special case where (G_N) is regular, the universal cover $\widetilde{G_N}$ does not depend on N (it is the $(q+1)$-regular tree); the Green function $\tilde{g}_N^\gamma(\tilde{x}, \tilde{y})$ coincides with the limiting Green function $\mathcal{G}^\gamma(\tilde{x}, \tilde{y})$ on the regular tree. Moreover, $\mathcal{G}^\gamma(\tilde{x}, \tilde{x}) = \mathcal{G}^\gamma(o, o)$ for all $\tilde{x} \in \widetilde{G_N}$. It follows that μ_γ^N is the uniform probability measure on V_N (for every γ). So (18) implies that $\|\mathbb{1}_{\Lambda_N} \phi_j^{(N)}\|^2 \approx \alpha$ for most $\phi_j^{(N)}$. This shows that most $\phi_j^{(N)}$ are uniformly distributed, in the sense that if we consider any $\Lambda_N \subset V_N$ containing half the vertices, we find half the mass of $|\phi_j^{(N)}|^2$.

For general models, we cannot assert that $\mu_\gamma^N(\Lambda_N) = \alpha$ if $|\Lambda_N| \approx \alpha N$. Still, we prove that there exists $c_\alpha > 0$ such that for any $\Lambda_N \subset V_N$ with $|\Lambda_N| \geq \alpha N$, we have

$$\inf_{\eta_0 \in (0,1)} \liminf_{N \longrightarrow \infty} \inf_{\lambda \in I} \mu^N_{\lambda + i\eta_0}(\Lambda_N) \geq 2c_\alpha . \tag{19}$$

Combined with (18), this implies

Corollary 13. *For any $\alpha \in (0,1)$, there exists $c_\alpha > 0$ such that for any $\Lambda_N \subset V_N$ with $|\Lambda_N| \geq \alpha N$, we have*

$$\frac{1}{N} \# \left\{ \lambda_j^{(N)} \in I : \left\| \mathbb{1}_{\Lambda_N} \phi_j^{(N)} \right\|^2 < c_\alpha \right\} \underset{N \longrightarrow +\infty}{\longrightarrow} 0 .$$

Hence, while in the regular case we had $\|\mathbb{1}_{\Lambda_N} \phi_j^{(N)}\|^2 \approx \alpha$ for most $\phi_j^{(N)}$, in the general case, we can still assert that $\|\mathbb{1}_{\Lambda_N} \phi_j^{(N)}\|^2 \geq c_\alpha > 0$ for most $\phi_j^{(N)}$. This corollary indicates that our theorem can truly be interpreted as a delocalization theorem.

We also prove that that for any continuous $F : \mathbb{R} \to \mathbb{R}$, we have uniformly in $\lambda \in I$,

$$\frac{1}{N} \sum_{x \in V_N} F\left(N \mu^N_{\lambda + i\eta_0}(x) \right) \underset{N \longrightarrow +\infty}{\longrightarrow} \mathbb{E}\left(F\left(\frac{\operatorname{Im} \mathcal{G}^{\lambda + i\eta_0}(o,o)}{\mathbb{E}\left(\operatorname{Im} \mathcal{G}^{\lambda + i\eta_0}(o,o) \right)} \right) \right) . \tag{20}$$

This says that the empirical distribution of $\left(N \mu^N_{\lambda + i\eta_0}(x) \right)$ (when x is chosen uniformly at random in V_N) converges to the law of $\left(\frac{\operatorname{Im} \mathcal{G}^\gamma(o,o)}{\mathbb{E}(\operatorname{Im} \mathcal{G}^\gamma(o,o))} \right)$. This is a second way of saying

that $\mu^N_{\lambda+i\eta_0}(x)$ is of order $1/N$: when multiplied by N, it has a non-trivial limiting distribution.

Remark 14. *The results proven in Anantharaman and Sabri [2017a] actually hold for more general Schrödinger operators than adjacency matrices: one can consider weighted Laplacians (with conductances on the edges) and add a potential; in other words, on each G_N, we can consider a discrete Schrödinger operator $\mathcal{H}_N$. The limiting object in assumption **(BSCT)** is now a random rooted tree $[\mathcal{T}, o]$ endowed with a random Schrödinger operator $\mathcal{H}$. Assumption **(Green)** has to be modified, replacing the adjacency matrix $\mathcal{A}$ by the operator $\mathcal{H}$. Similarly, in the statement of the theorem, the Green functions $\tilde{g}^\gamma_N$ to be considered are those of $\mathcal{H}_N$ lifted to the universal cover $\widetilde{G_N}$.*

Remark 15. *In particular, our result applies to the case where the limiting system $([\mathcal{T}, o], \mathcal{H})$ is $\mathcal{T} = \mathcal{X}$ (the $(q+1)$-regular tree) with an arbitrary origin o, and $\mathcal{H} = \mathcal{H}_\epsilon = \mathcal{A} + \epsilon\mathcal{W}$ where $\mathcal{W}$ is a random real-valued potential on $\mathcal{X}$. More precisely the values $\mathcal{W}(x)$ ($x \in \mathcal{X}$) are i.i.d. random variables of common law ν. This is known as the Anderson model on $\mathcal{X}$. It was shown by Klein [1998] that the spectrum of $\mathcal{H}_\epsilon$ is a.s. purely absolutely continuous on $I = (-2\sqrt{q}+\delta, 2\sqrt{q}-\delta)$, provided ϵ is small enough (depending on δ). This just assumes a second moment on ν. Under stronger regularity assumptions on ν, one can show that Assumption **(Green)** holds on I (see Anantharaman and Sabri [n.d.], following Aizenman and Warzel [2012]). Examples of sequences of expander regular graphs G_N with discrete Schrödinger operators $\mathcal{H}_N$ converging to $([\mathcal{X}, o], \mathcal{H}_\epsilon)$ are given in Anantharaman and Sabri [2017b].*

Remark 16. *Examples of sequences of* non-regular graphs *satisfying our three assumptions were investigated in Anantharaman and Sabri [n.d.]. In the examples considered there, the limiting trees $\mathcal{T}$ are* trees of finite cone type*; roughly speaking, those are trees where the local geometry can only take a finite number of values. If $\mathcal{A}$ is the adjacency matrix of such a tree, we showed in Anantharaman and Sabri [ibid.] that the spectrum σ of $\mathcal{A}$ is a finite union of closed intervals, and that there are a finite number of points $y_1, \dots, y_\ell$ in σ such that Assumption **(Green)** holds on any I of the form $\sigma \setminus ([y_1-\delta, y_1+\delta] \cup \dots \cup [y_\ell-\delta, y_\ell+\delta])$ (for any $\delta > 0$). We showed – extending Remark 15 – that on such trees, Assumption **(Green)** remains true after adding a small random potential to $\mathcal{A}_\mathcal{T}$. Finally, we showed the existence of sequences (G_N) converging to $\mathcal{T}$ and satisfying the **(EXP)** condition.*

4 Perspectives and link with other work

4.1 Random regular graphs. It is important to stress the fact that Theorem 11 holds for *deterministic* sequences of graphs. For any sequence (G_N) satisfying the assumptions of the theorem, the conclusion holds for any observable a. As already noted, the result only says something about the delocalization of "most" eigenfunctions, where the "good" eigenfunctions exhibiting delocalization may depend on the choice of the observable a.

In the past years, there has been tremendous interest in spectral statistics and delocalization of eigenfunctions of *random* sequences of graphs and Schrödinger operators. Many papers consider *random* regular graphs, with degree going slowly to infinity or fixed, sometimes adding a random i.i.d potential (Tran, Vu, and Wang [2013], Dumitriu and Pal [2012], Bauerschmidt, Knowles, and Yau [2017], Bauerschmidt, Huang, Knowles, and Yau [2015], Geisinger [2015], Bauerschmidt, Huang, and Yau [2016], and Geisinger [2015]). A (labelled) random regular graph on N vertices is produced as follows: given the vertex set $\{1, \dots, N\}$, consider all the ways to draw edges between those vertices, that produce a $(q+1)$-regular graph (without self-loops and multiple edges); note that $(q+1)N$ has to be an even integer. Pick a graph at random for the uniform probability measure on all possible configurations.

The very impressive papers Bauerschmidt, Knowles, and Yau [2017], Bauerschmidt, Huang, Knowles, and Yau [2015], and Bauerschmidt, Huang, and Yau [2016] show "quantum unique ergodicity" for the *adjacency matrix* of random regular graphs: given an observable $a_N : \{1, \dots, N\} \longrightarrow \mathbb{R}$, for most $(q+1)$-regular graphs on the vertices $\{1, \dots, N\}$ we have that $\sum_{x=1}^{N} a_N(x)|\phi_j^{(N)}(x)|^2$ is close to $\langle a_N \rangle$ for *all* indices j, with an excellent control of the remainder term:

Theorem 17 (Bauerschmidt, Huang, and Yau [2016]). *Let ω be such that $\sqrt{q} \geq (\omega + 1)2^{2\omega+45}$.*

(i) With probability $\geq 1 - o(N^{-\omega+8})$ on the choice of the graph,

$$\|\phi_j\|_\infty \leq \frac{(\log N)^{121}}{\sqrt{N}}$$

for all eigenfunctions associated to eigenvalues such that $|\lambda_j \pm 2\sqrt{q}| > (\log N)^{-3/2}$.

(ii) (Quantum Unique Ergodicity for random regular graphs) Given an observable $a_N : \{1, \dots, N\} \longrightarrow \mathbb{R}$, we have, with probability $\geq 1 - o(N^{-\omega+8})$ on the choice of the graph, for N large enough,

$$\left| \sum_{x=1}^{N} a_N(x)|\phi_j^{(N)}(x)|^2 - \langle a_N \rangle \right| \leq \frac{(\log N)^{250}}{N} \sqrt{\sum_x |a_N(x)|^2}, \tag{21}$$

for all eigenfunctions associated to eigenvalues $\lambda_j \in (-2\sqrt{q} + \epsilon, 2\sqrt{q} - \epsilon)$ (bulk eigenvalues).

In particular, if $a_N = 1\!\!1_{\Lambda_N}$ where $\Lambda_N \subset \{1, \dots, N\}$, we find

$$\left| \sum_{x \in \Lambda_N} |\phi_j^{(N)}(x)|^2 - \frac{|\Lambda_N|}{N} \right| \leq \frac{(\log N)^{250}}{N} \sqrt{|\Lambda_N|}.$$

(Note in passing that $\omega > 8$ implies that $q > 2^{128}$).

So the ℓ^∞-norm of bulk eigenfunctions is as small as can be, and QUE takes place on sets of size $|\Lambda_N| > (\log N)^{500}$. By comparison, in Theorem 11, for graphs whose girth goes to ∞, our proof would never do better than

$$\left| \sum_{x \in \Lambda_N} |\phi_j^{(N)}(x)|^2 - \frac{|\Lambda_N|}{N} \right| \leq \frac{1}{\sqrt{N \log N}} \sqrt{|\Lambda_N|}.$$

So Theorem 17 is a considerable strengthening of (18), that only said something for *most* indices j and for $|\Lambda_N| > N^{1/2}$. This possibility to prove QUE is, of course, due to the fact that a_N is probabilistically independent of the choice of the graph; in Theorem 11 and (18), a_N could depend on the graph. It might well be that a positive proportion of graphs contradicts QUE, if we were allowed to choose observables a_N depending on the graph (this is a completely open question). Note also that if we are given a deterministic sequence of regular graphs (for instance, say, the Ramanujan graphs defined by Lubotzky, Phillips, and Sarnak [1988]), we do not know if Theorem 17 applies to it, as it is an *almost sure* conclusion.

Remark 18. *Note that we emphasized Theorem 17 from Bauerschmidt, Huang, and Yau [2016] because our main concern here is the delocalization of eigenfunctions. The main focus of Bauerschmidt, Huang, and Yau [ibid.] is however on the universality of the local spectral statistics for random regular graphs. This would deserve a separate paper.*

The recent paper by Backhausz and Szegedy [2016] proves a very important result, saying that for almost all random regular graphs G_N on N vertices, and *all* eigenvectors $\phi_j^{(N)}$s, the value distribution of $\sqrt{N}\phi_j(x)$ as x runs over $\{1, \dots, N\}$ is close to some Gaussian $\mathfrak{N}(0, \sigma_j^2)$ with $0 \le \sigma_j \le 1$. More generally, for any $R \ge 0$, picking x uniformly at random in $\{1, \dots, N\}$ and looking at the values of $(\sqrt{N}\phi_j(y))_{y, d(y,x) \le R}$ in the vicinity of x in G_N, the obtained random function is close in distribution to a gaussian process on $B_{\mathfrak{X}}(o, R)$. The covariance function has to be of the form $\sigma_j^2 \Phi_{\lambda_j}(d(x, y))$ where Φ_{λ_j} is the spherical function of parameter λ_j on the $(q+1)$-regular tree $\mathfrak{X}$. Proving that $\sigma_j = 1$, or even just that $\sigma_j \neq 0$, is a challenge; it would amount to proving that eigenfunctions cannot be localized on $o(N)$ vertices. Theorem 10 does not say this, it only says that eigenfunctions cannot be localized on N^α vertices. Our Theorem 11, or the random version Theorem 17 do not say this either, because we can only test one observable a_N at a time. The indices j for which (18) holds, or the set of graphs satisfying (21), depend on a_N. If we wanted to have a common set that does the job for all observables (whose number is exponential in N), we would need to have exponential error bounds in (18) or (21).

4.2 From graph Laplacians to Hecke operators. What do these discrete results teach us about the problem we were originally interested in, namely the eigenfunctions of the Laplace–Beltrami operators on a riemannian manifold ? A natural question that comes to mind is to try to adapt Theorem 11 to sequences of graphs that are finer and finer triangulations of some given Riemann surface. With the appropriate choice of conductances on the edges, the corresponding discrete Laplacians approximate the continuous Laplacian. At the present time, we are unable to say anything about such graphs, because they have many loops and this is excluded by the hypotheses of Theorem 11.

But let us throw a look in a different direction, that of "Arithmetic quantum ergodicity".

Consider $\mathbb{S}^2$, the 2-dimensional sphere with its usual, round metric. The eigenfunctions of $\Delta_{\mathbb{S}^2}$ are spherical harmonics, i.e. restrictions to $\mathbb{S}^2 \subset \mathbb{R}^3$ of harmonic homogeneous polynomials in 3 variables. Harmonic homogeneous polynomials of degree ℓ give rise to eigenfunctions of $\Delta_{\mathbb{S}^2}$ for the eigenvalue $-\ell(\ell+1)$ (the dimension of the eigenspace is $2\ell+1$).

The Laplacian $\Delta_{\mathbb{S}^2}$ commutes with the infinitesimal rotation

$$\mathbf{J}_{12} = \frac{1}{i}\left(x_2\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1}\right)$$

Note that J_{12} is a differential operator of order 1, and that its principal symbol is the kinetic momentum around the vertical axis.

The basis $(\phi_n) = (Y_\ell^m)_{\ell\geq 0, |m|\leq \ell}$ of joint eigenfunctions of $\Delta_{\mathbb{S}^d}$ and J_{12} cannot satisfy the conclusions of Theorem 5. In fact, using the same notation as in Section 2.4, let us consider a subsequence (ϕ_{n_k}) such that $\langle \phi_{n_k}, A\phi_{n_k}\rangle$ converges for all A; the limit is of the form $\int_{S^*M} \sigma^0(A)d\mu$, where μ is a probability measure on S^*M. The fact that ϕ_{n_k} is an eigenfunction of J_{12} is converted into the property that μ is carried by a level set of the kinetic momentum (which is a submanifold of positive codimension in S^*M); thus μ cannot be the Lebesgue measure.

Because the spectrum of the Laplacian has huge multiplicities, one can wonder whether other bases of eigenfunctions on the sphere satisfy Theorem 5. Zelditch had the idea of considering random eigenbases (Zelditch [1992]). He showed that "almost every" choice of eigenbasis satisfies Theorem 5 (this was strengthened to Quantum Unique Ergodicity by VanderKam [1997]).

Brooks, Le Masson, and Lindenstrauss [2015] showed quantum ergodicity for an explicit basis of eigenfunctions of $\Delta_{\mathbb{S}^2}$, that are also eigenfunctions of a kind of "discrete" Laplacian on $\mathbb{S}^2$: for $g_1, \dots, g_k$ a finite set of rotations in $SO(3)$,

$$T_k f(x) = \sum_{j=1}^{k} (f(g_j x) + f(g_j^{-1}x))$$

commutes with $\Delta_{\mathbb{S}^2}$.

Theorem 19 (Brooks, Le Masson, and Lindenstrauss [ibid.]). *Assume that $g_1, \dots, g_k$ generate a free subgroup of $SO(3)$.*

For each ℓ, let $(\psi_j^{(\ell)})_{j=1}^{2\ell+1}$ be an orthonormal family of eigenfunctions of $-\Delta_{\mathbb{S}^2}$ of eigenvalue $\ell(\ell+1)$, that are also eigenfunctions of T_k.

Then for any continuous function a on $\mathbb{S}^2$, we have

$$\frac{1}{2\ell+1}\sum_{j=1}^{2\ell+1}\left|\int_M a(x)|\psi_j^{(\ell)}(x)|^2 d\mathrm{Vol}(x) - \int_M a(x)d\mathrm{Vol}(x)\right|^2 \underset{\ell\to\infty}{\longrightarrow} 0.$$

Restricting T_k to the space of spherical harmonics of degree ℓ is shown to be roughly the same as letting T_k act on a discretization of the sphere by an ℓ^{-1}-net. This is the same as studying the Laplacian on a $2k$-regular graph with $N \sim \ell^2$ vertices, and if $g_1, \dots, g_k$ generate a free subgroup, this graph has few short loops. Thus, Theorem 19

is similar to Theorem 11. The theorem on regular graphs can serve as a canvas to prove Theorem 19.

Remark 20. *T_k is not a pseudodifferential operator, so the argument sketched above to show that the basis $(Y_\ell^m)_{\ell\geq 0, |m|\leq \ell}$ could not satisfy quantum ergodicity does not apply here.*

Remark 21. *We note that for very special choices of rotations – rotations that correspond to norm n elements in an order in a quaternion division algebra, the operators T_k are called Hecke operators. It has been conjectured by Böcherer, Sarnak, and Schulze-Pillot [2003] that such joint eigenfunctions satisfy the much stronger quantum unique ergodicity property. This conjecture is still open.*

The idea of adapting a result on discrete graphs to the realm of Hecke operators on arithmetic manifolds had already been used in Brooks and Lindenstrauss [2014]. In 2000, Bourgain and Lindenstrauss had considered the measures μ obtained in Theorem 7, when the eigenfunctions (ϕ_n) are joint eigenfunctions of Δ and of the *infinite* family of Hecke operators on an arithmetic hyperbolic surface (e.g., the modular surface). They were able to show that μ has positive entropy on *almost-every ergodic component*, and this fact was a key ingredient in the proof of Arithmetic Quantum Unique Ergodicity by Lindenstrauss [2006]. In Brooks and Lindenstrauss [2014], the authorthe authors used the fact that a Hecke operator, restricted to a net in the manifold M, acts similarly to the discrete Laplacian on a regular graph with few short loops, to adapt Theorem 10 and show that μ has positive entropy on almost-every ergodic component, using only *one* Hecke operator.

In the next paragraph, we mention another continuous adaptation of Theorem 11: instead of thinking of discrete Laplacians living in a riemannian manifold and restricted to a finer and finer net, we look at riemannian manifolds that get larger and larger:

4.3 Quantum ergodicity on Riemann surfaces of high genus. Theorems 11 and 12 were dubbed as "quantum ergodicity" theorems in reference to the historical Theorem 5. However, we already noted a difference in the meaning of these results. Theorem 5 holds in the high-frequency régime, whereas the graph-results deal with the large-scale régime. So, a continuous analogue of Theorem 11 would be to consider compact Riemannian manifolds whose volume goes to infinity. Such a result was obtained by Le Masson and Sahlsten for Riemann surfaces of high genus:

Theorem 22 (Le Masson and Sahlsten [n.d.])**.** *Let (S_N) be a sequence of hyperbolic surfaces, whose genus (equivalently, volume) goes to ∞.*

(EXP) *Assume the first eigenvalue $\lambda_1(N)$ of $-\Delta$ on S_N is bounded away from 0 as $N \longrightarrow \infty$.*

(BSH)*Assume there are few short geodesics; in other words, (S_N) converges in the Benjamini–Schramm sense to the hyperbolic disc: for any $R > 0$,*

$$\lim_{N\longrightarrow+\infty} \frac{\mathrm{Vol}\{x \in S_N, \rho(x) < R\}}{\mathrm{Vol}(S_N)} = 0$$

where $\rho(x)$ means the injectivity radius at x.

Fix an interval $I \subset (1/4, +\infty)$.
Let $(\phi_i^{(N)})$ be an orthonormal basis of eigenfunctions of the Laplacian on S_N.
Let $a = a_N : S_N \longrightarrow \mathbb{C}$ be such that $|a(x)| \leq 1$ for all $x \in S_N$. Then

$$\lim_{N \longrightarrow +\infty} \frac{1}{Vol(S_N)} \sum_{\lambda_i(N) \in I} \left| \int_{S_N} a(x) |\phi_i^{(N)}(x)|^2 dx - \langle a \rangle \right|^2 = 0$$

where $\langle a \rangle = \frac{1}{Vol(S_N)} \int_{S_N} a(x) dx$.

We note that $(1/4, +\infty)$ is the L^2-spectrum of the Laplacian on the hyperbolic disc. This spectrum is purely absolutely continuous. So, like in the graph case, we are working with the sequence of compact S_N converging to an infinite-volume simply connected manifold, with purely absolutely continuous spectrum. It would be interesting to find more examples of such manifolds (and to extend Theorem 22 to that more general setting), but we have already mentioned in Section 2.5.1 the difficulty of proving absolutely continuous spectrum.

A tremendously interesting question would to put this result in the framework of *random* Riemann surfaces:

- does Quantum Unique Ergodicity hold for large *random* Riemann surfaces, in the spirit of Theorem 17 ?
- for a typical random Riemann surface, is the value distribution of the eigenfunctions $(\phi_i^{(N)})$ asymptotically gaussian, similarly to the case of random regular graphs recently treated by citeBaSz16 ? This would come very close to justifying the Random Wave Ansatz of Berry [1977] – the latter was formulated in the high-frequency régime, but a version in the large-scale limit would also be of high interest.

The most natural notion of random Riemann surface of genus g is obtained by putting the Weil–Petersson volume measure on their moduli spaces. The volume of the moduli space of Riemann surface of genus g was computed by Mirzakhani (see Mirzakhani [2010, 2013] and references therein), and she could give its asymptotic behaviour as $g \longrightarrow +\infty$. She showed that a random Riemann surface has a uniform spectral gap in the spectrum of the Laplacian, as $g \longrightarrow +\infty$; this is similar to what is known for random regular graphs. She also obtained asymptotic information about the law of the length of the shortest closed geodesic and the shortest separating geodesic. However, this model of random Riemann surfaces does not seem flexible enough to allow for a direct transposition of the wonderful result of Backhausz and Szegedy [2016]. This is a very intriguing topic to explore.

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MATHEMATICS OF MACHINE LEARNING: AN INTRODUCTION

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Abstract

Machine learning is the subfield of computer science concerned with creating machines that can improve from experience and interaction. It relies upon mathematical optimization, statistics, and algorithm design. Rapid empirical success in this field currently outstrips mathematical understanding. This elementary article sketches the basic framework of machine learning and hints at the open mathematical problems in it.

The dictionary defines the act of "learning" as *gaining or acquiring knowledge or skill (in something) by study, experience, or being taught.* Machine learning, a field in computer science, seeks to design machines that learn. This may seem to fly in contradiction to the usual view of computers as fixed and logic-based devices whose behavior is completely fixed by their programmer. But this view is simplistic because it is in fact straightforward to write programs that learn new capabilities from new experiences and new data (images, pieces of text, etc.). This learned capability can become part of its program, and of course, any newly learnt capabilities can also be trivially copied from one machine to another.

Machine learning is related to *artificial intelligence*, but somewhat distinct because it does not seek to recreate only human-like skills in a machine. Some skills —e.g., detecting patterns in millions of images from a particle accelerator, or in billions of Facebook posts— may be easy for a machine, but beyond the cognitive abilities of humans. (In fact, lately machines can go beyond human capabilities in some image recognition tasks.) Conversely, many human skills such as composing good music and proving math theorems seem beyond the reach of current machine learning paradigms.

The quest to imbue machines with learning abilities rests upon an emerging body of knowledge that spans computer science, mathematical optimization, statistics, applied math, applied physics etc. It ultimately requires us to mathematically formulate nebulous concepts such as the " meaning" of a picture, or a newspaper story. This article provides a brief introduction to machine learning.

The mathematical notion closest to machine learning is *curve-fitting*, which has long been a mainstay of science and social science. For example, the supposed inverse relationship between an economy's inflation and unemployment rates, called the *Philips curve*, was discovered by fitting a curve to economic data over a few decades. Machine

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learning algorithms do something similar, except the settings are more complicated and with many more —sometimes, tens of millions—variables. This raises many issues, computational as well as statistical. Let's introduce them with a simple example.

1 Introduction: the linear model

Suppose a movie review consists of a paragraph or two of text, as well as a numerical score in $[0, 1]$ ($0 =$ worst and $1 =$ best). The machine is trying to learn how to predict the numerical score when given only the text part of the review. As training data, it is given N movie reviews and their scores; that is, $(x^1, y^1), (x^2, y^2), \ldots, (x^N, y^N))$ where x^i is a piece of text and y^i is a score. From this dataset it has to figure out the rule for predicting the score from the text.

If the English vocabulary has V words, then each x^i can be seen as a vector in $\Re^V$, where the j'th coordinate is the number of times the j'th word appears in this piece of text. Note that V is large, say $100,000$, so this vector representation is very sparse (i.e., has very few nonzero entries) when the text review consists of a few dozen words.

The simplest approach for prediction involves a linear model. To simplify the description, assume each review has the same length, namely, has k words. The model assumes that each word has an associated *sentiment weight*, which is a scalar. The model says that the review's score can be predicted by adding up the sentiment weights of all words in the review. Note that if a word occurs k times then it contributes k times its weight.

In other words, if $\vec{\theta}$ is the vector of sentiment weights for all V dictionary words, then the machine tries to predict y^i from $\vec{\theta} \cdot x^i$. The learning algorithm consists of finding the best fit for $\vec{\theta}$ via the classic *least squares* method.

$$\min_{\theta} \quad \sum_{i=1}^{N} (\vec{\theta} \cdot x^i - y^i)^2 \tag{1}$$

After training we expect to find that the weights assigned to words are meaningful. Positive words like *terrific, enjoyable, loved* etc. get high weights and negative words like *terrible, hated, avoid* get low or negative weights.

To finish our discussion we need to address two important issues.

1.1 Computational efficiency. The first important issue is: how efficiently can we find such a vector of weights $\vec{\theta}$? Such questions about *computational complexity* are important. Luckily, here the algorithmic task can be solved very efficiently to optimality. The reason is that the optimization problem in (1) happens to be *convex*, a notion we define below. Under fairly general conditions, convex optimization problems can be solved efficiently.

1.2 Statistical efficiency. The second question is statistical: how do we quantitatively measure the success of learning after training with N datapoints? The end result

of training is the learnt weight vector $\vec{\theta}$, and it is useful only if the machine is able to use it to predict the ratings for reviews that it *hasn't seen during training*. This is called *generalization* and it is a nontrivial issue. For instance, suppose $V = 100,000$ and we are only given $10,000$ reviews. Then by simple linear algebra of underdetermined systems, there *always* exists a weight vector such that $\vec{\theta} \cdot x^i = y^i$ for all i. Surely such a generic solution doesn't do well on unseen reviews? (This is analogous to interpolating a degree 20 curves to only 10 datapoints, and expecting it to fit unseen datapoints.) Surprisingly, in real-life it can, provided we change the above objective to the following, where λ is a scalar that is discovered by experimenting with the data, as explained below.

$$\min_{\theta} \quad \sum_i (\vec{\theta} \cdot x^i - y^i)^2 + \lambda \|\vec{\theta}\|_2^2 \tag{2}$$

To obtain guarantees on generalization, we make a key assumption: reviews used for training are *independent* samples from a *fixed distribution* on all possible reviews. This raises inconvenient philosophical questions about whether there even *exists* such an invariant distribution across all reviews —e.g., surely last year's movie reviews come from a different distribution than this year's? We brush away such questions, while noting in passing that it is an active area of research to formulate learning in more realistic settings —such as when the learner and teacher are allowed to interact, or when teacher is allowed to tailor the examples to speed up learning.

Having made that assumption, we are trying to prove that

$$\|\frac{1}{N}(\sum_i \theta^* \cdot x^i) - E_x[\theta^* \cdot x]\| \leq \epsilon, \tag{3}$$

where the expectation in the second term is over the entire distribution of reviews. At first glance this appears to be a trivial matter of bounding the difference between the population average and the sample average, in other words, to use measure concentration bounds. But actually there is a complication: the solution θ^* was computed using the sample, and thus depends intimately upon it. We handle this complication by taking a union bound over all possible θ^*.

First, we can discretize θ^* by rounding off entries in θ^* to the nearest integer multiple of ϵ, since this can affect the predicted score by at most $\epsilon/2$. Now all entries in θ^* are at least ϵ, which means there are at most $m = \|\theta^*\|^2/\epsilon^2$ of them. The number of possible choices for such vectors is at most $T = \binom{V}{m}(1/\epsilon)^m$ where recall that V denotes the number of words in the dictionary. Now (3) follows from standard concentration bounds provided the number of training samples exceed $c_0 \log T/\epsilon^2$ for some suitable (and explicit) constant c_0. This number grows roughly as $\|\theta^*\|^2 \log V/\epsilon^2$, which is usually much smaller than V.

By now it should be clearer what role the tunable λ multiplier plays in (2). For best generalization we wish to find a solution θ^* that minimizes the ℓ_2 norm. Increasing λ penalizes solutions θ with higher ℓ_2 norm, so it serves to balance the ℓ_2 norm against the total ℓ_2 error on training data. So the algorithm can start with a high value of λ (which rules out all θ except those with very low norm) and then perform binary search to home in on a value that balances the error in (3) and the ℓ_2 norm just exactly so that the we end up with the minimum norm solution.

The above simple argument can be strengthened in various ways and ultimately connects with broader questions in statistics Hastie, Tibshirani, and Friedman [2009] as well as beautiful parts of discrete mathematics such as VC dimension and Rademacher complexity Ben-David and Shalev-Schwartz [2014].

2 Supervised learning

The above simple example illustrates a more general paradigm: *supervised learning*, which concerns learning to classify data-points after seeing many labeled examples. This is the most well-known and successful paradigm of machine learning. To illustrate it we use a famous and empirically successful example, *image recognition*. Imagine we have divided everyday objects into k classes: *chair, building, dog, drink* etc. and want to train the machine to assign the correct label when given an image. Here each image is in pixel format, so assume it is a point in $\Re^d$ The training data contains N images of each class, where N is some modest number (such as 1000). Let the labels be $\{1, 2, \ldots, k\}$. In formalizing the learning problem, it helps to think of the label y^i of x^i as a vector in $\Re^k$: it has an entry 1 in coordinate y^i and zero in other coordinates. Ideally, the learning algorithm would learn to produce labels with only one nonzero coordinate as well, which we encourage by appropriately setting up the optimization problem.

The machine has to learn a function $f_\theta : \Re^d \to \Re^k$ that classifies the images correctly, where θ are the parameters in the description of f_θ. The training objective – variously called *loss function* and *empirical risk*—is

$$\min_\theta \sum_{i=1}^{N} (f_\theta(x^i) - y^i)^2. \qquad (\ell_2 \text{ loss}). \tag{4}$$

Variations of this formulation are used as well, for example the following where y_j denotes jth coordinate of y:

$$\min_\theta \sum_{i=1}^{N} \sum_{j=1}^{k} y_j^i \log(f_\theta(x^i)_j) \qquad (\text{cross entropy loss}). \tag{5}$$

This framework for supervised learning goes by the name *Empirical Risk Minimization (ERM)* N. Vapnik [1998]. The learning *generalizes* if the expected loss of the optimum solution θ^* on the entire distribution is close to that on the samples. The flip side of this issue is statistical efficiency —determining the minimum number of samples that lead to good generalization—as was discussed earlier.

Regularization. Often the performance of gradient descent on an objective $g(\theta)$—both with regards to optimization speed and generalization—is greatly aided by adding a *regularization* term h to the objective, turning it into $g(\theta) + \lambda h(\theta)$. This $h(\theta)$ term shapes the optimization landscape, and its effect can be tuned by varying the multiplier λ. The term $\lambda \|\theta\|_2^2$ in (2) is in fact a form of regularization, and aids generalization as we saw.

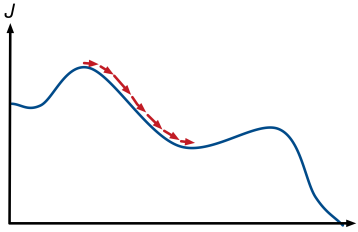

Figure 1: Gradient descent on a nonconvex function is not guaranteed to reach the global minimum.

2.1 Mathematical optimization in machine learning. The problems in (2) (4) (5) are instances of the following general problem where $g\colon \Re^n \to \Re$ and K is a compact subset of $\Re^n$.

$$\begin{aligned} \min \quad & g(\theta) \\ & \theta \in K \end{aligned}$$

The minimum exists, but can we find it efficiently? One could imagine using a variety of algorithms to solve such an optimization problem —optimization theory is quite well-developed! Usually design of such algorithms needs to assume that the objects in question are efficiently computable. Specifically, given a θ we need to be able to (a) efficiently *compute* $f(\theta)$ and (b) check if $\theta \in K$. Both assumptions are easily true in machine learning setting.

In practice, machine learning algorithms often use some variant of gradient descent, which seems to give the best balance between performance and scalability. Basically the same algorithm that is covered in freshman calculus, this algorithm iteratively improves the solution, starting at initial point θ^0 and then finding $\theta^1, \theta^2, \ldots,$ such that at step t

$$s^{t+1} \leftarrow \theta^t - \eta \nabla g(\theta^t)) \tag{6}$$

$$\theta^{t+1} \leftarrow \mathrm{Proj}(s^{t+1}, K) \tag{7}$$

where $\eta > 0$ is called *learning rate* and $\mathrm{Proj}(s^{t+1}, K)$ is the point in K closest to s^{t+1}, also called *projection* of s^{t+1} on K. Pythagoras theorem implies monotonicity: $g(\theta^{t+1}) \leq g(\theta^t)$. In general, gradient descent started with arbitrary θ^0 is not guaranteed to reach the minimum, as is clear from the figure. It converges to a stationary point where $\nabla(f) = 0$, and at best we can hope this is a local optimum.

A well-behaved special case is when g is a *convex* function and K is a convex body, as is the case in (2). Then gradient descent does reach the global optimum if run long enough. Under modest conditions —e.g., a bound on the Lipschitz constant—it approaches the global optimum quite quickly. A comprehensive survey of such *convex optimization* procedures appears in Boyd and Vandenberghe [2008].

But in general, problems (4), (5) are not convex and gradient descent can converge, at best, to a local optimum. A nonconvex problem may have multiple local optima, with some having lower objective values than others. So it is unclear which ones gradient

descent ends up at. Nevertheless, in practice gradient descent works quite well: the solutions found are generally of good quality. Explaining why this happens is an important open problem. It is known that regularization can help, and a cottage industry of tricks has sprung up for regularizing the problem. Another important trick that helps is *stochastic gradient descent*, whereby one estimates the gradient of ERM objective 4 via a small sample of training samples: this improves the running time, and also seems to act as a regularizer.

2.2 Nonconvex models and deep learning. Clearly, the linear model studied above is simplistic. It associates a sentiment score with each word, and sums up the sentiment scores of the words in a review to get an idea of the numerical score. Thus the score only depends upon the multi-set of words in it and completely ignores linguistic structure: " Good, is it not?" gets the same score as " It is not good." Clearly, a fuller understanding of the text must involve more nuanced consideration of larger units such as phrases and sentences. One could try to hand-design features that the machine should pay attention to, e.g., those involving antonyms, synonyms etc. While these can help to some extent, empirically the best results are obtained by just letting the machine automatically figure out the features that it finds most useful. The most powerful current technique for doing this is to train a deep net. A thorough treatment of deep learning appears in the text I. Goodfellow, Bengio, and Courville [2016].

Deep net is a modern name for *neural net*, a notion from the 1940s. It is loosely inspired by the neurons of human brain, specifically the way they are interconnected via wiring that transmits electric signals and their mode of producing an output depending upon the sum of the incoming signals. A *deep net with d hidden layers* consists of d matrices $A_1, A_2, \ldots, A_d$, and a specific function $\sigma : \Re \to \Re$ called the *nonlinearity*. The most popular nonlinearity σ these days is the *rectilinear linear* function $\mathrm{RELU}_b(x) = \max\{0, x - b\}$. Here b is called the *bias*, and it is also a parameter of the network together with the A_i's. Defining $y^0 = x^0$ this net computes $y^1, y^2, \ldots, y^d$ where $y^{i+1} = \sigma(A_i y^i)$. Here $\sigma(z)$ denotes the vector obtained by applying σ to each coordinate of z. Also we are assuming that the dimensions of y^i's and A_i's match so that the matrix-vector products are well-defined. Each coordinate of a computed vector y^i is referred to as a *node* of the net, and each entry of one of the A_i's is refered to as an *edge*. The *output* of the net is y^d. The *size* of the net is the number of nodes in it. The number of parameters is the number of edges plus the number of nodes.

A deep net thus defines an input-output behavior, mapping the input vector x^0 to the output vector $y^d = f_{A_1, A_2, \ldots, A_d, \vec{b}}(x^0)$ where A_i's are the layer matrices and $\vec{b}$ is the vector of all bias values at the nodes. Thus this model can be used to do supervised learning, where the trainable parameters are the matrices and the biases. (An important subcase of a deep net is a *convolutional deep net* where the matrices A_i's have a specific compact representation whereby the same weight is reused in a fixed pattern across the input. These are easier to train in practice especially on data such as images which have patterns that are well-represented by such nets. We will ignore convolution in this survey.)

How does depth help in deep nets? While a net with a single hidden layer (i.e., depth 2) can in principle express any function computed by a net with more layers, doing so may come at a cost of requiring vastly more nodes Eldan and Shamir [2016] and Telgarsky [2016]. Training such a vast net would be computationally infeasible. Thus increasing depth allows a more succinct net to do interesting classification tasks.

To train d-layer deep nets for supervised learning using the above-mentioned *Empirical Risk Minimization* paradigm, we need to solve an optimization problem that solves for the matrices $A_1, A_2, \ldots, A_d$ and the bias vector $\vec{b}$. Writing out the expression for Empirical Risk we find it to be nonconvex in the variables. Nevertheless, we can plough ahead and try to solve it using some variant of gradient descent.

Backpropagation: To do gradient descent millions of times we need a quick way to compute the gradient of the objective. Since the final output is obtained by applying a composition of single layers, computing the gradient is a simple matter of applying the chain rule. Anybody who's taken freshman calculus can write this gradient. The tricky issue is to do so efficiently, meaning given the matrix entries and the bias values, to compute the gradient using as few basic operations —additions and multiplications— as possible. (An elementary operation like addition and multiplication is, simplistically speaking, a unit of effort for the computer's CPU.) Applying chain rule naïvely would require a number of operations that grows *quadratically* in the number of parameters. Since modern deep nets are often trained with tens of millions of parameters, quadratic in that number would be rather large even for today's computers. A clever algorithm called *backpropagation* can compute the gradient with number of operations that is *linear* in the number of parameters. This is a crucial saving that enables deep learning to get off the ground, so to speak. An elementary exposition of backpropagation and its variants appears at Arora and Ma [2016].

Computational and statistical complexity. It can be shown that finding the optimum deep net is in general computationally intractable. However, this refers to computational complexity for unnatural, worst-case instances. Real-life instances are better behaved, and clearly good training is possible. Furthermore, there is evidence that *overparametrizing* the network with many more parameters than necessary can simplify the training. Consequently, today's deep nets are often trained with many more parameters than the number of training examples. *A priori* this raises fears that overparametrization would lead to lack of generalization but in practice generalization does not appear to suffer. Explaining why generalization happens is an open problem, unlike in the linear case described earlier.

What fueled deep learning's rise? While the basic ingredients of deep learning were known for several decades, a confluence of factors around 2011 led to its rapid progress and adoption. The first was availability of large labeled datasets. Datasets for training image recognition software used to be created in academia, and it was just not feasible for a small academic team to hand-label a very large number of images. Starting a decade ago, researchers could use crowd-sourcing to create datasets containing millions

of humanly-labeled images, such as ImageNet Deng, Dong, Socher, L.-J. Li, K. Li, and Fei-Fei [2009]. The second factor was availability of extremely fast *Graphical Processing Units* (GPUs) that brought the power of supercomputers to grad student desktops and fed a wave of experimentation that led to deep learning's resurgence. The third factor is developments in the theory of optimization for machine learning. The new generation of researchers understand notions such as regularization and acceleration and were able to employ them effectively —as well as design new ideas such as batch normalization, dropout, AdaGrad, Adam, etc.—to improve optimization —specifically, what things to try when training a large net fails initially.

Finally. enormous corporate interest in uses of deep learning leads to enormous research effort in industry as well.

3 Unsupervised learning

The techniques discussed thus far can train machines to do *classification tasks* where the output is a scalar (or small number of scalars) and there is plentiful training data that has been labeled by humans. But this captures only a small part of what we humans consider as learning. One suspects that a big part of our learning is *unsupervised*, whereby we passively observe the world around us and notice patterns in it. When we see a new animal or bird while visiting a new continent, we do not need to be told its name to already be able to describe it, and relate it to animals we've seen in the past. Efforts to endow machines with such capabilities have not been as successful.

Viewed from a distance, all methods for unsupervised learning try to formalize a notion of " high level" descriptor of data. If the training datapoints are $x^1, x^2, \ldots,$, one assumes that each has an implicit (i.e., unknown) high level descriptor $h^1, h^2, \ldots$. To give an (advanced) example, x^i could be a pixel-level description of a photo of an unknown bird, and h^i could say in some form " white bird with long legs and long beak." Clearly, each h^i corresponds to multiple (even infinitely many) images and conversely even an image can have multiple high level descriptions. Methods for unsupervised learning allow for this possibility. They define some (possibly loose) way to go from x^i to h^i and vice versa. The following is a non-exhaustive list of ideas that have been tried for many years.

3.1 Dimension reduction of some sort. Dimension reduction amounts to finding low-dimensional vectors $y^1, y^2, \ldots,$ that capture the " essential properties" of x^1, x^2, $\ldots$. The simplest example is to try to approximate the distance: for all i, j the distance between y^i and y^j is approximately the same as between x^i and x^j.

Specific formulations include *Principal Component Analysis* (project to top k eigendirections of $\sum_i x^i \otimes x^i$), *Manifold Learning* (assume there is an unknown low-dimensional manifold $\mathfrak{M}$ such that each $x^i = h^i + noise$ where h^i is a point on the manifold), tSNE, etc.

3.2 Fitting a bayesian model to the data. This method assumes that there is a distribution $p_\theta(x, h)$ from which the sample x^i's were generated. Here θ is a vector of

parameters that describe the distribution, and p_{theta} comes from a specific family of distributions. To give a simple example, a multivariate gaussian distribution is given by the density function $p_\Sigma(x,h) = \exp(\frac{(x-h)^T\Sigma(x-h)}{2})$ where h is the mean and Σ^{-1} is the covariance matrix. Hence we can think of x as h with some added noise.

Examples of bayesian models in unsupervised learning include *topic models*, *hidden Markov models*, *mixed membership models*, *Indian buffet process*, *hierarchical topic models*, *Restricted Boltzmann Machines* etc.

There are two important problems associated with this approach to unsupervised learning. We assume that the machine is given independent samples $x^1, x^2, \ldots, x^N$ from the distribution $p_\theta(x) = \int p_\theta(x,h)dh$. (In words, " pick a sample (x,h) from $p_\theta(x,h)$, and discard the h.") It is customary to assume $p_\theta(x,h)$ factors as $p_\theta(x|h)p_\theta(h)$ where $p_\theta(h)$ has some simple functional form that is known. (Note that such a $p_\theta(h)$ always exists by Bayes' rule, but in general may not have a simple functional form.)

Parameter learning consists of estimating the best θ that explains the data. The method used is classical *maximum likelihood*: select the θ that assigns the maximum probability to the data. Since the data $x^1, x^2, \ldots, x^N$ were independent samples from the distribution, this amounts to

$$\text{argmax}_\theta \prod_i p_\theta(x^i). \tag{8}$$

It is customary to take logarithms and re-express as

$$\text{argmax}_\theta \sum_i \log p_\theta(x^i), \tag{9}$$

which is the so-called *cross-entropy* loss.

Inference involves constructing h given x, where θ is assumed to be known. This involves sampling from the conditional distribution $p_\theta(h|x)$, which is given by Bayes rule.

While the problems are clear enough, the calculations are not easy. For fairly simple models, inference and parameter learning can be computationally intractable. It is customary to use heuristic approaches such as *Expectation Maximization* and *variational inference*. Recently there has been success in designing provably efficient algorithms for parameter learning via *tensor decomposition* methods; see Anandkumar, Ge, Hsu, Kakade, and Telgarsky [2014] for a comprehensive introduction.

3.3 Learning to generate portion of a datapoint from the rest. As mentioned, a full bayesian treatment of unsupervised learning runs into difficult computational problems that have not been easy to solve for large-scale problems. A more successful approach is to treat unsupervised learning more analogously to supervised learning, by observing that there is *implicit supervision* in the data itself.

Concretely, in many settings the datapoint x is much larger (i.e, has many more coordinates) than the latent h, which after all is a meant to be a high-level description. Thus h in principle could be inferred from (say) the first $3/4$th of coordinates of x.

And given h we could predict (at least in a probabilistic sense) the last $1/4$th of the coordinates of x. This train of thought suggests that the last $1/4$th coordinates of x can be predicted from its first $3/4$th coordinates. Thus if we try to set ourselves the task of predicting the last $1/4$th coordinates of x from its first $3/4$th coordinates, implicitly we must need to learn the underlying structure, in other words, some version of h.

Concretely, if the input x is written as $x_1 x_2$ where x_1 contains the first $3/4$th of coordinates and x_2 the last $1/4$th then such a learning approach assumes there is a mapping f_θ such that $f_\theta(x_1) \approx x_2$ where $\approx$ is formalized using some measure of closeness, e.g., ℓ_p norm. Here θ is a vector of parameters. For example, θ could describe a multilayer deep net that maps x_1 to x_2, and the deep net could be found via something like

$$\text{argmin}_\theta \sum_i |x_2^i - f_\theta(x_1^i)|_2^2. \tag{10}$$

This is very analogous to the Empirical Risk Minimization paradigm mentioned above.

Application: Word embeddings. How can we mathematically capture the meaning of an English word? From a mathematical viewpoint one is tempted to reach for mathematical notions such as model theory, which codifies semantics for formal logic. However, the meaning of a word is much more elusive. For one, the word may have multiple meanings (*bank* can refer to a financial institution or the side of a river), and each meaning may have many shades of meaning (is *paint* used in the same sense in *he painted the wall* and *he painted a mural on the wall*?)

In machine learning it has been more useful to represent the meaning of the word with a vector. This started with work in information retrieval (Turney and Pantel [2010]) but recent techniques resort to the general idea sketched above. Specifically, it assumes that every word w is represented by a vector $v_w \in \Re^d$ for some d which is not too large or too small. (Depending upon the application, d is chosen to be a few hundred to a few thousand. There is no good theory explaining the choice.) Thus the model parameters θ consists of these vectors, one for every word in the English dictionary. The model is trained by assuming that if we black out a word in a text corpus, then we can typically figure out the missing word by looking at say 5 words to the left and to the right. For example in the famous **word2vec** method Mikolov, Sutskever, Chen, Corrado, and Dean [2013], the precise functional form assumed is

$$\Pr[w \mid w_1, w_2, \ldots, w_5] \propto \exp(v_w \cdot (\frac{1}{5}\sum_i v_{w_i})). \tag{11}$$

Training such a model requires some tricks, which we won't cover here. Note that the trained embeddings have fascinating properties. One of them is the ability to solve *word analogy* tasks. To solve the analogy problem *man : woman :: king : ??*, one tries to find the word w such that $v_w - v_{king}$ is most similar to $v_{woman} - v_{man}$, that is to say, minimizes $\|v_w - v_{king} + v_{woman} - v_{man}\|_2^2$. Among all $100,000$ words in the English dictionary, the minimizer word happens to be *queen*. This simple idea can solve many simple word analogies, though success rate is far from perfect. This and related

discoveries have made word embeddings a useful tool in natural language processing. A theoretical explanation for the above method for analogy solving appears in Arora, Y. Li, Liang, Ma, and Risteski [2016].

3.4 Deep Generative Models. Deep nets, which were mentioned above, have also been used for unsupervised learning although the successes here are not as spectacular so far. A deep generative model G consists of a deep net that is defined completely analogously as before, which maps $\Re^d$ to $\Re^n$ for some d, n. It maps a random seed s, usually assumed to be a sample from the standard Gaussian distribution in $\Re^d$, to a vector x in $\Re^n$ that is supposed to be a random sample from the target distribution that we are trying to learn. This model is trained using a set of samples from the target distribution $\mathfrak{D}$ (for example, real-life images).

Thus the deep net implicitly defines a probability distribution $\mathfrak{U}$, which we are trying to make close to $\mathfrak{D}$. This technically is a subcase of the setting in Section 3.2, and the main idea in training is to do some form of gradient descent on the objective (9). Some notable notions in this line of work include Restricted Boltzman Machines Hinton and Salakhutdinov [2006], Variational Autoencoders Kingma and Welling [2014], and Generative Adversarial Nets I. J. Goodfellow, Pouget-Abadie, Mirza, Xu, Warde-Farley, Ozair, Courville, and Bengio [2015].

4 Reinforcement learning

Reinforcement learning concerns design of autonomous agents that take a sequence (potentially of unbounded length) of actions. For example, a self-driving car that has to take a dozens of actions every second, and maintain a safe course on the road. Such an agent may be trained a long time in various ways, but once trained has to be autonomous. Another setting where similar issues arise is in playing a complicated game like Chess or Go, where machines now outplay humans.

To formulate the goals of such learning, let's identify key aspects of such a system. (a) It needs to maintain some *state* at every time step, to allow it to store relevant information from previous steps (e.g., current speed, direction, separation from nearby vehicles) that will be needed in future steps. We denote the set of all possible states by S. (b) There is uncertainty in every measurement and action, which will be modeled via *probabilities*. (c) In each state the agent has the choice of some actions. Let A denote the set of possible actions. When the agent takes action $a \in A$ in a state, it makes a probabilistic transition to another state. (d) The agent moves from state to state as follows. Upon reaching a state, it takes an action, which causes it to transition probabilistically to another state, and in the process get some internal *reward*. This reward is its " internal motivation," so to speak. For example, reward function for a self-driving car may be a simple function of distances from the nearest vehicles in all four directions. The agent is trying to maximise this reward, as formalized later.

Similar frameworks have been well-studied in the past century in fields such as *control theory*, *finance*, *economic theory*, *operations research*, etc. In machine learning the above framework is called a *Markov Decision Process (MDP)*. As sketched above, it

consists of the following components: a finite set of *states* S; a set of actions A (each action can be taken in each state); a *probabilistic transition function* that gives for each pair of states (s, s') and action a a probability $p(s, a, s')$ of transitioning to s' when action a is taken in state s (for all s, a it satisfies $\sum_{s'} p(s, a, s') = 1$); and a *reward function* that gives for each pair of states (s, s') and action a a reward $r(s, a, s')$ which is obtained when an action a is taken in state s followed by a transition to state s'.

The goal of the learner is to identify a policy π, which maps states to actions. Once an agent decides a policy $\pi : S \to A$, the MDP turns effectively into a Markov chain, where $p(s, \pi(s), s')$ is the probability of transitioning to s' at the next step if the agent is currently at state s. Thus if it is started in a state s_0, the agent's trajectory is a random sample from the distribution of random walks starting from s_0. It is customary to assume for convenience that this Markov chain is ergodic. Thus if $s_0, s_1, s_2, \ldots,$ are random variables listing am infinite sequence of states that are visited during a random walk starting from s_0 then the expected reward is

$$E[\sum_{i=0}^{\infty} R(s_i, \pi(s_i), s_{i+1})].$$

In general this can be infinite, so it is customary to use a *discounting* whereby rewards obtained t steps into the future are treated as if they were multiplied by a factor γ^t where $\gamma < 1$ is the *discount factor*. Then total expected reward

$$E[\sum_{i=0}^{\infty} \gamma^i R(s_i, \pi(s_i), s_{i+1})]$$

stays finite. (The discounting idea is borrowed from economics, where this is a formalization of the familiar human instinct to treat *a bird in hand as better than two in the bush*.) The policy is optimum if this discount reward is optimum for every choice of s_0. The optimum policy can be computed using *dynamic programming* or *linear programming* in time that is a fixed polynomial of the number of states.

However, in practice today the set of states is often very large, or even infinite. For example, perhaps a state is a vector in $\Re^d$ and an action is a vector in $\Re^k$, which makes a policy a function from $\Re^d$ to $\Re^k$. Now there is no known efficient algorithm for finding an optimum policy, and in fact the task is known to be NP-hard. In practice, various heuristics are known such as *policy iteration* and *value interacction*, where the policy being computed is represented implicitly via a suitable representation, often a deep net. Usually the machine does not know the underlying MDP and has to learn it while coming up with the policy. For a detailed introduction see Sutton and Barto [1998]. Providing theoretical support for this heuristic work is an important open problem, since obvious ways to formalize it run into NP-hard problems. A start would be to formalize what it means for training to generalize here, since the above algorithms such as policy iteration do an *exploration* to progressively improve the policy, which takes us far afield from the independent sample framework utilized in our treatment of supervised learning in Section 1.

We note that the above framework can be changed in various ways to provide other well-studied frameworks that we will not describe here, such as *online computation*, *bandit optimization*, etc.. These capture less general types of sequential decision-making, which retain aspects of classical optimization by restricting attention to convex functions. For an introduction see Hazan [2016].

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HARMONIC ANALYTIC GEOMETRY ON SUBSETS IN HIGH DIMENSIONS – EMPIRICAL MODELS

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Abstract

We describe a recent evolution of Harmonic Analysis to generate analytic tools for the joint organization of the geometry of subsets of $\mathbb{R}^n$ and the analysis of functions and operators on the subsets. In this analysis we establish a duality between the geometry of functions and the geometry of the space. The methods are used to automate various analytic organizations, as well as to enable informative data analysis. These tools extend to higher order tensors, to combine dynamic analysis of changing structures.

In particular we view these tools as necessary to enable automated empirical modeling, in which the goal is to model dynamics in nature, *ab initio*, through observations alone. We will illustrate recent developments in which physical models can be discovered and modelled directly from observations, in which the conventional Newtonian differential equations, are replaced by observed geometric data constraints. This work represents an extended global collaboration including, recently, A. Averbuch, A. Singer, Y. Kevrekidis, R. Talmon, M. Gavish, W. Leeb, J. Ankenman, G. Mishne and many more.

1 Introduction

We describe developments in Harmonic Analysis on subsets of $\mathbb{R}^n$, methodologies which integrate geometry, combinatorics, probability and Harmonic analysis, both linear and nonlinear. We view the emerging structures, as providing natural settings to enable data driven Empirical models for observed dynamics.

Our initial focus is on methods applicable to discrete subsets viewed here as data samples on a continuous structure, a varifold, an infinite dimensional metric space etc. These samples could be generated through a discretization of a stochastic differential equation or through observations of natural or human driven processes.

The challenges of high dimensions, and the need to process massive amounts of seemingly unstructured clouds of points in $\mathbb{R}^n$ (sometimes data) forces us to introduce automated analytic methodologies to reveal the geometry of natural data, understand natural function spaces, or operators on such functions.

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A basic insight is that the geometry of a subset is intimately connected to the geometry of functions on its points, (or sometimes operators on functions) not just the coordinate functions which are linear functions, or exponentials $e^{ix\cdot w}$, with random w or band limited functions and corresponding prolate functions or, more generally, the eigenvectors of natural operators such as graph Laplacians on the subset.

We exploit the fact that eigenvectors of the Laplace Beltrami operator on a manifold (or their discrete approximations), provide, both a high dimensional embedding of the manifold, and a coordinate system, opening the door to analysis.

Some of these ideas are well known classically, for riemannian manifolds, where the Laplace operator, Dirac operators, pseudo differential operators, enable the passage from local properties, to global geometric invariants (as in Atiyah–Singer theories). In Harmonic Analysis, well known theorems, of G. David, S. Semmes and Peter Jones, show the equivalence of the existence of a bi-Lipschitz parameterization of a subset of $\mathbb{R}^n$ and the boundedness of the restriction of Calderon Zygmund operators, as well as some geometric multi-scale Carleson measure type deviation estimates.

The program described here, can be viewed as describing "unsupervised geometric machine learning", and parallels some of the goals and methodologies of Deep Neural nets (such as variational auto encoders), and Recurrent Neural nets, where a variety of algorithms strive to build generative models. for data clouds, see an overview by LeCun, Bengio, and Hinton [2015]. The duality (or triality) point of view described here can be seen as complementary, and necessary to provide better understanding of internal dependence structures.

One of our goals in this paper is to describe the interplay of such analytic tools with the geometry and combinatorics of data and information. We will provide a range of illustrations and application to the analysis of operators, as well as to the analysis of documents, questionnaires, and higher dimensional data bases viewed as tensors.

As will become apparent, the data geometry, or document organization point of view, can illuminate and inspire fundamental questions of geometry, such as duality and Heisenberg principles in riemannian geometry, Carnot geometry etc, defining "dual metric" structures on the set of eigenvectors of the Laplace operator, (or sub-Laplace operator). Similarly the abstract organization of data bases, can inspire deep geometric organization of operators, their decompositions and analysis (following the Calderon Zygmund 'hard' Harmonic Analysis paradigm). In particular the tuning of the geometry to the nature of an operator, as well as the 3 tensor geometry that we discuss, could illuminate the variable geometric structures, which arise in solving nonlinear partial differential equations, defining "naturally evolving metric spaces".

The following topics are interlaced in this presentation:

(a) Geometries of point clouds, and their graphs.

(b) From local to global, the role of eigenfunctions as integrators.

(c) Diffusion geometries in original coordinates, and organization in "intrinsic coordinates".

(d) Coupled dual geometries, Matrices of Data and Operators, duality between rows and columns, tensor product geometries.

(e) Harmonic Analysis, Haar systems, tensor Besov and bi-Hölder functions, Calderon Zygmund decompositions.

(f) Sparse grids and efficient processing of data.

(g) Applications; to Mathematics, organization of operators, the dual geometries of eigenvectors,

(h) Application to empirical modeling of natural dynamical systems through observations alone, defining **intrinsic latent variables**. Triality or, extensions of duality to 3 tensors.

2 Geometries of point clouds in $\mathbb{R}^n$

2.1 Illustrative example. Usually when considering a data set, each item or document is converted into a vector in high dimensional Euclidean space. For example a text document could be converted to the vector, whose coordinates are, the list of occurrence frequencies of words in a lexicon. A particularly illuminating example carrying the complexity of issues we wish to address is a Corpus of text documents represented as a collection, or list of points in $\mathbb{R}^n$.

They have to be organized according to their mutual relevance. We can view this list either as a single cloud of documents or as a database matrix, in which each column is a document and each row, is the list of probabilities of occurrence of a given word in the various documents. We view the words as functions on the documents, and the documents as functions on the words. We will describe an ab initio geometric methodology to jointly assemble the language and the documents into a "smooth" coherent structure, in which documents are organized by context or topic, and vocabulary is organized conceptually by contextual occurrences. We will later describe an adapted tensor geometry and harmonic analysis of rows and columns that links concept and context by duality.

The naïve approach to use the distance (or similarity) between two documents through their Euclidean distance or their inner product, is bound to fail, as already in moderate dimensions most points are far away, or essentially orthogonal. The distances in high dimensions are informative only when they are quite small, leading to the "connect the dots" diffusion geometry.

For this example if the distribution of the vocabulary in two documents are extremely close, we can infer that they deal with a similar topic. In this case we can link the two documents and weigh the link by a weight reflecting the probability that the documents are dealing with the same topic. This construction builds a graph of documents, as well as a corresponding random walk (or diffusion process) on the graph. The analogy with riemannian geometry, in which we have a local metric, which defines a Laplace operator or a heat diffusion process is quite obvious, and will drive much of the initial discussion.

However; this approach to organize the documents as a cloud of points is by itself faulty as it does not account directly for the conceptual similarity, and dependencies between words, or between documents and their content. In order to untangle these relations we view the collection of documents as a matrix in row columns duality.

The columns are viewed as functions on the rows and the rows as functions on the columns. We organize the columns into a hierarchy of topics, (a partition tree of subsets.) These topical groups are then used to organize the vocabulary (rows) into a graph by their co-occurrence in various document topics. This enables the organization of the vocabulary into a hierarchy of conceptual groups, which themselves can be reused to redefine the affinity between documents, (this process can be iterated as long as we gain in efficiency and precision of the representation) Coupling the construction of the two partition tree Hierarchies – on the columns and the rows – takes us away from the representation of the dataset as a point cloud in Euclidean space, towards representation of the dataset as a function on the product set $\{rows\} \times \{columns\}$. This natural document organization is quite abstract and will be quantified below, in particular it will become clear that the construction generalizes various methods of organization in Numerical Analysis, and Harmonic Analysis, and extends naturally to higher order tensorial structures.

2.2 Calculus. The first fundamental point is that there is a natural reformulation of the basic concepts of Differential Calculus (or PDE) in terms of eigenvectors of appropriate linear transformations that will enable us to go from this local or infinitesimal description to an integrated global view of a data cloud. More generally it explains the ability to build data driven empirical models, without the use of calculus. We start from a simple reformulation of the fundamental theorem of calculus, which is an observation of Amit Singer. A basic problem already posed by Cauchy is the following:

> **Sensor Localization Problem.** Assume we know some of the distances between a set of points in Euclidean space and assume these distances are known to determine the system, how does one map the points?

Think of the particular example where you know the distances of each city of a country to a few nearest neighbors: how would one manage to condense that information into a map of that country? There is a trivial answer: if enough local triangles with known lengths are given, then we can compute a local map which can be assembled bit by bit like a puzzle: this can be thought of as an analogue of integration. A more powerful method is obtained by writing each point p_i as the center of mass of its known neighbors, i.e.

$$p_i = \sum_{p_j \sim p_i} w_{ij}\, p_j \quad \text{where} \quad \sum_j w_{ij} = 1.$$

Observe that these equations are invariant under rigid motion and scaling. This tells us that the vector of $x-$coordinates of all points is an eigenvector corresponding to eigenvalue 1 of the matrix W. Similarly, the vector of $y-$coordinates and the vector all of whose coordinates are 1 are also in the same space. We thus see easily that the solution to the sensor localization problem is obtained by finding a basis of this eigenspace and expressing three points in this basis (using their mutual distances). Similarly, if we are given a set of points $(n, f(n)) \in \mathbb{R}^2$ and we know the differences $|f(n) - f(n-1)|$

and $|f(n) - f(n-2)|$, then we can determine f (which is a simple variant of the fundamental theorem of Calculus).

2.3 Diffusion. We now return to point clouds in $\mathbb{R}^n$. We can define a notion of local affinity, or similarity between elements of a set of points $\{p_1, \ldots, p_n\} \subset \mathbb{R}^n$ via the matrix

$$A_{ij} = \frac{\exp(-|p_i - p_j|^2/\varepsilon)}{\sum_{k=1}^n \exp(-|p_i - p_k|^2/\varepsilon)}.$$

This matrix can be interpreted as collecting the transition probabilities of a Markov process. $\varepsilon > 0$ is a parameter controlling the scale of influence (with small ε making a transition to very close neighbors likely while ε large allows for medium- and long-range transitions). Alternatively, it may be preferable to consider a notion of similarity given by

$$A_{ij} = \frac{\exp(-|p_i - p_j|^2/\varepsilon)}{\omega_i \omega_j}$$

where the weights ω_i, ω_j are chosen such that A is Markov matrix in both rows and columns (see N. Marshall and Coifman [2017]). Later we will correct it, or select a graph structure optimized for efficient analysis of functions on the data cloud, or to discover intrinsic riemannian metrics. It is easy to verify that in the case that the points are uniformly distributed on a smooth submanifold of Euclidean space $\Delta = (I - A)/\varepsilon$ is an approximation (in a weak topology) of the Laplace–Beltrami operator on the manifold. Moreover, eigenvectors of A approximate the eigenvectors of the Laplace operator and powers of A correspond to diffusion on the manifold scaled by ε. See Belkin and Niyogi [2001], Lafon [2004], and Marshall and Coifman [2017].

Another more generic (non manifold) example consists of data generated through a stochastic Langevin equation, (a stochastic gradient descent differential equation) this kind of data can be also organized as above, with $\Delta = (I - A)/\epsilon$ approximating the corresponding Fokker Plank operator. Coifman [2005] and Coifman, Lafon, Lee, Maggioni, Nadler, Warner, and Zucker [2005]

We can diagonalize A and use the eigenvectors of A to define powers of the diffusion

$$A^t(p_i, p_j) = \sum_{k=1}^n \lambda_k^t \phi_k(p_i)\phi_k(p_j).$$

This one-parameter family of diffusion defines an embedding Φ_t in $\mathbb{R}^n$ as follows:

$$\Phi_t(p_i) = \left\{\lambda_k^t \phi_k(p_i) : 1 \le k \le n\right\}.$$

We see that this embedding can be computed to any precision by restricting the eigenvector expansion to the first few eigenvectors (depending on the decay of the eigenvalues (λ_k)). This enables a lower-dimensional embedding of the data through what we call the diffusion map. The eigenvectors can also provide natural local coordinates on the manifold, see Jones, Maggioni, and Schul [2008]. In the case of stochastic data the eigenfunctions approximate the eigenvectors of the Fokker Plank operator, they are

supported on the main diffusion trails, and reveal latent variables. See Nadler, Lafon, Coifman, and I. G. Kevrekidis [2006]

The diffusion distance at time t is given, in the bi-stochastic symmetric case as

$$d_t^2(p,q) = A^t(p,p) + A^t(q,q) - 2A^t(p,q) = |\Phi_t(p) - \Phi_t(q)|^2$$

Where A^t represents the t power of A or the diffusion at time t.

3 Harmonic-Analysis of Databases-Matrices, and Tensors

3.1 Matrix organization in high dimensional Data analysis. Our claim is that when dealing with a subset of $\mathbb{R}^n$ where n is large but the subset locally is of much lower dimension, exhibiting local correlations, for example if the subset is a subset of a Varifold, or the cloud is formed by stochastic orbits of dynamical systems, one wishes to understand and encapsulate the local constraints. Moreover linear functions such as the coordinates are not linear as functions on the set. In fact any collection of functions can provide us more coordinates. In particular, band limited functions such as $\exp(i\langle x,\xi\rangle)$ where $|\xi| < C$ are quite informative in revealing the geometry. More general plane waves as generated through deep neural nets can serve similar modeling functions.

As illustrated before on the example of a Corpus of text documents. It becomes productive to view the points as a matrix of data, or a discretized version of a kernel, both rows and columns could correspond to real-world variables or entities of enduring interest. The values of n (dimension) and p (number of points) are often of comparable magnitude, may both be large, and in an asymptotic analysis, may both be allowed to grow to infinity. The correlation or codependence structure of *both* rows and columns is of interest, this has been a main point of analysis, when viewing the data as a matrix of an operator, such as a Green operator, or an eigendecomposition transform, discussed below.

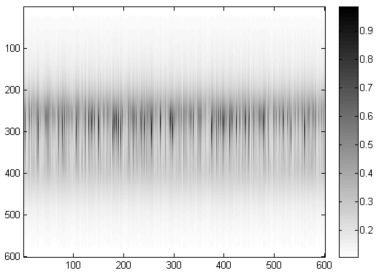
3.2 Matrix organization in numerical analysis. A bottleneck in many numerical analysis tasks involves the need to store very large matrices, apply them to vectors and compute functions of the operators they represent. For example the Fast Fourier Transform and the Fast Multipole Methods are explicitly based on exploiting the known geometrical organization of the row set and the column set of the transformation. A corresponding paradigm in Harmonic Analysis is the organization of an operator as in Calderon–Zygmund theory, (Here we derive automatically the C-Z organization directly from the kernel of the operator or the data matrix).

Consider V. Rokhlin's Fast Multipole Method algorithm Greengard and Rokhlin [1987], which organizes a matrix

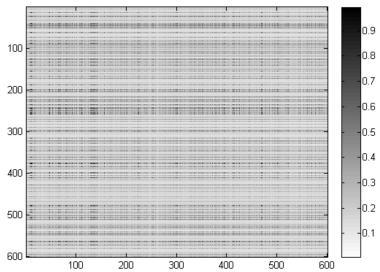
$$M_{i,j} = \|x_i - y_j\|^{-1}$$

of electrostatic or gravitational interactions between a known set of sources $\{x_i\} \subset \mathbb{R}^3$ and a known set of receivers $\{y_j\} \subset \mathbb{R}^3$, by exploiting the known geometry of the source set (the column set, say) and the receiver set (the row set). A similar approach

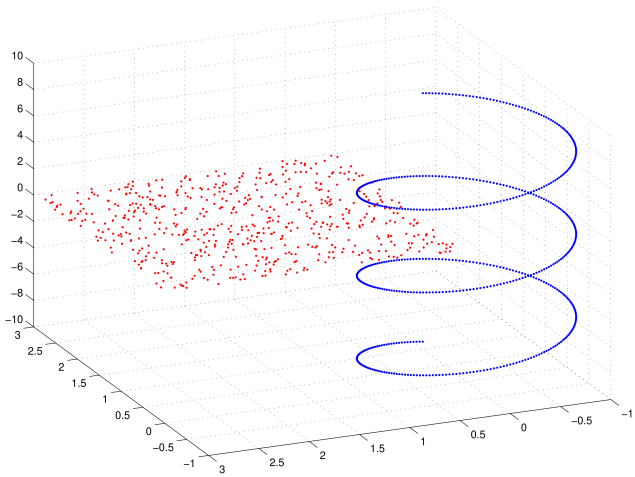
yields fast wavelet transforms of linear operators Alpert, Beylkin, Coifman, and Rokhlin [1993]. There, too, the known organization of matrix rows and columns leads to efficient algorithms for storing, applying and computing functions of certain linear operators. Suppose however that we wish to apply an analog of the Fast Multipole method



(a) Original matrix



(b) "Scrambled" matrix



(c) Charge layout

Figure 1: Geometric unravelling of a scrambled matrix (random labels) of potential interactions (b). Charges are on the spiral, receivers in the plane. Our matrix organization reveals the two geometries and their internal structures.

to a given matrix of electrostatic interactions, $M_{i,j} = \|x_i - y_j\|^{-1}$, where the sets $\{x_i\}$ and $\{y_j\}$ themselves are unknown. The order in which rows and columns are given is meaningless, yet the locations $\{x_i\}$ and $\{y_j\}$ remain encoded in M. (Figure 1) In this context, the theory developed below leads to data agnostic organizational methods which are able, even for some oscillatory potentials, such as $M(x_i, y_j) = M_{i,j} = cos(100\|x_i - y_j\|)\|x_i - y_j\|^{-1}$ to recover the underlying coupled source and receiver

"geometric optics", (in the case of points sampled on a surface or a curve), and furthermore leads to orthonormal bases enabling the implementation of a corresponding fast transform, analogous to Alpert, Beylkin, Coifman, and Rokhlin [1993], the ℓ_1 norm of matrix coefficients in this basis measures the compression rate it is able to achieve: It can be easily proved that this norm controls the mixed smoothness of the matrix.

3.3 Setup. Let M be a matrix, we denote its column set by X and its row set by Y. M can be viewed as a function on the product space, namely

$$M : X \times Y \to \mathbb{R}$$

Our first step in processing M, regardless of the particular problem, is to simultaneously organize X and Y, or in other words, to construct a product geometry on $X \times Y$ in which proximity (in some appropriate metrics) implies predictability of matrix entries. Equivalently, we would like the function M to be "smooth" with respect to the tensor product geometry in its domain. As we will see, smoothness, compressibility, having low entropy, are all interlinked in this organization. We start by redefining the classical notions of smoothness in the context of tree metrics.

3.4 Brief description of Haar Bases. A hierarchical partition tree on a dataset X is an ordered collection of (finite) disjoint covers of the set where each cover is a refinement of the preceding cover, Such a structure allows harmonic analysis of real-valued functions on X, as it induces special orthonormal *Haar bases* Gavish, Nadler, and Coifman [2010]. The elements of the cover will be denoted as folders or nodes of the tree connecting a folder to the coarser folder containing it.

A Haar basis is obtained from a partition tree as follows. Suppose that a node (subset or folder) in the tree has n children, that is, that the set described by the node decomposes into n subsets in the next, more refined, level. Then this node contributes $n-1$ functions to the basis. These functions are all supported on the set described by the node, are piecewise constant on its n subsets, all mutually orthogonal, and are orthogonal to the constant function on the set.

Observe that just like the classical Haar functions, coefficients of an expansion in a Haar basis measure variability of the conditional expectations of the function in sub nodes of a given node.

Tasks such as compression of functions on the data set, as well as subsampling, denoising and "learning" such functions, can be performed in Haar coefficient space using methods familiar from Euclidean harmonic analysis and signal processing Gavish, Nadler, and Coifman [ibid.].

Some results for the classical Haar basis on $[0,1]$ extend to generalized Haar bases. Recall that the classical Haar functions based on the dyadic tree are given by

$$h_I(x) = \left(|I|^{-\frac{1}{2}}\right)(\chi_- - \chi_+) \ ,$$

where χ_- is the indicator of the left half of I and χ_+ is the indicator of the right half of I.

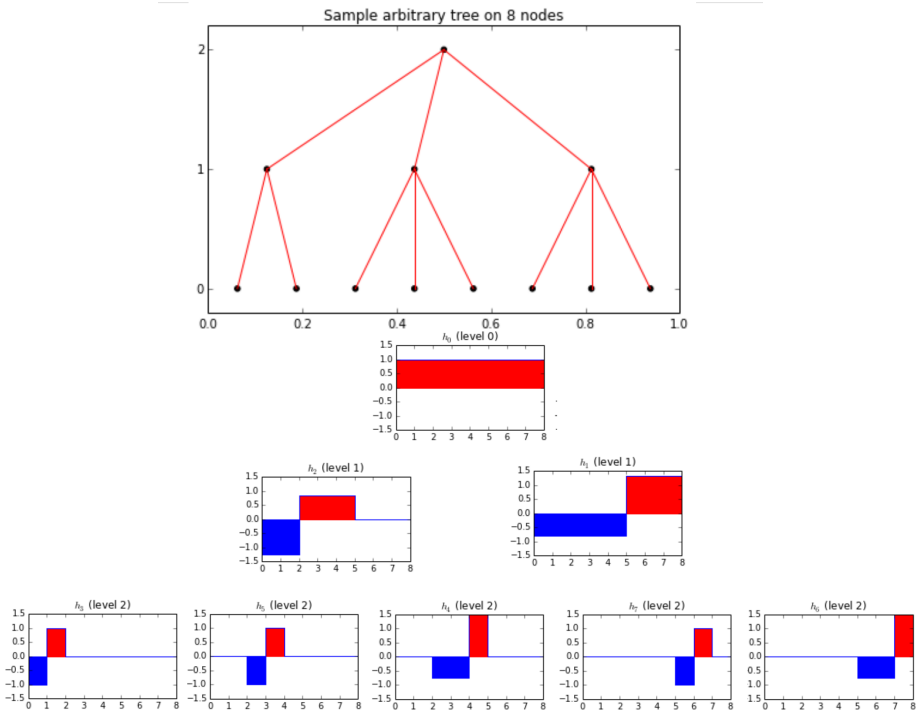


Figure 2: A partition tree on the unit interval starting with a partition into three subintervals, one of which is further divided in two and the other two into three subintervals. The corresponding Haar functions are orthogonal, measuring the variation of averages among neighbors, with the color corresponding to their sign.

The classical Haar basis on $[0,1]$ is induced by the partition tree of dyadic subintervals of $[0,1]$. This tree defines a natural dyadic distance $d(x,y)$ on $[0,1]$, defined as the length of the smallest dyadic interval containing both points. Hölder classes in the metric d are characterized by the Haar coefficients $a_I = \int f(x)h_I(x)dx$:

$$|a_I| < c|I|^{\frac{1}{2}+\beta} \quad \Leftrightarrow \quad |f(x) - f(x')| < c \cdot d(x,x')^{\beta} .$$

A natural partition tree on a set of points in $\mathbb{R}^d$, is the vector quantization tree i.e. a hierarchical organization into disjoint covers by subsets (folders) of approximate diameter $(1/2)^n$. We define a hierarchical tree distance between two points as being the diameter of the smallest folder containing both points.

The characterization of smoothness property holds for any Haar basis when d is the tree metric induced by the partition tree, and $|I| = \frac{\#I}{\#X}$ is the normalized size of the subset (folder) I. (We remark also that for $\beta < 1$ the usual Holder condition is equivalent to dyadic Holder for all shifted dyadic trees.)

We note that there are multiple ways to build partition trees (and corresponding smoothness spaces). The different construction methods can be divided into two classes: bottom-up construction and top-down construction. Broadly, a bottom-up construction

begins with the definition of the lower levels, initially by grouping the leaves/samples, e.g., using k-means in the diffusion embedding. Then, these groups are further grouped in an iterative procedure to create the next levels, ending at the root, in which all the samples are placed under a single folder.

A top-down construction is typically implemented by an iterative clustering method, initially applied to the entire set of samples, then refined over the course of the iterations, starting with the root of the tree and ending at the leaves.

A simple blend is achieved by using the first few diffusion eigenvectors, to split the data into two groups using the first non-trivial eigenvector (approximate max-cut) then repeating on each subgroup using its own first non-trivial eigenvector, since the eigenvector computation is a bottom up iteration, this results in a binary tree, which is often well tuned to the internal data structures.

3.5 Matrix organization through coupled partition trees. To illustrate the basic concept underlying the simultaneous row-column organization, consider the case of a vector (namely, a matrix with one row). In this case, the only reasonable organization would be to bin the entries in decreasing order, (or in binary quantization tree). This decreasing function is obviously smooth outside a small exceptional set (being of bounded variation). Our approach extends this simple construction – which can be viewed as just a one-dimensional quantization tree – to a coupled quantization tree.

We now digress briefly to indicate a simple mathematical framework for joint row and column organization and analysis of a matrix.(quantization bi-trees) which renders an arbitrary matrix into a bi-Holder matrix, (extending the one row example). We start a hierarchical vector quantization tree on the set of columns, X,(as vectors in Euclidean space) with tree metric ρ_X.

The tree metric ρ_X is such that the rows are (tautologically) Lipschitz smooth in the tree metric, as functions of the columns. This implies that the Haar coefficients of the rows, relative to the tree on the columns, scale with the diameter. A similar hierarchical organization on the rescaled Haar coefficients of Y (the rows) as a function of the variable x, induces a similar tree metric ρ_Y on the rows with a similar smoothness property of the columns.

As we will see this implies that the full matrix is a bi-Lipschitz function i.e. it satisfies a *Mixed Lipschitz Hölder* condition

$$|M(x_0, y_0) - M(x_0, y_1) - M(x_1, y_0) + M(x_1, y_1)| \leq C \cdot \rho_X(x_0, x_1)^{\alpha} \cdot \rho_Y(y_0, y_1)^{\alpha}$$

This condition enables the estimation of one value in terms of three neighbors with a higher order error in the two metrics. (For the square in two dimensions, this would be a relaxation of the bounded mixed derivative condition $\left|\frac{\partial^2 M}{\partial x \partial y}\right| \leq C$, which has been studied in the context of approximation in high dimensions Smoljak [1963], Gavish and Coifman [2012], Bungartz and Griebel [2004], and Strömberg [1998].

This simple organization is not very effective in high dimension, as most points are far away from each other, leading us to explore various constructions of more intrinsic data driven metrics and trees, such as the diffusion metrics described above, or the

corresponding "earth mover" metrics. One of our goals is to achieve higher efficiency in representing the matrix, and develop a Harmonic Analysis, or signal processing of functions on $X \times Y$. In particular we will see this as an automatic process to build a multiscale Harmonic Analysis of an operator, or Matrix.

We describe briefly elementary analysis of the Mixed Hölder function classes (as well as their Besov space duals) on an abstract product set equipped with a partition tree pair. A useful tool is an orthogonal transform for the space of matrices (functions on $X \times Y$), naturally induced by the pair of partition trees (or the tensor product of the corresponding martingale difference transforms). Specifically, we take the tensor product of the *Haar bases* induced on X and on Y by their respective partition trees,

The Mixed-Hölder arises naturally in several different ways. First, as seen above for vector quantization trees, any matrix can be given Mixed Hölder structure. Second, it can be shown that any bounded matrix decomposes into a sum of a Mixed Hölder part and a part with small support (as for the one row example). (of course the constants are pretty bad for random data in high dimensions)

3.6 Coupled partition trees, optimized duality. Our goal is to build coupled partition trees to optimize compression of the original function (Matrix) expanded in the tensor Haar basis, say by minimizing an l^1 norm of the tensor Haar coefficients. Such a task requires the discovery of both systems of Haar functions, it is clear that a unique minimizer does not exist in general. Moreover, the appropriate structure is a function of context and precision, as will become clear for various examples, in mathematics and beyond.

We now consider a matrix M and assume two partition trees – one on the column set of M and one on the row set of M – have already been constructed. Each tree induces a Haar basis and a tree metric as above. The tensor product of the Haar bases is an orthonormal basis for the space of matrices of the same dimensions as M. We review some analysis of M in this basis.

Denote by $|R| = |I \times J|$ a "rectangle" of entries of M, where I is a folder in the column tree and J is a folder in the row tree. Denote by $|R| = |I||J|$ the volume of the "rectangle" R. Indexing Haar functions by their support folders, we write $h_I(x)$ for a Haar function on the rows. This allows us to index basis functions in the tensor product basis by rectangles and write $h_R(x, y) = h_I(x)h_J(y)$.

Analysis and synthesis of the matrix M is in the tensor orthonormal Haar basis is simply

$$a_R = \int M(x, y)h_R(x, y)dxdy$$
$$M(x, y) = \sum_R a_R h_R(x, y)\,.$$

The characterization of Hölder functions mentioned above extends to mixed-Hölder matrices Coifman and Gavish [2011] and Gavish and Coifman [2012]:

$$\left|a_R\right| < c\left|R\right|^{1/2+\beta} \Leftrightarrow$$
$$\left|M(x,y) - M(x',y) - M(x,y') + M(x',y')\right| \leq c\rho_X(x,x')^\beta \rho_Y(y,y')^\beta$$

where ρ_X and ρ_Y are the tree metrics induced by the partition trees on the rows and columns, respectively. Observe that his condition implies the conventional two dimensional Holder condition

$$\left|M(x,y) - M(x',y')\right| \leq \rho_X(x,x')^\beta + \rho_Y(y,y')^\beta$$

Simplicity or sparsity of an expansion is quantified by an "entropy" such as

$$e_\alpha(M) = \left(\sum \left|a_R\right|^\alpha\right)^{1/\alpha}$$

for some $\alpha < 2$. We comment that this norm is just a tensor Besov norm that is easily seen to generalize Earth mover distances when scaled correctly, adding flexibility to our construction below. This norm can be generalized to the following family of Besov norms

$$e_{\alpha,\beta}(M) = \left(\sum \left|R\right|^\beta \left|a_R\right|^\alpha\right)^{1/\alpha}$$

for some α, β. Useful relations between this "entropy", efficiency of the representation in tensor Haar basis and the mixed-Hölder condition, is given by the following two propositions valid for "balanced trees" Coifman and Gavish [2011] and Gavish and Coifman [2012].

Proposition. *Assume $e_\alpha(M) = \left(\sum \left|a_R\right|^\alpha\right) \leq 1$. Then the number of coefficients needed to approximate the expansion to precision $\varepsilon^{1-\alpha/2}$ does not exceed $\varepsilon^{-\alpha}\log(\varepsilon^{-1})$ and we need only consider large coefficients corresponding to Haar functions whose support is large. Specifically, we have*

$$\int \left|M - \sum_{|R|>\varepsilon,\, |a_R|>\varepsilon} a_R h_R(x)\right|^\alpha dx < \varepsilon^{1-\alpha/2}$$

The next proposition shows that $e_\alpha(M)$ estimates the rate at which M can be approximated by Hölder functions outside sets of small measure.

Proposition. *Let f be such that $e_\alpha \leq 1$. Then there is a decreasing sequence of sets E_ℓ such that $|E_\ell| \leq 2^{-\ell}$ and decompositions of Calderon Zygmund type $f = g_\ell + b_\ell$. Here, b_ℓ is supported on E_ℓ and g_ℓ is bi-Hölder $\beta = 1/\alpha - 1/2$ with constant $2^{(\ell+1)/\alpha}$. Equivalently, g_ℓ has Haar coefficients satisfying $|a_R| \leq 2^{(\ell+1)/\alpha}|R|^{1/\alpha}$.*

Thus we can decompose any matrix into a "good", or mixed-Hölder part, and a "bad" part with small support.

Mixed-Hölder matrices indeed deserve to be called "good" matrices, as they can be substantially sub-sampled. To see this, note that the number of samples needed to recover the functions to a given precision is of the order of the number of tensor Haar coefficients needed for that precision. For balanced partition trees, this is approximately the number of bi-folders R, whose area exceeds the precision ε. This number is of the order of $(1/\varepsilon)^{\alpha} \log(1/\varepsilon)$.

These remarks imply that the entropy condition quantifies the compatibility between the pair of partition trees (on the rows and on the columns) and the matrix on which they are constructed. In other words, to construct useful trees we should seek to minimize the entropy in the induced tensor Haar basis.

For a given matrix M, finding a partition tree pair, which is a global minimum of the entropy, is computationally intractable and not sensible, as the matrix could be the superposition of different structures, corresponding to conflicting organizations. At best we should attempt to peel off organized structured layers.

The iterative procedures for building tree pairs described previously for the text documents example, perform well in practice. These procedures alternate between construction of partition trees on rows and on columns. Each tree defines a Besov norms its dual (i.e. functions on its nodes) which is used to reorganize the dual into a tree leading to a new tree on the original nodes.

A nice example in mathematics, is to view the matrix of eigenvectors of the Laplace operator on a compact riemannian manifold as a data base, in which the columns are the points on the manifold and the rows are the values at the point of different eigenvectors.

We can organize the riemannian geometry in a multiscale geometry, The construction described before builds Besov norms on functions on the manifold, which can be used to measure a distance between eigenvectors (the L^2 distance is useless being = $\sqrt{2}$), thereby inducing a distance on the "Fourier dual" of Laplace eigenvectors. Of course different geometries on the space, will give rise to different dual geometries.

To conclude, we see emerging an analysis or "Signal processing toolbox" for digital data as a first step to analyse the geometry of large data sets in high-dimensional space and analyse functions defined on such data sets. The ideas described above are strongly related to nonlinear principal component analysis, kernel methods, spectral graph embedding, and many more, at the intersection of several branches of mathematics, computer science and engineering. They are documented in literally hundreds of papers in various communities. For a basic introduction to many of these ideas and more, as they relate to diffusion geometries. We refer the interested reader to the July 2006 special issue of Applied and Computational Harmonic Analysis, and references therein Coifman and Lafon [2006].

4 Empirical dynamics, or higher dimensional tensors

The purpose of this section is to show that a corresponding generalization of the analysis to 3-tensors **by triality** enables the organization of dynamical systems as well as purely empirical modeling of natural dynamics. A particular implementation of these

algorithms will allow a systematic realization of all these steps – inferring "natural geometries" from data, using just the data organization counterpart of the above discussion: similarity between nearby observations/measurements.

Recovering the underlying structure of nonlinear dynamical systems from data ("system identification") has attracted significant research efforts over many years, and several ingenious techniques have been proposed to address different aspects of this problem. These include methods to find nonlinear differential equations to discover governing equations from time-series or video sequences equation-free modeling approaches, and methods for empirical dynamic modeling. We present methods extending our prior discussion building upon the work of I. G. Kevrekidis, Gear, Hyman, P. G. Kevrekidis, Runborg, and Theodoropoulos [2003] and Mezic [2016]. Our goal is the organization of observations originating from many different types of dynamical systems into a joint coherent structure, which should parametrize the various dynamical regimes and build empirical models of the whole observation space. Since we are comparing dynamical observations, which are distorted versions of each other, we are forced to discover variants of the *EMD* Shirdhonkar and Jacobs [2008], which go beyond classical transforms in enabling data-driven comparisons between trajectories and their dynamics, see Ankenman [2014] and Coifman and Leeb [2013]

4.1 Problem Formulation and Toy Examples. In our data agnostic setting, we think of time-dependent measurements which are the result of a number of experiments that we will call *trials*; during each trial, the (unknown "state") parameter values remain constant.

In this *black box* setting, the dynamical system is unknown, nonlinear and *autonomous*, and is given by

$$\frac{d\boldsymbol{x}}{dt} = f(\boldsymbol{x};\boldsymbol{p}) \tag{1}$$

$$\boldsymbol{y} = h(\boldsymbol{x}) \tag{2}$$

We do not have access to its state $\boldsymbol{x}$ nor to its parameter values $\boldsymbol{p}$; we also do not know the evolution law f, nor the measurement function h. We only have measurements (observations) $\boldsymbol{y}$ labelled by time t.

The black box is endowed with "knobs" that, in an unknown way, change the values of the parameters $\boldsymbol{p}$; so in every trial, for a new, but unknown, set of parameter values $\boldsymbol{p}$, we can observe $\boldsymbol{y}$ coming out of the box without knowing $\boldsymbol{x}$ or f or even h. We want to characterize the system dynamics by systematically organizing our observations (collected over several trials) of its outputs.

More specifically, we want to (a) organize the observations by finding a set of *state variables* and a set of *system parameters* that jointly preserve the essential features of the dynamics; and then (b) find the corresponding *intrinsic* geometry of this combined variable-parameter space, thus building a sort of normal form for the problem. Small changes in this *jointly intrinsic space* will correspond to small changes in dynamic behavior (i.e. to robustness). Having discovered a useful "joint geometry" we can then inspect its individual constituents. Inspecting, for example, the geometry of the discovered parameter space, will help identify regimes of different qualitative behavior.

This might be different dynamic behavior, like hysteresis, or oscillations, separated by bifurcations; alternatively, we might observe transitions between different sizes of the minimal realizations: regimes where the number of minimal variables/parameters necessary in the realization changes.

We can also inspect the identified state variable geometry, which will help us organize the temporal measurements in coherent phase portraits. In addition, if there exist regimes where the system becomes *singularly perturbed*, we expect we will be able to realize that the requisite minimal phase portrait dimension changes (reduces), and that the reduction in the number of state variables is linked with the reduction in the number of intrinsic parameters.

As an illustrative example, consider the following dynamical system, arising in the unfolding of the Bogdanov–Takens bifurcation Guckenheimer and Holmes [1983]:

$$\begin{aligned} \frac{dx_1}{dt} &= x_2 \\ \frac{dx_2}{dt} &= \beta_1 + \beta_2 x_1 + x_1^2 - x_1 x_2. \end{aligned} \tag{3}$$

This set of differential equations defines a dynamical system with two parameters $\boldsymbol{p} = (\beta_1, \beta_2)$, two state variables $\boldsymbol{x} = (x_1, x_2)$, and two observables $\boldsymbol{y} = (y_1, y_2)$; at first we choose the observable to be the state variables themselves, i.e., $(y_1, y_2) = (x_1, x_2)$ with $h(\boldsymbol{x})$ being the identity function. It is known that the parameter space of this system (β_1, β_2) can be divided into 4 different regimes separated by one-parameter bifurcation curves Guckenheimer and Holmes [ibid.]. Figure 1. shows this "ground truth" bifurcation diagram for our simulated 2D grid of parameter values. Each point $\boldsymbol{p} = (\beta_1, \beta_2)$ on the grid is colored according to its respective dynamical regime.

Our goal in this case would be to discover an accurate bifurcation map of the system in a data-driven manner purely from observations. These observations consist of several samples, where each sample is a single trajectory $\boldsymbol{y}(t)$ of the system initialized with unknown (possibly different) parameter values and initial values. In addition, we would like to deduce from these large number of realizations of trajectories $\boldsymbol{y}(t)$ arbitrarily and differently initialized that the system depends on only two parameters and can be realized with only two state variables; and to reconstruct the bifurcation diagram with its phase portraits.

4.2 Learning dynamic structures and latent variables from observations. Consider data arising from an autonomous dynamical system; we view the observations as entries in a three-dimensional tensor. One axis of the tensor corresponds to variations in the problem parameters, one to variations in the problem variables, and the third axis corresponds to time evolution along trajectories.

Formally, let $\mathcal{P}$ denote an ensemble of N_p sets of the d_p system parameters. Let $\mathcal{V}$ be a ensemble of N_v sets of initial condition values of the d_v state variables. For each $\boldsymbol{p} \in \mathcal{P}$ and $\boldsymbol{v} \in \mathcal{V}$, we observe a trajectory $Y(\boldsymbol{v}, \boldsymbol{p}, t)$ of length N_t in $\mathbb{R}^{d_v}$ of the system variables, where $t = 1, \ldots, N_t$ denotes the time sample. In summary, $\boldsymbol{p}$ is a label of the particular differential equations of the dynamical system, $\boldsymbol{v}$ is a label of the observations trajectory, and t is the time label.

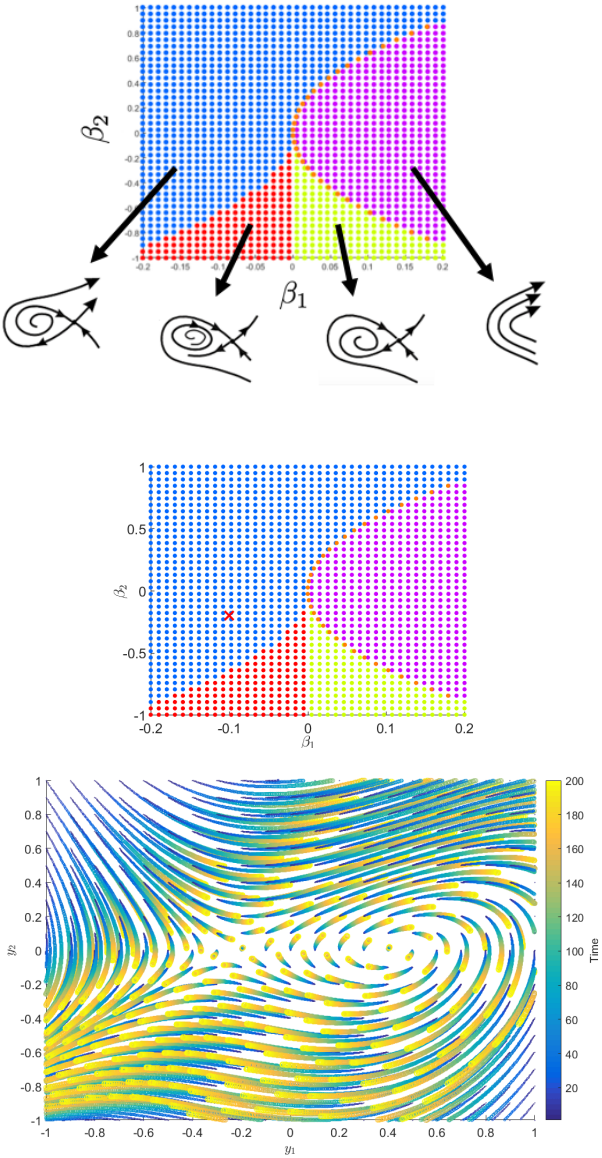


Figure 3: (up) The Bogdanov–Takens bifurcation maps with insets illustrating the typical phase-portraits in each dynamical regime. (left) The Bogdanov–Takens bifurcation map. (right) An example of the phase-portrait of the simulated trajectories of the Bogdanov–Takens system corresponding to the parameter set $(\beta_1, \beta_2) = (-0.1, -0.2)$, marked by red 'x' on the left.

Let $\mathbf{Y}$ denote the entire 3D tensor of observations of dimension $N_p \times N_v \times N_t$ consisting of all the data at hand. With respect to the black box setting described in the

introduction, we emphasize that the identity of the parameters and variables is hidden; we only have trajectories of observations corresponding to various trials with possibly different hidden parameter values and with different hidden initial input coordinates.

To make the problem definition concrete we describe the setting of a specific example. Recall the Bogdanov–Takens dynamical system of two variables and two parameters, introduced in (3). We generate a set $\mathcal{P}$ of $N_p = 400$ different parameter values $\boldsymbol{p} = (\beta_1, \beta_2)$ from a regular fixed 2D grid, where $\beta_1 \in [-0.2, 0.2]$ and $\beta_2 \in [-1, 1]$, and additional 10 parameter values located exactly on the bifurcation. Similarly, we generate a set $\mathcal{V}$ of $N_v = 441$ different initial conditions $\boldsymbol{v} = (y_1(0), y_2(0))$ from a fixed 2D grid in $[-1, 1]^2$. For each $\boldsymbol{p} \in \mathcal{P}$ and $\boldsymbol{v} \in \mathcal{V}$, we observe a trajectory of the system for $N_t = 200$ time steps, where the interval between two adjacent time samples is $\Delta t = 0.004\,[\text{sec}]$ and collect all the trajectories into a single 3D tensor $\mathbf{Y}$. In this example, $N_p = 410$, $N_v = 441$ and $N_t = 200$ so overall we have $\mathbf{Y} \in \mathbb{R}^{410\times441\times200}$. For illustration purposes, Figure 3. (right) depicts $\beta_1 = -0.1$ and $\beta_2 = -0.2$ (marked by a red '×' in Figure 3 (left)).

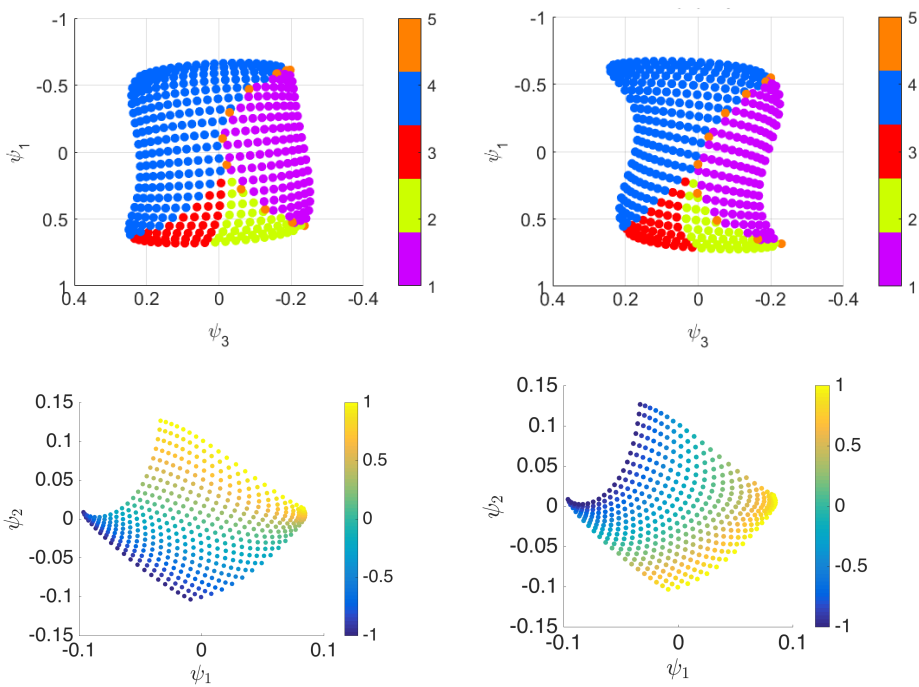


Figure 4: (left) Data-driven embedding of the parameters axis of the observations collected from the Bogdanov–Takens system (colored according to the true bifurcation map). Embeddings built from (a) state variable observations; and (b) observations through a nonlinear invertible function. (right) Data-driven embedding of the state variables axis (c) colored by the initial conditions of x_1, and (d) by the initial conditions of x_2.

We note that the trajectories (as illustrated in Figure 3) are long enough to partially overlap in phase space. Such an overlap induces the coupling between the time and variables axes, which is captured and exploited by our analysis. We wish to find a reliable representation of the hidden parameters, of the hidden variables, and of the time axis.

Define $\boldsymbol{y}_p = \{Y(\boldsymbol{v}, \boldsymbol{p}, t) | \forall \boldsymbol{v}, \forall t\}$ for each of the N_p vectors of hidden parameter values $\boldsymbol{p}$ in $\mathcal{P}$, namely, a data sample consisting of all the trajectories from a single trial. For simplicity of notation, we will use subscripts to denote both the appropriate axis and a specific set of entries values on the axis. We refer to $\{\boldsymbol{y}_p\}$, $\boldsymbol{p} \in \mathcal{P}$ as the data samples from the parameters axis viewpoint. In the Bogdanov–Takens example, Figure 3 depicts $\boldsymbol{y}_p$ for $\boldsymbol{p} = (\beta_1, \beta_2) = (-0.1, -0.2)$.

Similarly, let $\boldsymbol{y}_v$ and $\boldsymbol{y}_t$ be the samples from the viewpoints of the variables axis and the time axis, respectively, which are defined by

$$\begin{aligned} \boldsymbol{y}_v &= \{Y(\boldsymbol{v}, \boldsymbol{p}, t) | \forall \boldsymbol{p}, \forall t\}, \quad \boldsymbol{v} \in \mathcal{V} \\ \boldsymbol{y}_t &= \{Y(\boldsymbol{v}, \boldsymbol{p}, t) | \forall \boldsymbol{v}, \forall \boldsymbol{p}\}, \quad t = 1, \ldots, N_t. \end{aligned}$$

One way to accomplish our goal is to process the data *three successive times*, each time from a different viewpoint.

Here, we use a data-driven parametrization approach based on a kernel. From the trials (effectively, parameters) axis point of view, a typical kernel is defined by

$$k(\boldsymbol{y}_{p_1}, \boldsymbol{y}_{p_2}) = e^{-\|\boldsymbol{y}_{p_1} - \boldsymbol{y}_{p_2}\|^2/\epsilon}, \forall \boldsymbol{p}_1, \boldsymbol{p}_2 \in \mathcal{P} \tag{4}$$

based on distances between any pair of samples, where the Gaussian function induces a sense of locality relative to the kernel scale ϵ. To aggregate the pairwise affinities comprising the kernel into a global parametrization, traditionally, the eigenvalue decomposition (EVD) is applied to the kernel, and the eigenvalues and eigenvectors are used to construct the desired parametrization. The specific initial parametrization method that is used here is diffusion maps Coifman and Maggioni [2006].

From three *separate* diffusion maps applications to the sets $\{\boldsymbol{y}_p\}$, $\{\boldsymbol{y}_v\}$, and $\{\boldsymbol{y}_t\}$, we can obtain three mappings as in (**??**), denoting the associated eigenvectors by $\{\boldsymbol{\psi}_\ell^{\mathcal{P}}\}$, $\{\boldsymbol{\psi}_\ell^{\mathcal{V}}\}$, and $\{\boldsymbol{\psi}_\ell^{\mathcal{T}}\}$, respectively.

However, such mappings do not take into account the strong correlations and co-dependencies between the parameter values and the dynamics of the variables which arise in typical dynamical systems. For example, in the Bogdanov–Takens system, the dynamical regime changes significantly depending on the values of the parameters.

To incorporate such co-dependencies, we extend the mutual metric learning algorithm described for matrices in order to build flexible EMD like distances on each axes of this 3 tensor. In the introduction of the affinity matrix in (4), we deliberately did not specify the norm used to compare between two samples. Common practice is to use the Euclidean norm. However, as pointed out by Lafon [2004] anisotropic diffusion maps can be computed by using different norms. This issue has been extensively studied recently, and several norms and metrics have been developed for this purpose by Mishne, Talmon, Meir, Schiller, Dubin, and Coifman [2015], Dsilva, Talmon, Gear, Coifman, and I. G. Kevrekidis [2016].

Here, following Mishne, Talmon, Meir, Schiller, Dubin, and Coifman [2015], we describe the 3- tensor extension of the preceding metric learning construction for matrix organization where the different axis geometries evolve together.

4.3 Tensor Metric Construction, or "informed metrics" .

Partition Trees, and 3 tensor Besov spaces. The construction described previously for matrices, is easily extended to higher dimensional tensors, the only constraint is to define appropriate metrics on each coordinate axis, in the three tensor case a coordinate label defines a sub matrix, we match two labels 1 and 2 through the tensor Besov distance between them,

$$d_{\alpha,\beta}(M_1 - M_2) = \left(\sum \left|R|^{\beta}|a_R^1 - a_R^2\right|^{\alpha}\right)^{1/\alpha}$$

for some α, β.) Observe that the Besov distance for $\beta > 0$, can be computed without using the Haar functions, simply by replacing the Haar coefficient on the submatrix R by the average on R, see Coifman and Leeb [2013] and Ankenman [2014] We build a partition tree for each axis based on this tensor product metric. Observe that this is a flexible metric generalizing earth mover to the context of matrices, where rows and columns have different smoothness geometries, it is not a conventional transportation metric.

4.4 Iterative Metric Construction. The construction of the partition tree described above relies on a learning a metric between the samples on the different axis coordinates (sample labels), in which the construction of the tree relies on an iteratively evolving "metric" induced by partition trees on the coordinates of the samples. Namely, the construction of $\mathcal{T}_v$ relies on a metric between the samples $\boldsymbol{y}_v$ which are matrices in the p, t labels, i.e., it uses the 2 tensor Besov distance or EMD. and the construction of $\mathcal{T}_t$ relies on a metric between the samples v, p Given $\mathcal{T}_v$ and $\mathcal{T}_t$, the informed metric between the samples $\boldsymbol{y}_p$ is constructed, and then, used to build a new partition tree $\mathcal{T}_p$ of the samples $\boldsymbol{y}_p$. In the second substep within the iteration, $\mathcal{T}_p$ can be used to construct refined metrics between $\boldsymbol{y}_v$ and between $\boldsymbol{y}_t$.

Once the metric is constructed, it can be used to build a partition tree $\mathcal{T}_p$ on the samples $\boldsymbol{y}_p$.

The construction of the informed metric between the samples $\boldsymbol{y}_p$ described above is repeated in an analogous manner to build informed metrics between the samples $\boldsymbol{y}_v$ and between the samples $\boldsymbol{y}_t$. Proving convergence for this "iterative, self-consistent re-normalization" of the coordinates, is the subject of current research.

We note that the particular choice of the specific Besov norm is explained in detail in Coifman and Leeb [2013] yet other L^2 type norms can be used depending on the application at hand.

First, the recursive procedure described above repeats in iterative manner, where in each iteration, three informed metrics are constructed one by one, based on the metrics *from the preceding iteration*. As the iterations progress, the metrics are gradually refined, and the dependency on the initialization is reduced.

Our method is applied to the 3D tensor of trajectories $\mathbf{Y}$ collected from the Bogdanov–Takens system. As described above, $\mathbf{Y}$ consists of (short) trajectories of observations arising from the system initialized with various initial conditions and with various parameters. We emphasize that the knowledge of the different regimes and the bifurcation map *were not taken into account* in the analysis; only the time-dependent data $\mathbf{Y}$ were considered.

Figure 4 (a) depicts the scatter plot of the two dominant eigenvectors representing the parameters axis. It consists of N_p points (the length of the eigenvectors), where each point corresponds to a single sample $\boldsymbol{y}_p \in \mathbb{R}^{N_v \times N_t}$, which is associated with parameters values $\boldsymbol{p} = (\beta_1, \beta_2)$ on the 2D grid depicted in Figure 3. Moreover, each point in Figure 1 is colored by the same color-coding used in Figure 2. We observe that our method discovers an *empirical bifurcation mapping of the system*. Indeed, the obtained representation of the parameters through the eigenvectors establishes a new coordinate system with a geometry, built solely from observations, which reflects the organization of the parameters space according to the true underlying bifurcation map – the "visual homeomorphism" (stopping short of claiming visual isometry) is clear.

To illustrate the generality of our method, we now apply a *nonlinear (yet invertible)* observation function

$$\boldsymbol{z}(t) = h(\boldsymbol{x}(t))$$

with $h_k(\boldsymbol{x}(t)) = \sqrt{\boldsymbol{a}_k^T \boldsymbol{x}(t) + \alpha_k}$, $k = 1, 2$, where $\boldsymbol{a}_k$ is a random observation vector and α_k is a constant set to guarantee positivity. Figure 4 (b) depicts the scatter plot of the two dominant eigenvectors representing the parameters axis obtained from the new set of nonlinear observations. An equivalent organization is clearly achieved.

Figure 4 (c) depicts the scatter plot of the two dominant eigenvectors representing the state variable axis. The plot consists of N_v points (the length of the eigenvectors), where each point corresponds to a single sample $\boldsymbol{y}_v \in \mathbb{R}^{N_p \times N_t}$, which is associated with a particular set of initial condition values $\boldsymbol{v} = (y_1(0), y_2(0))$. The embedded points are colored in Figure 4 (c) by the initial conditions of the variable y_1, and in Figure 4 (d) by the initial conditions of the variable y_2. The color-coding implies that the recovered 2D space corresponds to the 2D space of the true variables of the system. In other words, the high dimensional samples $\boldsymbol{y}_v$ are embedded in a 2D space, which recovers a 2D structure accurately representing the true directions of the hidden, minimal, two state variables of the system.

4.5 Two Coupled Pendula. To demonstrate the ability to extract true physical parameters we simulate a system of two simple coupled pendula with equal lengths L and equal masses m, connected by a spring with variable constant $k(t)$, (corresponding to variable dynamics that we need recover)

To highlight the broad scope of our approach from a data analysis perspective, we assume that we do not have direct access to the horizontal displacement. Instead, we generate *movies* of the motion of the coupled pendula). We now apply a *fixed, invertible, random projection* to each frame of the movie. In other words, each frame of the movie was multiplied by a fixed matrix, whose columns were independently sampled

from a multivariate normal distribution and normalized to have a unit norm. The resulting movie with the projected frames can be found on YouTube under the title Coupled Pendulum Random Projection.

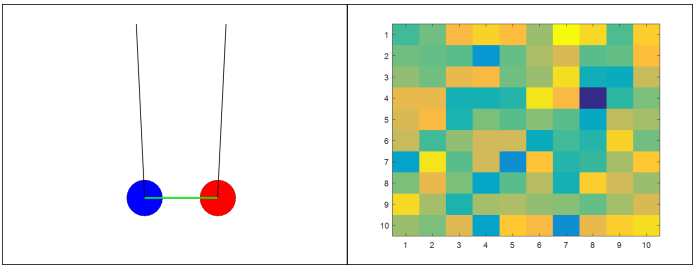


Figure 5: An example of a snapshot of the coupled pendulum system paired with its random projection counterpart.

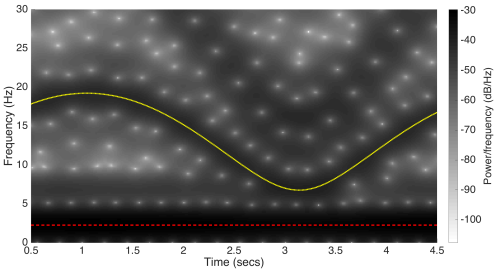


Figure 6: The Fourier spectrogram of the principal eigenvector representing the time axis. These results are based on the random projections of the movies frames with the same time-varying spring constant. The two frequencies ω_1 and $\omega_2(t)$ are overlayed on the spectrogram. The dashed red line corresponds to the fixed oscillation frequency ω_1 and the dotted yellow line corresponds to the time-varying oscillation frequency ω_2.

The pendulum model above was designed as an analogy to calcium fluorescence measuring neuronal activity in the motor cortex of a mouse repeating a task on multipls trials, the fluctuating pixels of the pendulum codes led us to latent variable which are the time variable normal modes. Identical processing on neuronal fluctuations should reveal internal latent controls

The setting in the figure below is identical to the one described above, see Figure 7, but we don't assume any equations and follow Mishne, Talmon, Meir, Schiller, Dubin, and Coifman [2015] in which the project is described

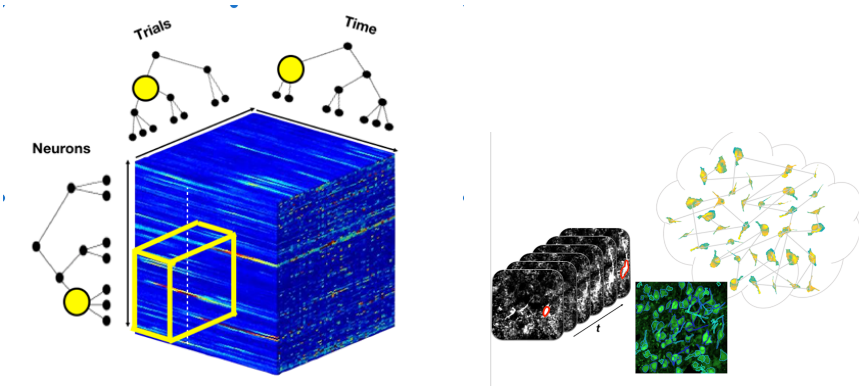


Figure 7: This figure illustrates the setting of our tri-geometry analysis of the collected trial-based neuronal activity from the motor cortical region. These measurements were taken from a behaving mouse in a single day of experiments. The data is composed of 60 trials. A single trial consists of 12 seconds acquired at 10Hz. The recordings are taken from 121 neurons located in M1 cortex. The entire data set of neuronal activity is therefore viewed as a 3-dimensional tensor(left), measuring a (121-dimensional) vector of neuronal activity at each time frame within each trial, and the neuronal activity is represented by the intensity level of the image (blue – no activity, red – high activity) The three trees in triality are plotted and a sub box in yellow corresponds to a group of neurons coactivity on a group of trials, at a fixed period. The data is visualized on the right as 2D slices of temporal neural images, with a clean neural map extracted in green, and the neuron graph above.

5 Learning Empirical Intrinsic Geometry, EIG

In the preceding sections we have glossed over the basic problem of initializing the geometric affinity, we ignored the dependence of the eigenvectors on the coordinate system. We now describe with more detail, a simpler setup where empirical analysis reveals the underlying intrinsic latent geometric coordinates on which data is measured.

Our basic assumption is, that we are observing a stochastic time series governed by a Langevin equation on a riemannian manifold, these observations are transformed through a nonlinear transformation into high dimensions in an ambient unknown independent noisy environment. Our goal is to show that we can recover the original riemannian manifold as well as the potential, driving the dynamics of the observations. Moreover by building a geometry capturing the normalized variabilities of local statistical histograms, we eliminate the effect of external noise interferences.

As an example consider a molecule (Alanine Dipeptide) consisting of 10 atoms and oscillating stochasticallly in water. It is known that the configuration at any given time is essentially described by two angle parameters. We assume that we observe five atoms of the molecule for a certain period of time, and five other atoms in the remainder time. The task is to describe the position of all atoms at all times, or more precisely, discover the angle parameters and their relation to the position of all atoms. See Figure 8. The

main point is that the observations are quite different, perhaps using completely different sensors in different environments (but same dynamic phenomenon) and that we derive an identical intrinsic "natural" manifold parameterizing the observations.

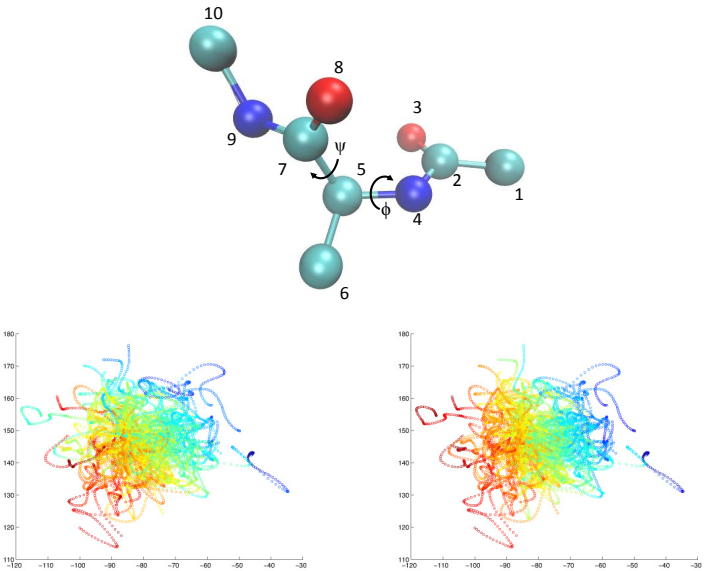


Figure 8: (a) A representative molecular structure of Alanine Dipeptide, excluding the hydrogens. The atoms are numbered and the two dihedral angles ϕ and ψ are indicated. (b)-(c): A 2-dimensional scatter plot of random trajectories of the dihedral angles ϕ and ψ. Based on observations of the corresponding random trajectories of merely five out of ten atoms of the molecule, we infer a model describing one of the angles. The points are colored according to the values of the inferred model from the five even atoms (b) and the five odd atoms (c). We observe that the gradient of the color is parallel to the x-axis, indicating an adequate representation of one of the angles. In addition, the color patterns are similar, indicating that the models are independent of the particular atoms observed, and describe the common intrinsic parameterization of the molecule dynamics.

An important remark is that we observe stochastic data constrained to lie on an unknown riemannian manifold, that we need somehow to reconstruct explicitly, not having any coordinate system on the manifold. This is achieved through the explicit construction of the eigenvectors of an intrinsic Laplace operator on the manifold (observations), these can be used to obtain a low dimensional canonical embeddings independent of observation modality, and obtain local charts on the manifold. This invariant description of the dynamics, is similar to the reformulation of Newton's law through invariant Hamiltonian equations see Talmon and Coifman [2013].

Broad outline.

To achieve this task and learn an intrinsic riemannian manifold structure, we assume that we observe stochastic clouds of points corresponding to some unknown standard brownian ensemble (as in the example of the Alanine molecule for short time intervals). More specifically this process has three scales:

The first identifies "local micro clouds" and converts them to statistical histograms.

The second relates clouds of histograms to each other using the affine invariant Mahalanobis metric between histograms, this metric is immune to the distortion due to independent noise.

The third builds the whole riemannian manifold by integrating the local metrics

.

This provides an intrinsic riemannian manifold that is both insensitive to noise and, invariant to changes of variables.

(This construction is a data driven version of information geometry see Ollivier [n.d.])

5.1 detailed description. Specifically and for simplicity of exposition, we consider a flat manifold for which we adopt the state-space formalism to provide a generic problem formulation that may be adapted to a wide variety of applications.

Let $\boldsymbol{\theta}_t$ be a d-dimensional underlying coordinates of a process in time index t. The dynamics of the process are described by normalized stochastic differential equations as follows[1]

$$d\theta_t^i = a^i(\theta_t^i)dt + dw_t^i,\ i = 1, \ldots, d, \tag{5}$$

where a^i are unknown drift functions and $\dot{w}_t^i$ are independent white noises. For simplicity, we consider here normalized processes with unit variance noises. Since a^i are any drift functions, we may first apply normalization without effecting the following derivation. See Singer and Coifman [2008] and Talmon and Coifman [2013] for details. We note that the underlying process is equivalent to the system state in the classical terminology of the state-space approach.

Let $\mathbf{y}_t$ denote an n-dimensional observation process in time index t, drawn from a probability density function (pdf) $f(\mathbf{y}; \boldsymbol{\theta})$. The statistics of the observation process are time-varying and depend on the underlying process $\boldsymbol{\theta}_t$. We consider a model in which the clean observation process is accessible only via a noisy n-dimensional measurement process $\mathbf{z}_t$, given by

$$\mathbf{z}_t = g(\mathbf{y}_t, \mathbf{v}_t) \tag{6}$$

where g is an unknown (possibly nonlinear) measurement function and $\mathbf{v}_t$ is a corrupting n-dimensional measurement noise, drawn from an unknown stationary pdf $q(\mathbf{v})$ and independent of $\mathbf{y}_t$.

[1] x^i denotes access to the ith coordinate of a point $\mathbf{x}$.

The description of $\boldsymbol{\theta}_t$ constitutes a parametric manifold that controls the accessible measurements at-hand. Our goal is to reveal the underlying process $\boldsymbol{\theta}_t$ and its dynamics based on a sequence of measurements $\{\mathbf{z}_t\}$.

Let $p(\mathbf{z};\boldsymbol{\theta})$ denote the pdf of the measured process $\mathbf{z}_t$ controlled by $\boldsymbol{\theta}_t$, it satisfies the following property.

Lemma 1. *The pdf of the measured process $\mathbf{z}_t$ is a linear transformation of the pdf of the clean observation component $\mathbf{y}_t$.*

The proof is obvious, relying on the independence of $\mathbf{y}_t$ and $\mathbf{v}_t$, the pdf of the measured process is given by

$$p(\mathbf{z};\boldsymbol{\theta}) = \int\limits_{g(\mathbf{y},\mathbf{v})=\mathbf{z}} f(\mathbf{y};\boldsymbol{\theta})q(\mathbf{v})d\mathbf{y}d\mathbf{v}. \tag{7}$$

We note that in the common case of additive measurement noise, i.e., $g(\mathbf{y},\mathbf{v}) = \mathbf{y}+\mathbf{v}$, only a single solution $\mathbf{v}(\mathbf{z}) = \mathbf{z}-\mathbf{y}$ exists. Thus, $p(\mathbf{z};\boldsymbol{\theta})$ in (7) becomes a linear convolution

$$p(\mathbf{z};\boldsymbol{\theta}) = \int\limits_{\mathbf{y}} f(\mathbf{y};\boldsymbol{\theta})q(\mathbf{z}-\mathbf{y})d\mathbf{y} = f(\mathbf{z};\boldsymbol{\theta}) * q(\mathbf{z}).$$

The dynamics of the underlying process are conveyed by the time-varying pdf of the measured process. Thus, this pdf may be very useful in revealing the desired underlying process and its dynamics. Unfortunately, the pdf is unknown since the underlying process and the dynamical and measurement models are unknown. Assume we have access to a class of estimators of the pdf over discrete bins which can be viewed as linear transformations. Let $\mathbf{h}_t$ be such an estimator with m bins which is viewed as an m-dimensional process and is given by

$$p(\mathbf{z};\boldsymbol{\theta}_t) \overset{\mathcal{T}}{\mapsto} \mathbf{h}_t, \tag{8}$$

where $\mathcal{T}$ is a linear transformation of the density $p(\mathbf{z};\boldsymbol{\theta})$ from the infinite sample space of $\mathbf{z}$ into a finite interval space of dimension m. By Lemma 1 and by definition (8) we get the following results.

The process $\mathbf{h}_t$ is a linear transformation of the pdf of the clean observation component $\mathbf{y}_t$.

The process $\mathbf{h}_t$ can be described as a deterministic nonlinear map of the underlying process $\boldsymbol{\theta}_t$.

We can use histograms as estimates of the pdf, and we assume that a *sequence* of measurements is available. Accordingly, let $\mathbf{h}_t$ be the empirical local histogram of the measured process $\mathbf{z}_t$ in a short-time window of length L_1 at time t. Let $\mathcal{Z}$ be the sample space of $\mathbf{z}_t$ and let $\mathcal{Z} = \bigcup_{j=1}^{m} \mathcal{H}_j$ be a finite partition of $\mathcal{Z}$ into m disjoint histogram bins. Thus, the value of each histogram bin is given by

$$h_t^j = \frac{1}{|\mathcal{H}_j|}\frac{1}{L_1}\sum_{s=t-L_1+1}^{t} \mathbf{1}_{\mathcal{H}_j}(\mathbf{z}_s), \tag{9}$$

where $\mathbf{1}_{\mathcal{H}_j}(\mathbf{z}_t)$ is the indicator function of the bin $\mathcal{H}_j$ and $|\mathcal{H}_j|$ is its cardinality. By assuming (unrealistically) that infinite number of samples are available and that their density in each histogram bin is uniform, (9) can be expressed as

$$h_t^j = \frac{1}{|\mathcal{H}_j|} \int_{\mathbf{z}\in\mathcal{H}_j} p(\mathbf{z};\boldsymbol{\theta})d\mathbf{z}. \tag{10}$$

Thus, ideally the histograms are *linear transformations* of the pdf. In addition, if we shrink the bins of the histograms as we get more and more data, the histograms converge to the pdf

$$\mathbf{h}_t \xrightarrow[|\mathcal{H}_j|\to 0]{L_1\to\infty} p(\mathbf{z};\boldsymbol{\theta}). \tag{11}$$

In practice, since the computation of high-dimensional histograms is challenging, we preprocess high-dimensional data by applying random filters in order to reduce the dimensionality without corrupting the information.

5.2 Mahalanobis Distance. We view $\mathbf{h}_t$ (the linear transformation of the local densities, e.g. the local histograms) as feature vectors for each measurement $\mathbf{z}_t$. The process $\mathbf{h}_t$ satisfies the dynamics given by Itô's lemma

$$\begin{aligned} h_t^j &= \sum_{i=1}^{d}\left(\frac{1}{2}\frac{\partial^2 h^j}{\partial\theta^i\partial\theta^i} + a^i\frac{\partial h^j}{\partial\theta^i}\right)dt \\ &\quad + \sum_{i=1}^{d}\frac{\partial h^j}{\partial\theta^i}dw_t^i,\ j=1,\dots,m. \end{aligned} \tag{12}$$

For simplicity of notation, we omit the time index t from the partial derivatives. According to (12), the (j,k)th element of the $m\times m$ covariance matrix $\mathbf{C}_t$ of $\mathbf{h}_t$ is given by

$$C_t^{jk} = \mathrm{Cov}(h_t^j, h_t^k) = \sum_{i=1}^{d}\frac{\partial h^j}{\partial\theta^i}\frac{\partial h^k}{\partial\theta^i},\ j,k=1,\dots,m. \tag{13}$$

In matrix form, (13) can be rewritten as

$$\mathbf{C}_t = \mathbf{J}_t\mathbf{J}_t^T \tag{14}$$

where $\mathbf{J}_t$ is the $m\times d$ Jacobian matrix, whose (j,i)th element is defined by

$$J_t^{ji} = \frac{\partial h^j}{\partial\theta^i},\ j=1,\dots,m,\ i=1,\dots,d.$$

Thus, the covariance matrix $\mathbf{C}_t$ is a semi-definite positive matrix of rank d.

We define a nonsymmetric $\mathbf{C}$-dependent squared distance between pairs of measurements as

$$a_{\mathbf{C}}^2(\mathbf{z}_t,\mathbf{z}_s) = (\mathbf{h}_t-\mathbf{h}_s)^T\mathbf{C}_s^{-1}(\mathbf{h}_t-\mathbf{h}_s) \tag{15}$$

and a corresponding symmetric distance as

$$d_{\mathbf{C}}^2(\mathbf{z}_t, \mathbf{z}_s) = 2(\mathbf{h}_t - \mathbf{h}_s)^T (\mathbf{C}_t + \mathbf{C}_s)^{-1} (\mathbf{h}_t - \mathbf{h}_s). \tag{16}$$

Since usually the dimension d of the underlying process is smaller than the number of histogram bins m, the covariance matrix is singular and non-invertible. Thus, in practice we use the pseudo-inverse to compute the inverse matrices in (15) and (16).

The distance in (16) is known as the *Mahalanobis distance* with the property that it is invariant under linear transformations. Thus, by Lemma 1, it is invariant to the measurement noise and functional distortion (e.g., additive noise or multiplicative noise). We note however that the linear transformation employed by the measurement noise on the observable pdf (7) may degrade the available information.

In addition, by Lemma 3.1 in Kushnir, Haddad, and Coifman [2012b], the Mahalanobis distance in (16) approximates the Euclidean distance between samples of the underlying process. Let $\boldsymbol{\theta}_t$ and $\boldsymbol{\theta}_s$ be two samples of the underlying process. Then, the Euclidean distance between the samples is approximated to a second order by a local linearization of the nonlinear map of $\boldsymbol{\theta}_t$ to $\mathbf{h}_t$, and is given by

$$\|\boldsymbol{\theta}_t - \boldsymbol{\theta}_s\|^2 = d_{\mathbf{C}}^2(\mathbf{z}_t, \mathbf{z}_s) + O(\|\mathbf{h}_t - \mathbf{h}_s\|^4). \tag{17}$$

For more details see Singer and Coifman [2008] and Kushnir, Haddad, and Coifman [2012b]. Assuming there is an intrinsic map $i(\mathbf{h}_t) = \boldsymbol{\theta}_t$ from the feature vector to the underlying process, the approximation in (17) is equivalent to the inverse problem defined by the following nonlinear differential equation

$$\sum_{i=1}^{m} \frac{\partial \theta^j}{\partial h^i} \frac{\partial \theta^k}{\partial h^i} = \left[C_t^{-1}\right]^{jk}, \quad j, k = 1, \ldots, d. \tag{18}$$

This equation which is nothing more than a discrete formulation of the definition of a riemannian metric on the manifold is empirically solved through the eigenvectors of the corresponding discrete Laplace operator. The approximation in (17) recovers the intrinsic distances on the parametric manifold and is obtained empirically from the noisy measurements by "infinitesimally" inverting the measurement function.

For further illustration, see Fig. 9.

5.3 Local Covariance Matrix Estimation. Let t_0 be the time index of a "pivot" sample $\mathbf{h}_{t_0}$ of a "cloud" of samples $\{\mathbf{h}_{t_0,s}\}_{s=1}^{L_2}$ of size L_2 taken from a local neighborhood in time. Here we assume that a sequence of measurements is available, the temporal neighborhoods can be simply short windows in time centered at time index t_0.

The pdf estimates and the local clouds implicitly define two time scales on the sequence of measurements. The fine time scale is defined by short-time windows of L_1 measurements to estimate the temporal pdf. The coarse time scale is defined by the local neighborhood of L_2 neighboring feature vectors in time. Accordingly, we note that the approximation in (17) is valid as long as the statistics of the noise are locally fixed in the short-time windows of length L_1 (i.e., slowly changing compared to the fast

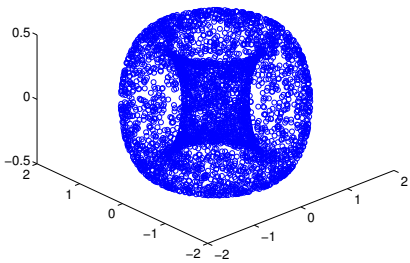


Figure 9: Consider a set of points on a 2-dimensional torus in $\mathbb{R}^3$ ("the manifold") which are samples of a Brownian motion on the torus. The geometric interpretation of the intrinsic notion is the search for a canonical description of the set, which is independent of the coordinate system. For example, the points can be written in 3 cartesian coordinates, or in the common parameterization of a torus using two angles, however, the intrinsic model (constructed based on the points) describing the torus should be the same. The Mahalanobis distance attaches to each point a riemannian metric that corresponds to a probability measure that is driven by the underlying dynamics (the Brownian motion in this particular case), and therefore, it is invariant to the coordinate system.

variations of the underlying process) and the fast variations of the underlying process can be detected in the difference between the feature vectors in windows of length L_2.

According to the dynamical model in (5) and (12), the samples in the local cloud can be seen as small perturbations of the pivot sample created by the noise $\mathbf{w}_t$. Thus, we assume that the samples share similar local probability densities[2] and may be used to estimate the local covariance matrix, which is required for the construction of the Mahalanobis metric (16). The empirical covariance matrix of the cloud is estimated by

$$\begin{aligned}\hat{\mathbf{C}}_{t_0} &= \frac{1}{L_2}\sum_{s=1}^{L_2}\left(\mathbf{h}_{t_0,s}-\hat{\boldsymbol{\mu}}_{t_0}\right)\left(\mathbf{h}_{t_0,s}-\hat{\boldsymbol{\mu}}_{t_0}\right)^T \\ &\simeq \mathbb{E}\left[\left(\mathbf{h}_{t_0}-\mathbb{E}[\mathbf{h}_{t_0}]\right)\left(\mathbf{h}_{t_0}-\mathbb{E}[\mathbf{h}_{t_0}]\right)^T\right] = \mathbf{C}_{t_0}\end{aligned} \tag{19}$$

where $\hat{\boldsymbol{\mu}}_{t_0}$ is the empirical mean of the set.

As the rank of the matrix d is usually smaller than the covariance matrix dimension m, in order to compute the inverse matrix we use only the d principal components of the matrix. This operation "cleans" the matrix and filters out noise. In addition, when the empirical rank of the local covariance matrices of the feature vectors is lower than d, it indicates that the available feature vectors are insufficient and a larger cloud should be used.

[2]We emphasize that we consider the statistics of the feature vectors and not the feature vectors themselves, which are estimates of the varying statistics of the raw measurements.

6 Concluding remarks and bibliography

This overview of various methodologies to learn and extract natural geometries, and latent variables from point clouds generated by, observations, computations, or mathematical processes, is by necessity superficial, and neglects to cover the massive amount of literature and algorithms around machine learning. We refer to Ollivier [n.d.] who has pursued invariant geometric ideas, like the EIG approach, in the context of information geometry and natural deep learning. The use of tensors to extract features analogous to principal components ("tensor PCA") in the context of machine learning, or data processing is also quite extensive, see Anandkumar, Ge, Hsu, Kakade, and Telgarsky [2014].

Our goal here was to emphasize on the one hand the co-dependent geometries in duality or triality (duality in Besov spaces enables generalizing flexible earth mover distances), and on the other hand to illustrate the essential interplay between geometry of point clouds with various analytic measures of smoothness. This construction enables both effective Harmonic analysis and dynamic metric constructions. We point out that in the case of matrices, or even convolution operators on functions, it is not generally effective to use their eigenvectors or Fourier transform, in order to unravel its effect on functions.

The Calderon–Zygmund decompositions were introduced to gain an intimate understanding of the Hilbert transform, they have their wavelet analogs. We try to convey here, that this basic geometric organizationn philosophy is natural in the context of mathematical geometric learning. More generally given a class of geometric structures, such as curves or embedded surfaces it is natural to relate them through the properties of various operators intrinsic operators, such as a Diffusion, or other functions of the Laplace operator, and then use a distance between these operators, as a way of measuring similarity between the structures see Berard et al, who show that the distance between riemannian manifolds can be measured Bérard, Besson, and Gallot [1994]. This is our approach in the 3 tensor case, where we can view one axis as the label for the structures, and the other two as representing the corresponding operators, the metrics so defined are quite remarkable. A similar vision is developed to achieve "shape" matching by G. Peyre, M. Cuturi, J. Solomon. They measure the distance between affinity matrices, or diffusions through appropriate Earth mover distances, see Peyré, Cuturi, and Solomon [2016] and their references.

We should also mention that alternative multiscale data models were developed by Allard, Chen, and Maggioni [2012], as well as R Lederman and Lederman and Talmon [2018]who developed a methodology to extract common latent variables between disparate sets of observations, this method can have a profound impact on the scientific discovery process.

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SOME RECENT DEVELOPMENTS IN KÄHLER GEOMETRY AND EXCEPTIONAL HOLONOMY

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Abstract

This article is a broad-brush survey of two areas in differential geometry. While these two areas are not usually put side-by-side in this way, there are several reasons for discussing them together. First, they both fit into a very general pattern, where one asks about the existence of various differential-geometric structures on a manifold. In one case we consider a complex Kähler manifold and seek a distinguished metric, for example a Kählera–Einstein metric. In the other we seek a metric of exceptional holonomy on a manifold of dimension 7 or 8. Second, as we shall see in more detail below, there are numerous points of contact between these areas at a technical level. Third, there is a pleasant contrast between the state of development in the fields. These questions in Kähler geometry have been studied for more than half a century: there is a huge literature with many deep and far-ranging results. By contrast, the theory of manifolds of exceptional holonomy is a wide-open field: very little is known in the way of general results and the developments so far have focused on examples. In many cases these examples depend on advances in Kähler geometry.

1 Kähler geometry

1.1 Review of basics. Let X be a compact complex manifold of complex dimension n. We recall that a Hermitian metric on X corresponds to a positive 2-form ω_0 of type $(1,1)$ That is, in local complex co-ordinates z_a

$$\omega_0 = i \sum_{a,b} g_{ab} dz_a d\overline{z}_b,$$

where at each point (g_{ab}) is a positive-definite Hermitian matrix. The Hermitian metric is *Kähler* if ω_0 is a closed form, so then X is simultaneously a complex, Riemannian and symplectic manifold. The form ω_0 defines a cohomology class $[\omega_0] \in H^2(X;\mathbf{R})$ and any other Kähler metric in this class can be represented by a potential function $\omega_\phi = \omega_0 + i\,\partial\overline{\partial}\phi$. In local co-ordinates this changes g_{ab} to

$$g_{ab} + \frac{\partial^2 \phi}{\partial z_a \partial \overline{z}_b}.$$

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Recall that in Riemannian geometry the *Ricci tensor* is a symmetric 2-tensor given by a contraction of the full Riemann curvature. Geometrically, it represents the infinitesimal behaviour of the volume form of the manifold, in geodesic co-ordinates. In the Kähler case the Ricci curvature also corresponds to a $(1,1)$ form and has another interpretation as the curvature of the induced connection on the *anti-canonical line bundle* $K_X^{-1} = \Lambda^n TX$. In local co-ordinates this Ricci form is given by $\rho = i\partial\bar{\partial}L$, where L is the logarithm of the volume element

$$L = \log\det\left(g_{ab} + \frac{\partial^2\phi}{\partial z_a \partial \bar{z}_b}\right).$$

The trace of the Ricci curvature is the *scalar curvature* S, which is a function on X.

The aspect of Kähler geometry which is our concern here is the existence of particular metrics, in a given Kähler class, which correspond to the solutions of partial differential equations for the potential function ϕ. The four equations we have in mind are as follows.

- *Kähler–Einstein metrics* with $\rho = \lambda\omega$ for $\lambda = 1, -1$ or 0. These can only occur when $c_1(X)$ (a topological invariant of the complex structure) is the corresponding multiple of the Kähler class.
- *Constant scalar curvature Kähler* (CSCK) metrics with $S =$ constant. The constant is determined by the topological data, $c_1(X)$, $[\omega]$. When $c_1(X)$ is a multiple of $[\omega]$ an integral identity shows that a CSCK metric is Kähler–Einstein.
- *Extremal metrics*. The satisfy the condition that the gradient of S, a vector field on the manifold, is holomorphic. Obviously CSCK metrics (where the gradient is zero) are extremal and in general the holomorphic vector field is determined by the topological data.
- *Kähler–Ricci solitons*. These satisfy the condition that

  $$\rho - \lambda\omega = L_v\omega$$

  where v is a holomorphic vector field and L_v denotes the Lie derivative. Obviously Kähler–Einstein metrics satisfy this condition (with $v = 0$) and, as before, v is determined by the topological data.

Extremal metrics and Kähler–Ricci solitons arise naturally in the study of associated parabolic equations. The *Kähler–Ricci flow* is the evolution equation

$$\frac{\partial\omega}{\partial t} = \lambda\omega - \rho.$$

A Kähle–Ricci soliton represents a fixed point of this flow modulo the holomorphic diffeomorphisms—the flow starting from a Kähler–Ricci soliton evolves as a family of geometrically equivalent metrics. The *Calabi flow* is the evolution equation

$$\frac{\partial\omega}{\partial t} = i\partial\bar{\partial}S,$$

and extremal metrics appear in the same way.

1.2 The YTD conjecture. The existence of Kähler–Einstein metrics in the cases when $\lambda = -1$ and $\lambda = 0$ was established in the 1970's in breakthrough work of Yau [1978] confirming conjectures of Calabi from the 1950's Calabi [1957],Calabi [1956]. (The case when $\lambda = -1$ was treated independently by Aubin.) The positive case, when $\lambda = 1$ (which arises when X is a Fano manifold) is more subtle and, beginning with a theorem of Matsushima—also from the 1950's—various obstructions were found to the existence of a Kähler–Einstein metric. In the 1980's Yau suggested that the existence of these metrics should be equivalent to some algebro-geometric notion of "stability". One motivation for this idea came from the theory of *Hermitian Yang–Mills connections*. Let E be a holomorphic vector bundle over a Kähler manifold (X, ω_0) and suppose for simplicity that $c_1(E) = 0$ and that E is indecomposable (cannot be written as a direct sum). A Hermitian Yang–Mills connection on E is a unitary connection, compatible with the holomorphic structure, such that the inner product of the curvature with the Kähler form vanishes. The bundle E is called *stable* if any proper subsheaf $\mathcal{S} \subset \mathcal{O}(E)$ has $c_1(\mathcal{S}) < 0$. This is a purely complex-geometric notion (or algebro-geometric, in the case when X is a projective variety), which arose in the study of moduli problems for holomorphic bundles. The main result is that stability is the necessary and sufficient condition for the existence of a Hermitian Yang–Mills connection. This was proved by Uhlenbeck and Yau [1986] in 1986 (with an independent treatment by Donaldson in the algebraic case), confirming conjectures made by Kobayashi and Hitchin. Many extensions of the result have been found, for example to coupled equations with connections and additional "fields". Yau's proposal was that the Kähler–Einstein question could be put in a similar framework.

Considerable progress towards confirming Yau's proposal was made by Tian during the 1990's, In particular, Tian gave a definition of the notion of stability for Fano manifolds, called *K-stability*, which turns out to be the correct one Tian [1997]. Around the turn of the century, Donaldson suggested that the whole conjectural picture could be extended to the question of existence of constant scalar curvature metrics in a given rational Kähler class on an projective variety, with a suitable extension of the definition of K-stability Donaldson [1997], Donaldson [1999]. Later, Székelyhidi proposed further extensions to the existence of extremal Kähler metrics Székelyhidi [2007].

We now recall the definition of K-stability. We restrict attention to the case when the Kähler class is the first Chern class of a positive line bundle L, so in algebro-geometric terms we have a polarised manifold (X, L). The definition is founded on the notions of *test configurations* and *Futaki invariants*. A test configuration for (X, L) is a normal variety $\mathcal{X}$ which forms a flat family $\pi : \mathcal{X} \to \mathbf{C}$ with a line bundle $\mathcal{L} \to \mathcal{X}$ such that:

- $\mathcal{L}$ is ample on the fibres of π;
- there is a $\mathbf{C}^*$-action on $(\mathcal{X}, \mathcal{L})$ covering the standard action on $\mathbf{C}$;
- the fibre $\pi^{-1}(1)$ is isomorphic to X and $\mathcal{L}|_{\pi^{-1}(1)} = L^r$ for some r.

Of course the $\mathbf{C}^*$-action means that all non-zero fibres are isomorphic to X and the essential point is that a test configuration gives a *degeneration* of X to a typically-different object: the scheme-theoretic central fibre $X_0 = \pi^{-1}(0)$. For example we might degenerate a smooth conic in the plane to a pair of lines or a "double line".

Given such a test configuration, the central fibre $X_0 = \pi^{-1}(0)$, with the restriction $\mathfrak{L}_0$, is a polarised scheme with $\mathbf{C}^*$-action. In such a situation there is a numerical invariant, the Futaki invariant, which can be defined in various ways. One is through the induced action on a "CM line" associated to $(X_0, \mathfrak{L}_0)$; another is through the asymptotics as $k \to \infty$ of the induced action on the vector spaces $H^0(X_0, \mathfrak{L}_0^k)$. The upshot is that we get a numerical invariant $\mathrm{Fut}(\mathfrak{X})$ of a test configuration. We say that (X, L) is K-stable if $\mathrm{Fut}(\mathfrak{X}) \geq 0$ for all test configurations and equality occurs only if $\mathfrak{X}$ is a product $X \times \mathbf{C}$, with a possibly non-trivial $\mathbf{C}^*$ action on the X factor. The "standard conjecture" in the field, which has come to be called the Yau–Tian–Donaldaon (YTD) conjecture, is that there is a constant scalar curvature metric on X in the class $c_1(L)$ if and only if (X, L) is K-stable. An inkling of the connection between the algebro-geometric and differential geometric sides is given by considering the case when the central fibre X_0 is smooth. Choose a Kähler form in the class $c_1(\mathfrak{L}_0)$ on X_0 invariant under the action of the subgroup $S^1 \subset \mathbf{C}^*$. The action is generated by a Hamiltonian function H on X_0, with respect to this Kähler form, regarded as a symplectic structure. One then has a formula

$$\mathrm{Fut}(\mathfrak{X}) = \int_{X_0} (S - \hat{S})\, H\, d\mu,$$

where S is the scalar curvature of X_0 and $\hat{S}$ is the average value of S.

This YTD conjecture has close parallels with the Kobayashi–Hitchin conjecture for Hermitian–Yang–Mills connections discussed above and in fact the definition of stability for holomorphic bundles can be put in a similar shape, involving suitable degenerations of the vector bundle.

The YTD conjecture has been confirmed in certain cases, in particular for toric surfaces Donaldson [2009] and for Fano manifolds, as we will discuss below. However there are reasons to think that in general the conjecture may not be exactly true as formulated and that the correct conjecture should involve a slightly different definition of stability. One such notion has been proposed by Székelyhidi [2015]. This involves an algebraic generalisation of the notion of a test configuration having the form of a filtration of the co-ordinate ring

$$R_{X,L} = \bigoplus_k H^0(X, L^k).$$

The set of filtrations can be thought of roughly as a "completion" of the set of test configurations. Székelyhidi obtained a slightly stronger notion called $\hat{K}$-stability, using an extension of the definition of the Futaki invariant to filtrations.

Another important variant is "uniform K-stability" which also goes back to work of Székelyhidi, with later developments by Dervan [2016] and Boucksom, Hisamoto, and Jonsson [2017]. This notion depends upon a choice of a "norm" $\|\mathfrak{X}\|$ on test configurations. For example in the case of a smooth central fibre considered above this could be given by an L^p norm of the Hamiltonian function H, normalised to have integral zero:

the definition in the general case is more technical. A pair (X, L) with no non-trivial automorphisms is called uniformly K-stable if there is some $\epsilon > 0$ such that for all test configurations $\mathcal{X}$

$$\mathrm{Fut}(\mathcal{X}) \geq \epsilon \|\mathcal{X}\|.$$

(The definition needs to be modified slightly in the case when $\mathrm{Aut}(X, L)$ is non-trivial.)

The relation between these different notions of stability can be illustrated by considering the case of toric pairs (X, L). Such pairs corresponds to certain closed polytopes $P \subset \mathbf{R}^n$, which come with a distinguished measure $d\sigma$ on the boundary. Then we can define a linear functional on functions on P,

$$L_P(f) = \int_{\partial P} f d\sigma - A \int_P f d\mu,$$

where the constant A is fixed by requiring that L_P vanishes on constant functions. We say that a function f is "normalised" if

$$\int_P f \, d\mu = 0, \int_P f \, x_i \, d\mu = 0$$

for each co-ordinate function x_i. We say that f is a *convex rational piecewise-linear* function if $f = \max\{\lambda_r\}$, where $\{\lambda_r\}$ is a finite collection of affine-linear functions with rational coefficients. Write QPL for the set of such functions f. Then we may consider three properties of the polytope P.

1. For all non-zero normalised $f \in QPL$ we have $L_P(f) > 0$.
2. For all non-zero normalised continuous convex functions f on P we have $L_P(f) > 0$.
3. There is an $\epsilon > 0$ such for all normalised $f \in QPL$ we have

$$L_P(f) \geq \epsilon \int_P |f| d\mu.$$

These three properties correspond respectively to K-stability, $\hat{K}$-stability and uniform K-stability of the pair (X, L) determined by P. (Notice that in the last item it does not matter whether we consider rational piecewise linear or general continuous convex functions, since the former are a dense subset of the latter.)

1.3 Fano manifolds and Kähler–Einstein metrics. The YTD conjecture in the case of Fano manifolds and Kähler–Einstein metrics (the case originally considered by Yau) was confirmed in X. Chen, Donaldson, and Sun [2015]. Since this work was discussed by Székelyhidi in his contribution to the 2014 ICM Székelyhidi [2014] we will not say much about it here. By now, four different proofs of (essentially) this result have appeared; the references for the other three being Datar and Székelyhidi [2016], X. Chen, Sun, and Wang [2015], K. Liu and Ma [2007]. Three of these proofs follow somewhat similar strategies. The approach is to take some deformation or continuity procedure

and show that either it produces a Kähler–Einstein metric on our Fano manifold X or it produces (roughly speaking) the central fibre X_0 in a test configuration which contradicts K-stability. These three proofs depend heavily on the Cheeger–Colding convergence theory for manifolds with Ricci curvature bounds which we will come back to in Section 3.4 below. In the first proof, by Chen, Donaldson and Sun, the deformation process involves Kähler–Einstein metrics with cone singularities along a divisor in X and increasing the cone angle from some small initial value. If we achieve cone angle 2π we have a smooth metric. In the second proof, Datar and Székelyhidi used the Aubin–Yau continuity method with the 1-parameter family of equations

$$\mathrm{Ric}(\omega_s) = (1-s)\alpha + s\omega_s,$$

where α is a fixed positive $(1,1)$-form. In the third proof, X. Chen and Wang [2012] study the Kähler–Ricci flow ω_t starting with some initial metric on X. The flow exists for all positive time and Chen and Wang show that for any sequence $t_i \to \infty$ there is a subsequence t_i' such that the Riemannian manifolds $(X, \omega_{t_i'})$ converge geometrically to a Kähler–Ricci soliton metric on some possibly singular space Z, which is a complex algebraic Q-Fano variety. In X. Chen, Sun, and Wang [2015] Chen, Sun and Wang showed further that this limit is unique, so in fact the Kähler–Ricci flow (X, ω_t) converges in this sense as $t \to \infty$. The case when $Z = X$ gives the desired Kähler–Ricci soliton metric on X (which is Kähler–Einstein if X is K-stable), and if Z is not equal to X then Chen, Sun and Wang show how to produce a destabilising test configuration from Z.

The information which these proofs using convergence theory provide about the precise way in which the possible failure of the PDE construction strategies relates to K-stability has independent interest. There are parallel questions in the case of the Hermitian-Yang–Mills equations which have been fully settled relatively recently. Jacob [2015] and Sibley [2015] study the Hermitian Yang–Mills flow (analogous to the Kähler–Ricci flow) and show that the limit of this produces the algebro-geometric "Harder–Narasimhan stratification" of a holomorphic bundle, thus refining the basic existence result for Hermitian-Yang–Mills connections on stable bundles.

The fourth proof of the YTD conjecture for Fano manifolds, by Berman, Boucksom and Jonsson, uses very different techniques. It relies on foundations from algebraic geometry and pluripotential theory, rather than from Riemannian geometry, and they exploit the variational point of view which we discuss in 2.4 below. The result proved is slightly weaker: they show when X has finite automorphism group the existence of a Kähler–Einstein metric is equivalent to uniform K-stability.

Over the past four years the interaction between "Kähler–Einstein geometry" and algebraic geometry has blossomed, with many related developments. One important theme is given by various notions related to "volume". A beautiful example is a result of Fujita [2015]. In differential geometric language this states that if a compact complex n-manifold X has a Kähler–Einstein metric with Ricci $= \omega$ then the volume of the manifold is at most $(2\pi)^n(n+1)^n$, with equality if and only if $X = \mathbf{CP}^n$ (with its usual Fubini–Study metric, suitably scaled). This can be set alongside the standard Bishop

comparison result, which states that among all Riemannian manifolds with Ricci $= g$ (in fact with Ricci $\geq g$) the round sphere (of the right scale) maximises the volume. So Fujita's result is a Kähler analogue. But while the statement is differential geometric, Fujita's proof is entirely algebro-geometric, using the equivalence with K-stability. In the other direction, differential geometric results on Kähler–Einstein metrics have had important consequences for the construction of moduli spaces of Fano manifolds in algebraic geometry, giving information which cannot at present be obtained by purely algebro-geometric methods. We will not go into this important topic further here but refer to the contribution of Song Sun to these ICM Proceedings Sun [2018].

Another line of development involves singularities. An important part of the Cheeger–Colding convergence theory is that the relevant limit spaces have metric *tangent cones* at each point. In a setting (such as the limits of the Ricci flow discussed above, or in considering compactified moduli spaces) where one has such a limit space Z which is a singular complex algebraic variety one can ask how this metric theory is related to the algebraic geometry of the singularities. This question was studied by the author and Sun Donaldson and Sun [2017], who showed that the metric tangent cone at a point $p \in Z$ is unique and can be obtained from a valuation ν_p on the local ring of Z at p. This valuation records the order of vanishing of a function, with respect to the metric on Z:

$$\nu_p(f) = \lim_{r \to 0} \frac{\log \max_{B_r} |f|}{\log r},$$

where B_r is the r-ball in Z centred at p. The work of Donaldson and Sun left open the question of whether the valuation can be determined entirely algebro-geometrically. This was settled by Hein and Sun [2017] in the case of singular Calabi–Yau varieties with certain singularities, including ordinary double points (which is the most important case for applications). Much progress in the general case has been made by Li, Liu and Xu, who show that the valuation can be characterised as minimising a certain algebro-geometrically defined normalised volume C. Li [2015], C. Li and Y. Liu [2016],Y. Liu and Xu [2017], C. Li and Xu [2017] Again we refer to Sun [2018] for a more detailed account of these developments.

The tangent cones that appear in this context are, differential-geometrically, *Sasaki–Einstein cones* (possibly with singularities). The algebro-geometric setting is to consider an action of a torus $T = (\mathbf{C}^*)^r$ on $\mathbf{C}^N$ and a T-invariant affine variety $W \subset \mathbf{C}^N$ with trivial canonical bundle. The general existence question in this setting asks if there is a Sasaki–Einstein metric on W with radial vector field generated by a vector in the Lie algebra of T. One case is when $r = 1$ and we have the standard action of $\mathbf{C}^*$, so W is a cone in the algebro-geometric sense over a projective variety and the existence problem reduces to the existence of a Kähler–Einstein metric on this projective variety. The analogue of the YTD conjecture in this Sasaki–Einstein setting (for varieties W with an isolated singularity at 0) was established by Collins and Székelyhidi [2015].

A glaring problem in this field is that it is usually very hard to decide if a manifold is K-stable. On the face of it, the definition requires one to check all possible test configurations. (This is in contrast to the situation for bundles, where the criterion for stability is very explicit.) We mention three lines of recent progress in this direction.

- The proofs of the YTD conjecture show that in the Fano case it suffices to check test configurations with normal central fibres (as in Tian's original proposal). This was also proved algebro-geometrically a little earlier by C. Li and Xu [2014].
- For explicit examples of interest such as cubic 3-folds in $\mathbf{P}^4$ Y. Liu and Xu [2017], and intersections of two quadrics Spotti and Sun [2017], K-stability is completely understood (including for the singular varieties that are added to compactify moduli spaces).
- Various cases of manifolds with large symmetry groups have been analysed Ilten and Süss [2017], Delcroix [2016].
- Fujita and Odaka [2016] defined a numerical invariant $\delta(X)$ of a Fano manifold and conjectured that uniform K-stability is equivalent to $\delta(X) > 1$, which was later confirmed by Blum and Jonsson [n.d.]. This invariant δ is obtained from the theory of the log canonical threshold (lct) (which in this context is a numerical invariant measuring how singular a divisor is). In fact $\delta(X) = \limsup_{k\to\infty} \delta_k(X)$ with
$$\delta_k = \inf_D \operatorname{lct}(X, D).$$
Here, for each k, the divisor D ranges over sums $\sum_i D_i$, of the divisors associated to a basis s_i of $H^0_k = H^0(X, K_X^{-k})$ and with a normalising factor $\left(k \dim H^0_k\right)^{-1}$. Fujita and Odaka also relate their invariant δ to another definition of "Gibbs stability" proposed by Berman.

1.4 Geometry in the space of Kähler metrics. The existence questions in Kähler geometry which we have been discussing have appealing formulations in terms of infinite-dimensional geometry. Given the Kähler class $[\omega_0]$ we write $\mathcal{H}$ for the space of Kähler potentials
$$\mathcal{H} = \{\phi : \omega_0 + i\,\partial\bar{\partial}\phi > 0\}.$$
(When the Kähler class is integral this has a more invariant description in terms of metrics on the corresponding ample line bundle.) We define a Riemannian metric, making $\mathcal{H}$ into an infinite-dimensional Riemannian manifold
$$\|\delta\phi\|^2_\phi = \int_X (\delta\phi)^2 d\mu_\phi,$$
where $d\mu_\phi$ is the usual volume form $\omega_\phi^n/n!$. This metric was discovered by Mabuchi [1987] and rediscovered later by Semmes and Donaldson. It has the remarkable property that $\mathcal{H}$ becomes an infinite-dimensional symmetric space of non-positive curvature,

formally the dual of the group of Hamiltonian symplectomorphisms of (X, ω_0) Donaldson [1999]. A geodesic segment in $\mathcal{H}$ corresponds to an S^1-invariant degenerate $(1,1)$ form Ω on $(0,1) \times S^1 \times X$ with $\Omega^{n+1} = 0$, which in turn corresponds to a solution of a certain homogeneous complex Monge–Ampère equation.

Various functionals on $\mathcal{H}$ play an important role in the theory. Typically these are defined through their first variations (that is, what is defined is a closed 1-form on $\mathcal{H}$). The *Mabuchi functional* $\mathcal{F}$ is defined by

$$\delta\mathcal{F} = \int_X (\delta\phi)(S - \hat{S})\, d\mu_\phi,$$

where S is the scalar curvature and $\hat{S}$ is the average Mabuchi [1986] By construction, the Euler–Lagrange equation associated to the Mabuchi functional is the constant scalar curvature condition. The Mabuchi functional is convex along geodesics in $\mathcal{H}$; so in a formal way the search for a constant scalar curvature metric is the search for a critical point (in fact a global minimum) of a convex function on a space of negative curvature. This formal picture can be fitted into a general framework involving "stability" and moment maps—see the exposition in Donaldson [2017b], for example.

The first steps in making this formal picture of real use in Kähler geometry were taken by X. Chen [2000]. He showed that any two points in $\mathcal{H}$ can be joined by a geodesic segment, provided that the definitions were extended to allow non-smooth potentials, roughly speaking of class $C^{1,1}$. Calabi and Chen showed in Calabi and X. X. Chen [2002] that there is a genuine induced metric on $\mathcal{H}$, and that it is a space of negative curvature in a generalised sense of triangle comparison. Chen also found an important formula for the Mabuchi functional. Let V_ϕ be the volume element of ω_ϕ in terms of the fixed reference metric ω_0:

$$V_\phi \omega_0^n = \omega_\phi^n$$

and let ρ_0 be the Ricci form of ω_0. Then Chen's formula is

$$\mathcal{F}(\phi) = \int_X V_\phi \log V_\phi \, d\mu_0 + J_{\rho_0}(\phi),$$

where the "J-functional" is defined by

$$\delta J_{\rho_0} = \int_X (\delta\phi)\, \rho_0 \wedge \omega_\phi^{n-1}.$$

One important consequence of this formula is that it gives a definition of $\mathcal{F}$ for $C^{1,1}$ potentials.

The connection with K-stability and algebraic geometry comes from the fact that under certain conditions a test configuration gives a geodesic ray in $\mathcal{H}$ and the asymptotic behaviour of the Mabuchi functional along the ray can be related to the Futaki invariant of the test configuration. Statements along these lines, under various technical hypotheses, have been proved by Phong and Sturm [2007] and many other authors.

There have been important developments in this area over the past few years. One is due to Berman and Berndtsson [2017], who showed that $\mathcal{F}$ is convex along generalised geodesics. This leads to a short and conceptual proof of various foundational results, such as the uniqueness of constant scalar curvature metrics. (All of this discussion can be modified to apply to extremal metrics.) Another is due to Darvas [2015], who showed that a metric completion of $\mathcal{H}$ has a good analytical meaning. In fact it turns out to be best to start with a Finsler metric on $\mathcal{H}$, defined by the L^1-norm rather than the L^2-norm. Then Darvas showed that the completion can be identified with a space $\mathcal{E}^1$ of "finite-energy" currents which was introduced previously in Eyssidieux, Guedj, and Zeriahi [2009] and which is important in pluripotential theory.

These ideas form some of the background for the version of the YTD conjecture proved by Berman *et al*. In fact they work with both the Mabuchi functional and the *Ding functional*, which has a similar character but is only defined in the Kähler–Einstein context. Very roughly, the strategy is as follows.

- Existence of the minimum follows from properness of the Ding functional;
- If the functional is not proper there is a geodesic ray on which the functional is bounded above;
- This geodesic ray can be approximated by test configurations and the boundedness gives a contradiction to uniform K-stability.

The same kind of strategy can be applied beyond the Kähler–Einstein case. For example Darvas and Rubinstein show that (for any Kähler class), properness of the Mabuchi functional (with respect to the L^1-distance) implies the existence of "weak" minimisers, in the completion $\mathcal{E}^1$ Darvas and Rubinstein [2017]. This brings to the fore the question of regularity of the weak minimisers. In a similar vein, Streets [2014] defined a generalised "minimising movement" version of the Calabi flow (exploiting on the favourable metric geometry of $\mathcal{H}$), and proved that this satisfies long time existence. So there are parallel question about the regularity of this Streets flow.

Very recent work of Chen and Cheng makes important progress in this variational and PDE framework X. Chen and Cheng [n.d.]. In connecting with these developments with algebraic geometry, one central question is whether uniform K-stability implies properness of the Mabuchi functional.

2 Exceptional holonomy

2.1 Background. Let (M, g) be a Riemannian manifold of dimension m and $p \in M$ a basepoint. Parallel transport around a loop based at p defines an orthogonal transformation of the tangent space at p. The set of all these transformations arising from contractible loops is a closed, connected, subgroup of the orthogonal group called the *holonomy group* of (M, g). This can be viewed, up to conjugacy, as a subgroup G of $SO(m)$. Work of Berger from the 1950's, with refinements by later authors, gives a complete classification of all groups that can arise. Leaving aside reducible subgroups (where the manifold has a local product structure) and symmetric spaces (which are

completely classified) there are five standard families and two exceptional cases. The standard families fall into three pairs.

- **Real**: $G = SO(m)$.
- **Complex**: m is even and $G = U(m/2)$ or $G = SU(m/2)$.
- **Quaternionic**: m is divisible by 4 and $G = Sp(m/4)$ or $G = Sp(m/4).Sp(1)$

The first case is that of a generic Riemannian metric. In the second we make the standard identification $\mathbf{R}^m = \mathbf{C}^{m/2}$. In the third we make the standard identification $\mathbf{R}^m = \mathbf{H}^{m/4}$ where $\mathbf{H}$ denotes the quaternions and the group $Sp(n).Sp(1)$ is given by left multiplication by quaternionic matrices in $Sp(n)$ and right multiplication by unit quaternions $Sp(1)$.

The exceptional cases occur in dimensions 7 and 8 and correspond to the exceptional Lie group $G_2 \subset SO(7)$ and the subgroup $\mathrm{Spin}(7) \subset SO(8)$ given by the spin representation in dimension 7. Just as the standard families are built on the algebra of the three fields $\mathbf{R}, \mathbf{C}, \mathbf{H}$, the exceptional cases are built on "exceptional" algebraic phenomena, associated to the Octonions, triality, etc. The basic definitions can be approached in various ways: a convenient approach for differential geometers emphasises exterior algebra.

Let V be a real vector space of dimension m. For $m \geq 9$ and $2 < p < m-2$ the dimension of the exterior power $\Lambda^p V^*$ exceeds that of the general linear group $GL(V)$. But in low dimensions there are a few cases where $GL(V)$ acts with an open orbit. In particular this happens when $m = 7$ and $p = 3$. For an oriented 7-dimensional V and $\phi \in \Lambda^3 V^*$ we consider the quadratic form on V

$$v \mapsto i_v(\phi) \wedge i_v(\phi) \wedge \phi. \tag{1}$$

This takes values in the oriented line $\Lambda^7 V^*$, so it makes sense to say that the form is positive definite. In such a case we say that ϕ is a positive 3-form. The basic facts are:

- the positive 3-forms make up a single orbit for the action of $GL^+(V)$;
- the stabiliser in $GL^+(V)$ of a positive 3-form is a compact Lie group isomorphic to the exceptional Lie group G_2.

With regard to the second item; a positive 3-form ϕ determines a Euclidean form g_ϕ on V, so the stabiliser is a subgroup of the orthogonal group for this Euclidean structure. Indeed the quadratic form (1) gives a conformal class of Euclidean structures, so the definition of g_ϕ is just a matter of fixing the scale. This can be achieved by requiring that $|\phi|^2 = 7$ in the norm induced by g_ϕ. Let $*_\phi$ be the $*$-operator determined by g_ϕ. Then we have a 4-form $*_\phi\phi$ which is also preserved by the stabiliser of ϕ.

We now go to dimension 8 by considering the vector space $W = V \oplus \mathbf{R}$ with coordinate t in the $\mathbf{R}$ factor. Given a positive 3-form ϕ as above we consider the 4-form

$$\Omega = \phi \wedge dt + *_\phi\phi \in \Lambda^4(W^*).$$

The basic fact is that the stabiliser of Ω in $GL^+(W)$ is a compact Lie group isomorphic to Spin(7) (the double cover of $SO(7)$). We write $\mathcal{A}$ for the orbit under $GL^+(W)$ of Ω in $\Lambda^4 W^*$, so $\mathcal{A}$ has dimension $64 - 21 = 43$. The forms in $\mathcal{A}$ are called *admissible*.

Moving to differential geometry, we have a notion of a positive 3-form on an oriented 7-manifold and an admissible 4-form on an oriented 8-manifold. The link with exceptional holonomy is provided by a result of Fernández and Gray [1982]. To state this precisely it is useful to make a small change of viewpoint and to talk about a "torsion-free G-structure", which is to say a reduction of the structure group of the tangent bundle of the manifold to a group $G \subset SO(m)$ and a torsion-free G-connection on the tangent bundle. This gives a Riemannian metric with holonomy contained in G and the notion takes care of technical complications when considering holonomy around non-contractible loops and when the holonomy is strictly smaller than G. The result of Fernández and Gray is:

- Giving a torsion-free G_2-structure on an oriented 7-manifold is equivalent to giving a positive 3-form ϕ such that $d\phi = 0$ and $d *_\phi \phi = 0$.

- Giving a torsion free Spin(7)-structure on an oriented 8-manifold is equivalent to giving a admissible 4-form Ω with $d\Omega = 0$.

There are many other points of view on these two exceptional holonomy groups. For example, they also can be characterised by the existence of a covariant constant spinor field. A very important feature is that the Ricci curvature is identically zero in both cases.

For the rest of this article we will concentrate on the 7-dimensional case. For euphony here we will call a torsion-free G_2-structure simply a G_2-structure.

2.2 Examples of manifolds with holonomy G_2. While the Berger classification goes back to the 1950's, the modern developments in this area begin with work of Bryant in the mid 1980's Bryant [1987b], giving the first local examples of metrics with exceptional holonomy–see also Bryant's lecture in ICM 1986 Bryant [1987a]. The global theory took off about 10 years later when Joyce established the existence of compact examples Joyce [1996b], Joyce [1996a]. Joyce's approach was to start with a singular quotient space such as T/Γ where T is a flat torus and Γ is a finite group and then remove the singularities by a "gluing" construction. This work was described in Joyce's 1998 ICM lecture Joyce [1998] and in a monograph Joyce [2000] which gives a comprehensive treatment of the field up to the turn of the century. There is at present essentially only one other source of examples of compact manifolds with G_2-holonomy, given by the "twisted connected sum" construction introduced by Kovalev in Kovalev [2003].

We give a brief outline of these Joyce and Kovalev constructions. One might think of these as being based on two models for a positive 3-form, corresponding to the two subgroups: $SO(4) \subset G_2$ and $SU(3) \subset G_2$. For the first model we take $\mathbf{R}^4$, regarded as an oriented 4-dimensional Euclidean vector space, and choose a standard basis $\omega_1, \omega_2, \omega_3$ for the space of self-dual 2-forms Λ^2_+ on $\mathbf{R}^4$. Then we take $\mathbf{R}^3$ with co-ordinates t_1, t_2, t_3

and write down the positive 3-form

$$\phi = dt_1 dt_2 dt_3 - \sum \omega_i dt_i \tag{2}$$

on the 7-dimensional vector space $\mathbf{R}^4 \oplus \mathbf{R}^3$. In more invariant terms, there is a canonical positive 3-form on $\mathbf{R}^4 \oplus \Lambda^2_+$, preserved by the action of $SO(4)$. Now let Q be a *hyperkähler 4-manifold*, which is the same as saying that there is a orthonormal frame of closed self-dual forms ω_i on Q. The same formula (2) defines a closed and co-closed positive 3-form on on $Q \times \mathbf{R}^3$ or $Q \times T^3$, which of course corresponds to a metric with holonomy strictly contained in G_2. The simplest case of Joyce's construction arises from a quotient T^7/Γ where the singular set is a disjoint union of flat 3-tori and a neighbourhood of each component is modelled on $T^3 \times \mathbf{R}^4/\pm 1$. The Eguchi–Hanson manifold Q is a non-compact hyperkähler 4-manifold asymptotic to $\mathbf{R}^4/\pm 1$. Joyce removes neighbourhoods of each of the components of the singular set and replaces them by corresponding neighbourhoods in $T^3 \times Q_\epsilon$, where Q_ϵ is the same manifold Q with the metric scaled by a factor ϵ. He defines a positive 3-form—the "approximate solution"—on the resulting 7-manifold by gluing the structures on each piece and then shows, using a PDE and implicit function theorem argument, that for small ϵ this approximate solution can be deformed slightly to obtain a G_2-structure.

For the second model we start with $\mathbf{C}^3$, regarded as having symmetry group $SU(3)$, so we have the standard Hermitian metric with $(1,1)$ form ω and a standard complex 3-form Θ. Thus, in the usual co-ordinates $z_a = x_a + iy_a$:

$$\omega = \sum dx_a dy_a \quad , \quad \Theta = dz_1 dz_2 dz_3.$$

We write down a positive 3-form

$$\omega \wedge dt + \mathrm{Re}(\Theta), \tag{3}$$

on the 7-dimensional real vector space $\mathbf{C}^3 \oplus \mathbf{R}$, where t is the coordinate on the $\mathbf{R}$-factor. Let Z be a 6-manifold with holonomy $SU(3)$. This is the same as saying that Z is a three complex-dimensional Kähler manifold with Kähler form ω and a holomorphic 3-form Θ of constant norm: at each point we can choose co-ordinates so that the forms match up with the standard models above. (This is also called a *Calabi–Yau* structure on Z. Up to coverings, it is equivalent to saying that Z has a Kähler–Einstein metric with zero Ricci curvature: the case $\lambda = 0$ in the language of the first part of this article.) The same formula (3) defines a G_2 structure on the product $Z \times \mathbf{R}$ or $Z \times S^1$.

Kovalev's construction depends on a supply of suitable examples of Calabi–Yau manifolds Z, and these come from a variant of the Kähler theory discussed in the first part of this article. Kovalev takes a Fano 3-fold W with a smooth anticanonical divisor $D \subset W$. By standard algebraic geometry theory, D is a complex $K3$ surface. (An example is $W = \mathbf{CP}^3$ and D a smooth quartic surface.) Also by standard theory, we can choose a curve $C \subset D$ so that the proper transform $\tilde{D}$ of D in the blow-up $\tilde{W}$ of W along C has trivial normal bundle. (Of course, the proper transform $\tilde{D}$ is isomorphic to D.) Now let Z be the complement of $\tilde{D}$ in $\tilde{W}$. A result of Tian and Yau, in the general

vein of the theory discussed in the first part of this article, gives the existence of a complete Calabi–Yau structure on Z. Kovalev showed (with later clarification by Haskins, Hein, and Nordström [2015]) that this Tian–Yau metric is *asymptotically cylindrical*—it is asymptotic to a cylinder $\tilde{D} \times S^1 \times \mathbf{R}$ where the K3 surface $\tilde{D} = D$ is endowed with the Ricci-flat Kähler–Einstein metric given by Yau's theorem. Thus, from the preceding discussion, we get an asymptotically cylindrical 7-manifold $Z \times S^1$ with holonomy strictly contained in G_2 and with asymptotic cross-section

$$\tilde{D} \times S^1 \times S^1. \tag{4}$$

Next, Kovalev considers two manifolds Z_1, Z_2 constructed by the above procedure. These have asymptotic cross-sections $\tilde{D}_i \times S^1 \times S^1$. He cuts off the ends of these manifolds, taking a large parameter L which will be roughly the diameter of the remaining pieces, and constructs a compact 7-manifold M by gluing these pieces together. The "twist" in the construction is that in making this gluing he interchanges the two S^1 factors in the cross-sections (4) , so there is no global S^1 action on M. To match up (asymptotically, for large L) the positive 3-forms on the two pieces the $K3$ surfaces $\tilde{D}_i$ must satisfy a certain matching condition involving "hyperkähler rotation" of the complex structures. This matching problem can be studied through the highly-developed Torelli theory on $K3$ surfaces, and the upshot was that Kovalev was able to find many examples satisfying the condition. Gluing the positive three forms on the two pieces in M gives an approximate solution and Kovalev showed that this could be deformed to a genuine G_2-structure on M, for large values of the parameter L.

The Joyce and Kovalev constructions have similarities. They both start with building blocks constructed using better-understood geometry—with holonomy strictly contained in G_2— and then apply analytical gluing or singular perturbation techniques. Such techniques have been used very widely in global differential geometry over the past three decades and in particular dominate much of the work so far in areas related to exceptional holonomy. There is also a contrast between the two constructions. As the parameter ϵ tends to 0 the diameters in the Joyce examples are bounded but the maximum of the curvature tends to infinity; while in the Kovalev examples, as $L \to \infty$ the curvature stays bounded but not the diameter.

2.3 The Hitchin functional, moduli and general existence questions. The condition for G_2 holonomy comes in two parts $d\phi = 0, d *_\phi \phi = 0$ and so we may consider the weaker condition of a *closed positive 3-form*. This structure is somewhat analogous to a symplectic structure: it is a closed differential form satisfying an open condition pointwise. In this framework, the G_2-holonomy condition has a variational formulation, due to Hitchin [2000], Hitchin [2001]. We first return to the elementary geometry of 3-forms on a 7-dimensional vector space V. A positive 3-form ϕ defines a volume form $\nu(\phi) \in \Lambda^7 V^*$. In this setting, the 4-form $*_\phi \phi$ appears as the derivative of the function ν:

$$\delta\nu = \frac{1}{3} *_\phi \phi \wedge (\delta\phi). \tag{5}$$

Now let M be a compact oriented 7-manifold. For $c \in H^3(M;\mathbf{R})$ write $\mathcal{P}_c$ for the set of closed positive 3-forms representing c (of course this could be the empty set). Hitchin considers the functional on $\mathcal{P}_c$ given by the total volume:

$$V(\phi) = \int_M \nu(\phi).$$

A variation within $\mathcal{P}_c$ is give by the exterior derivative of a 2-form: $\delta\phi = d\alpha$ and after applying (5) and integration by parts we have

$$\delta V = \frac{1}{3}\int_M \alpha \wedge d(*_\phi \phi)),$$

so the condition $d *_\phi \phi = 0$ is the Euler–Lagrange equation defining the critical points of the volume functional V. By Hodge theory considerations, Hitchin showed that any critical point of is a strict local maximum, modulo the action of the diffeomorphisms of M. That is, the Hessian of V is negative-definite transverse to the orbits of the diffeomorphism group acting on $\mathcal{P}_c$.

This approach leads to a simple treatment of the deformation theory of G_2-structures (which was analysed first by Bryant). The nondegeneracy of the Hessian means that for c' close to c in $H^3(M;\mathbf{R})$ there is a small perturbation of the critical point in $\mathcal{P}_c$ to one in $\mathcal{P}_{c'}$, and this is unique up to diffeomorphisms. Let Diff_0 be the identity component of the diffeomorphism group of M and $\mathcal{T}$ the "moduli space" of G_2-structures on M modulo Diff_0. Then the discussion above leads to the fact that the period map $\phi \mapsto [\phi]$ induces a local homeomorphism from $\mathcal{T}$ to $H^3(M;\mathbf{R})$.

This approach also gives motivation for the *Laplacian flow* on closed positive 3-forms, which was introduced by Bryant [2006]. This the evolution equation

$$\frac{\partial\phi}{\partial t} = \Delta_\phi \phi, \tag{6}$$

where Δ_ϕ is the Laplacian on the metric g_ϕ defined by ϕ. This flow has a similar character to the Ricci flow: in fact under the flow (6) the metric evolves by

$$\frac{\partial g}{\partial t} = -2\mathrm{Ricci} + Q(\tau),$$

where $Q(\tau)$ is a quadratic expression in the torsion tensor $\tau = d *_\phi \phi$. The leading term (involving two derivatives of ϕ) is the same as the Ricci flow. Under the Laplacian flow the volume evolves as

$$\frac{dV}{dt} = \|d *_\phi \phi\|^2,$$

and the flow can be viewed as the ascending gradient flow of the volume functional.

This variational point of view leads to many questions. The most optimistic, naïve, hope might be that there is a unique G_2-structure (up to diffeomorphism) for each connected component of $\mathcal{P}_c$; that this is a global maximum of the volume functional and that the Laplacian flow starting from any initial point exists for all time and converges to this maximum. If all this were true the existence problem for G_2-structures would

essentially be reduced to understanding the existence of closed positive 3-forms. Examples of Fernández show that this most naïve picture does not always hold (Fernández [1987], and see also the discussion in Bryant [2006]). Fernández considers left-invariant structures on a 7-dimensional nilpotent Lie groups, which descend to compact quotient manifolds N/Γ. She gives an example where:

- There are closed positive 3-forms but no G_2 structures;
- The volume functional is not bounded above;
- The flow exists for all time but the volume tends to infinity and the Riemannian curvature tends to 0.

But it remains possible that some modified version of the naïve picture is true—for example it could (as far as the author knows) be true for manifolds M with $H^1(M;\mathbf{R}) = 0$. Or it could be that one needs to allow singularities in the flow and the limits. But at present there is very little known about any of these questions, both for the existence of closed positive 3-forms and for their relation to the existence of G_2-structures. (For example, it is not known—as far as the author is aware—if there is a closed positive 3-form on the sphere S^7.)

There is a symmetry in the differential geometric foundations between the 3-form ϕ and the 4-form $*_\phi\phi$. One can equally well start with a "positive" 4-form and use that to define the 3-form. Thus we can ask analogous questions on the relation between "co-closed" 3-forms and G_2-structures. By contrast to the closed case, the existence of co-closed structures is completely understood through work of Crowley and Nordström [2015]. They show that these structures obey an "h-principle", so the question is reduced to the homotopy theory of reductions of the structure group of the tangent bundle to G_2, which we will discuss in the next section.

2.4 Some recent developments. We will now outline some developments in the study of G_2-structures from the past 5 years or so. One such development came in the work of Corti, Haskins, Nordström, and Pacini [2015], which clarified and greatly extended Kovalev's work on twisted connected sums. As we sketched in Section 3.2 above, Kovalev's construction begins with "building blocks" which are 3-dimensional Fano manifolds. Recall that a complex manifold is Fano if its anticanonical line bundle K_Z^{-1} is ample. Corti *et al* considered more generally *semi-Fano* 3-folds, where K_Z^{-1} is *big* and *nef*, and showed that these could be used in the twisted connected sum construction. Fano 3-folds, up to deformation, are completely classified and there are 105 deformation types. While there is no complete classification, the number of deformation types of semi-Fano 3-folds is known to be many orders of magnitude larger and so the work of Corti *et al* lead to a huge increase in the number of known examples of compact manifolds with G_2-structures. The original work of Joyce lead to about 250 examples and Kovalev's original construction gave a number of a similar order. Corti et al found at least 50 million—and probably many, many more—deformation classes of matching pairs which could be used to construct G_2-manifolds. They were also able to adjust the construction to give examples with various interesting features and they gave a detailed analysis of the topology of the manifolds constructed.

The topological theory is relatively straightforward in the case of 2-connected 7-manifolds M with $H_3(M;\mathbf{Z})$ torsion-free. Up to homeomorphism the only invariants are the third Betti number b_3 and the divisibility of the Pontryagin class $p_1(M) \in H^4(M;\mathbf{Z})$. Any such manifold is homeomorphic to a connected sum

$$\widetilde{S^3 \times S^4} \sharp S^3 \times S^4 \ \ldots \sharp S^3 \times S^4,$$

where the factor $\widetilde{S^3 \times S^4}$ (which carries the Pontryagin class) is one of standard list of S^3 bundles over S^4. This also gives the diffeomorphism classification up to connected sums with one of the 27 exotic 7-spheres. Corti *et al* show that many hundreds of manifolds of this kind can be constructed as twisted connected sums. Moreover they show that the same differentiable manifold can often be constructed in a many different ways, leading to the question whether the resulting G_2-structures are in the same connected component of the moduli space.

Topological aspects of G_2-manifold theory were developed in a series of papers of Crowley and Nordström [2015]Crowley and Nordström [2014] and Crowley, Goette, Nordström Crowley, Goette, and Nordström [2015]. The quotient $\mathrm{Spin}(7)/G_2$ is a 7-sphere and it follows immediately that any spin 7-manifold has a compatible topological G_2-structure (in the sense of reduction of the structure group of the tangent bundle). But there are different homotopy classes of such reductions. For example if the tangent bundle is trivial the reduction corresponds to a map

$$M \to \mathrm{Spin}(7)/G_2 = S^7$$

which has an integer degree. Taking account of the action of the diffeomorphisms, Crowley and Nordstrom showed that this integer is cut down to an invariant $\nu \in \mathbf{Z}/48$. If the divisibility of the Pontrayagin class divides 224 they show that this is a complete invariant (up to homotopy and diffeomorphism), and in general they introduced another more refined invariant to complete the classification. The ν invariant can be defined as the reduction mod 48 of $\chi(W) - 3\sigma(W)$, where W is an 8-manifold with boundary M and with a $\mathrm{Spin}(7)$ structure compatible with the G_2 structure on the boundary. They show that $\nu = 24$ for all the structures obtained from twisted connected sums, but that there is an example from the Joyce construction with ν odd. In Crowley, Goette, and Nordström [ibid.], Crowley, Goette and Nordström develop an analytical approach based on the Atiyah, Patodi, Singer theory and define a $\mathbf{Z}$-valued lift $\hat{\nu}$ of ν which is a deformation invariant of G_2-structures. The arguments hinge on the fact that the kernel of the Dirac operator defined by such a structure has fixed dimension 1, spanned by the covariant constant spinor field. They apply this to the question raised at the end of the previous paragraph, giving an example of a pair of G_2-structures on the same differentiable manifold which are equivalent at the homotopy level but which can not be joined by a path of torsion-free structures. In another direction, Crowley and Nordström define a "generalised Eels–Kuiper invariant" of the smooth structure, and show that different twisted connected sums can realise different values of this invariant on the same topological manifold Crowley and Nordström [2014].

Another line of activity makes contact with Riemannian convergence theory. As we noted in 3.2 above, the Joyce and Kovalev constructions can both be seen as describing regions near the "boundary of moduli space"—or in other words different kinds of

degeneration of sequences of torsion-free G_2-structures. Understanding all such degenerations is an important general question. Since these structures define metrics with zero Ricci curvature the substantial body of Riemannian convergence theory, as developed by Cheeger, Colding and others, can be brought to bear on this question and fit them into a wider context. We recall the basic Gromov compactness theorem for a sequence of complete Riemannian m-manifolds (M_i, g_i) with (say) zero Ricci curvature and with chosen base points $p_i \in M_i$. After passing to a subsequence there is a Gromov–Hausdorff limit (M_∞, p_∞) which is a based metric space. If the sequence has bounded diameter then there is no need to choose base points. In this theory there is a fundamental distinction between *non-collapsing* and *collapsing* cases. In the non-collapsing case the limit is a Riemannian m-manifold outside a singular set of codimension at least 4. This occurs for Joyce's examples, when the parameter ϵ tends to zero and the limit is an orbifold. The twisted sum construction, with parameter $L \to \infty$, is also non-collapsing, but there are three possible choices of limits depending on the choice of base-points. In a collapsing situation the Gromov–Hausdorff limit is a space of dimension less than m and one can hope that it can be endowed with some vestige of the structure on the sequence M_i.

In Donaldson [2017a] the author proposed a programme to study collapsing of G_2-structures along co-associative fibrations. This develops ideas of Kovalev [2005] and Baraglia [2010], and is in a similar vein to seminal work of Gross and Wilson in the case of $K3$ surfaces collapsing along elliptic fibrations Gross and Wilson [2000]. Recall that a 4-dimensional submanifold X of a G_2 manifold (M, ϕ) is called co-associative if the restriction of ϕ to X vanishes. We consider a smooth map $\pi : M \to B$ where B is a 3-manifold and π is a fibration outside a link $L \subset B$, with fibres diffeomorphic to the K3 surface. Transverse to L, the map is required to be modelled on a non-degenerate complex quadratic form. so the fibres over L are modelled on complex surfaces with ordinary double points. A particular example of this set-up is when $M = Y \times S^1$, $B = S^2 \times S^1$ for a Calabi–Yau 3-fold Y with a Lefschetz fibration over S^2. The proposal of Donaldson [2017a] is a sequence of structures of this kind in which the fibres collapse should have an "adiabatic limit". Locally, on a small ball $B_0 \subset B \setminus L$, the limiting data is given by a family of hyperkähler structures on a fixed 4-manifold X (diffeomorphic to a K3 surface), parametrised by B_0. By the Torelli Theorem for K3 surfaces this data is a map

$$f : B_0 \to \mathrm{Gr}_3^+(H^2(X, \mathbf{R})),$$

where the target is the space of positive 3-dimensional subspaces of $H^2(X)$, the latter being endowed with its cup product form of signature $(3, 19)$. The proposal is that the vestige of the G_2-structures is the requirement that f be the Gauss map of a *maximal submanifold* in $H^2(X; \mathbf{R})$ (i.e. a critical point of the induced volume functional on 3-dimensional submanifolds). This local discussion can be extended to the whole of B but the set-up is more complicated and brings in the monodromy action on the cohomology of the fibres. The main analytical difficulty in carrying through this proposal involves the behaviour around the singular fibres. Indeed just the same difficulty arises when studying the analogous question for fibred Calabi–Yau 3-folds (a problem which has been studied by many authors, for example Gross, Tosatti, and Zhang [2016]).

Significant progress in this direction comes from recent work of Y. Li [2017], who constructed a new Ricci-flat metric on $\mathbf{C}^3$ which makes a likely model for the collapsing sequence. Li's work was extended soon after by Conlon and Rochon [2017] and Székelyhidi [2017], leading to many new families of non-compact Ricci-flat Kähler metrics. One can expect that these metrics will be important both in Kähler geometry and in exceptional holonomy.

Other recent work involving collapsing comes in work of Foscolo, Haskins, and Nordström [2017] on codimension one collapse. They consider a circle bundle $\pi : M \to Y$ over a non-compact Calabi–Yau 3-fold Y and S^1-invariant G_2-structures on M. There is a parameter ϵ related to the length of the circle fibres and they obtain a well-defined adiabatic limit of the equations as $\epsilon \to 0$. They give general conditions under which this limiting equation can be solved and show that one can go back to construct a solution of the original problem, for small ϵ. They also find explicit examples of Calabi–Yau 3-folds satisfying the conditions, so the result is that they get new examples of families of G_2-structures, collapsing to a 6-dimensional limit. The existence of the relevant metrics on Y depends on recent developments in the Kähler theory, of the kind discussed in the first part of this article.

Further developments have taken place in the study of the Laplacian flow for closed positive 3-forms. Lotay and Wei develop may foundational results and show that the flow can be continued for all time provided that the torsion and curvature are bounded Lotay and Wei [2017]. G. Chen showed that the deep results of Perelman for Ricci flow can be adapted to the Laplacian flow G. Chen [2017]. This was used by Fine and Yao to obtain long time existence of a 4-dimensional reduction of the flow under the assumption that the scalar curvature remains bounded Fine and Yao [2017].

2.5 Gauge Theory and Calibrated geometry. There are two particularly interesting classes of differential geometric objects which can be considered on manifolds with the exceptional holonomy groups G_2, Spin(7):

- special "instanton" solutions of the Yang–Mills equation;
- special "calibrated" solutions of the minimal submanifold equation.

In the Yang–Mills case, for a Riemannian manifold M with holonomy group G we get a sub-bundle Λ^2_G of the 2-forms corresponding to the Lie algebra of G (that is, Λ^2 can be viewed as the Lie algebra of the orthogonal group of the tangent bundle and G is a subgroup of that orthogonal group). So we can define an "instanton" to be a connection on a bundle $E \to M$ whose curvature lies in $\Lambda^2_G \otimes \mathrm{End}E$. For general holonomy groups G this may not be an interesting notion, but in the two exceptional cases it yields an elliptic equation (suitably interpreted) for the connection. In Donaldson and Thomas [1998] the author and R. Thomas pointed out the strong formal analogies between these equations in dimensions 8 and 7 and the usual instanton theory and Floer theory in dimensions 4 and 3. In fact the discussion can be extended to link dimensions 8,7,6 and the holonomy groups Spin(7), G_2, $SU(3)$. The "instanton" equation over a manifold with holonomy $SU(3)$ is just the Hermitian Yang–Mills equation discussed in Section

2.2 above, whose solutions (over a compact manifold) correspond to stable holomorphic vector bundles.

In the submanifold case we have the calibrated submanifolds:

- Spin(7): codimension-4 *Cayley submanifolds*;
- G_2: codimension 4 *associative submanifolds*, codimension 3 *co-associative submanifolds*
- $SU(3)$: codimension 4 *holomorphic curves*, codimension 3 *special Lagrangian submanifolds*

The calibrated condition means that these submanifolds are absolutely volume minimising in their homology class. This theory goes back to the seminal paper of Harvey and Lawson from the early 1980's (which was influential in reviving interest generally in exceptional holonomy). A notable feature of the cases above is that the calibrated condition is an elliptic equation, suitably interpreted. The suggestion in Donaldson and Thomas [1998] was that there could be "enumerative" theories built on these geometric objects (connections and submanifolds), yielding numerical invariants or other structures such as Floer homology groups. In the two decades since Donaldson and Thomas [ibid.] only one such theory has been developed rigorously, leading to what are sometimes called Donaldson–Thomas invariants of Calabi–Yau 3-folds. This development has been entirely algebro-geometric, working with stable holomorphic bundles, or more generally sheaves, rather than Hermitian Yang–Mills connections. In fact the main algebro-geometric interest has been in the rank 1 case, where the Hermitian Yang–Mills theory does not directly apply.

Going back to differential geometry: there has been substantial activity and advances. A first question is the very existence of these higher dimensional instantons. This has been addressed by Walpuski [2013],Walpuski [2017a] Sá Earp and Walpuski [2015], Menet, Nordström, and Earp [2015] through gluing constructions which parallel the Joyce and Kovalev constructions of the underlying manifolds. Sá Earp showed that a stable holomorphic bundle over a Fano building block whose restriction to the divisor D is stable gives rise to a Hermitian-Yang–Mills connection on the cylindrical-end manifold. Two such bundles can be used to construct an instanton the twisted connected sum if they satisfy a suitable matching condition over the divisors. In Walpuski [2013], Walpuski proved the existence of instantons over one of Joyce's examples by gluing a connnection constructed from a 4-dimensional instanton over the Eguchi–Hanson manifold to a flat connection over the orbifold. Walpuski used related techniques to construct instantons over manifolds with Spin(7)-holonomy Walpuski [2017a].

The fundamental problem in developing enumerative theories is the potential failure of compactness. The foundations to treat this question were developed by Tian [2000], leading to a theory which has close analogies with the Riemannian convergence theory. For any sequence of solutions of the instanton equation on a fixed bundle there is a subsequence which converges outside a set S of codimension at least 4 and the codimension 4 component of S satisfies a generalised form of the corresponding calibrated condition. Restricting to the G_2 case, the basic problem occurs in a generic 1-parameter family ϕ_t of G_2-structures. We expect that for each t there are a finite number of G_2 instantons

(on a given bundle E), but at some parameter value t_0 this number may change because a family of solutions A_t for $t < t_0$ (say) may diverge as $t \to t_0$ with "bubbling" along some associative submanifold P. Thus, starting on the gauge theory side, we are inevitably drawn to consider the submanifold theory as well. This fits in with the close analogies between the theories at the formal level and also with the algebro-geometric point of view in the Calabi–Yau case (where we associate an ideal sheaf to a curve). Under reasonable assumptions one can show that this divergent phenomenon occurs exactly for the parameter values t where there is a solution of a certain "Fueter equation" over an associative submanifold P Walpuski [2017b].

Haydys and Walpuski have proposed an approach to overcome this fundamental compactness problem Haydys and Walpuski [2015]. They consider a variant of the Seiberg–Witten equations over a 3-manifold and their proposal is to count both instantons and associative submanifolds, the latter weighted by a count of the solutions of the Seiberg–Witten equations. The reason for this is that compactness can also fail for these Seiberg–Witten equations and an analysis of this failure of compactness suggests that it occurs exactly when there is a solution of the same Fueter equation. So one can hope that (with a careful choice of signs) the jump in the count of instantons should be exactly compensated by the jump in the Seiberg–Witten count.

There are many technical problems in carrying through this Haydys and Walpuski programme (particularly in the case when P is a homology sphere and reducible solutions present difficulties, of a kind familiar in the gauge theory literature). The programme motivates a better understanding of higher codimension singularities, making contact with deep work of Taubes [2013]. As shown by Doan and Walpuski [2017], one has to consider also solutions of the Fueter equation which have singularities along 1-dimensional sets in P and it seems likely that these correspond to instantons on the 7-manifold with 1-dimensional singular sets. Once again there is link here with the complex geometry. Bando and Siu showed that the Hermitian-Yang–Mills theory can be extended to reflexive sheaves and one expects that, transverse to the 1-dimensional singular set in the 7-manifold, the geometry of the connection should be modelled by a reflexive sheaf on a complex 3-fold, with a point singularity.

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DYNAMICS, NUMERICAL ANALYSIS, AND SOME GEOMETRY

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Abstract

Geometric aspects play an important role in the construction and analysis of structure-preserving numerical methods for a wide variety of ordinary and partial differential equations. Here we review the development and theory of symplectic integrators for Hamiltonian ordinary and partial differential equations, of dynamical low-rank approximation of time-dependent large matrices and tensors, and its use in numerical integrators for Hamiltonian tensor network approximations in quantum dynamics.

1 Introduction

It has become a commonplace notion in all of numerical analysis (which here is understood as comprising the construction and the mathematical analysis of numerical algorithms) that a good algorithm should "respect the structure of the problem" — and in many cases the "structure" is of *geometric* nature. This immediately leads to two basic questions, which need to be answered specifically for each problem:

- How can numerical methods be constructed that "respect the geometry" of the problem at hand?
- What are benefits from using a structure-preserving algorithm for this problem, and how do they come about?

In this note we present results in the numerical analysis of *dynamic* (evolutionary, time-dependent) ordinary and partial differential equations for which geometric aspects play an important role. These results belong to the area that has become known as *Geometric Numerical Integration*, which has developed vividly in the past quarter-century, with substantial contributions by researchers with very different mathematical backgrounds. We just refer to the books (in chronological order) Sanz-Serna and Calvo [1994], Hairer, Lubich, and Wanner [2002], Suris [2003], Leimkuhler and Reich [2004], Hairer, Lubich, and Wanner [2006], Lubich [2008], Feng and Qin [2010], Faou [2012], Wu, You, and Wang [2013], and Blanes and Casas [2016] and to the Acta Numerica review articles Sanz-Serna [1992], Iserles, Munthe-Kaas, Nørsett, and Zanna [2000], Marsden and

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West [2001], McLachlan and Quispel [2002], Hairer, Lubich, and Wanner [2003], Deckelnick, Dziuk, and Elliott [2005], Bond and Leimkuhler [2007], Chu [2008], Hochbruck and Ostermann [2010], Wanner [2010], Christiansen, Munthe-Kaas, and Owren [2011], Abdulle, E, Engquist, and Vanden-Eijnden [2012], and Dziuk and Elliott [2013]. In this note we restrict ourselves to some selected topics to which we have contributed.

In Section 2 we begin with reviewing numerical methods for approximately solving *Hamiltonian systems* of ordinary differential equations, which are ubiquitous in many areas of physics. Such systems are characterized by the *symplecticity* of the flow, a geometric property that one would like to transfer to the numerical discretization, which is then called a *symplectic integrator*. Here, the two questions above become the following:

- How are symplectic integrators constructed?
- What are favourable long-time properties of symplectic integrators, and how can they be explained?

The first question relates numerical methods with the theories of Hamilton and Jacobi from the mid-19th century, and the latter question connects numerical methods with the analytical techniques of Hamiltonian perturbation theory, a subject developed from the late 19th throughout the 20th century, from Lindstedt and Poincaré and Birkhoff to Siegel and Kolmogorov, Arnold and Moser (KAM theory), to Nekhoroshev and further eminent mathematicians. This connection comes about via *backward error analysis*, which is a concept that first appeared in numerical linear algebra Wilkinson [1960]. The viewpoint is to interpret the numerical approximation as the exact (or almost exact) solution of a modified equation. In the case of a symplectic integrator applied to a Hamiltonian differential equation, the modified differential equation turns out to be again Hamiltonian, with a Hamiltonian that is a small perturbation to the original one. This brings Hamiltonian perturbation theory into play for the long-time analysis of symplectic integrators. Beyond the purely mathematical aspects, it should be kept in mind that symplectic integrators are first and foremost an important tool in computational physics. In fact such numerical methods appeared first in the physics literature de Vogelaere [1956], Verlet [1967], and Ruth [1983], in such areas as nuclear physics and molecular dynamics, and slightly later Wisdom and Holman [1991] in celestial mechanics, which has been the original motivation in the development of Hamiltonian perturbation theory Poincaré [1892] and Siegel and Moser [1971]. It was not least with the use of symplectic integrators that the centuries-old question about the stability of the solar system was finally answered negatively in the last decade by Laskar; see Laskar [2013] and compare also with Moser [1978].

In Section 3 we consider numerical methods for finite-dimensional *Hamiltonian systems with multiple time scales* where, in the words of Fermi, Pasta, and Ulam [1955], "the non-linearity is introduced as a perturbation to a primarily linear problem. The behavior of the systems is to be studied for times which are long compared to the characteristic periods of the corresponding linear problem." The two basic questions above are reconsidered for such systems. Except for unrealistically small time steps, the backward error analysis of Section 2 does not work for such systems, and a different technique of

analysis is required. *Modulated Fourier expansions* in time were originally developed (since 2000) for studying numerical methods for such systems and were subsequently also recognized as a powerful analytical technique for proving new results for continuous systems of this type, including the original Fermi–Pasta–Ulam system. While the canonical transformations of Hamiltonian perturbation theory transform the system into a normal form from which long-time behaviour can be read off, modulated Fourier expansions embed the system into a high-dimensional system that has a Lagrangian structure with invariance properties that enable us to infer long-time properties of the original system. Modulated Fourier expansions do not use nonlinear coordinate transformations, which is one reason for their suitability for studying numerical methods, which are most often not invariant under nonlinear transformations.

In Section 4 we present long-time results for suitable numerical discretizations of *Hamiltonian partial differential equations* such as nonlinear wave equations and nonlinear Schrödinger equations. A number of important results on this topic have been obtained in the last decade, linking the numerical analysis of such equations to recent advances in their mathematical analysis. The viewpoint we take here is to consider the Hamiltonian partial differential equation as an infinite-dimensional system of the oscillatory type of Section 3 with infinitely many frequencies, and we present results on the long-time behaviour of the numerical and the exact solutions that have been obtained with modulated Fourier expansions or with techniques from infinite-dimensional Hamiltonian perturbation theory. We mention, however, that there exist other viewpoints on the equations considered, with different geometric concepts such as multisymplecticity Bridges [1997] and Marsden, Patrick, and Shkoller [1998]. While multisymplectic integrators, which preserve this geometric structure, have been constructed and favourably tested in numerical experiments Bridges and Reich [2001] and Ascher and McLachlan [2004] (and many works thereafter), as of now there appear to be no proven results on the long-time behaviour of such methods.

In Section 5 we consider *dynamical low-rank approximation*, which leads to a different class of dynamical problems with different geometric aspects. The problem here is to approximate large (or rather too large, huge) time-dependent matrices, which may be given explicitly or are the unknown solution to a matrix differential equation, by matrices of a prescribed rank, typically much smaller than the matrix dimension so that a data-compressed approximation is obtained. Such problems of data and/or model reduction arise in a wide variety of applications ranging from information retrieval to quantum dynamics. On projecting the time derivative of the matrices to the tangent space of the manifold of low-rank matrices at the current approximation, this problem leads to a differential equation on the low-rank manifold, which needs to be solved numerically. We present answers to the two basic questions formulated at the beginning of this introduction, for this particular problem. The proposed "geometric" numerical integrator, which is based on splitting the orthogonal projection onto the tangent space, is robust to the (ubiquitous) presence of small singular values in the approximations. This numerically important robustness property relies on a geometric property: The low-rank manifold is a ruled manifold (like a hyperboloid). It contains flat subspaces along which one can pass between any two points on the manifold, and the numerical integrator does just that. In this way the high curvature of the low-rank manifold at matrices with small

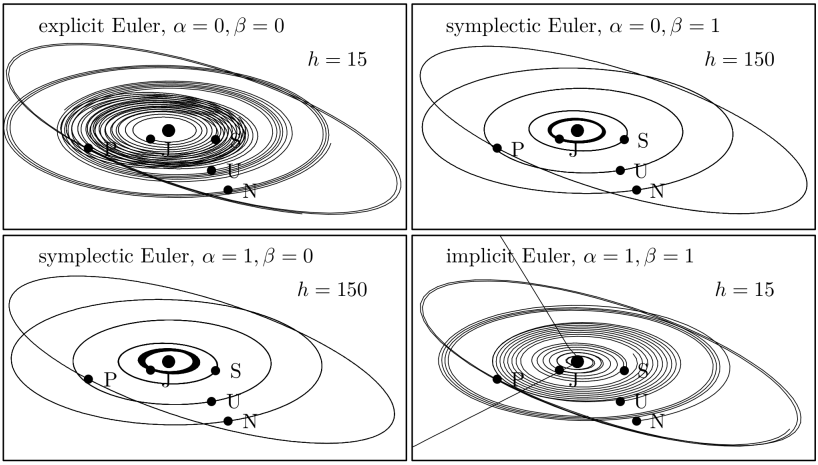

Figure 2.1: Numerical simulation of the outer solar system.

singular values does not become harmful. Finally, we address the nontrivial extension to tensors of various formats (Tucker tensors, tensor trains, hierarchical tensors), which is of interest in time-dependent problems with several spatial dimensions.

Section 6 on tensor and tensor network approximations in *quantum dynamics* combines the worlds of the previous two sections and connects them with recent developments in computational quantum physics. The reduction of the time-dependent many-particle Schrödinger equation to a low-rank tensor manifold by the Dirac–Frenkel time-dependent variational principle uses a tangent-space projection that is both orthogonal and symplectic. It results in a (non-canonical) Hamiltonian differential equation on the tensor manifold that can be discretized in time by the projector-splitting integrator of Section 5, which is robust to small singular values and preserves both the norm and the energy of the wavefunction.

2 Hamiltonian systems of ordinary differential equations

2.1 Hamiltonian systems. Differential equations of the form (with $\dot{} = \mathrm{d}/\mathrm{d}t$)

$$\dot{p} = -\nabla_q H(p,q), \quad \dot{q} = +\nabla_p H(p,q) \tag{2-1}$$

are fundamental to many branches of physics. The real-valued Hamilton function H, defined on a domain of $\mathbb{R}^{d+d}$ (the phase space), represents the total energy and $q(t) \in \mathbb{R}^d$ and $p(t) \in \mathbb{R}^d$ represent the positions and momenta, respectively, of a conservative system at time t. The total energy is conserved:

$$H(p(t),q(t)) = H(p(0),q(0))$$

along every solution $(p(t),q(t))$ of the Hamiltonian differential equations.

Numerical example: We consider four variants of the Euler method, which for a given (small) step size $h > 0$ compute approximations $p_n \approx p(nh)$, $q_n \approx q(nh)$ via

$$p_{n+1} = p_n - h\nabla_q H(p_{n+\alpha}, q_{n+\beta}), \qquad q_{n+1} = q_n + h\nabla_p H(p_{n+\alpha}, q_{n+\beta}),$$

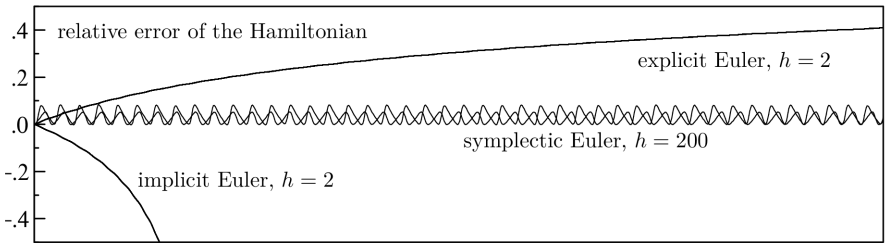

Figure 2.2: Relative error of the Hamiltonian on the interval $0 \le t \le 200\,000$.

with $\alpha, \beta \in \{0, 1\}$. For $\alpha = \beta = 0$ this is the explicit Euler method, for $\alpha = \beta = 1$ it is the implicit Euler method. The partitioned methods with $\alpha \neq \beta$ are known as the *symplectic Euler methods*. All four methods are of order $r = 1$, that is, the error after one step of the method is $\mathcal{O}(h^{r+1})$ with $r = 1$.

We apply these methods to the outer solar system, which is an N-body problem with Hamiltonian

$$H(p,q) = \frac{1}{2}\sum_{i=0}^{N} \frac{1}{m_i}\,|p^i|^2 - G\sum_{i=1}^{N}\sum_{j=0}^{i-1} \frac{m_i m_j}{|q^i - q^j|},$$

where $p = (p^0, \dots, p^N)$, $q = (q^0, \dots, q^N)$ and $|\cdot|$ denotes the Euclidean norm, and the constants are taken from Hairer, Lubich, and Wanner [2006, Section I.2.4]. The positions $q^i \in \mathbb{R}^3$ and momenta $p^i \in \mathbb{R}^3$ are those of the sun and the five outer planets (including Pluto). Figure 2.1 shows the numerical solution obtained by the four versions of the Euler method on a time interval of 200 000 earth days. For the explicit Euler method the planets spiral outwards, for the implicit Euler method they spiral inwards, fall into the sun and finally are ejected. Both symplectic Euler methods show a qualitatively correct behaviour, even with a step size (in days) that is much larger than the one used for the explicit and implicit Euler methods. Figure 2.2 shows the relative error of the Hamiltonian, $\big(H(p_n,q_n) - H(p_0,q_0)\big)\big/|H(p_0,q_0)|$, along the numerical solution of the four versions of Euler's method on the time interval $0 \le nh \le 200\,000$. Whereas the size of the error increases for the explicit and implicit Euler methods, it remains bounded and small, of a size proportional to the step size h, for both symplectic Euler methods.

2.2 Symplecticity of the flow and symplectic integrators. The time-t *flow* of a differential equation $\dot{y} = f(y)$ is the map φ_t that associates with an initial value y_0 at time 0 the solution value at time t: $\varphi_t(y_0) = y(t)$. Consider now the Hamiltonian system (2-1) or equivalently, for $y = (p,q)$,

$$\dot{y} = J^{-1}\nabla H(y) \quad \text{with} \quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

The flow φ_t of the Hamiltonian system is *symplectic* (or *canonical*), that is, the derivative matrix $D\varphi_t$ with respect to the initial value satisfies

$$D\varphi_t(y)^\top\, J\, D\varphi_t(y) = J$$

for all y and t for which $\varphi_t(y)$ exists. This quadratic relation is formally similar to orthogonality, with J in place of the identity matrix I, but it is related to the preservation of areas rather than lengths in phase space.

There is also a local converse: If the flow of some differential equation is symplectic, then there exists locally a Hamilton function for which the corresponding Hamiltonian system coincides with this differential equation.

A numerical one-step method $y_{n+1} = \Phi_h(y_n)$ (with step size h) is called symplectic if the numerical flow Φ_h is a symplectic map:

$$D\Phi_h(y)^\top \, J \, D\Phi_h(y) = J.$$

Such methods exist: the "symplectic Euler methods" of the previous subsection are indeed symplectic. This was first noted, or considered noteworthy, in an unpublished report by de Vogelaere [1956]. The symplecticity can be readily verified by direct calculation or by observing that the symplectic Euler methods are symplectic maps with the h-scaled Hamilton function taken as the generating function of a canonical transformation in Hamilton and Jacobi's theory. More than 25 years later, Ruth [1983] and Feng [1985, 1986] independently constructed higher-order symplectic integrators using generating functions of Hamilton–Jacobi theory. These symplectic methods require, however, higher derivatives of the Hamilton function. Symplectic integrators began to find widespread interest in numerical analysis when in 1988 Lasagni [1988], Sanz-Serna [1988], and Suris [n.d.] independently characterized symplectic Runge–Kutta methods by a quadratic relation of the method coefficients. This relation was already known to be satisfied by the class of Gauss–Butcher methods (the order-preserving extension of Gaussian quadrature formulae to differential equations), which include methods of arbitrary order. Like the Euler methods, Runge–Kutta methods only require evaluations of the vector field, but no higher derivatives.

The standard integrator of molecular dynamics, introduced to the field by Verlet [1967] and used ever since, is also symplectic. For a Hamiltonian

$$H(p,q) = \frac{1}{2} p^\top M^{-1} p + V(q)$$

with a symmetric positive definite mass matrix M, the method is explicit and given by the formulas

$$\begin{aligned} p_{n+1/2} &= p_n - \frac{h}{2} \nabla V(q_n) \\ q_{n+1} &= q_n + h M^{-1} p_{n+1/2} \\ p_{n+1} &= p_{n+1/2} - \frac{h}{2} \nabla V(q_{n+1}). \end{aligned}$$

Such a method was also formulated by the astronomer Störmer in 1907, and can even be traced back to Newton's *Principia* from 1687, where it was used as a theoretical tool in the proof of the preservation of angular momentum in the two-body problem (Kepler's second law), which is preserved by this method (cf. Wanner [2010]). The above method is referred to as the Störmer–Verlet method, Verlet method or leapfrog method in different communities. The symplecticity of this method can be understood

in various ways by relating the method to classes of methods that have proven useful in a variety of applications (cf. Hairer, Lubich, and Wanner [2003]): as a *composition method* (it is a composition of the two symplectic Euler methods with half step size), as a *splitting method* (it solves in an alternating way the Hamiltonian differential equations corresponding to the kinetic energy $\frac{1}{2}p^\top M^{-1}p$ and the potential energy $V(q)$), •and as a *variational integrator*: it minimizes the discrete action functional that results from approximating the action integral

$$\int_{t_0}^{t_N} L(q(t), \dot{q}(t))\,\mathrm{d}t \quad \text{with} \quad L(q, \dot{q}) = \frac{1}{2}\dot{q}^\top M\dot{q} - V(q)$$

by the trapezoidal rule and using piecewise linear approximation to $q(t)$. The Störmer–Verlet method can thus be interpreted as resulting from a discretization of the Hamilton variational principle. Such an interpretation can in fact be given for every symplectic method. Conversely, symplectic methods can be *constructed* by minimizing a discrete action integral. In particular, approximating the action integral by a quadrature formula and the positions $q(t)$ by a piecewise polynomial leads to a symplectic partitioned Runge–Kutta method. With Gauss quadrature, this gives a reinterpretation of the Gauss–Butcher methods (cf. Suris [1990], Marsden and West [2001], and Hairer, Lubich, and Wanner [2006]).

2.3 Backward error analysis. The numerical example of Section 2.1, and many more examples in the literature, show that symplectic integrators behave much better over long times than their non-symplectic counterparts. How can this be explained, or put differently: How does the geometry lead to favourable dynamics? There is a caveat: As was noted early on Gladman, Duncan, and Candy [1991] and Calvo and Sanz-Serna [1992], all the benefits of symplectic integrators are lost when they are used with variable step sizes as obtained by standard step size control. So it is not just about preserving symplecticity.

Much insight into this question is obtained from the viewpoint of *backward analysis*, where the result of one step of a numerical method for a differential equation $\dot{y} = f(y)$ is interpreted as the solution to a modified differential equation (or more precisely formal solution, having the same expansion in powers of the step size h)

$$\dot{\widetilde{y}} = f(\widetilde{y}) + hf_1(\widetilde{y}) + h^2 f_2(\widetilde{y}) + h^3 f_3(\widetilde{y}) + \dots .$$

The question then is how geometric properties of the numerical method, such as symplecticity, are reflected in the modified differential equation. It turns out that in the case of a *symplectic* integrator applied to a Hamiltonian differential equation, each of the perturbation terms is a *Hamiltonian* vector field, $f_j(y) = J^{-1}\nabla H_j(y)$ (at least locally, on simply connected domains). The formal construction was first given by Moser [1968], where the problem of interpolating a near-identity symplectic map by a Hamiltonian flow was considered. For the important class of symplectic partitioned Runge–Kutta methods (which includes all the examples mentioned in Section 2.2), a different construction in Hairer [1994], using the theory of P-series and their associated trees, showed that the perturbation Hamiltonians H_j are indeed *global*, defined on the

same domain on which the Hamilton function H is defined and smooth. Alternatively, this can also be shown using the explicit generating functions for symplectic partitioned Runge–Kutta methods as derived by Lasagni; see Hairer, Lubich, and Wanner [2006, Sect. IX.3]. This global result is in particular important for studying the behaviour of symplectic integrators for near-integrable Hamiltonian systems, which are considered in neighbourhoods of tori. It allows us to bring the rich arsenal of Hamiltonian perturbation theory to bear on the long-time analysis of symplectic integrators.

The step from a formal theory (with the three dots at the end of the line) to rigorous estimates was taken by Benettin and Giorgilli [1994] (see also Hairer and Lubich [1997] and Reich [1999] and Hairer, Lubich, and Wanner [2006, Chapter IX] for related later work), who showed that in the case of an analytic vector field f, the result $y_1 = \Phi_h(y_0)$ of one step of the numerical method and the time-h flow $\widetilde{\varphi}_h(y_0)$ of the corresponding modified differential equation, suitably truncated after $N \sim 1/h$ terms, differ by a term that is exponentially small in $1/h$:

$$\|\Phi_h(y_0) - \widetilde{\varphi}_h(y_0)\| \le C h \, \mathrm{e}^{-c/h},$$

uniformly for y_0 varying in a compact set. The constants C and c can be given explicitly. It turns out that c is inversely proportional to a local Lipschitz constant L of f, and hence the estimate is meaningful only under the condition $hL \ll 1$. We note that in an oscillatory Hamiltonian system, L corresponds to the highest frequency in the system.

A different approach to constructing a modified Hamiltonian whose flow is exponentially close to the near-identity symplectic map is outlined by Neishtadt [1984], who exactly embeds the symplectic map into the flow of a non-autonomous Hamiltonian system with rapid oscillations and then uses averaging to obtain an autonomous modified Hamiltonian.

2.4 Long-time near-conservation of energy. The above results immediately explain the observed near-preservation of the total energy by symplectic integrators used with constant step size: Over each time step, and as long as the numerical solution stays in a fixed compact set, the Hamilton function $\widetilde{H}$ of the optimally truncated modified differential equation is almost conserved up to errors of size $\mathcal{O}(h\mathrm{e}^{-c/h})$. On writing $\widetilde{H}(y_n) - \widetilde{H}(y_0)$ as a telescoping sum and adding up the errors, we thus obtain

$$\widetilde{H}(y_n) - \widetilde{H}(y_0) = \mathcal{O}(\mathrm{e}^{-c/2h}) \quad \text{for} \quad nh \le \mathrm{e}^{c/2h}.$$

For a symplectic method of order r, the modified Hamilton function $\widetilde{H}$ is $\mathcal{O}(h^r)$ close to the original Hamilton function H, uniformly on compact sets, and so we have *near-conservation of energy over exponentially long times*:

$$H(y_n) - H(y_0) = \mathcal{O}(h^r) \quad \text{for} \quad nh \le \mathrm{e}^{c/2h}.$$

Symplecticity is, however, not necessary for good energy behaviour of a numerical method. First, the assumption can clearly be weakened to conjugate symplecticity, that is, the one-step method $y_{n+1} = \Phi_h(y_n)$ is such that $\Phi_h = \chi_h^{-1} \circ \Psi_h \circ \chi_h$ where the map Ψ_h is symplectic. But then, for some methods such as the Störmer–Verlet method,

long-time near-conservation of energy can be proved by an argument that does not use symplecticity, but just the time-symmetry $\Phi_{-h} \circ \Phi_h = \mathrm{id}$ of the method Hairer, Lubich, and Wanner [2003]. That proof is similar in spirit to proving the conservation of the energy $\frac{1}{2} p^\top M^{-1} p + V(q) = \frac{1}{2} \dot{q}^\top M \dot{q} + V(q)$ for the second-order differential equation $M\ddot{q} + \nabla V(q) = 0$ by taking the inner product with $\dot{q}$ and noting that there results a total differential: $\frac{\mathrm{d}}{\mathrm{d}t}(\frac{1}{2}\dot{q}^\top M \dot{q} + V(q)) = 0$. This kind of argument can be extended to proving long-time near-conservation of energy and momentum for symmetric multistep methods Hairer and Lubich [2004, 2017], which do not preserve symplecticity and are not conjugate to a symplectic method. Arguments of this type are also basic in studying the long-time behaviour of numerical methods in situations where the product of the step size with the highest frequency is not very small, contrary to the condition $hL \ll 1$ above. We will encounter such situations in Sections 3 and 4.

2.5 Integrable and near-integrable Hamiltonian systems. Symplectic integrators enjoy remarkable properties when they are used on integrable and perturbed integrable Hamiltonian systems. Their study combines backward error analysis and the perturbation theory of integrable systems, a rich mathematical theory originally developed for problems of celestial mechanics Poincaré [1892], Siegel and Moser [1971], and V. I. Arnold, Kozlov, and Neishtadt [1997].

A Hamiltonian system with the (real-analytic) Hamilton function $H(p,q)$ is called *integrable* if there exists a symplectic transformation $(p,q) = \psi(a,\theta)$ to *action-angle variables* (a,θ), defined for actions $a = (a_1,\dots,a_d)$ in some open set of $\mathbb{R}^d$ and for angles θ on the d-dimensional torus $\mathbb{T}^d = \{(\theta_1,\dots,\theta_d);\ \theta_i \in \mathbb{R} \bmod 2\pi\}$, such that the Hamiltonian in these variables depends only on the actions:

$$H(p,q) = H(\psi(a,\theta)) = K(a).$$

In the action-angle variables, the equations of motion are simply $\dot{a} = 0, \dot{\theta} = \omega(a)$ with the *frequencies* $\omega = (\omega_1,\dots,\omega_d)^T = \nabla_a K$. For every a, the torus $\{(a,\theta) : \theta \in \mathbb{T}^d\}$ is thus invariant under the flow. We express the actions and angles in terms of the original variables (p,q) via the inverse transform as

$$(a,\theta) = (I(p,q), \Theta(p,q))$$

and note that the components of $I = (I_1,\dots,I_d)$ are first integrals (conserved quantities) of the integrable system.

The effect of a small perturbation of an integrable system is well under control in subsets of the phase space where the frequencies ω satisfy Siegel's *diophantine condition*:

$$|k \cdot \omega| \geq \gamma |k|^{-\nu} \quad \text{for all } k \in \mathbb{Z}^d,\ k \neq 0,$$

for some positive constants γ and ν, with $|k| = \sum_i |k_i|$. For $\nu > d-1$, almost all frequencies (in the sense of Lebesgue measure) satisfy this non-resonance condition for some $\gamma > 0$. For any choice of γ and ν the complementary set is, however, open and dense in $\mathbb{R}^d$.

For general numerical integrators applied to integrable systems (or perturbations thereof) the error grows quadratically with time, and there is a linear drift in the actions

I_i along the numerical solution. Consider now a symplectic partitioned Runge–Kutta method of order r (or more generally, a symplectic method that has a globally defined modified Hamilton function), applied to the integrable system with a sufficiently small step size $h \le h_0$. Then, there is the following result on *linear error growth* and *long-time near-preservation of the actions* Hairer, Lubich, and Wanner [2002, Sect. X.3]: every numerical solution (p_n, q_n) starting with frequencies $\omega_0 = \omega(I(p_0, q_0))$ such that $\|\omega_0 - \omega^*\| \le c|\log h|^{-\nu-1}$ for some $\omega^* \in \mathbb{R}^d$ that obeys the above diophantine condition, satisfies

$$\begin{aligned} \|(p_n, q_n) - (p(t), q(t))\| &\le C\, t\, h^r \\ \|I(p_n, q_n) - I(p_0, q_0)\| &\le C\, h^r \end{aligned} \qquad \text{for } t = nh \le h^{-r}\,.$$

(The constants h_0, c, C depend on d, γ, ν and on bounds of the Hamiltonian.) Under stronger conditions on the initial values, the near-preservation of the action variables along the numerical solution holds for times that are exponentially long in a negative power of the step size Hairer and Lubich [1997]. For a Cantor set of initial values and a Cantor set of step sizes this holds even perpetually, as the existence of invariant tori of the numerical integrator close to the invariant tori of the integrable system was shown by Shang [1999, 2000].

The linear error growth persists when the symplectic integrator is applied to a *perturbed integrable system* $H(p, q) + \varepsilon G(p, q)$ with a perturbation parameter of size $\varepsilon = \mathcal{O}(h^\alpha)$ for some positive exponent α. Perturbed integrable systems have KAM tori, i.e., deformations of the invariant tori of the integrable system corresponding to diophantine frequencies ω, which are invariant under the flow of the perturbed system. If the method is applied to such a perturbed integrable system, then the numerical method has almost-invariant tori over exponentially long times Hairer and Lubich [1997]. For a Cantor set of non-resonant step sizes there are even truly invariant tori on which the numerical one-step map reduces to rotation by $h\omega$ in suitable coordinates Hairer, Lubich, and Wanner [2002, Sect. X.6.2].

In a very different line of research, one asks for integrable discretizations of integrable systems; see the monumental treatise by Suris [2003].

2.6 Hamiltonian systems on manifolds. In a more general setting, a Hamiltonian system is considered on a symplectic manifold, which is a manifold $\mathfrak{M}$ with a closed, non-degenerate alternating two-form ω, called the symplectic form. Given a smooth Hamilton function $H : \mathfrak{M} \to \mathbb{R}$, the corresponding Hamiltonian differential equation is to find $u : [0, T] \to \mathfrak{M}$ such that

$$\omega_{u(t)}(\dot u(t), v) = \mathrm{d}H(u(t))[v] \qquad \text{for all } v \in T_{u(t)}\mathfrak{M},$$

where $T_u\mathfrak{M}$ denotes the tangent space at u of $\mathfrak{M}$, for a given initial value $u(0) = u_0 \in \mathfrak{M}$. On inserting $v = \dot u(t)$ it is seen that the total energy $H(u(t))$ is constant in time. We write again $u(t) = \varphi_t(u_0)$ to indicate the dependence on the initial value. The flow map φ_t is *symplectic* in the sense that the symplectic form ω is preserved along the flow: for all t and u_0 where $\varphi_t(u_0)$ exists,

$$\omega_{\varphi_t(u_0)}(\mathrm{d}\varphi_t(u_0)[\xi], \mathrm{d}\varphi_t(u_0)[\eta]) = \omega_{u_0}(\xi, \eta) \quad \text{for all } \xi, \eta \in T_{u_0}\mathfrak{M}; \qquad \text{or } \varphi_t^*\omega = \omega.$$

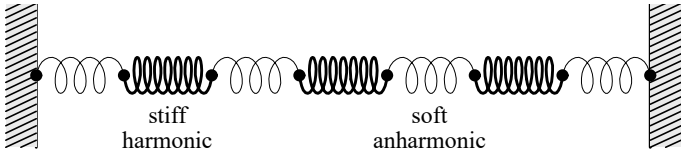

Figure 3.1: Chain of alternating stiff harmonic and soft anharmonic springs.

Contrary to the canonical Hamiltonian systems considered before, no general prescription is known how to construct a symplectic numerical integrator for a general Hamiltonian system on a general symplectic manifold.

However, for the important class of Hamiltonian systems with holonomic constraints, there exist symplectic extensions of the Störmer–Verlet method Andersen [1983] and Leimkuhler and Reich [1994] and of higher-order partitioned Runge–Kutta methods Jay [1996]. Here the symplectic manifold $\mathfrak{M}$ is the submanifold of $\mathbb{R}^{2d}$ given by constraints $g(q) = 0$, which constrain only the positions, together with the implied constraints for the momenta, $Dg(q)\nabla_p H(p, q) = 0$.

Apart from holonomic mechanical systems, there exist specially tailored symplectic integrators for particular problem classes of non-canonical Hamiltonian systems. These are often splitting methods, as for example, for rigid body dynamics Dullweber, Leimkuhler, and McLachlan [1997] and Benettin, Cherubini, and Fassò [2001], for Gaussian wavepackets in quantum dynamics Faou and Lubich [2006], and for post-Newtonian equations in general relativity Lubich, Walther, and Brügmann [2010].

3 Hamiltonian systems with multiple time scales

3.1 Oscillatory Hamiltonian systems. The numerical experiment by Fermi, Pasta and Ulam in 1955, which showed unexpected recurrent behaviour instead of relaxation to equipartition of energy in a chain of weakly nonlinearly coupled particles, has spurred a wealth of research in both mathematics and physics; see, e.g., Gallavotti [2008], Berman and Izrailev [2005], Ford [1992], and Weissert [1997]. Even today, there are only few rigorous mathematical results for large particle numbers in the FPU problem over long times Bambusi and Ponno [2006] and Hairer and Lubich [2012], and rigorous theory is lagging behind the insight obtained from carefully conducted numerical experiments Benettin, Christodoulidi, and Ponno [2013].

Here we consider a related class of oscillatory Hamiltonian systems for which the long-time behaviour is by now quite well understood analytically both for the continuous problem and its numerical discretizations, and which show interesting behaviour on several time scales. The considered multiscale Hamiltonian systems couple high-frequency harmonic oscillators with a Hamiltonian of slow motion. An illustrative example of such a Hamiltonian is provided by a Fermi–Pasta–Ulam type system of point masses interconnected by stiff harmonic and soft anharmonic springs, as shown in Figure 3.1; see Galgani, Giorgilli, Martinoli, and Vanzini [1992] and Hairer, Lubich, and Wanner [2006, Section I.5]. The general setting is as follows: For positions $q = (q_0, q_1, \dots, q_m)$ and momenta $p = (p_0, p_1, \dots, p_m)$ with $p_j, q_j \in \mathbb{R}^{d_j}$, let the Hamilton function be given

by

$$H(p,q) = H_\omega(p,q) + H_{\text{slow}}(p,q),$$

where the oscillatory and slow-motion energies are given by

$$H_\omega(p,q) = \sum_{j=1}^{m} \frac{1}{2}\Big(|p_j|^2 + \omega_j^2\,|q_j|^2\Big), \qquad H_{\text{slow}}(p,q) = \frac{1}{2}|p_0|^2 + U(q)$$

with high frequencies

$$\omega_j \geq \varepsilon^{-1}, \qquad 0 < \varepsilon \ll 1.$$

The coupling potential U is assumed smooth with derivatives bounded independently of the small parameter ε. On eliminating the momenta $p_j = \dot{q}_j$, the Hamilton equations become the system of second-order differential equations

$$\ddot{q}_j + \omega_j^2 q_j = -\nabla_j U(q), \qquad j = 0, \ldots, m,$$

where ∇_j denotes the gradient with respect to q_j, and where we set $\omega_0 = 0$. We are interested in the behaviour of the system for initial values with an oscillatory energy that is bounded independently of ε:

$$H_\omega(p(0), q(0)) \leq \text{Const.}$$

This system shows different behaviour on different time scales:

(i) almost-harmonic motion of the fast variables (p_j, q_j) $(j \neq 0)$ on time scale ε;

(ii) motion of the slow variables (p_0, q_0) on the time scale ε^0;

(iii) energy exchange between the harmonic oscillators with the same frequency on the time scale ε^{-1};

(iv) energy exchange between the harmonic oscillators corresponding to frequencies in 1:2 or 1:3 resonance on the time scale ε^{-2} or ε^{-3}, respectively;

(v) near-preservation of the jth oscillatory energy $E_j = \frac{1}{2}(|p_j|^2 + \omega_j^2\,|q_j|^2)$ for a non-resonant frequency ω_j beyond the time scale ε^{-N} for arbitrary N; and

(vi) near-preservation of the total oscillatory energy H_ω over intervals that are beyond the time scale ε^{-N} for arbitrary N, uniformly for $\omega_j \geq \varepsilon^{-1}$ without any non-resonance condition (but depending on the number m of different frequencies). Hence, there is nearly no energy exchange between the slow and the fast subsystem for very long times irrespective of resonances, almost-resonances or non-resonances among the high frequencies.

The long-time results (iii)–(vi) require in addition that q_0 stays in a compact set, which is ensured if the potential $U(q) \to +\infty$ as $|q| \to \infty$. These results can be proved by two alternative techniques:

(H) using canonical coordinate transformations of Hamiltonian perturbation theory to a normal form; or

(F) using modulated Fourier expansions in time.

The latter technique was developed by the authors of the present paper (in part together with David Cohen) and will be outlined in the next subsection.

Motivated by the problem of relaxation times in statistical mechanics, item (v) was first shown using (H) by Benettin, Galgani, and Giorgilli [1987] for the single-frequency case, even over times exponentially long in ε^{-1} for a real-analytic Hamilton function, and in Benettin, Galgani, and Giorgilli [1989] for the multi-frequency case over times exponentially long in some negative power of ε that depends on the diophantine non-resonance condition; it was subsequently shown using (F) in Cohen, Hairer, and Lubich [2003] over exponentially long times in ε^{-1} for the single frequency-case, and in Cohen, Hairer, and Lubich [2005] over times ε^{-N} with N depending on the chosen non-resonance condition on the frequencies.

Item (vi) was first shown using (F) in Gauckler, Hairer, and Lubich [2013] and subsequently using (H) in Bambusi, Giorgilli, Paleari, and Penati [2013], where the result was extended to exponentially long time scales.

The relationship between the two techniques of proof, (H) and (F), is not clear at present. The proofs look very different in the basic arguments, in the geometric content and in the technical details, yet lead to very similar results about the long-time behaviour of the continuous problem.

3.2 Modulated Fourier expansion. Modulated Fourier expansions in time have proven useful in the long-time analysis of differential equations where the nonlinearity appears as a perturbation to a primarily linear problem (as laid out in the programme of Fermi, Pasta, and Ulam [1955] cited in the introduction). This encompasses important classes of Hamiltonian ordinary and partial differential equations. The approach can be successfully used for the analysis of the continuous problems as well as for their numerical discretizations, as is amply shown in the corresponding references in this and the next section. In particular for the analysis of numerical methods, it offers the advantage that it does not require nonlinear coordinate transforms. Instead, it embeds the original system in a high-dimensional system of modulation equations that has a Lagrangian / Hamiltonian structure with invariance properties. In addition to the use of modulated Fourier expansions as an analytical technique, they have been used also as a numerical approximation method in Hairer, Lubich, and Wanner [2002, Chapter XIII] and Cohen [2004], Condon, Deaño, and Iserles [2009, 2010], Faou and Schratz [2014], Bao, Cai, and Zhao [2014], and Zhao [2017].

We now describe the basic steps how, for the problem of the previous subsection, a simple ansatz for the solution over a short time interval leads to long-time near-conservation results for the oscillatory energies $E_j = \frac{1}{2}(|p_j|^2 + \omega_j^2\,|q_j|^2)$. We approximate the solution q_j of the second-order differential equation of the previous section as a *modulated Fourier expansion*,

$$q_j(t) \approx \sum_k z_j^k(t)\,\mathrm{e}^{\mathrm{i}(k\cdot\omega)t} \qquad \text{for short times } 0 \le t \le 1,$$

with modulation functions z_j^k, all derivatives of which are required to be bounded independently of ε. The sum is taken over a finite set of multi-indices $k = (k_1, \dots, k_m) \in \mathbb{Z}^m$, and $k \cdot \omega = \sum k_j \omega_j$. The slowly changing modulation functions are multiplied with the highly oscillatory exponentials $\mathrm{e}^{\mathrm{i}(k\cdot\omega)t} = \prod_{j=1}^m \big(\mathrm{e}^{\mathrm{i}\omega_j t}\big)^{k_j}$, which are products of solutions to the linear equations $\ddot{x}_j + \omega_j^2 x_j = 0$. Such products can be expected to be introduced into the solution q_j by the nonlinearity.

Similar multiscale expansions have appeared on various occasions in the literature. The distinguishing feature here is that such a short-time expansion is used to derive long-time properties of the Hamiltonian system.

3.2.1 Modulation system and non-resonance condition. When we insert this ansatz into the differential equation and collect the coefficients to the same exponential $\mathrm{e}^{\mathrm{i}(k\cdot\omega)t}$, we obtain the infinite system of modulation equations for $\mathbf{z} = (z_j^k)$

$$(\omega_j^2 - (k\cdot\omega)^2)\, z_j^k + 2\mathrm{i}(k\cdot\omega)\dot{z}_j^k + \ddot{z}_j^k = -\frac{\partial\, \mathcal{U}}{\partial z_j^{-k}}\,(\mathbf{z}).$$

The left-hand side results from the linear part $\ddot{q}_j + \omega_j^2 q_j$ of the differential equation. The right-hand side results from the nonlinearity and turns out to have a gradient structure with the modulation potential

$$\mathcal{U}(\mathbf{z}) = U(z^0) + \sum_{\ell\ge 1} \sum_{k^1+\dots+k^\ell=0} \frac{1}{\ell!}\, U^{(\ell)}(z^0)\big[z^{k^1}, \dots, z^{k^\ell}\big],$$

with $0 \ne k^i \in \mathbb{Z}^m$. The sum is suitably truncated, say to $\ell \le N$, $|k^i| \le N$.

The infinite modulation system is truncated and can be solved approximately (up to a defect ε^N) for modulation functions z_j^k with derivatives bounded independently of ε under a non-resonance condition that ensures that $\omega_j^2 - (k\cdot\omega)^2$ is the dominating coefficient on the left-hand side of the modulation equation, except when k is plus or minus the jth unit vector. For example, we can assume, as in Cohen, Hairer, and Lubich [2005], that there exists $c > 0$ such that

$$|k\cdot\omega| \ge c\,\varepsilon^{-1/2} \quad \text{for} \quad k \in \mathbb{Z}^m \quad \text{with} \quad 0 < |k| \le 2N,$$

where $|k| = \sum_{j=1}^m |k_j|$. Under such a non-resonance condition one can construct and appropriately bound the modulation functions z_j^k, and the modulated Fourier expansion is then an $\mathcal{O}(\varepsilon^N)$ approximation to the solution over a short time interval $t = \mathcal{O}(1)$.

3.2.2 Lagrangian structure and invariants of the modulation system. With the multiplied functions $y_j^k(t) = z_j^k(t)\mathrm{e}^{\mathrm{i}(k\cdot\omega)t}$ that appear as summands in the modulated Fourier expansion, the modulation equations take the simpler form

$$\ddot{y}_j^k + \omega_j^2 y_j^k = -\frac{\partial\, \mathcal{U}}{\partial y_j^{-k}}\,(\mathbf{y}).$$

The modulation potential $\mathcal{U}$ has the important invariance property

$$\mathcal{U}(S_\ell(\theta)\mathbf{y}) = \mathcal{U}(\mathbf{y}) \quad \text{for} \quad S_\ell(\theta)\mathbf{y} = (\mathrm{e}^{\mathrm{i}k_\ell\theta} y_j^k)_{j,k} \quad \text{with} \quad \theta \in \mathbb{R},$$

as is directly seen from the sum over $k^1 + \ldots + k^m = 0$ in the definition of $\mathcal{U}$. We have thus embedded the original differential equations in a system of modulation equations that are Lagrange equations for the Lagrange function

$$\mathcal{L}(\mathbf{y}, \dot{\mathbf{y}}) = \frac{1}{2}\sum_{j,k} \dot{y}_j^{-k}\dot{y}_j^k - \frac{1}{2}\sum_{j,k} \omega_j^2 y_j^{-k} y_j^k - \mathcal{U}(\mathbf{y}),$$

which is invariant under the action of the group $\{S_\ell(\theta) : \theta \in \mathbb{R}\}$. We are thus in the realm of Emmy Noether's theorem from 1918, which states that invariance under a continuous group action (a geometric property) yields the existence of conserved quantities of the motion (a dynamic property). By Noether's theorem, the modulation equations thus conserve

$$\mathcal{E}_\ell(\mathbf{y}, \dot{\mathbf{y}}) = -\mathrm{i}\sum_j \sum_k k_\ell \omega_\ell \, y_j^{-k} \, \dot{y}_j^k.$$

Since the modulation equations are solved only up to a defect $\mathcal{O}(\varepsilon^N)$ in the construction of the modulated Fourier expansion, the functions $\mathcal{E}_\ell$ are almost-conserved quantities with $\mathcal{O}(\varepsilon^{N+1})$ deviations over intervals of length $\mathcal{O}(1)$. They turn out to be $\mathcal{O}(\varepsilon)$ close to the oscillatory energies E_ℓ. By patching together many short time intervals, the drift in the almost-invariants $\mathcal{E}_\ell$ is controlled to remain bounded by $Ct\varepsilon^{N+1} \le C\varepsilon$ over long times $t \le \varepsilon^{-N}$, and hence also the deviation in the oscillatory energies E_ℓ is only $\mathcal{O}(\varepsilon)$ over such long times. We thus obtain long-time near-conservation of the oscillatory energies E_ℓ.

3.3 Long-time results for numerical integrators. Modulated Fourier expansions were first developed in Hairer and Lubich [2000b] and further in Hairer, Lubich, and Wanner [2002, Chapter XIII] to understand the observed long-term near-conservation of energy by some numerical methods for step sizes for which the smallness condition $hL \ll 1$ of the backward error analysis of Section 2.3 is not fulfilled. For the numerical solution of the differential equation of Section 3.1, we are interested in using numerical integrators that allow large step sizes h such that $h/\varepsilon \ge c_0 > 0$. In this situation, the one-step map of a numerical integrator is no longer a near-identity map, as was the case in Section 2.

For a class of time-symmetric *trigonometric integrators*, which are exact for the uncoupled harmonic oscillator equations $\ddot{x}_j + \omega_j^2 x_j = 0$ and reduce to the Störmer–Verlet method for $\omega_j = 0$, the following results are proved for step sizes h that satisfy a *numerical non-resonance condition*:

$$h\omega_j \text{ is bounded away (by } \sqrt{h}) \text{ from a multiple of } \pi.$$

Under just this condition it is shown in Cohen, Gauckler, Hairer, and Lubich [2015], using modulated Fourier expansions, that the slow energy H_{slow} is nearly preserved

along the numerical solution for very long times $t \leq h^{-N}$ for arbitrary $N \geq 1$ provided the total energy remains bounded along the numerical solution. If in addition,

sums of $\pm h\omega_j$ with at most $N+1$ terms are
bounded away from non-zero multiples of 2π,

then also the total and oscillatory energies H and H_ω are nearly preserved along the numerical solution for $t \leq h^{-N}$ for the symplectic methods among the considered symmetric trigonometric integrators. Modified total and oscillatory energies are nearly preserved by the non-symplectic methods in this class. These results yield the numerical version of property (vi) above. A numerical version of property (v) was shown in Cohen, Hairer, and Lubich [2005]. The single-frequency case was previously studied in Hairer and Lubich [2000b]. For the *Störmer–Verlet method*, which can be interpreted as a trigonometric integrator with modified frequencies, related long-time results are given in Hairer and Lubich [2000a] and Cohen, Gauckler, Hairer, and Lubich [2015].

The numerical version of the energy transfer of property (iii) was studied in Hairer, Lubich, and Wanner [2002, Section XIII.4] and in Cohen, Hairer, and Lubich [2005] and McLachlan and Stern [2014]. Getting the energy transfer qualitatively correct by the numerical method turns out to put more restrictions on the choice of methods than long-time energy conservation.

While we concentrated here on long-time results, it should be mentioned that fixed-time convergence results of numerical methods for the multiscale problem as $h \to 0$ and $\varepsilon \to 0$ with $h/\varepsilon \geq c_0 > 0$ also pose many challenges; see, e.g., García-Archilla, Sanz-Serna, and Skeel [1999], Hochbruck and Lubich [1999], Grimm and Hochbruck [2006], and Buchholz, Gauckler, Grimm, Hochbruck, and Jahnke [2017] for systems with constant high frequencies and also Lubich and Weiss [2014] and Hairer and Lubich [2016] for systems with state-dependent high frequencies, where near-preservation of adiabatic invariants is essential. We also refer to Hairer, Lubich, and Wanner [2006, Chapters XIII and XIV] and to the review Cohen, Jahnke, Lorenz, and Lubich [2006].

4 Hamiltonian partial differential equations

There is a vast literature on the long-time behaviour of nonlinear wave equations, nonlinear Schrödinger equations and other Hamiltonian partial differential equations; see, e.g., the monographs Kuksin [1993], Bourgain [1999], Craig [2000], Kuksin [2000], Kappeler and Pöschel [2003], and Grébert and Kappeler [2014] where infinite-dimensional versions of Hamiltonian perturbation theory are developed. Here we consider a few analytical results that have recently been transferred also to numerical discretizations.

4.1 Long-time regularity preservation. We consider the nonlinear wave equation (or nonlinear Klein–Gordon equation)

$$\partial_t^2 u = \partial_x^2 u - \rho u + g(u), \qquad u = u(x,t) \in \mathbb{R}$$

with 2π-periodic boundary condition in one space dimension, a positive mass parameter ρ and a smooth nonlinearity $g = G'$ with $g(0) = g'(0) = 0$. This equation

is a Hamiltonian partial differential equation $\partial_t v = -\nabla_u H(u,v)$, $\partial_t u = \nabla_v H(u,v)$ (where $v = \partial_t u$) with Hamilton function

$$H(u,v) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2}\Big(v^2 + (\partial_x u)^2 + \rho u^2\Big) - G(u)\right)\mathrm{d}x$$

on the Sobolev space H^1 of 2π-periodic functions.

Written in terms of the Fourier coefficients u_j of $u(x,t) = \sum_{j\in\mathbb{Z}} u_j(t)\mathrm{e}^{\mathrm{i}jx}$, the nonlinear wave equation takes the form of the oscillatory second-order differential equation of Section 3.1, but the system is now infinite-dimensional:

$$\ddot{u}_j + \omega_j^2 u_j = \mathfrak{F}_j g(u), \qquad j \in \mathbb{Z},$$

where $\mathfrak{F}_j$ gives the jth Fourier coefficient and $\omega_j = \sqrt{j^2+\rho}$ are the frequencies.

The following result is proved, using infinite-dimensional Hamiltonian perturbation theory (Birkhoff normal forms), by Bambusi [2003], for arbitrary $N \geq 1$: Under a non-resonance condition on the frequencies ω_j, which is satisfied for almost all values of the parameter ρ, and for initial data (u^0, v^0) that are ε-small in a Sobolev space $H^{s+1}\times H^s$ with sufficiently large $s = s(N)$, the harmonic energies $E_j = \frac{1}{2}(|\dot{u}_j|^2 + \omega_j^2|u_j|^2)$ are nearly preserved over the time scale $t \leq \varepsilon^{-N}$, and so is the $H^{s+1}\times H^s$ norm of the solution $(u(t), v(t))$.

An alternative proof using modulated Fourier expansions was given in Cohen, Hairer, and Lubich [2008b] with the view towards transferring the result to numerical discretizations with trigonometric integrators as done in Cohen, Hairer, and Lubich [2008a], for which in addition also a numerical non-resonance condition is required.

Related long-time near-conservation results are proved for other classes of Hamiltonian differential equations, in particular for nonlinear Schrödinger equations with a resonance removing convolution potential, in Bourgain [1996], Bambusi and Grébert [2006], and Grébert [2007] using Birkhoff normal forms and in Gauckler and Lubich [2010] using modulated Fourier expansions. These results are transferred to numerical discretization by Fourier collocation in space and splitting methods in time in Faou, Grébert, and Paturel [2010a,b] and L. Gauckler and C. Lubich [2010].

For small initial data where only one pair of Fourier modes is excited (that is, $E_j(0) = 0$ for $|j| \neq 1$, which would yield a plane wave solution in the linear wave equation), higher Fourier modes become excited in the above nonlinearly perturbed wave equation to yield, within short time, mode energies $E_j(t)$ that decay geometrically with $|j|$ and then remain almost constant for very long times. Using modulated Fourier expansions, this is proved in Gauckler, Hairer, Lubich, and Weiss [2012] for the continuous problem and extended to numerical discretizations in Gauckler and Weiss [2017].

4.2 Long-time near-conservation of energy for numerical discretizations. In Section 2 we have seen that symplectic integrators approximately preserve the energy of a Hamiltonian ordinary differential equation over long times. It is then not far-fetched to expect the same for full (space and time) discretizations of Hamiltonian partial differential equations. However, the standard backward error analysis argument does *not* carry over from ODEs to PDEs:

– Backward error analysis requires that the product of the time step size, now denoted Δt, and the local Lipschitz constant L be small. After space discretization with a meshwidth Δx, we have L proportional to $1/\Delta x$ in the case of wave equations and proportional to $1/\Delta x^2$ in the case of Schrödinger equations. The condition $\Delta t\, L \ll 1$ then requires unrealistically small time steps as compared with the familiar Courant–Friedrichs–Lewy (CFL) condition $\Delta t\, L \leq \text{Const}$.

– Even when a symplectic integrator is used with a very small time step size such that $\Delta t\, L \ll 1$, the numerical method will nearly preserve the Hamilton function of the spatially discretized system, not that of the PDE. The two energies are close to each other only as long as the numerical solution is sufficiently regular, which usually cannot be guaranteed *a priori*.

The above hurdles are overcome in Cohen, Hairer, and Lubich [2008a] for the nonlinearly perturbed wave equation of the previous subsection discretized by Fourier collocation in space and symplectic trigonometric integrators in time. Here, high regularity of the numerical solution and near-conservation of energy are proved simultaneously using modulated Fourier expansions. In L. Gauckler and C. Lubich [2010], this technique and the energy conservation results are taken further to a class of nonlinear Schrödinger equations with a resonance-removing convolution potential (in arbitrary space dimension) discretized by Fourier collocation in space and a splitting method in time.

Long-time near-conservation of energy for symplectic splitting methods applied to the nonlinear Schrödinger equation in one space dimension (without a resonance-removing convolution potential) is shown in Faou and Grébert [2011] and Faou [2012] with a backward error analysis adapted to partial differential equations and, under weaker step size restrictions, in Gauckler [2016] with modulated Fourier expansions. In contrast to the aforementioned results, these results are not restricted to initial values in higher-order Sobolev spaces.

Apart from these results in the weakly nonlinear regime, the basic question of long-time approximate conservation of the total energy under numerical discretization remains wide open in view of the difficulties addressed above.

4.3 Orbital stability results for nonlinear Schrödinger equations and their numerical discretizations. We consider the cubic nonlinear Schrödinger equation

$$i\,\partial_t u = -\Delta u + \kappa |u|^2 u$$

near special solutions (ground states and plane waves) and describe results about orbital stability that have been obtained for the continuous problem and for numerical discretizations.

Ground state. The cubic nonlinear Schrödinger equation on the real line in the focusing case $\kappa = -1$ admits the solitary wave solution $u(x,t) = \mathrm{e}^{\mathrm{i}\lambda t}\eta(x)$ with $\eta(x) = \frac{1}{\sqrt{2}}\operatorname{sech}(x/2)$ with an appropriate real parameter λ. It is known from Weinstein [1985] that this solution is orbitally stable in the sense that for a small H^1-perturbation of the initial data, the exact solution remains close to the orbit of η for all times. For the case restricted to symmetric initial conditions, orbital stability of a full discretization

with a finite-difference space discretization and a splitting method as time discretization is shown by Bambusi, Faou, and Grébert [2013].

Plane waves. We now consider the cubic nonlinear Schrödinger equation on a torus $\mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d$ of arbitrary dimension $d \geq 1$ with real κ. It admits the plane wave solution $u(x,t) = \rho e^{i(m\cdot x)} e^{-i(|m|^2+\kappa|\rho|^2)t}$ for arbitrary $\rho > 0$ and $m \in \mathbb{Z}^d$. It is shown in E. Faou, L. Gauckler, and C. Lubich [2013] that for almost all $\rho > 0$, plane wave solutions are orbitally stable for long times $t \leq \varepsilon^{-N}$ under ε-perturbations in higher-order Sobolev spaces H^s, $s = s(N) \gg 1$, if $1 + 2\kappa|\rho|^2 > 0$ (which is the condition for linear stability). This result is given with two different proofs, one based on Birkhoff normal forms and the other one on modulated Fourier expansions. The latter technique is used in E. Faou, L. Gauckler, and C. Lubich [2014] to transfer the result to numerical discretization using Fourier collocation in space and a splitting method for time discretization. The long-time orbital stability under smooth perturbations is in contrast to the instability under rough perturbations shown in Hani [2014].

5 Dynamical low-rank approximation

Low-rank approximation of too large matrices and tensors is a fundamental approach to data compression and model reduction in a wide range of application areas. Given a matrix $A \in \mathbb{R}^{m\times n}$, the best rank-$r$ approximation to A with respect to the distance given by the Frobenius norm (that is, the Euclidean norm of the vector of entries of a matrix) is known to be obtained by a truncated singular value decomposition: $A \approx \sum_{i=1}^r \sigma_i u_i v_i^\top$, where $\sigma_1, \dots, \sigma_r$ are the r largest singular values of A, and $u_i \in \mathbb{R}^m$ and $v_i \in \mathbb{R}^n$ are the corresponding left and right singular vectors, which form an orthonormal basis of the range and corange, respectively, of the best approximation. Hence, only r vectors of both length m and n need to be stored. If $r \ll \min(m,n)$, then the requirements for storing and handling the data are significantly reduced.

When $A(t) \in \mathbb{R}^{m\times n}$, $0 \leq t \leq T$, is a time-dependent family of large matrices, computing the best rank-r approximation would require singular value decompositions of $A(t)$ for every time instance t of interest, which is often not computationally feasible. Moreover, when $A(t)$ is not given explicitly but is the unknown solution to a matrix differential equation $\dot{A}(t) = F(t, A(t))$, then computing the best rank-r approximation would require to first solve the differential equation on $\mathbb{R}^{m\times n}$, which may not be feasible for large m and n, and then to compute the singular value decompositions at all times of interest, which may again not be feasible.

5.1 Dynamical low-rank approximation of matrices. An alternative — and often computationally feasible — approach can be traced back to Dirac [1930] in a particular context of quantum dynamics (see also the next section). Its abstract version can be viewed as a nonlinear Galerkin method on the tangent bundle of an approximation manifold $\mathfrak{M}$ and reads as follows: Consider a differential equation $\dot{A}(t) = F(t, A(t))$ in a (finite- or infinite-dimensional) Hilbert space $\mathcal{H}$, and let $\mathfrak{M}$ be a submanifold of $\mathcal{H}$. An approximation $Y(t) \in \mathfrak{M}$ to a solution $A(t)$ (for $0 \leq t \leq T$) is determined by choosing the time derivative $\dot{Y}(t)$ as the orthogonal projection of the vector field $F(t, Y(t))$ to

the tangent space $T_{Y(t)}\mathfrak{M}$ at $Y(t) \in \mathfrak{M}$:

$$\dot{Y}(t) = P_{Y(t)} F(t, Y(t)), \tag{5-1}$$

where P_Y denotes the orthogonal projection onto the tangent space at $Y \in \mathfrak{M}$. Equation (5-1) is a differential equation on the approximation manifold $\mathfrak{M}$, which is complemented with an initial approximation $Y(0) \in \mathfrak{M}$ to $A(0) \in \mathcal{H}$. When $\mathfrak{M}$ is a flat space, then this is the standard Galerkin method, which is a basic approximation method for the spatial discretization of partial differential equations. When $\mathfrak{M}$ is not flat, then the tangent space projection P_Y depends on Y, and (5-1) is a nonlinear differential equation even if F is linear.

For the dynamical low-rank approximation of time-dependent matrices, (5-1) is used with $\mathfrak{M}$ chosen as the manifold of rank-r matrices in the space $\mathcal{H} = \mathbb{R}^{m\times n}$ equipped with the Frobenius inner product (the Euclidean inner product of the matrix entries). This approach was first proposed and studied in Koch and Lubich [2007a]. The rank-r matrices are represented in (non-unique) factorized form as

$$Y = USV^\top,$$

where $U \in \mathbb{R}^{m\times r}$ and $V \in \mathbb{R}^{n\times r}$ have orthonormal columns and $S \in \mathbb{R}^{r\times r}$ is an invertible matrix. The intermediate small matrix S is not assumed diagonal, but it has the same non-zero singular values as $Y \in \mathfrak{M}$. Differential equations for the factors U, S, V can be derived from (5-1) (uniquely under the gauge conditions $U^\top \dot{U} = 0$ and $V^\top \dot{V} = 0$). They contain the inverse of S as a factor on the right-hand side. It is a typical situation that S has small singular values, because in order to obtain accurate approximability, the discarded singular values need to be small, and then the smallest retained singular values are usually not much larger. Small singular values complicate the analysis of the approximation properties of the dynamical low-rank approximation (5-1), for a geometric reason: the curvature of the rank-r manifold $\mathfrak{M}$ at $Y \in \mathfrak{M}$ (measured as the local Lipschitz constant of the projection map $Y \mapsto P_Y$) is proportional to the inverse of the smallest singular value of Y. It seems obvious that high curvature of the approximation manifold can impair the approximation properties of (5-1), and for a general manifold this is indeed the case. Nevertheless, for the manifold $\mathfrak{M}$ of rank-r matrices there are numerical and theoretical results in Koch and Lubich [ibid.] that show good approximation properties also in the presence of arbitrarily small singular values.

5.2 Projector-splitting integrator. The numerical solution of the differential equations for U, S, V encounters difficulties with standard time integration methods (such as explicit or implicit Runge–Kutta methods) when S has small singular values, since the inverse of S appears as a factor on the right-hand side of the system of differential equations.

A numerical integration method for these differential equations with remarkable properties is given in Lubich and Oseledets [2014]. It is based on splitting the tangent space projection, which at $Y = USV^\top$ is an alternating sum of three subprojections:

$$P_Y Z = ZVV^\top - UU^\top ZVV^\top + UU^\top Z.$$

Starting from a factorization $Y_n = U_n S_n V_n^\top$ at time t_n, the corresponding splitting integrator updates the factorization of the rank-r approximation to $Y_{n+1} = U_{n+1} S_{n+1} V_{n+1}^\top$ at time t_{n+1}. It alternates between solving (approximately if need be) matrix differential equations of dimensions $m \times r$ (for US), $r \times r$ (for S), $n \times r$ (for $VS^\top$) and doing orthogonal decompositions of matrices of these dimensions. The inverse of S does not show up in these computations.

The projector-splitting integrator has a surprising exactness property: if the given matrix $A(t)$ is already of rank r for all t, then the integrator reproduces $A(t)$ exactly Lubich and Oseledets [ibid.]. More importantly, the projector-splitting integrator is robust to the presence of small singular values: it admits convergent error bounds that are independent of the singular values Kieri, Lubich, and Walach [2016]. The proof uses the above exactness property and a geometric peculiarity: in each substep of the algorithm, the approximation moves along a flat subspace of the manifold $\mathfrak{M}$ of rank-r matrices. In this way, the high curvature due to small singular values does no harm.

5.3 Dynamical low-rank approximation of tensors. The dynamical low-rank approximation and the projector-splitting integrator have been extended from matrices to tensors $A(t) \in \mathbb{R}^{n_1 \times \cdots \times n_d}$ such that the favourable approximation and robustness properties are retained; see Koch and Lubich [2010], Lubich, Rohwedder, Schneider, and Vandereycken [2013], A. Arnold and Jahnke [2014], Lubich, Oseledets, and Vandereycken [2015], and Lubich, Vandereycken, and Walach [2017]. The dynamical low-rank approximation can be done in various tensor formats that allow for a notion of rank, such as Tucker tensors, tensor trains, hierarchical tensors, and general tensor tree networks; see Hackbusch [2012] and Uschmajew and Vandereycken [2013] for these concepts and for some of their geometric properties.

6 Quantum dynamics

6.1 The time-dependent variational approximation principle. The time-dependent Schrödinger equation for the N-particle wavefunction $\psi = \psi(x_1, \ldots, x_N, t)$,

$$\mathrm{i}\partial_t \psi = H\psi,$$

posed as an evolution equation on the complex Hilbert space $\mathcal{H} = L^2((\mathbb{R}^3)^N, \mathbb{C})$ with a self-adjoint Hamiltonian operator H, is not accessible to direct numerical treatment in the case of several, let alone many particles. "One must therefore resort to approximate methods", as Dirac [1930] noted already in the early days of quantum mechanics. For a particular approximation scheme, which is nowadays known as the time-dependent Hartree–Fock method, he used the tangent space projection (5-1) for the Schrödinger equation. Only later was this recognized as a general approximation approach, which is now known as the (Dirac–Frenkel) time-dependent variational principle in the physical and chemical literature: Given a submanifold $\mathfrak{M}$ of $\mathcal{H}$, an approximation $u(t) \in \mathfrak{M}$ to the wavefunction $\psi(\cdot, t) \in \mathcal{H}$ is determined by the condition that

$$\dot{u} \text{ is chosen as that } w \in T_u\mathfrak{M} \text{ for which } \|\mathrm{i}w - Hu\| \text{ is minimal.}$$

This is precisely (5-1) in the context of the Schrödinger equation: $\dot{u} = P_u \frac{1}{\mathrm{i}} H u$. If we assume that the approximation manifold $\mathfrak{M}$ is such that for all $u \in \mathfrak{M}$,

$$T_u \mathfrak{M} \text{ is a complex vector space,}$$

(so that with $v \in T_u \mathfrak{M}$, also $\mathrm{i}v \in T_u \mathfrak{M}$), then the orthogonal projection P_u turns out to be also a *symplectic* projection with respect to the canonical symplectic two-form on $\mathcal{H}$ given by $\omega(\xi, \eta) = 2\,\mathrm{Im}\langle \xi, \eta \rangle$ for $\xi, \eta \in \mathcal{H}$, and $\mathfrak{M}$ is a symplectic manifold. With the Hamilton function $H(u) = \langle u, Hu \rangle$, the differential equation for u can then be rewritten as

$$\omega(\dot{u}, v) = \mathrm{d}H(u)[v] \qquad \text{for all } v \in T_u \mathfrak{M},$$

which is a Hamiltonian system on the symplectic manifold $\mathfrak{M}$; cf. Section 2.6. The total energy $H(u)$ is therefore conserved along solutions, and the flow is symplectic on $\mathfrak{M}$. The norm is conserved if $\mathfrak{M}$ contains rays, i.e., with $u \in \mathfrak{M}$ also $\alpha u \in \mathfrak{M}$ for all $\alpha > 0$. We refer to the books Kramer and Saraceno [1981] and Lubich [2008] for geometric, dynamic and approximation aspects of the time-dependent variational approximation principle.

6.2 Tensor and tensor network approximations. In an approach that builds on the time-honoured idea of separation of variables, the multi-configuration time-dependent Hartree method (MCTDH) Meyer, Manthe, and Cederbaum [1990] and Meyer, Gatti, and Worth [2009] uses the time-dependent variational principle to determine an approximation to the multivariate wavefunction that is a linear combination of products of univariate functions:

$$u(x_1, \dots, x_N, t) = \sum_{i_1=1}^{r_1} \cdots \sum_{i_N=1}^{r_N} c_{i_1,\dots,i_N}(t)\, \varphi_{i_1}^{(1)}(x_1, t) \dots \varphi_{i_N}^{(N)}(x_N, t).$$

The time-dependent variational principle yields a coupled system of ordinary differential equations for the coefficient tensor $\big(c_{i_1,\dots,i_N}(t)\big)$ of full multilinear rank and low-dimensional nonlinear Schrödinger equations for the single-particle functions $\varphi_{i_n}^{(n)}(x_n, t)$, which are assumed orthonormal for each $n = 1, \dots, N$. Well-posedness and regularity for this nonlinear system of evolution equations is studied in Koch and Lubich [2007b], and an asymptotic error analysis of the MCTDH approximation for growing ranks r_n is given in Conte and Lubich [2010].

The projector-splitting integrator of Section 5.2 is extended to MCTDH in Lubich [2015]. The nonlinear MCTDH equations are thus split into a chain of linear single-particle differential equations, alternating with orthogonal matrix decompositions. The integrator conserves the L^2 norm and the total energy and, as is proved in Lubich, Vandereycken, and Walach [2017], it is robust to the presence of small singular values in matricizations of the coefficient tensor.

In the last decade, tensor network approximations, and in particular matrix product states, have increasingly come into use for the description of strongly interacting quantum many-body systems; see, e.g., Verstraete, Murg, and Cirac [2008], Cirac and

Verstraete [2009], and Szalay, Pfeffer, Murg, Barcza, Verstraete, Schneider, and Legeza [2015]. Matrix product states (known as tensor trains in the mathematical literature Oseledets [2011]) approximate the wavefunction by

$$u(x_1, \dots, x_N, t) = G_1(x_1, t) \cdot \dots \cdot G_N(x_N, t)$$

with matrices $G_n(x_n, t)$ of compatible (low) dimensions. This approach can be viewed as a non-commutative separation of variables. Its memory requirements grow only linearly with the number of particles N, which makes the approach computationally attractive for many-body systems. The approximability of the wavefunction or derived quantities by this approach is a different issue, with some excellent computational results but hardly any rigorous mathematical theory so far.

For the numerical integration of the equations of motion that result from the time-dependent variational approximation principle, the projector-splitting integrator has recently been extended to matrix product states in Lubich, Oseledets, and Vandereycken [2015] and Haegeman, Lubich, Oseledets, Vandereycken, and Verstraete [2016], with favourable properties like the MCTDH integrator. The important robustness to the presence of small singular values is proved in Kieri, Lubich, and Walach [2016], again using the property that the integrator moves along flat subspaces within the tensor manifold.

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LONG-TERM HISTORY AND EPHEMERAL CONFIGURATIONS

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Abstract

Mathematical concepts and results have often been given a long history, stretching far back in time. Yet recent work in the history of mathematics has tended to focus on local topics, over a short term-scale, and on the study of ephemeral configurations of mathematicians, theorems or practices. The first part of the paper explains why this change has taken place: a renewed interest in the connections between mathematics and society, an increased attention to the variety of components and aspects of mathematical work, and a critical outlook on historiography itself. The problems of a long-term history are illustrated and tested using a number of episodes in the nineteenth-century history of Hermitian forms, and finally, some open questions are proposed.

"Mathematics is the art of giving the same name to different things," wrote Henri Poincaré at the very beginning of the twentieth century (Poincaré [1908, p. 31]). The sentence, to be found in a chapter entitled "The future of mathematics" seemed particularly relevant around 1900: a structural point of view and a wish to clarify and to firmly found mathematics were then gaining ground and both contributed to shorten chains of argument and gather together under the same word phenomena which had until then been scattered (Corry [2004]). Significantly, Poincaré's examples included uniform convergence and the concept of group.

1 Long-term histories

But the view of mathematics encapsulated by this — that it deals somehow with "sameness" — has also found its way into the history of mathematics. It has been in particular a key feature (though often only implicitly) in the writing of most long-term histories. In the popular genres of the history of π or of the Pythagorean theorem from Antiquity to present times, is hidden the idea that, despite changes in symbolism, despite the use or not of figures, tables or letters, despite the presence or not of proofs, some mathematical thing is indeed the same. That, for example, it is interesting to extract, from behind what would be then only its masks, the computations of a certain quantity, say the ratio of a circle's circumference to its diameter, before the quantity may have

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been baptised π, before it has been described as being a number or a well-defined ratio, even before any relation has been established between it and the computation of the area of a disk. In more sophisticated versions, by telling the story of a series of past events which have led to finally define an object in our present, history of mathematics may convey the idea that the series of past objects associated with these events were already more or less the same in the past, and have only been subsumed under the same name. That there is an identity to be detected through or behind contingent disguises concerns not only numbers or simple statements but also whole domains like algebra or statistics, or advanced concepts like group or methods of proof. In such a history, the present of mathematics speaks for its past on various scales: besides the present formulation of a statement or an object, it also defines the implicit norms for mathematical activities, for instance that they should involve proofs (even if unwritten), or that disciplines are more or less fixed. Debates of course may have been launched and corrections been made about the point of departure, the real origin: did Euclid's *Elements* or Babylonian tablets contain algebra, should we begin the history of algebra only with al-Khwārizmī or François Viète? But there were rarely doubts raised about the relevance of the question itself and the linear character of the history constructed under the key premise of an identifiable thing running all the way long. Many would agree with André Weil that: "More often than not, what makes [history] interesting is precisely the early occurrence of concepts and methods destined to emerge only later into the conscious mind of mathematicians; the historian's task is to disengage them and trace their influence or lack of influence on subsequent developments" (Weil [1980, pp. 231–232]).

There are several good reasons to adopt such a point of view. For one, it is close to the "spontaneous history" of the working mathematician, the chronology often given at the beginning of a mathematical article in order to motivate the results it establishes.[1] The topics of such a history are also more easily those of primary interest to mathematicians (recent concepts or theorems, in particular). That mathematics deals with long-lived objects may also help to consolidate its status, at times when the importance of mathematics is questioned or the population of students in mathematics is decreasing; that "mathematical truths have been called eternal truths …[because] in very different expressions, one can recognise the same truths," in Hieronymus Zeuthen's terms (Lützen and Purkert [1994, p. 17]), guarantees a particular value for the discipline as a whole. Such a conviction may also reinforce (or be reinforced by) Platonist views of mathematics, and as such has been taken over by some philosophers. For instance, the philosopher of mathematics Jacques-Paul Dubucs, after pointing out the differences between two presentations of a proof that there are infinitely many prime numbers, one in Euclid, one modern by Godfrey H. Hardy and Edward M. Wright, claimed that proper epistemological investigation should focus on what is perceived as a stable and identical core and in particular has "no reason to discriminate the two texts which propose *fundamentally* the same proof," emphasising his agreement on this issue with the authors of the modern text (Dubucs [1991, p. 41]).

[1] The expression "spontaneous history" along with Louis Althusser's "spontaneous philosophy" is now often used in history and philosophy of science to designate casual remarks made by working scientists about the past of their work (resp. their general views on mathematics), see for instance Rheinberger [1994].

2 Ephemeral configurations

However, William Aspray and Philip Kitcher noticed in 1988 that "a new and more sophisticated historiography has arisen […] This historiography measures events of the past against the standards of their time, not against the mathematical practices of today" (Aspray and Kitcher [1988, pp. 24–25]). Indeed, innovative approaches in the history of mathematics of the last decades have often expressed misgivings over a cleaned-up history, based on a too-rapid identification of a concept or a problem, and over its historiographical consequences. In Thomas Hawkins's words: "The challenge to the historian is to depict the origins of a mathematical theory so as to capture the diverse ways in which the creation of that theory was a vital part of the mathematics and mathematical perceptions of the era which produced it" (Hawkins [1987]). Consequently, the focus of recent history of mathematics has been much more on localised issues, short-term interests and ephemeral situations, on "the era which produced" the mathematics in question; and moreover it has centred on diversity, differences and changes.

Confluent factors are here at stake. One has been largely advertised. It is linked to contemporary debates in the history of science in the large and comes with the wish to take into account social aspects of mathematics and "how they shape the form and the content of mathematical ideas" (Aspray and Kitcher [1988, p. 25]), while dimming the line between so-called internal history (that of concepts and results) and external history (that of institutions or scientific politics). Given the quantity of recent historical writing on these issues, I shall only mention a few examples. The unification of Italian states during the nineteenth century and the cultural *Risorgimento* which accompanied it favoured a flourishing of mathematics, in particular a strong renewal of interest in geometry in all its forms, with Luigi Cremona, Corrado Segre, Guido Castelnuovo or Eugenio Beltrami and their followers (Bottazzini [1994], Bottazzini and Nastasi [2013], and Casnati, Conte, Gatto, Giacardi, Marchisio, and Verra [2016]). The Meiji Restoration in Japan witnessed a multifaceted confrontation between the then extremely active, traditional Japanese mathematics (*wasan*) and its Western counterparts (Horiuchi [1996]). The First World War, a "war of guns and mathematics" as one soldier described it, did not just kill hundreds of mathematicians on the battlefields, among many millions of others: it also launched entire domains on a vast new scale, such as fluid mechanics or probability theory, and completely reconfigured international mathematical exchanges (for instance fostering a development of set theory, logic and real analysis in newly independent Poland) (Aubin and Goldstein [2014]). One might also think of the variety of national circumstances which preceded the creation of mathematical societies in the late nineteenth and early twentieth centuries (Parshall [1995]) or the various reforms in mathematical education (Gispert and Schubring [2011] and Karp and Schubring [2014]).

At a smaller scale, specific opportunities at specific times, putting mathematicians in close contact with certain milieux, have hosted particular, sometimes unexpected, mathematical work, be it analysis in administrative reforms (Brian [1994]), number theory in the textile industry (Decaillot [2002]) and in computer hardware (De Mol and Bullynck [2008]), or convexity in the military (Kjeldsen [2002]). In such cases (and in many others studied in detail over the last decades), it is not a question of superficial analogies or obvious applications; very often the ways these connections were made, the concrete

manner of transmission of knowledge through personal or institutional links, the objectives pursued, are what provides impulse to a mathematical investigation, explains the formulation it takes or the particular balance between computations and theory it displays (Kjeldsen [2010] and Tournès [2012]). This is, by definition, ephemeral in the sense that it implies links with social situations which have their own time scale and are most certainly not "eternal truths."

However, another reason for this shorter-term focus has recently become even more decisive, to wit, a more acute sensibility to the multi-layered structure of mathematics and the need to study more carefully its variegated components. At first sight, it seems simple enough: mathematics is most often inscribed in texts—although ethnomathematicians also study it directly in strings or sand (Ascher [1991], d'Ambrosio [2000], and Petit and Vandendriessche [2014])—, it uses words, symbols and drawings, it defines or studies certain objects, it states results and justifies them. It could seem that all we have to do is to decipher the texts and explain the objects, the results and the proofs. Of course, that there were at times debates among mathematicians about certain proofs or objects is well-known: even as late as early modern times, some would not accept a proof based on an algebraically-expressed relation and required geometric proofs in the Euclidean style, as more solid; others dismissed proofs by contradiction or, later, non-constructive proofs; the legitimacy of negative numbers or of functions without derivatives or of sets has been put into question. And around 1900, how to found mathematics—on axiomatics, on integers, or on logic—was a topic of heated controversy among mathematicians, which in turn has been studied by historians.

But more recently, other aspects have been explored, aspects which are not necessarily linked to public and noisy debates, but are part and parcel of ordinary mathematical activities. What is or could be an object or a result, how are they chosen or defined or justified, has changed in time; it has also been seen as depending on the place or the author. What sort of question is considered interesting, by whom and why; which criteria are required to make an argument convincing or a solution satisfactory, again for whom and why; all these aspects and their relations are worth being studied for their own sake. Historians have, for instance, shown that arguments in words or symbols may rely upon, or be inspired by, or sometimes even been replaced by figures, diagrams, tables, instruments.[2] That an acceptable answer to a problem may be, at times, and for certain groups of mathematicians, a single number, an explicit description of all the solutions, an equation, an existence theorem, or the creation of a new concept (Goldstein [2001], Chorlay [2010], and Ehrhardt [2012]). The way various domains are defined and interact, or are perceived as distinct, has also changed within mathematics, but also between mathematics and other domains, in particular physics (Archibald [1989], Gray [1999], and Schlote and M. Schneider [2011]).

[2]For illustrations of these different cases, see for instance Netz [1999], Lorenat [2014], Mumma, Panza, and Sandu [2012], Campbell-Kelly, Croarken, Flood, and Robson [2003], Durand-Richard [2010], Tobies and Tournès [2011], and Flament and Nabonnand [2011]. References being too numerous for exhaustivity, those given are only illustrative.

What are called "epistemic values," that is the internalised criteria of what constitutes good mathematics at one time, have also been studied: rigour is the most obvious perhaps and has a complex history, but universalism or effectiveness or generality or naturalness may appear at some moments to be even more decisive (Mehrtens [1990], Rowe [1992], Schubring [2005], Corry [2004], and Chemla, Chorlay, and Rabouin [2016]). Working mathematicians have usually their own answer to these questions, but the point here is to reconstruct the whole range of positions at a given time, in a given milieu, and to understand their effect in mathematical work. For instance, in the dispute between Leopold Kronecker and Camille Jordan in 1874 about what we now see as the *same* reduction theorem for matrices, differences in formulation (elementary divisors on one side, canonical forms on the other), and in disciplines (invariant theory vs group theory) were at play, but also differences in conceptions of generality (Brechenmacher [2016]).

Last, but not least, the way mathematics is made public and circulates has been proved to be both a serious constraint on its form and an essential factor in its transmission; the organization of correspondence among mathematicians (indeed the mere form of a mathematical letter), the creation of mathematical research journals in the nineteenth century and their different organisation through time, the advent of academies, seminars and conferences, teaching programs and textbooks, for instance, have all been scrutinised (Peiffer [1998], Rowe [2004], Schubring [1985a,b], Ausejo and Hormigón [1993], Verdier [2009], Gérini [2002], and Remmert and U. Schneider [2010]). The last two aspects could also be considered as links between mathematical developments and general cultural issues, but here the emphasis is on the combination of these components inside mathematical texts themselves.

Depending on one's own tastes, this sheer variety in the course of time may appear fascinating or an irrelevant antiquarian interest. However, we now have enough evidence that all these aspects may count for understanding the development of mathematics. Mathematics weaves together objects, techniques, signs of various kinds, justifications, professional lifestyles, epistemic ideas. Recent biographies, indeed, offer successful examples of the study of such articulations (Parshall [2006], Crilly [2006], and Alfonsi [2011]). But historians have also studied these components separately, in a comparative way, in order to display their range and their evolution in time. These components have distinct time-scales and changes do not occur simultaneously. Even when one is able to understand a long-term development of one component (for instance, of mathematical publishing), its articulation with other components is generally stable only over a shorter period. A further difficulty is that if concepts or theorems are aspects of which the mathematician is aware (and very much so), some of the components I mentioned are much more implicit, or are operational at a collective, not at an individual, level; they can be best detected and analysed for an entire group (Goldstein [1999]). All this explains the interest in studying what I am calling here "configurations," a word borrowed from the sociologist Norbert Elias. Elias wanted to set himself apart from previous sociological theories based on an a priori hierarchic opposition between individuals and society, and he promoted the idea of first studying configurations formed by interactions between persons in interdependence, at different scales, be they players

in a game or workers in an enterprise. For us, configurations organise texts and persons, coordinating some of the components we have mentioned (we shall see concrete examples later).

A last reason has favoured more localised studies: a critical outlook by historians of mathematics on their own practice. Words like "discipline" or "school" have been used in the past without further ado, in particular because they were terms inherited from mathematicians themselves. Recent work has shown that to define and use them more carefully gave a better grasp for describing the past. For instance, using a characterisation of a discipline as a list of internal elements (core concepts, proof system, etc), Norbert Schappacher and I were able to distinguish in the lineage of C. F. Gauss's *Disquisitiones Arithmeticae* those parts which fused into a existing discipline (his treatment of the cyclotomic equation, for instance, which had a potent effect on the theory of equations), those parts which emancipated themselves as autonomous disciplines with their own programmes and priorities (quadratic forms in the middle of the nineteenth century, reciprocity laws in the theory of number fields later), and those parts which, in the nineteenth century, were taken over in an isolated manner by some mathematicians (primality tests).[3] Caution also applies to common descriptors of historical phenomena themselves, such as "context" (Ritter [2004]), "*longue durée*" (Aubin and Dahan Dalmedico [2002]) or "revolution" (Gillies [1992]). This reflexivity has also permitted historians to find counter-examples to overly-crude hypotheses on the long-term development of mathematics (Goldstein and Ritter [2003] and Gilain and Guilbaud [2015]).

Thus, in the last decades, historians have used these analyses of configurations to deconstruct the identifications provided by mathematical works in the course of time. Examples include statements such as the fundamental theorem of algebra (Gilain [1991]), Sturm's theorem (Sinaceur [1991]), Fermat's theorem that "the area of a right-angled triangle in integers cannot be a square" (Goldstein [1995]) or the decomposition theorem of matrices (Brechenmacher [2007]); concepts like ideals (Edwards [1980, 1992]) or points (Schappacher [2010]); or even whole domains like Galois theory (Ehrhardt [2012]). As explained by André Weil, for instance, Fermat's 1640 statement, and even his proof, can be identified and seen as *the same* as a special case of the Mordell–Weil theorem, according to which the group of rational points of the elliptic curve defined by $y^2z = x^3 - xz^2$ is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. But we can also see this identification as a historical problem: it requires first to reconstruct various configurations involving each of the statements, and then to understand how they have come to be identified—in other words, to study also the mathematical *work* that provided such retrospective identification.

Here, differences are obvious, but some cases are more delicate. This can be illustrated by the theorem that there are exactly twenty-seven lines on a non-singular cubic projective surface; since its statement (and proof) in 1849 by both Arthur Cayley and George Salmon, its formulation has remained remarkably stable for more than a century. But what changed is its association with other problems: as shown in (Lê [2015]), it is for instance in tandem with the fact that there are 9 inflection points on a cubic projective plane curve and other analogous statements that it played a decisive role for the

[3] Goldstein and Schappacher [2007], see also Gauthier [2009, 2010]. On the issue of "school," see the synthesis (Rowe [2003]).

assimilation of group-theoretical methods by geometers *before* Felix Klein's Erlangen Program; this specific configuration of questions and disciplinary issues, around the so-called "equations of geometry," lasted only a few years, but was a key feature in the transmission of the theorem.

In what follows, I would like to illustrate these issues with what could be described as a minimal example: the concrete case of a rather technical and apparently stable concept, that of an Hermitian form,[4] over a short period of time, the second half of the nineteenth century.

3 Did Hermite invent Hermitian forms?

From all accounts, for instance by Vahlen [1900, pp. 612–613] or Dickson [1919–1923, vol. 3, p. 269], forms of this type first publicly appear in an article authored by Charles Hermite and it is thus legitimate to ask if Hermite indeed invented Hermitian forms, and how.

At the very beginning of this paper (Hermite [1854]), Hermite explained:

> One knows how easily one can extend the most fundamental arithmetical concepts coming from *real integers* to complex numbers of the type $a + b\sqrt{-1}$. Thus, starting from elementary propositions concerning divisibility, one quickly reaches those deeper and more hidden properties which rely upon the consideration of quadratic forms, without changing anything essential in the principle of methods which are proper to real numbers. In certain circumstances, however, this extension seems to require new principles and one is led to follow in several different directions the analogies between the two orders of arithmetical considerations. We would like to offer an example to which we have been led while studying the representation of a number as a *sum of four squares*.

No references were provided in this introduction, but the allusions seem rather clear, both then and now. Carl Friedrich Gauss, searching for an extension to higher powers of the reciprocity law for squares that he had proved in his 1801 *Disquisitiones Arithmeticae*, launched an arithmetical study of what he called "complex integers" (now Gaussian integers), that is "complex numbers of the type $a + b\sqrt{-1}$," with a and b ordinary integers (Gauss [1831/1863]). Among them he defined prime complex numbers and units, proved the factorisation of the "complex integers" into a product of these prime numbers (unique up to units and to the order of the factors), showed how to extend Euclidean division and congruences to these numbers: in short, "the elementary propositions which concern divisibility." In 1842, Peter Gustav Lejeune Dirichlet began to study "those deeper and more hidden properties which rely upon the consideration of quadratic forms," in particular the representation of Gauss's "complex integers" by what will come to be known as Dirichlet forms at the end of the century, that is, quadratic

[4] For our purpose, such a form will be simply an expression of the type $\sum_{i,j=1}^{n} a_{ij} x_i \overline{x_j}$, with coefficients a_{ij} in $\mathbb{C}$, such that $a_{ji} = \overline{a_{ij}}$ (here, the bar designates the complex conjugation); in particular, the diagonal coefficients a_{ii} are real numbers.

forms $f(x, y) = ax^2 + 2bxy + cy^2$, where the coefficients a, b, c, and eventually the values taken by the indeterminates x and y, are also "complex integers" (Dirichlet [1842]).

Such a discontinuous chronology (1801–1831–1842–1854) might lead us to the topic of Hermite's paper, but would not be sufficient to understand Hermite's background or point of view. First of all, Gauss's discussion of his complex integers appeared in Latin in the proceedings of the Göttingen Society of Science, with limited distribution. But as early as 1832, Dirichlet explained Gauss's work and completed it in what was at the time the only important journal entirely devoted to mathematics, August Leopold Crelle's *Journal für die reine und angewandte Mathematik*, created in 1826. Dirichlet, who had spent several years in Paris, was an important go-between for mathematics: his 1832 article, written in French, was clearly aimed at an international audience. In the same decade, he would use new tools developed by analysts, in particular Fourier series, to complete proofs of Gauss's statements and revisit a number of his arithmetical results, stressing their links to various areas of mathematics in a way that would draw greater attention to them. In 1840, for instance, a letter to Joseph Liouville, on the occasion of a French translation of one of his papers, announced to the French community his current interest for "extending to quadratic forms with complex coefficients and indeterminates, that is, of the form $t + u\sqrt{-1}$, the theorems which occur in the ordinary case of real integers. If one tries in particular to obtain the number of different quadratic forms which exist in this case for a given determinant, one arrives at this remarkable result, that the number in question depends on the division of the lemniscate; exactly as in the case of real forms with positive determinant, it is linked to the division of the circle" (Dirichlet [1840]). The lemniscate pointed to the integral $\int \frac{dx}{\sqrt{1-x^4}}$ and to elliptic functions, then at the forefront of research, and which will be soon the main topic of interest of the young Hermite.

Then, the ten years preceding Hermite's 1854 paper were turbulent years for complex functions and numbers, and to a lesser extent, for quadratic forms. In a footnote of his 1832 paper, Dirichlet announced, somehow optimistically, that numbers of the type $t + u\sqrt{a}$, for t and u integers and a an integer without square divisors, give rise to theorems analogous to those on Gaussian integers, and with similar proofs. In 1839 (with a French translation in Liouville's *Journal de mathématiques pures et appliquées* three years later), Carl Jacob Jacobi, again recalling Gauss's theory of complex integers, showed that a prime number $p = 8n + 1$ can be written as a product of *four* complex numbers, each of them a linear combination with integral coefficients of powers of a given 8th-root of unity, such that the three decompositions of p as $x^2 + y^2$, $x^2 + 2y^2$ and $x^2 - 2y^2$ be issued "from a common source." Announcing similar results for a prime $p = 5n + 1$ (and 5th-roots of unity), but with no hint of a proof, Jacobi provided the spur for decisive work by several younger mathematicians in the 1840s. These included Gotthold Eisenstein's approach to complex multiplication, and Ernst Eduard Kummer's theory of ideal numbers (in what we call now cyclotomic rings); Kummer's display that unique factorization failed in general certainly crushed Dirichlet's 1832 hopes and showed, as Hermite pointed out in 1854, that "in certain circumstances, this extension [of arithmetic to complex numbers] seems to require new principles." It also included

Hermite's own first research on quadratic forms, concerning which he wrote directly to Jacobi from 1847 on.[5] Personal relations here reinforced the circulation of the articles; Hermite was informed of Kummer's approach to the arithmetic of complex numbers by the mathematician Carl Wilhelm Borchardt during the latter's 1847 Parisian tour and he met Dirichlet and Eisenstein, among others, during his own trip to Berlin at the beginning of the 1850s.

Hermite's work on forms arose at least as much from his close reading of Jacobi—that on the decomposition of primes, but also that on elliptic functions—as from Dirichlet's articles on complex numbers. In his letters to Jacobi on quadratic forms, Hermite [1850] considered forms with any number of indeterminates and real coefficients (instead of the two indeterminates and integral coefficients of Gauss's *Disquisitiones*). His main theorem was to establish that there exists a (non-zero) value of the form, when evaluated on integers, which is less than a certain bound, depending only on the number of indeterminates and on the determinant of the form, but not of its coefficients.[6] That is:

Let $f(x_0, x_1 \ldots, x_n)$ a definite positive quadratic form with $n+1$ indeterminates, *real* coefficients, and determinant D. Then, there exist $n+1$ *integers* $\alpha, \beta, \ldots, \lambda$, such that

$$0 < f(\alpha, \beta, \ldots, \lambda) < \left(\frac{4}{3}\right)^{n/2} \sqrt[n+1]{|\ D\ |}. \tag{1}$$

Although the formulation does not make it obvious, this statement is closely related to the classification of forms. For definite binary forms, $f(x, y) = ax^2 + 2bxy + cy^2$, with determinant $D = ac - b^2$ and integral coefficients, Gauss's reduction theory stated that among all the forms arithmetically equivalent to a given form f (that is all the forms which are derived from f by an invertible linear transformation of the indeterminates with integral coefficients), there exists one, say F, whose first coefficient A is less than $2\sqrt{\frac{|D|}{3}}$. This coefficient $A = F(1, 0)$ is a value of F, and thus it is also (by linear transformation) a value at some integers of all the equivalent forms f. For these forms, this is exactly Hermite's main theorem for $n = 1$. Hermite used his general statement to prove that if one restricts oneself to forms with integral coefficients, the number of classes of arithmetically equivalent forms for a given determinant is finite, and in particular that, for $n = 1, 2, \ldots, 6$, there is only one class of forms with $|\ D\ | = 1$, represented in each case by the sum of n squares.

Classification was a key objective for Hermite: "[T]he task for number theory and integral calculus," he wrote to Jacobi, "[is] to penetrate into the nature of such a multitude of entities of reason, to classify them into mutually irreducible groups, to constitute them all individually through characteristic and elementary definitions" (Hermite [ibid., p. 286]). The striking echo of a quasi-botanical project in this quote is to be taken seriously: for Hermite and some of his contemporaries, the emphasis on classification

[5] This is explained in more detail in Goldstein and Schappacher [2007, pp. 39–51].

[6] It was common at the time not to distinguish explicitly among indeterminates, variables and values; or to give general statements without a clear list of exceptions. Moreover different conventions coexisted, for instance for the definition and the sign of determinants, etc. For reasons of space, and although these questions may be revealing and have been taken into account in historical work, I shall not in general discuss them here.

directly came from a view of mathematics as a natural science (Lê and Paumier [2016]). "Collecting and classifying" was also a very strong incentive for invariant theorists like Arthur Cayley (Crilly [2006, pp. 193–195]), and it was not limited to them, nor to the 1850s: in 1876, still, Leo Königsberger wrote for instance: "It seems to me that the main task now just as for descriptive natural history consists in gathering as much material as possible and in discovering principle by classifying and describing this material" (File H1850(6), Staatsbibliothek zu Berlin, Handschriftenabteilung).

Hermite's main theorem was quite versatile: with it, for instance, Hermite simultaneously showed how to approximate real numbers by rationals and proved two statements left unproved by Jacobi, that there is no complex function of one variable with three independent periods and, as announced above, that prime numbers of the form $p = 5n+1$ can be decomposed into a product of 4 linear combinations with integral coefficients of powers of a fifth root of 1. Characteristically, the unity of mathematics is here found in the bridges between analysis, algebra and arithmetic. In each case, the whole point is to choose correctly a quadratic form (or sometimes a continuous family of them) to encapsulate the phenomenon under scrutiny and to combine inequalities provided by the theorem with integrality properties.

Hermite also adapted this construction to discuss the divisors of forms of the type $x^2 + Ay^2$, and then, in 1854, took the natural step of testing it on the famous theorem that every integer is the sum of four squares.

Let A be a non-zero integer (without loss of generality, one may assume that 4 does not divide A). As a preliminary step, Hermite first showed how to find integers α and β such that $\alpha^2 + \beta^2 \equiv -1 \bmod (A)$.

He then introduced the quadratic form with 4 variables:

$$f(x, y, z, u) = (Ax + \alpha z + \beta u)^2 + (Ay - \beta z + \alpha u)^2 + z^2 + u^2,$$

with determinant A^4. When x, y, z, u are integers, the value $f(x, y, z, u)$ is also an integer and, thanks to the choice of α and β, it is divisible by A. On the other hand, Hermite's main theorem states that there exist integers x, y, z, u such that $0 < f(x, y, z, t) < (\frac{4}{3})^{\frac{3}{2}} \sqrt[4]{A^4}$, that is, $0 < f(x, y, z, u) < 1.54A$. The only possibility is that $f(x, y, z, u) = A$, which expresses A as a sum of 4 squares.

However, with classification in mind, Hermite decided to reformulate this proof: instead of f, he considered $\frac{1}{A} f$ (the determinant of which is then 1). It can be written, when rearranging terms, as

$$\frac{1}{A} f(x, y, z, u) = A(x^2 + y^2) + 2\alpha(zx + yu) + 2\beta(xu - zy) + \frac{\alpha^2 + \beta^2 + 1}{A}(z^2 + u^2). \tag{2}$$

As said above, Hermite had already proved that, up to arithmetic equivalence, there is only one quaternary quadratic form of determinant 1 with integral coefficients, the form $X^2 + Y^2 + Z^2 + U^2$. That is, there exist integers $m, m', \ldots, n, n', \ldots$ such that

the change of variables

$$\begin{aligned} X &= mx + m'y + m''z + m'''u \\ Y &= nx + n'y + n''z + n'''u \\ Z &= px + p'y + p''z + p'''u \\ U &= qx + q'y + q''z + q'''u \end{aligned}$$

transforms $X^2 + Y^2 + Z^2 + U^2$ into the form $\frac{1}{A} f$. By identification of the term in x^2, for instance, one obtains $A = m^2 + n^2 + p^2 + q^2$, as desired.

Viewing mathematics as a natural science also entailed the highlighting of specific practices. "[My own work] would very strikingly illustrate how much observation, divination, induction, experimental trial, and verification, causation, too [...] have to do with the work of the mathematician," claimed James Joseph Sylvester at the British Association for the Advancement of Science in 1869 (Parshall [2006]) and Hermite himself repeated several times that "the most abstract analysis is for the most part an observational science" (Goldstein [2007, p. 403]). Observed carefully, the very shape of the form $\frac{1}{A} f(x, y, z, u)$ (equation 2) suggests "complex numbers." At this stage, indeed, Hermite introduced a new set of indeterminates:

$$\begin{aligned} v &= x + \sqrt{-1}y & V &= X + \sqrt{-1}Y \\ v_0 &= x - \sqrt{-1}y & V_0 &= X - \sqrt{-1}Y \\ w &= z + \sqrt{-1}u & W &= Z + \sqrt{-1}U \\ w_0 &= z - \sqrt{-1}u & W_0 &= Z - \sqrt{-1}U \end{aligned}$$

and *restricted* the type of linear transformations to be taken into account.

> One can distinguish among all real transformations between the two groups of variables x, y, z, u on one hand, and X, Y, Z, U on the other, those which can be expressed as:
>
> $$\begin{aligned} v &= aV + bW \\ v_0 &= a_0 V_0 + b_0 W_0 \\ w &= cV + dW \\ v_0 &= c_0 V_0 + d_0 W_0 \end{aligned}$$
>
> where a, b, c, d are arbitrary imaginary numbers and a_0, b_0, c_0, d_0 their conjugates. Thus one obtains a perfectly defined class of real transformations.

In a modernised matrix notation, these transformations can be expressed as special transformations with 4 real variables:

$$\begin{pmatrix} x \\ y \\ z \\ u \end{pmatrix} = \begin{pmatrix} \mathrm{Re}\,(a) & -\mathrm{Im}\,(a) & \mathrm{Re}\,(b) & -\mathrm{Im}\,(b) \\ \mathrm{Im}\,(a) & \mathrm{Re}\,(a) & \mathrm{Im}\,(b) & \mathrm{Re}\,(b) \\ \mathrm{Re}\,(c) & -\mathrm{Im}\,(c) & \mathrm{Re}\,(d) & -\mathrm{Im}\,(d) \\ \mathrm{Im}\,(c) & \mathrm{Re}\,(c) & \mathrm{Im}\,(d) & \mathrm{Re}\,(d) \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ U \end{pmatrix}.$$

These transformations, in turn, preserve specific *quadratic* forms that Hermite introduced at this point (those we now call Hermitian forms):

$$f(v,w) = Avv_0 + Bvw_0 + B_0wv_0 + Cww_0 \tag{3}$$

where A et C are real numbers, and B a complex number (B_0 its complex conjugate).

If one replaces the complex variables v and w by their real and imaginary parts, that is by real variables, one finds:

$$f(x,y,z,u) = A(x^2+y^2) + 2\,\mathrm{Re}(B)(zx+yu) + 2\,\mathrm{Im}(B)(xu-zy) + C(z^2+u^2),$$

of which the form (2) used to prove the theorem of four squares is indeed a prototype. In Hermite's terms:

> Considered with respect to the original variables x, y, z, u, these forms are entirely real, but their study, with respect to the transformations we have defined previously, essentially relies upon the use of complex numbers. One is then led to attribute to them a mode of existence singularly analogous to that of binary quadratic forms, although they essentially contain four indeterminates (Hermite [1854, p. 346]).

Moreover, if the linear transformations previously considered have a determinant of complex norm 1, they leave invariant the quantity $\Delta = BB_0 - AC$, which plays the role of the determinant for the form (3). Hermite then undertook a classification of these forms.

Two important features should be underlined here. The first one is the central role played by the linear transformations, as key tools for classifications: this is the restriction on these transformations that defined the new type of forms. In several memoirs around 1850, Hermite and some of his contemporaries, Cayley, Borchardt, Eisenstein, Otto Hesse, for instance, described several types of equivalence among forms, depending on the transformations which were taken into account: either they looked for transformations keeping a given form or function invariant, or, fixing a group of transformations, they studied the forms which are left invariant by them. The classification of forms with real coefficients (through linear transformations with real coefficients), which led to the notion of signature, belonged to the same programme. In 1855, in a study on the transformations of Abelian functions with 2 variables (more precisely of their periods), Hermite again introduced other "particular forms with 4 indeterminates, where one does not use as analytical tool the most general transformations among 4 variables, but particular transformations …which reproduce analogous forms" (Hermite [1855b, p. 785]). These forms and transformations would be one of the origins of the symplectic forms and group (Brouzet [2004]).

This point of view, changing the group of transformations operating on the variables in order to delineate which type of forms or functions will be studied, was at the time tightly linked to invariant theory (Parshall [1989]) and its applications were varied. It is in the context of Sturm's theorem on the number of roots of an algebraic equation that belong to a given domain that Hermite generalised his 1854 construction to quadratic forms with $2n$ "pairwise conjugate" indeterminates (again the index 0 designates the complex conjugation),

$$f(x_1, x_2, \ldots, x_n, x_{1,0}, \ldots, x_{n,0}) = \sum_{i,j} a_{i,j} x_i x_{j,0},$$

with $a_{i,j}$ and $a_{j,i}$ complex conjugates (thus $a_{i,i}$ real numbers) (Hermite [1855a, 1856]); he called them simply "quadratic forms with conjugate imaginary indeterminates."

The second feature worth stressing concerns the way mathematical objects are introduced. Hermite's "forms with conjugate imaginary indeterminates" were for him quadratic forms of a specific type, not a new type of objects defined in an ad hoc way, or by simple analogy, to accommodate complex numbers. They are distinguished *among* a larger family of well-known objects because of their special properties (here their stability under a certain group of transformations), as a new species could have been. In Hermite's view, coherent with that of mathematics as a natural science, mathematicians do not, should not, create their objects: they "meet them or discover them and study them, like physicists, chemists and zoologists" (Hermite and Stieltjes [1905, vol. 2, p. 398]). Hermite's key role in a history of Hermitian forms cannot be doubted, but its description is thus delicate. Besides the problems raised by the word "invention" in this particular context, another issue is directly connected to our main point, that of sameness: Alfred Clebsch in 1860 and 1863 (Clebsch [1860, 1863]), Elwin Bruno Christoffel a bit later (Christoffel [1864]) also introduced, in completely different contexts, mathematical objects close to Hermite's one: Clebsch studied square *matrices* with complex entries, such that entries symmetric with respect to the diagonal are complex conjugates (that is, now, Hermitian matrices), Christoffel considered *bilinear forms* on two sets of indeterminates, $\phi = \sum [gh] u_g v_h$ and their values when u_g and v_g are set to be complex conjugate numbers, under the assumption that the coefficients $[gh]$ and $[hg]$ are complex conjugates. Both referred to Hermite's 1855 article (only as an afterthought, in 1864, for Clebsch) for specific results, but none identified his own construction to Hermite's one nor gave any special name to it. These various points of view on Hermitian forms will be unified much later in the century.

The configuration about the appearance of Hermitian forms I have briefly sketched thus includes local incentives, a series of specific mathematical themes, a collection of objects and the disciplinary tools available to study them (here for instance the reduction theory of forms), the state of the art on certain topics (for instance complex numbers), but also here a new emphasis on linear transformations and a mathematical world-view with an impact on the practice of mathematics and how an entity is constructed and accepted. None of these components is proper to Hermite alone, even if their coordination may be, and they would require to be studied as collective phenomena. Some of these components will evolve separately and at different rates, losing their connections with the development of our forms "with conjugate imaginary indeterminates."

Tracing, on the other hand, the fate of this particular type of forms is not obvious. In 1866, for instance, Cayley discussed "Hermitian matrices" and associated forms, but they were those attached to the transformation of Abelian functions, not "our" forms and transformations (Cayley [1866]). The variety of names or of notations (for complex conjugation, in particular) also forbids us to use simple criteria or visual display.

A technique which has recently demonstrated its effectiveness for other topics (Goldstein [1999], Brechenmacher [2007], and Roque [2015]) is first to systematise the search

for relevant writings in review journals (such as the *Jahrbuch über die Fortschritte der Mathematik*), then to reconstruct their own networks of references in order to track down how Hermitian forms circulated after 1854. Thamous, an electronic tool developed by Alain Herreman for collaboratively constructing relational databases, was used here extensively to register cross-references and other links between articles and to organise the corpus. For reasons of space, and with a view to the long-term question, I shall only briefly report on some of the configurations in which our forms occur before the First World War.

4 Picard's and Bianchi's groups

Both Émile Picard and Luigi Bianchi were born in 1856, at more or less the same time as Hermitian forms. And both would be instrumental in bringing them back center-stage, this time with a group-theoretical and geometrical apparatus. Although some of their results are very close (even leading to some tensions), their backgrounds are quite different. Picard is best known for his work on complex analysis and its application to algebraic surfaces (Houzel [1991, p. 245]) and, more generally, for extending to dimension 2 a number of results first established in dimension 1. At the beginning of the 1880s, for instance, he proved, in parallel with the 1-dimensional case, that of the elliptic curve, that surfaces which can be parametrised by Abelian functions have (under some restrictions on their singularities) a geometric genus $p_g \leq 1$ (Picard [1881]).

But at the very same time and in close proximity, a great enterprise was underway; Henri Poincaré had just developed his theory of "Fuchsian" and "Kleinian" functions (now both considered particular cases of automorphic functions), that is, meromorphic functions of one complex variable z inside a certain disk such that

$$f\left(\frac{az+b}{cz+d}\right) = f(z), \tag{4}$$

where the Moebius transformations $z \to \frac{az+b}{cz+d}$ belong to a discontinuous group of invertible transformations; the groups and the functions were "Fuchsian" for Poincaré when the coefficients a, b, c, d are real. Poincaré had interpreted the Moebius transformations on the unit disk (or equivalently on the upper half plane) as isometries in a non-Euclidean, hyperbolic, geometry and he had constructed fundamental domains for the Fuchsian groups. He had also shown, given a Fuchsian group, how to construct Fuchsian functions invariant under the group and how to connect them to the solution of differential equations (Gray [2000]).

Picard was at first in search of a first, analogous, example with two variables. For this he introduced the family of curves of equation $z^3 = t(t-1)(t-x)(t-y)$, with x and y two real parameters, and he studied the periods of $\int z^{-1}dt$ (Picard [1882a]). These periods, as functions of x and y, satisfy a system of partial differential equations, which admits a basis of three independent solutions A_1, A_2, A_3. The functions $u = \frac{A_2}{A_1}$ and $v = \frac{A_3}{A_1}$ of the two variables x and y can be inverted, providing Picard with what he was looking for, two uniform functions of the two variables u, v, defined on the

domain $2\,\mathrm{Re}(v) + \mathrm{Re}\,u^2 + \mathrm{Im}(u)^2 < 0$. He also computed a group of linear transformations under which his functions would remain invariant. But, as he explained later, in order to obtain more than this one isolated example, "the thought of a recourse to ternary quadratic forms with conjugate indeterminates showed [him] the way out" (Picard [1889, p. 36]).

A ternary form with conjugate indeterminates, analogous to (3),

$$f(x, y, z, x_0, y_0, z_0) = axx_0 + a'yy_0 + a''zz_0 + byz_0 + b_0y_0z + b'zx_0 + b_0'z_0x + b''xy_0 + b_0''x_0y \tag{5}$$

with real coefficients a, a', a'', and the index 0 designating the complex conjugation, as for Hermite, becomes becomes either $\pm(xx_0 + yy_0 + zz_0)$ or $\pm(xx_0 + yy_0 - zz_0)$ by an adequate linear transformation of the indeterminates. The first case corresponds to definite forms, the second to indefinite forms. In the 4-dimensional (real) space defined by the two complex variables $u = \frac{x}{z}, v = \frac{y}{z}$, the equation $f = 0$ represents a 3-dimensional (real) hypersurface (for instance, the form $xx_0 + yy_0 - zz_0$ corresponds to the hypersphere $uu_0 + vv_0 = 1$). Picard then restricted the coefficients of the form to be "complex integers" [7] and studied the linear transformations leaving the form invariant or, more precisely, their non-homogeneous versions operating on u and v. If the form is definite, the group that one obtains is finite, but Picard's hope was to use the groups obtained in the indefinite case as analogues of Fuchsian groups. He had first to justify their existence and main properties (Picard [1882b]).

This was done by showing how the interior of the hypersurface $f = 0$ can be cut into a tessellation of polyhedra, transformed one from another by an infinite discontinuous group. The procedure corresponded geometrically to Hermite's so-called continuous reduction process for quadratic forms: Picard associated to the indefinite form a 2-parameter family of definite forms, for which a well-known theory of reduction existed and could be expressed as in the classical binary case by conditions of inequalities on the coefficients of the forms. To each choice of the parameters, that is, to each definite form in the family, Picard could associate in a 4-dimensional space a point, which belongs to a specific domain; this domain is a fundamental polyhedron defined by the inequalities of the reduction theory, exactly if the form is reduced. By changing the parameters continuously, the associated definite form may cease to be reduced, but, as in the binary classical case, can be transformed into an equivalent reduced form by a linear transformation, and correspondingly the point associated to it is transformed into a point in the fundamental polyhedron. The initial ternary indefinite form thus gave rise to an infinite discontinuous group of transformations (that Picard called hyperfuchsian), associated to the tesselation of the hypersurface. In later papers, Picard constructed *hyperfuchsian functions* defined in the interior of the hypersurface and invariant by such a group and showed that hyperfuchsian functions corresponding to the same group could be expressed as rational functions of three of them, linked by an algebraic relation. Picard also developed an analogous arithmetical study of *binary* forms with conjugate indeterminates, interpreting their reduction geometrically in terms of domains on the

[7] It means, for most of his work, Gaussian integers. Picard did in fact consider complex quadratic numbers in general, but his definition of "integers" here was not correct.

plane limited by arcs of circles and, this way, constructed afresh Fuchsian groups (Picard [1883, 1884, 1891]).

When he picked up the topic in 1890, Luigi Bianchi was coming from a quite different background, which included number theory, group theory and some geometry, but not the analytical connections dear to Picard. Bianchi had studied with Felix Klein in Göttingen during his European post-doctoral tour and the main reference in his papers on the arithmetic of forms is Klein's *Vorlesungen über die Theorie der elliptischen Modulfunctionen*, completed by Robert Fricke, which had just appeared in 1890. As Bianchi [1891, p. 313] explained it:

> The geometrical method, on which Professor Klein bases the arithmetical theory of the ordinary binary quadratic forms, may be applied with the same success on a larger scale. To prove this is the aim of the following development which will treat in the same way the theory of Dirichlet forms with integral complex coefficients and indeterminates and of Hermitian forms with integral complex coefficients and conjugate indeterminates.

Indeed, Bianchi had just studied the arithmetic of Dirichlet forms with Gaussian-integer coefficients, in order to complete Dirichlet's results on the number of classes of such forms. He then proceeded to complete some points in Picard's study of the *arithmetic* of Hermitian forms, launching both an extension to forms with coefficients in any quadratic field and a detailed examination of the associated groups of transformations and their subgroups of finite index (Brigaglia [2007]). In particular, Bianchi would handle forms whose coefficients are integers in quadratic fields $\mathbb{Q}\sqrt{-D}$ for $D = 1, 2, 3, 5, 6, 7, 10, 11, 13, 15, 19$, displaying a good knowledge of Richard Dedekind's theory of ideals. He also extended the main group of transformations to include those whose determinant is any unity and, explicitly following an idea of Fricke, those of the type $z \to \frac{az_0+b}{cz_0+d}$ (the index, as before, indicating the complex conjugation) and he computed the corresponding fundamental polyhedra. As announced, all along, and in particular in his synthesis published in *Mathematische Annalen* in German, Bianchi handled side by side the arithmetic of Dirichlet and of Hermitian forms and their geometrical interpretations, in particular the determination of their fundamental polyhedra. The emphasis moved from a configuration where complex analysis plays a key role to one centered on number theory and group theory; Hermitian forms are then number-theoretical objects parallel to those introduced by Dirichlet.

Picard's and Bianchi's results were integrated by Klein and Fricke in their 1897 *Vorlesungen über die Theorie der automorphen Funktionen*, but through Bianchi's point of view. They would then be taken up and developed through different methods by a number of mathematicians in the following decades, Onorato Nicoletti, Otto Bohler, Leonard Dickson, Georges Humbert, Gaston Julia, Hel Braun, Hua Luogeng, and many others.

5 A theorem and three authors

The question of the *finite* subgroups of the linear groups returned to the forefront as the theorem: "For any finite group of n-ary linear homogeneous transformations, there

exists an n-ary positive definite Hermitian form, say, $\sum a_{ik} x_i \bar{x_k}$, that the group leaves absolutely invariant."[8]

For once, the date of this theorem is rather precise: July 1896! The day, on the other hand, and the author, are another story, which illustrates the new role of mathematical societies and seminars and their interaction with journals for the circulation of mathematics, as well as the wider internationalization of mathematics at the end of the nineteenth century (Parshall [1995]). On Monday July 20, 1896, the statement appeared in a note by Alfred Loewy presented to the French Academy of Sciences by Picard for insertion in the *Comptes rendus* (Loewy [1896]). Following the rule of this journal to accept only very short communications, it consisted mainly of an announcement and contained no proofs. Two years earlier, Loewy had obtained his thesis under Ferdinand Lindemann, with a work on the transformation of a quadratic form into itself. In his 1896 note, Loewy first gave a condition for a linear transformation to fix an arbitrary bilinear form with conjugate indeterminates, $\sum a_{ik} x_i x_k^0$, with arbitrary coefficients (the complex conjugation is indicated here by the exponent "0"). When restricted to a "quadratic form of M. Hermite," in Loewy's terms, that is, when $a_{ik} = a_{ki}^0$, it says that the form can be transformed into itself only by transformations whose characteristic polynomials have simple elementary divisors and roots of modulus 1. Loewy then stated the above theorem, and used it in particular to complete a previous study by Picard of the finite groups of ternary linear transformations (Picard [1887]). For each such group, except one, Picard had displayed a quadratic form left invariant by the group, and Loewy provided an explicit invariant Hermitian form for the remaining case. Following immediately after the publication of his note, there was unleashed a flood of publications.

On August 9, Lazarus Fuchs communicated to the French Academy a note pointing out that what Loewy had published "*without proof*" (Fuchs's emphasis) was a special case of his own results presented at the Berlin Academy exactly one month earlier, on July 9 (Fuchs [1896]). Now Fuchs had, for quite some time, been studying differential equations of the type:

$$\frac{d^n\omega}{dz^n} + q_1 \frac{d^{n-1}\omega}{dz^{n-1}} + \cdots + q_n\omega = 0$$

where the coefficients q_i are uniform functions of the variable z, with a finite number of poles. Choosing a set of n linearly independent solutions in the neighbourhood of a singularity, the new solutions one obtained when the variable z describes a circuit around the singularity can be expressed as the image by a linear transformation of the original ones, and Fuchs, among others, had studied these monodromy transformations, in particular their "fundamental equations" (for us, the characteristic equations of the matrices associated to these monodromy transformations) (Gray [1984]). In his July presentation, Fuchs had stated that, under several assumptions (in particular, quite unnecessarily, that the roots of at least one fundamental equation should be distinct), there

[8]Note that for the authors we are discussing, a linear transformation of the group operates on the indeterminates x_i, the transformation obtained by conjugation of the coefficients operating on the $\bar{x_i}$. Again, the terminology and the viewpoint vary slightly according to the authors: the word "substitution" instead of "transformation" is still widely used (see the titles of the papers), "homogeneous" to indicate elements of $GL_n(\mathbb{C})$ is sometimes omitted, etc.

exists a linear combination of a fundamental system of solutions ω_i of the differential equation

$$\phi = A_1\omega_1\omega_1' + A_2\omega_2\omega_2' + \cdots + A_n\omega_n\omega_n'$$

($\omega_i{}'$ being here the conjugate function), with determined real coefficients A_i, which is unaltered by the group of monodromy. For algebraically integrable differential equations, the group is finite and Fuchs had also used his theorem to complete Picard's work on ternary forms.

Felix Klein took the opportunity of the annual meeting of the Deutsche Mathematiker-Vereinigung, from September 21 to September 26, in Frankfurt to present a one-page paper which added another author and another filiation to the theorem (Klein [1896]). First of all, Klein recalled his own 1875 work (Klein [1875]) where he had explained how to interpret a finite group of complex linear transformations on two variables as a group of real quaternary collineations of the ellipsoid, $x^2 + y^2 + z^2 - w^2 = 0$; and that this group necessarily fixes a point within the ellipsoid, thus providing a finite group of (real) rotations around a fixed point. This was the basis of his own classification, for the binary case. The ternary case he attributed not only to Picard, but also to Hermann Valentiner. Valentiner, who after a thesis on space curves had gone to work for a Danish insurance company, while still contributing to mathematics, had indeed published in 1889 a book on the classification of finite binary and ternary groups of linear transformations (including the now-called Valentiner group) (Valentiner [1889]). Then, after a nod to Loewy's note, Klein devoted the remainder of his presentation to another proof that Eliakim Hastings Moore, from Chicago, had communicated to him: For any Hermitian definite form, the sum of its transformations by the (finitely) many elements of the group is still a Hermitian definite form and it is fixed by the group. In the written version of his communication, Klein added that Moore had indeed spoken about his theorem at a mathematical meeting in Chicago on July 10 (with a written version published locally on July 24)![9] He also alluded to Fuchs's work without more details (given the past tensions between Fuchs and Klein, one may think that this vague recognition was not completely satisfactory to Fuchs (Gray [1984])).

Both Moore and Loewy published an extended version of their respective work in 1898, in the same issue of the *Mathematischen Annalen*, of which Klein was editor-in-chief. Both men analysed the literature, and in particular underlined Fuchs's superfluous condition to refute his claim to the theorem (Loewy emphasising that, above all, the definite character of the invariant form was never even alluded to by Fuchs). Still their viewpoints were quite different, as can already been inferred from the proofs themselves and their backgrounds. Moore attributed to "the analytic phrasing in terms of binary groups of Klein's invariant point" (in Klein's 1875 paper) his own discovery of the universal invariant Hermitian form (Moore [1898]). After he had proved the existence of this form, he proceeded to deduce from it the theorem that a n-linear homogeneous transformation of finite order p can be written with adequate new variables as the multiplication by p^{th}-roots of unity, a theorem for which he quoted no less than five other proofs. One

[9] Archives of the Math Club, Box 1, folder 2, p.66; see (Parshall and Rowe [1994, p. 399]) for the Chicago environment of this theorem and a slightly different datation. On the argument, see (Hawkins [2013, pp. 511–512]); it is remarkable to us that Klein felt it necessary to explain that such a procedure would not necessarily work if the group was infinite.

of these, included just after his own paper in *Mathematische Annalen*, was due to his Chicago colleague Heinrich Maschke (Maschke [1898]); during this period, Maschke also worked on the theory of groups of linear transformations and he had even delivered a survey lecture at the Chicago Mathematical Club in May 1897. He would soon use Moore's theorem to study quaternary groups of transformations (Parshall and Rowe [1994, pp. 396–401]), before extending these results to his now celebrated statement on the representation of finite groups (Hawkins [2013, p. 512]).

Loewy (who developed his work even further in his 1898 *Habilitation* at the Albert-Ludwigs-Universität in Freiburg) focussed not on the finite groups, but on bilinear forms (Loewy [1898]); his aim was the study of the transformation of a bilinear form with conjugate complex variables (with non-zero determinant) into itself, in direct continuation of the theme of his thesis. His framework and his main reference is Georg Frobenius's work, to which he borrowed in particular his symbolic methods (Frobenius [1878]). Such a symbolism, although not linked to a matricial setting, would allow him for instance to characterise a Hermitian form S by the equation $\bar{S}' = S$ (the bar now designates the complex conjugation, the prime the transpose). In the last part of his paper, Loewy also addressed the issue of a reduction theory for Hermitian forms in n variables. It provided him with information on the characteristic polynomial of the linear transformations fixing a given Hermitian form, thus generalising a theorem of Frobenius [1883]. Although their name was not yet set, the bilinear and the quadratic viewpoints have thus merged and Hermitian forms have become at the end of the nineteenth century a familiar object in the nascent area of linear algebra.

6 Hermitian forms as geometric objects

As seen in Picard's work, Hermitian forms have been connected to a geometric setting already in the 1880s. But it is in the framework defined by Corrado Segre who, in 1890, opened new vistas in complex geometry, that they will take a key place as geometric objects (Segre [1890a,b]). A starting point for Segre was Karl Von Staudt's project "to make the geometry of position an autonomous science, which does not need measure" (Staudt [1847]). Instead of using cross-ratios, as his predecessors had done, von Staudt based his geometry only on the concept of harmonicity (corresponding to the case where the cross-ratio is -1), which he defined by a purely geometric construction. Two geometric entities whose respective elements are put into correspondence are then said to be projectively related if the elements of the second entity corresponding to a harmonic set of elements of the first entity are also an harmonic set. A key concept was that of two entities in involution: for instance, two projectively related ranges of points on the same line are said to be in involution if the corresponding point of a point A' corresponding to a point A is the point A itself; in modern terms, it means of course that the transformation from the first range of points to the second is of order 2, but for von Staudt, to be projectively related or to be in involution were *relations* between geometric entities ; there were no transformations, as mathematical objects *per se*. For some involutory relations, there exists fixed (real) elements, for others none. For instance, any line through a point O inside a circle—let us remind that in this projective setting, a

"circle" is simply a projective line—cuts the circle in two points A and A' ; the "sheaf" of all these lines, in von Staudt's terminology, defines an involutory relation between all the points A of the circle and their corresponding points A'. This relation has no fixed point. On the other hand, if the point O is outside the circle, the analogous relation has two fixed points, that is, the points of contact with the circle of the two tangents passing through O. In order "to give to projective geometry the same generality as analysis," von Staudt found it necessary to include imaginary (that is, complex) points into geometry, but: "Where, everybody asks, is the imaginary point when one abstracts oneself from the system of coordinates?" (Staudt [1847]). His first solution was to associate to an involutory situation on a real projective line with no real fixed points *a pair of conjugate imaginary points*; a few years later, he refined it, distinguishing between the two conjugates by taking into account an additional direction, restricted himself to projective relations leaving the direction invariant and thus extended all his constructions and incidence relations to complex points (Nabonnand [2008]).

Despite the uncompromising austerity of von Staudt's treatise and the complete absence of figures, the reception of this approach was particularly good as a prerequisite for courses in technical drawings and in engineering schools, as advocated in particular by Carl Culman and Theodor Reye at Zürich's Polytechnikum (the future ETH) (citeScholz1989):

> But still the engineer, and everybody else who wishes to become familiar with his ideas, must continually exert his power of imagination in order actually to see the object intended to be represented by the lines of a drawing which is not at all intelligible to the uninitiated. […] One principal object of geometrical study appears to me to be the exercise and the development of the power of imagination in the student, and I believe that this object is best attained in the way in which Von Staudt proceeds,

wrote Reye at the beginning of his own introduction to projective geometry (Reye [1866]). The topic gained a particular popularity in Italy after the *Risorgimento*, where the teaching of geometry was favoured both for national and epistemological reasons, while engineering training responded to the needs of modernization (Bottazzini [1994]). From Luigi Cremona's *Elementi di geometria projettiva* and Antonio Favaro's *Lezioni di statica grafica* in 1873 to Achille Sannia's *Lezioni di Geometria Proiettiva* (1891) to Francesco Severi 's *Geometria projettiva* (1904), courses blossomed in Turin, Padua, Bologna, Rome or Palermo (Capecchi and Ruta [2014]). In Turin, Corrado Segre, besides his own lectures on projective geometry,[10] significantly organised the translation into Italian of both von Staudt's *Geometry of position* and Klein's Erlangen program, by Mario Pieri (1889) and Gino Fano (1890) respectively. A decisive change with respect to von Staudt's work is that, for Segre (as for Klein), a projectivity is a transformation (e.g. a collineation, a homography), a mathematical object, and not only a way of relating entities. In complex geometry, projectivities preserve harmonic relations, but the reciprocal is false, for instance the complex conjugation is not a projectivity.

[10]His lecture notes are available on the beautiful website devoted to him by Livia Giaccardi, `http://www.corradosegre.unito.it/I21_30.php` see in particular Quadreno 22.

While von Staudt had chosen to restrict the relations he considered, Segre introduced what he called anti-projectivities, as well as their 2- or 3-dimensional companions (anticollineations, antipolarities,…) and studied projectivities and anti-projectivities alike. Analytically, a projectivity in dimension 1 corresponds to a homography

$$z \to z' = \frac{\alpha z + \beta}{\gamma z + \delta}$$

while an anti-projectivity corresponds to :

$$z \to z' = \frac{\alpha \bar{z} + \beta}{\gamma \bar{z} + \delta}$$

with $\alpha, \beta, \gamma, \delta$ complex numbers such that $\alpha\delta - \beta\gamma \neq 0$ (the bar indicates the complex conjugation).

It is then easy to see that, if one puts $z = \frac{x}{y}$, the equation giving the fixed points of such an antiprojectivity is of the type:

$$F(x, y) = ax\bar{x} + bx\bar{y} + \bar{b}\bar{x}y + cy\bar{y} = 0$$

Similarly, antipolarities are associated to forms with 4 variables. One recognises the forms that, in Segre's words, "have also already been introduced in number theory thanks to M. Hermite, M. Picard and others," that is, again, Hermitian forms (Brigaglia [2016, p. 275]). Segre explained in a letter to Adolf Hurwitz in June 1894: "I study there, among the other hyperalgebraic entities, those that I call hyperconic, hyperquadric [i.e. the locus of $F(x_1, x_2, x_3, x_4) = 0$], etc. which are analytically represented by the equations of Hermite; and thus they give the geometric equivalent of the forms of Hermite" (**Brigaglia 2016**).

According to Klein's Erlangen program, the group of projective and antiprojective transformations which fix, say, an hyperquadric, defines a geometry, and it is also possible to apply Cayley's idea, as extended by Klein, of using a (hyper)quadric to define a metric on this projective complex space. This was done almost simultaneously and independently by both Guido Fubini and Eduard Study. Fubini had been a student of Ulisse Dini and Luigi Bianchi at Pisa and Bianchi included results from Fubini's thesis on Clifford parallelism in his celebrated 1902 book *Lezioni di geometria differenziale*. A year later, Fubini, then in Catania, turned to Hermitian forms (Accademia delle Scienze [1982]). Starting from the interpretation of the group of linear transformations leaving invariant a quadratic form as a group of space motions, he was seeking an analogous construction for Hermitian forms. Fubini considered an Hermitian form algebraically equivalent to $x_1x_1^0 + x_2x_2^0 + \cdots + x_{n-1}x_{n-1}^0 - x_nx_n^0$ (here again the exponent 0 indicates the complex conjugation) and the associated group transforming a "hypervariety" of the type $\sum_1^{n-1} u_i u_i^0 - 1 = 0$ into itself. For two points, u and $\bar{u}$ (the bar here does *not* designate the complex conjugation), he introduced the quantity

$$R_{u\bar{u}} = \frac{\left(\sum_1^{n-1} u_i \bar{u}_1^0 - 1\right)\left(\sum_1^{n-1} u_i^0 \bar{u}_1 - 1\right)}{\left(\sum_1^{n-1} u_i u_1^0 - 1\right)\left(\sum_1^{n-1} \bar{u}_i^0 \bar{u}_1 - 1\right)} - 1. \tag{6}$$

This is invariant under the group of transformations, real and equal to 0 within the hypervariety only when the two points coincide; it was thus legitimate to call it a (pseudo)-distance between the two points. Fubini [1903] used it primarily with arithmetical and analytical applications in view, but he then also refined his construction in order to interpret the Hermitian form as a metric for a complex space (Fubini [1904]).

While Fubini refered mostly to Picard, Poincaré and Klein–Fricke, Study's idea was to develop this study within the framework defined by Segre. Study had worked before on invariant theory and quaternions and had just published in 1903 a book on *Geometrie der Dynamen*, using biquaternions and geometrical tools to study mechanical forces (Hartwich [2005]). He had an extensive program about the complex realm and in an article published in *Mathematische Annalen* (Study [1905]), he also defined Hermitian metrics and distances.

From a ternary indefinite Hermitian form, $(x\overline{x}) = x_1\overline{x_1} - x_2\overline{x_2} - x_3\overline{x_3}$ for instance (here the bar does designate the complex conjugation), Study, following Segre's concepts if not his terminology, defined a "Hermitian point-complex" by $(x\overline{x}) = 0$ (this is Segre's *iperconica*) and the inside of the point-complex by the condition $(x\overline{x}) > 0$ (by the very definition of a Hermitian form, its values are real). He was then able to define a hyperbolic Hermitian metric and the (real) distance of two points inside the point-complex. Under an adequate normalization, the distance between two points x and y is

$$(x, y) = 2\cosh^{-1}\frac{\sqrt{(x\overline{y})(\overline{x}y)}}{\sqrt{(x\overline{x})}\sqrt{(y\overline{y})}},$$

Study showing then that the distance between two points is the length of the geodesics linking them. He also developed the case of an elliptic Hermitian metric, based this time on a definite Hermitian form. This setting would then be developed by Julian Coolidge, Wilhelm Blaschke and of course Erich Kähler, Jan Schouten and Élie Cartan in the 1920s and 1930s.

7 Open questions: back to long-term histories

As we have seen, while in the 1850s, what will be called Hermitian forms were a subcategory of quadratic forms, they became later a category *parallel* to that of quadratic forms—and still later the encompassing category of those quadratic forms which are a particular case (those with complex conjugation reducing to identity). Alternatively, it may be tempting to summarise the history of Hermitian forms, of which we have just sketched some key episodes, as a thread from number-theoretical objects to algebraic ones to geometric ones. However, one would obviously lose several components we have met — be they pedagogical settings, or epistemological views, putting observations and classifications together at the core of mathematical activities, or disciplinary structures. In the configuration which allows Hermitian forms to appear as geometric objects, there appears the renewal of Italian mathematics after the unification of the country, a new program for a geometry as general as analysis, a concern for the representation of complex numbers by real objects, and an emphasis on transformations. If

close reading is required to display how these components were articulated, the question is still open of their extent, both in time and in scope, and of their precise role in the dynamics of the changes.

The difficulty is hardly reduced if one focuses on components which are obviously relevant, for instance here, the issue of geometrisations: several types of geometries occurred (projective, non Euclidean, differential) and they played several roles, from supporting intuition to providing a concrete existence to groups. As Fubini (Fubini [1904, p. 2]) explained, for instance, "the introduction of [his] new metric is something totally different from a simply formal thing. It allows us to have recourse to the intuition and the procedures of geometry to solve an algebraic question". Moreover, geometry did not go unchallenged: Valentiner remarked at the beginning of his book on finite groups of transformations that while Klein had already handled the binary case, he, Valentiner, was going to do it again, but this time purely algebraically, in order to free it from geometrical considerations (Valentiner [1889, p. 5]). The relation between geometry and classification programmes was established in Segre's case, according to his former student and colleague Alessandro (Terracini [1926]); it was however still an open question whether to connect it or not with more general views on mathematics as a science, as was the case for Hermite or Cayley (Lê and Paumier [2016]).

Besides these questions about specific components, an important range of open questions concerns the links which may be used to relate the various episodes and components, in particular those which can help us to understand how information or results were circulating. An obvious type of link, which has been used to delineate preceding configurations, are cross-references among articles. They indicate in particular the role played by Picard in the 1880s to put Hermitian forms back into the centre of the stage. One might also think of Cayley's definition of a metric in a projective space, linked to his work on invariants in the 1860s, and which will be, in a form mainly mediated by Klein, a model for the formulas used in the complex case at the end of the century.

Conflicts of priority often gave rise to such cross-references, but they were, at least in our case, particularly ephemeral. Issues were very soon settled in the literature: there will be a "Fubini–Study" metric, the theorem on the invariance of a Hermitian form by a finite group of linear transformations will be attributed to all the authors we have discussed above (Meyer [1899, p. 341], Miller, Blichfeldt, and Dickson [1916, p. 209]). Yet locating and studying such configurations remains of interest, as they help us reveal more decisive aspects: such as, for instance, the confluence of different approaches set in motion to describe the finite subgroups of $GL_n(\mathbb{C})$, or the problems raised by the extension of geometry to the complex realm, or the role of Klein as a intermediary between Göttingen, Italy and the United States (Parshall and Rowe [1994] and Casnati, Conte, Gatto, Giacardi, Marchisio, and Verra [2016]). That persons from different branches of mathematics could agree was even interpreted by some as a sign of modernity. Francesco Brioschi commented for instance in 1889: "The characteristic note of modern progress in mathematical studies can be recognised in the contribution that each special theory—that of functions, of substitutions, of forms, of transcendents, geometrical theories and so on—brings to the study of problems, where in other times only one seemed necessary" (Archibald [2011, p. 375]).

However, transmissions may also operate without explicit bibliographical indications, through correspondence or, as we have seen, through communications at the meetings of a mathematical society or via publication in a given journal (the role of *Mathematische Annalen* is here particularly interesting). Certain links functioned on a much larger scale and to pinpoint them concretely is more delicate: how a new object in mathematics is validated or how the new ideas of linear algebra irrigated a variety of topics during the second half of the nineteenth century are but two examples.

Further, breaks or discontinuities should also be studied more closely. This is particularly important because most theories in the dynamics of history of science rely on the issue of breaks (from Thomas Kuhn's scientific revolutions to Imre Lakatos's research programmes) and do not seem to be appropriate in the case of mathematics. It may happen that different aspects change simultaneously and there have been some attempts to speak in this case of a "revolution" (Gillies [1992]). Very often however the changes are not simultaneous (Gilain and Guilbaud [2015]). While indifference is of course the most obvious cause of decline, rediscovery or recycling into another theory are quite frequent in modern times and the circumstances of such a rebirth are often puzzling. We have only a few studies of such phenomena: that of invariant theory, dead (Fisher [1966]), but then born again, "like an Arabian phoenix arising from its ashes" (Rota [2001]). Or that of the "theory of order" that Louis Poinsot, inspired by the relations among roots of unity, promoted at the beginning of the century (Boucard [2011]), and which disappeared and reappeared several times (still attached to Poinsot's name), in the theory of equations, but also in ornamental architecture (Boucard and Eckes [2015]), intersecting the long-term changes in the theory of tactics, a field which mixed what we would now describe as combinatorics and group theory (Ehrhardt [2015])). In examples ranging from Descartes's curves to Fourier series, Alain Herreman [2013] analysed how concepts which had already appeared informally in mathematics receive, in what he calls "inaugural texts," a formal definition and a name, thus acquiring, so to speak, mathematical citizenship; but such a general study of a specific type of discontinuity, operating here in particular at a semiotic level, remains exceptional.

It may sometimes be necessary to change one's scale of observation in order to understand a discontinuity. A striking testimony is given by Max Born's 1914 study of the diamond (Born [1914]). Born used a determinant with complex elements, such that two elements symmetrical with respect to the main diagonal are complex conjugates, commented that the determinant is "real despite its complex appearance" and acknowledged in a footnote that it thus derived from a Hermitian form. As well-known, Born would remember this again a decade later, offering to Hermitian forms and matrices a spectacular role in the new quantum theory (Born and Jordan [1925]). In the 1850s, the program of classification and reduction of equations and forms had generated various concepts: characteristic polynomials and their roots, which we have met several times, linear transformations or matrices (Hawkins [1977, 1987], Brechenmacher [2010], and Hawkins [2013]). That the "secular equation" which describes the long-term perturbations of planetary motion is our characteristic polynomial for a symmetric matrix and has only real roots had indeed interesting physical applications; it was an incentive to work on the complex case for Clebsch [1860] and Christoffel [1864], whom I have mentioned above. Secular equations again appeared in a physical environment in Born's paper, but

without explicit link to these previous works. The connections between Born and Jacob Rosanes, who studied the transformation of quadratic forms in the 1870s and 1880s, are on the other hand well attested (Mehra and Rechenberg [1982, p. 119]) and Born also quoted more recent algebraic papers: this suggests a wider, less specific, relevant algebraic environment in which matrices and forms were now available independently of the narrow topic of the secular equation or of its other avatars.

In general history, following Fernand Braudel's suggestion to take into account not only the short-term events of political life, but also phenomena deployed over a more extended period of time, be they economic trends or even geological transformations, the "long term" has come to mean a time of immobility with respect to human actions. This is not the case in mathematics. A long term thread, based on a simple retrospective identification of concepts, or results, would be a snare. There is no easy line from Hermite to Picard to Moore and Study to Born. Long term histories thus require us to study not only local configurations but also the various ways in which they are, or not, connected. The links we have mentioned, whatever be their duration in time or the scale at which they operate, are constructed through the "art" of mathematicians; and this may be true also of the discontinuities. Insofar as they involve the work of mathematicians, they are—or should be—a primary concern for the history of mathematics.

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DYNAMICAL, SYMPLECTIC AND STOCHASTIC PERSPECTIVES ON GRADIENT-BASED OPTIMIZATION

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Abstract

Our topic is the relationship between dynamical systems and optimization. This is a venerable, vast area in mathematics, counting among its many historical threads the study of gradient flow and the variational perspective on mechanics. We aim to build some new connections in this general area, studying aspects of gradient-based optimization from a continuous-time, variational point of view. We go beyond classical gradient flow to focus on second-order dynamics, aiming to show the relevance of such dynamics to optimization algorithms that not only converge, but converge quickly.

Although our focus is theoretical, it is important to motivate the work by considering the applied context from which it has emerged. Modern statistical data analysis often involves very large data sets and very large parameter spaces, so that computational efficiency is of paramount importance in practical applications. In such settings, the notion of efficiency is more stringent than that of classical computational complexity theory, where the distinction between polynomial complexity and exponential complexity has been a useful focus. In large-scale data analysis, algorithms need to be not merely polynomial, but linear, or nearly linear, in relevant problem parameters. Optimization theory has provided both practical and theoretical support for this endeavor. It has supplied computationally-efficient algorithms, as well as analysis tools that allow rates of convergence to be determined as explicit functions of problem parameters. The dictum of efficiency has led to a focus on algorithms that are based principally on gradients of objective functions, or on estimates of gradients, given that Hessians incur quadratic or cubic complexity in the dimension of the configuration space (Bottou [2010] and Nesterov [2012a]).

More broadly, the blending of inferential and computational ideas is one of the major intellectual trends of the current century—one currently referred to by terms such as "data science" and "machine learning." It is a trend that inspires the search for new mathematical concepts that allow computational and inferential desiderata to be studied jointly. For example, one would like to impose runtime budgets on data-analysis algorithms as a function of statistical quantities such as risk, the number of data points and the model complexity, while also taking into account computational resource constraints such as number of processors, the communication bandwidth and the degree

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of asynchrony. Fundamental understanding of such tradeoffs seems likely to emerge through the development of lower bounds—by establishing notions of "best" one can strip away inessentials and reveal essential relationships. Here too optimization theory has been important. In a seminal line of research beginning in the 1970's, Nemirovskii, Nesterov and others developed a complexity theory of optimization, establishing lower bounds on rates of convergence and discovering algorithms that achieve those lower bounds (Nemirovskii and Yudin [1983] and Nesterov [1998]). Moreover, the model of complexity was a relative one—an "oracle" is specified and algorithms can only use information that is available to the oracle. For example, it is possible to consider oracles that have access only to function values and gradients. Thus the dictum of practical computational efficiency can be imposed in a natural way in the theory.

Our focus is the class of optimization algorithms known as "accelerated algorithms" Nesterov [1998]. These algorithms often attain the oracle lower-bound rates, although it is something of a mystery why they do so. We will argue that some of mystery is due to the historical focus in optimization on discrete-time algorithms and analyses. In optimization, the distinction between "continuous optimization" and "discrete optimization" refers to the configuration ("spatial") variables. By way of contrast, our discussion will focus on continuous time. In continuous time we can give acceleration a mathematical meaning as a differential concept, as a change of speed along a curve. And we can pose the question of "what is the fastest rate?" as a problem in variational analysis; in essence treating the problem of finding the "optimal way to optimize" for a given oracle itself as a formal problem of optimization. Such a variational perspective also has the advantage of being generative—we can derive algorithms that achieve fast rates rather than requiring an analysis to establish a fast rate for a specific algorithm that is derived in an adhoc manner.

Working in continuous time forces us to face the problem of discretizing a continuous-time dynamical system, so as to derive an algorithm that can be implemented on a digital computer. Interestingly, we will find that symplectic integrators, which are widely used for integrating dynamics obtained from variational or Hamiltonian perspectives, are relevant in the optimization setting. Symplectic integration preserves the continuous symmetries of the underlying dynamical system, and this stabilizes the dynamics, allowing step sizes to be larger. Thus algorithms obtained from symplectic integration can move more quickly through a configuration space; this gives a geometric meaning to "acceleration."

It is also of interest to consider continuous-time stochastic dynamics that are in some sense "accelerated." The simplest form of gradient-based stochastic differential equation is the Langevin diffusion. The particular variant that has been studied in the literature is an *overdamped* diffusion that is an analog of gradient descent. We will see that by considering instead an *underdamped* Langevin diffusion, we will obtain a method that is more akin to accelerated gradient descent, and which in fact provably yields a faster rate than the overdamped diffusion.

The presentation here is based on joint work with co-authors Andre Wibisono, Ashia Wilson, Michael Betancourt, Chi Jin, Praneeth Netrapalli, Rong Ge, Sham Kakade, Niladri Chatterji and Xiang Cheng, as well as other co-authors who will be acknowledged in specific sections.

1 Lagrangian and Hamiltonian Formulations of Accelerated Gradient Descent

Given a continuously differentiable function f on an open Euclidean domain $\mathcal{X}$, and given an initial point $\mathbf{x}_0 \in \mathcal{X}$, gradient descent is defined as the following discrete dynamical system:

$$\mathbf{x}_{k+1} = \mathbf{x}_k - \eta \nabla f(\mathbf{x}_k), \tag{1}$$

where $\eta > 0$ is a *step size* parameter.

When f is convex, it is known that gradient descent converges to the global optimum $\mathbf{x}^\star$, assumed unique for simplicity, at a rate of $O(1/k)$ Nesterov [ibid.]. This means that after k iterations, the function value $f(\mathbf{x}_k)$ is guaranteed to be within a constant times $1/k$ of the optimum value $f^\star = f(\mathbf{x}^\star)$. This is a worst-case rate, meaning that gradient descent converges as least as fast as $O(1/k)$ across the function class of convex functions. The constant hidden in the $O(\cdot)$ notation is an explicit function of a complexity measure such as a Lipschitz constant for the gradient.

In the 1980's, a complexity theory of optimization was developed in which rates such as $O(1/k)$ could be compared to lower bounds for particular problem classes (Nemirovskii and Yudin [1983]). For example, an oracle model appropriate for gradient-based optimization might consider all algorithms that have access to sequences of gradients of a function, and whose iterates must lie in the linear span of the current gradient and all previous gradients. This model encompasses, for example, gradient descent, but other algorithms are allowed as well. Nemirovskii and Yudin [ibid.] were able to prove, by construction of a worst-case function, that no algorithm in this class can converge at a rate faster than $O(1/k^2)$. This lower bound is better than gradient descent, and it holds open the promise that some gradient-based algorithm can beat gradient descent across the family of convex functions. That promise was realized by Nesterov [1983], who presented the following algorithm, known as *accelerated gradient descent*:

$$\begin{aligned} \mathbf{y}_{k+1} &= \mathbf{x}_k - \eta \nabla f(\mathbf{x}_k) \\ \mathbf{x}_{k+1} &= (1+\lambda_k)\mathbf{y}_{k+1} - \lambda_k \mathbf{y}_k, \end{aligned} \tag{2}$$

and proved that the algorithm converges at rate $O(1/k^2)$ for convex functions f. Here λ_k is an explicit function of the other problem parameters. We see that the acceleration involves two successive gradients, and the resulting dynamics are richer than those of gradient descent. In particular, accelerated gradient descent is not a descent algorithm—the function values can oscillate.

Nesterov's basic algorithm can be presented in other ways; in particular, we will also use a three-sequence version:

$$\begin{aligned} \mathbf{x}_k &= y_k + \lambda_k \mathbf{v}_k \\ \mathbf{y}_{k+1} &= \mathbf{x}_k - \eta \nabla f(\mathbf{x}_k) \\ \mathbf{v}_{k+1} &= \mathbf{y}_{k+1} - y_k. \end{aligned} \tag{3}$$

We note in passing that in both this version and the two-sequence version, the parameter λ_k is time-varying for some problem formulations and constant for others.

After Nesterov's seminal paper in 1983, the subsequent three decades have seen the development of a variety of accelerated algorithms in a wide variety of other problem settings. These include mirror descent, composite objective functions, non-Euclidean geometries, stochastic variants and higher-order gradient descent. Rates of convergence have been obtained for these algorithms, and these rates often achieve oracle lower bounds. Overall, acceleration has been one of the most productive ideas in modern optimization theory. See Nesterov [1998] for a basic introduction, and Bubeck, Y. T. Lee, and M. Singh [2015] and Allen-Zhu and Orecchia [2014] for examples of recent progress.

And yet the basic acceleration phenomenon has remained somewhat of a mystery. Its derivation and its analysis are often obtained only after lengthy algebraic manipulations, with clever but somewhat opaque upper bounds needed at critical junctures.

In Wibisono, Wilson, and Jordan [2016], we argue that this mystery is due in part to the discrete-time formalism that is generally used to derive and study gradient-based optimization algorithms. Indeed, the notion of "acceleration" seems ill-defined in a discrete-time framework; what does it mean to move more quickly along a sequence of discrete points? Such a notion seems to require an embedding in an underlying flow of time, such that acceleration can be viewed as a diffeomorphism. Moreover, if accelerated optimization algorithms are in some sense optimal, there must be something special about the curve that they follow in the configuration space, not merely the speed at which they move. Such a separate characterization of curve and speed also seem to require continuous time.

Wibisono, Wilson, and Jordan [ibid.] address these issues via a variational framework that aims to capture the phenomenon of acceleration in some generality. We review this framework in the remainder of this section, discussing the Lagrangian formulation that captures acceleration in continuous time, showing how this formulation gives rise to a family of differential equations whose convergence rates are the continuous-time counterpart of the discrete-time oracle rates. We highlight the problem of the numerical integration of these differential equations, setting up the symplectic integration approach that we discuss in Section 2.

We consider the general non-Euclidean setting in which the space $\mathcal{X}$ is endowed with a distance-generating function $h : \mathcal{X} \to \mathbb{R}$ that is convex and essentially smooth (i.e., h is continuously differentiable in $\mathcal{X}$, and $\|\nabla h(\mathbf{x})\|_* \to \infty$ as $\|\mathbf{x}\| \to \infty$). The

function h can be used to define a measure of distance in $\mathcal{X}$ via its *Bregman divergence*:

$$D_h(\mathbf{y}, \mathbf{x}) = h(\mathbf{y}) - h(\mathbf{x}) - \langle \nabla h(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle. \tag{4}$$

The Euclidean setting is obtained when $h(\mathbf{x}) = \frac{1}{2}\|\mathbf{x}\|^2$.

We use the Bregman divergence to construct a *Bregman kinetic energy* for a dynamical system. We do this by taking the Bregman divergence between a point $\mathbf{x}$ and its translation in the direction of the velocity $\mathbf{v}$ by a time-varying magnitude, $e^{-\alpha_t}$:

$$K(\mathbf{x}, \mathbf{v}, t) := D_h(\mathbf{x} + e^{-\alpha_t}\mathbf{v}, \mathbf{x}). \tag{5}$$

Using the definition in Eq. (4), we see that this kinetic energy can be interpreted as comparison between the amount that h changes under a finite translation, $h(\mathbf{x}+e^{-\alpha_t}\mathbf{v}) - h(\mathbf{x})$, versus an infinitesimal translation, $e^{-\alpha_t}\langle \nabla h(\mathbf{x}), \mathbf{v}\rangle$.

We now define a time-dependent potential energy, $U(\mathbf{x})$:

$$U(\mathbf{x}, t) := e^{\beta_t} f(\mathbf{x}), \tag{6}$$

and we subtract the potential energy from the kinetic energy to obtain the *Bregman Lagrangian*:

$$\begin{aligned} \mathfrak{L}(\mathbf{x}, \mathbf{v}, t) =:= e^{\alpha_t + \gamma_t}(K(\mathbf{x}, \mathbf{v}, t) - U(\mathbf{x}, t)) \\ = e^{\alpha_t + \gamma_t}(D_h(\mathbf{x} + e^{-\alpha_t}\mathbf{v}, \mathbf{x}) - e^{\beta_t} f(\mathbf{x})). \end{aligned} \tag{7}$$

In this equation, the time-dependent factors α_t, β_t and γ_t are algorithmic degrees of freedom that allow the Bregman–Lagrangian framework to encompass a range of different algorithms.

Although α_t, β_t and γ_t can be set independently in principle, we define a set of *ideal scaling conditions* that reduce these three degrees of freedom to a single functional degree of freedom:

$$\begin{aligned} \dot{\beta}_t &\leq e^{\alpha_t} \\ \dot{\gamma}_t &= e^{\alpha_t}. \end{aligned} \tag{8}$$

Wibisono, Wilson, and Jordan [ibid.] show that these conditions are needed to obtain differential equations whose rates of convergence are the optimal rates; see Theorem 1 below.

Given the Bregman Lagrangian, we use standard calculus of variations to obtain a differential equation whose solution is the path that optimizes the time-integrated Bregman Lagrangian. In particular, we form the Euler–Lagrange equations:

$$\frac{d}{dt}\left\{\frac{\partial \mathfrak{L}}{\partial \mathbf{v}}(\mathbf{x}_t, \dot{\mathbf{x}}_t, t)\right\} - \frac{\partial \mathfrak{L}}{\partial \mathbf{x}}(\mathbf{x}_t, \dot{\mathbf{x}}_t, t) = 0, \tag{9}$$

a computation which is easily done based on Eq. (7). Using the ideal scaling conditions in Eq. (8), the result simplifies to the following master differential equation:

$$\ddot{x}_t + (e^{\alpha_t} - \dot{\alpha}_t)\dot{x}_t + e^{2\alpha_t + \beta_t}\left[\nabla^2 h(x_t + e^{-\alpha_t}\dot{x}_t)\right]^{-1}\nabla f(x_t) = 0. \tag{10}$$

We see that the equation is second order and non-homogeneous. Moreover, the gradient $\nabla f(x_t)$ appears as a force, modified by geometric terms associated with the Bregman distance-generating function h. As we will discuss below, this equation is a general form of Nesterov acceleration in continuous time.

It is straightforward to obtain a convergence rate for the master differential equation. We define the following Lyapunov function:

$$\mathcal{E}_t = D_h(\mathbf{x}^\star, \mathbf{x}_t + e^{-\alpha_t}\dot{\mathbf{x}}_t) + e^{\beta_t}(f(\mathbf{x}_t) - f^\star)). \tag{11}$$

Taking a first derivative with respect to time, and asking that this derivative be less than or equal to zero, we immediately obtain a convergence rate, as documented in the following theorem, whose proof can be found in Wibisono, Wilson, and Jordan [2016].

Theorem 1. *If the ideal scaling in Eq. (8) holds, then solutions to the Euler–Lagrange equation Eq. (10) satisfy*

$$f(\mathbf{x}_t) - f^\star \leq O(e^{-\beta_t}).$$

For further explorations of Lyapunov-based analysis of accelerated gradient methods, see Wilson, Recht, and Jordan [2016].

Wibisono, Wilson, and Jordan [2016] studied a subfamily of Bregman Lagrangians with the following choice of parameters, indexed by a parameter $p > 0$:

$$\begin{aligned} \alpha_t &= \log p - \log t \\ \beta_t &= p \log t + \log C \\ \gamma_t &= p \log t, \end{aligned} \tag{12}$$

where $C > 0$ is a constant. This choice of parameters satisfies the ideal scaling condition in Eq. (8). The Euler–Lagrange equation, Eq. (10), reduces in this case to:

$$\ddot{x}_t + \frac{p+1}{t}\dot{x}_t + Cp^2 t^{p-2}\left[\nabla^2 h\left(x_t + \frac{t}{p}\dot{x}_t\right)\right]^{-1}\nabla f(x_t) = 0, \tag{13}$$

and, by Theorem 1, it has an $O(1/t^p)$ rate of convergence.

The case $p = 2$ of the Eq. (13) is the continuous-time limit of Nesterov's accelerated mirror descent (Krichene, Bayen, and Bartlett [2015]), the case $p = 3$ is the continuous-time limit of Nesterov's accelerated cubic-regularized Newton's method (Nesterov and Polyak [2006]). In the Euclidean case, when the Hessian $\nabla^2 h$ is the identity matrix, we recover the following differential equation:

$$\ddot{x}_t + \frac{3}{t}\dot{x}_t + \nabla f(x_t) = 0, \tag{14}$$

which was first derived by Su, Boyd, and Candes [2016] as the continuous-time limit of the basic Nesterov accelerated gradient descent algorithm in Eq. (2).

The Bregman Lagrangian has several mathematical properties that give significant insight into aspects of the acceleration phenomenon. For example, the Bregman Lagrangian is closed under time dilation. This means that if we take an Euler–Lagrange

curve of a Bregman Lagrangian and reparameterize time so that we travel the curve at a different speed, then the resulting curve is also the Euler–Lagrange curve of another Bregman Lagrangian, with appropriately modified parameters. Thus, the entire family of accelerated methods correspond to a single curve in spacetime and can be obtained by speeding up (or slowing down) any single curve. As suggested earlier, the Bregman Lagrangian framework permits us to separate out the consideration of the optimal curve from optimal speed of movement along that curve.

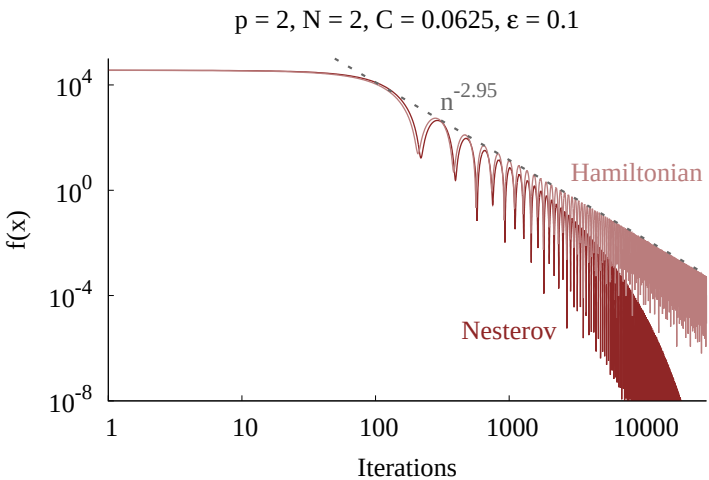
Finally, we turn to a core problem—how to discretize the master differential equation so that it can be solved numerically on a digital computer. Wibisono, Wilson, and Jordan [2016] showed that naïve discretizations can fail to yield stable discrete-time dynamical systems, or fail to preserve the fast oracle rates of the underlying continuous system. Motivated by the three-sequence form of Nesterov acceleration (see Eq. (3)), they derived the following algorithm, in the case of the logarithmic parameterization shown in Eq. (12):

$$\begin{aligned}
\mathbf{x}_{k+1} &= \frac{p}{k+p}\mathbf{z}_k + \frac{k}{k+p}\mathbf{y}_k \\
\mathbf{y}_k &= \operatorname*{argmin}_{\mathbf{y}\in\mathcal{X}} \left[f_{p-1}(\mathbf{y};\mathbf{x}_k) + \frac{N}{\epsilon^p\, p}||\mathbf{y}-\mathbf{x}_k||^p \right] \\
\mathbf{z}_k &= \operatorname*{argmin}_{\mathbf{z}\in\mathcal{X}} \left[C\, p\, k^{(p-1)} \langle \nabla f(\mathbf{y}_k), \mathbf{z}\rangle + \frac{1}{\epsilon^p} D_h(\mathbf{z}, \mathbf{z}_{k-1}) \right].
\end{aligned} \tag{15}$$

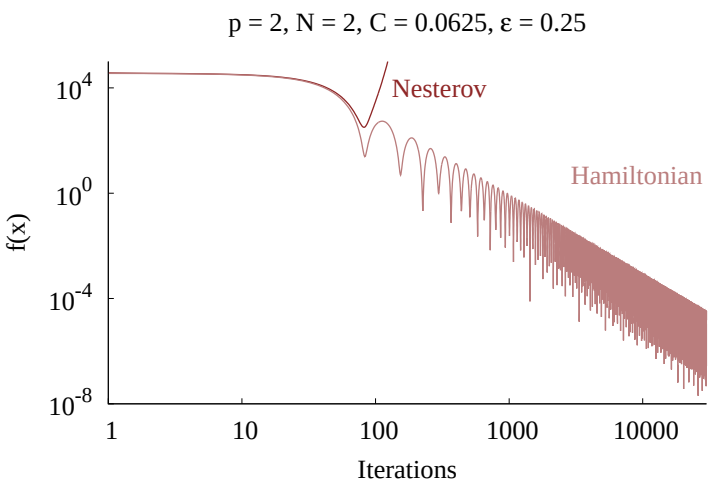
Here $f_{p-1}(\mathbf{y};\mathbf{x}_k)$ is the order-p Taylor expansion of the objective function around $\mathbf{x}_k$ and N and C are scaling coefficients and ϵ is a step size. Although Wibisono, Wilson, and Jordan [ibid.] were able to prove that this discretization is stable and achieves the oracle rate of $O(1/k^p)$, the discretization is heuristic and does not flow natural from the dynamical-systems framework. In the next section, we revisit the discretization issue from the point of view of symplectic integration.

2 A Symplectic Perspective on Acceleration

Symplectic integration is a general for the discretization of differential equations that preserves various of the continuous symmetries of the dynamical system (Hairer, Lubich, and Wanner [2006]). In the case of differential equations obtained from mechanics, these symmetries include physically-meaningful first integrals such as energy and momentum. Symplectic integrators exactly conserve these quantities even if the dynamical flow is only approximated. In addition to the appeal of this result from the point of view of physical conservation laws, the preservation of continuous symmetries means that symplectic integrators tend to be more stable than other integration schemes, such that it is possible to use larger step sizes in the discrete-time system. It is this latter fact that suggests a role for symplectic integrators in the integration of the differential equations associated with accelerated optimization methods. This idea has been pursued in recent work by Betancourt, Jordan, and Wilson [2018], whose results we review in this section.



(a)



(b)

Figure 1: (a) When appropriately tuned, both the leapfrog integrator and the three-sequence Nesterov algorithm simulate the same latent Bregman dynamics and hence achieve similar convergence rates, here approximately $\mathcal{O}(k^{-2.95})$. (b) Given a larger step size, the symplectic integrator remains stable and thus converges more quickly, whereas the three-sequence Nesterov algorithm becomes unstable.

While symplectic integrators can be obtained from a Lagrangian framework, they are most naturally obtained from a Hamiltonian framework. We thus begin by transforming the Bregman Lagrangian into a Bregman Hamiltonian. This is readily done via a Legendre transform, as detailed in Wibisono, Wilson, and Jordan [2016] and Betancourt, Jordan, and Wilson [2018]. The resulting Hamiltonian is as follows:

$$H(\mathbf{x}, \mathbf{r}, t) = e^{\alpha(t)+\gamma(t)} \left(D_{h^*}(e^{-\gamma(t)}\mathbf{r} + \frac{\partial h}{\partial \mathbf{x}}(\mathbf{r}), \frac{\partial h}{\partial \mathbf{x}}(\mathbf{x})) + e^{\beta(t)} f(\mathbf{x}) \right), \tag{16}$$

where

$$D_{h^*}(\mathbf{r}, \mathbf{s}) = h^*(\mathbf{r}) - h^*(\mathbf{s}) - \frac{\partial h^*}{\partial \mathbf{r}}(\mathbf{s}) \cdot (\mathbf{r} - \mathbf{s}),$$

and where h^* is the Fenchel conjugate:

$$h^*(r) \underset{\mathbf{v} \in T\mathcal{X}}{=} \sup \left(\langle \mathbf{r}, \mathbf{v} \rangle - h(v) \right).$$

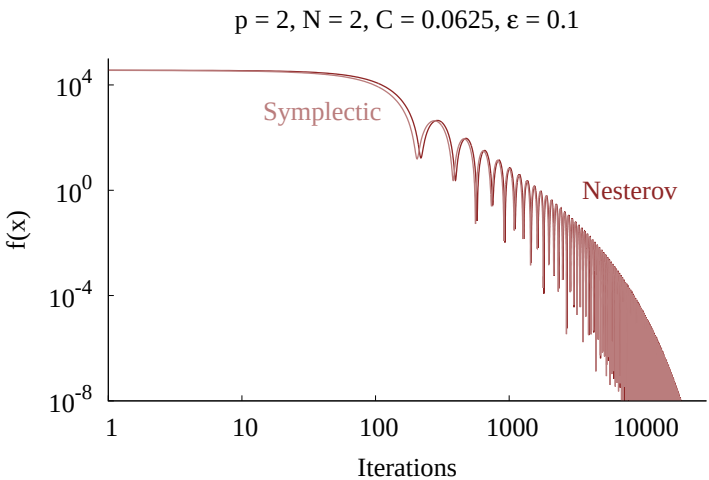
Given the Bregman Hamiltonian in Eq. (16), Betancourt, Jordan, and Wilson [ibid.] follow a standard sequence of steps to obtain a symplectic integrator. First, the Bregman Hamiltonian is time-varying, and it is thus lifted into a time-invariant Hamiltonian on an augmented configuration space that includes time as an explicit variable and includes a conjugate energy variable in the phase space. Second, the Hamiltonian is split into a set of component Hamiltonians, each of which can be solved analytically (or nearly so via simple numerical methods). Third, the component dynamics are composed symmetrically to form the full dynamics. In particular, Betancourt, Jordan, and Wilson [ibid.] illustrate how to form a symmetric leapfrog integrator (a particular kind of symplectic integrator) for the Bregman Hamiltonian. They prove that the error between this integrator and the true dynamics is of order $O(\epsilon^2)$, where ϵ is the step size in the discretization.

Betancourt, Jordan, and Wilson [ibid.] also present empirical results for a quadratic objective function, $f(x) = \langle \Sigma^{-1} x, x \rangle$, on a 50-dimensional Euclidean space, where

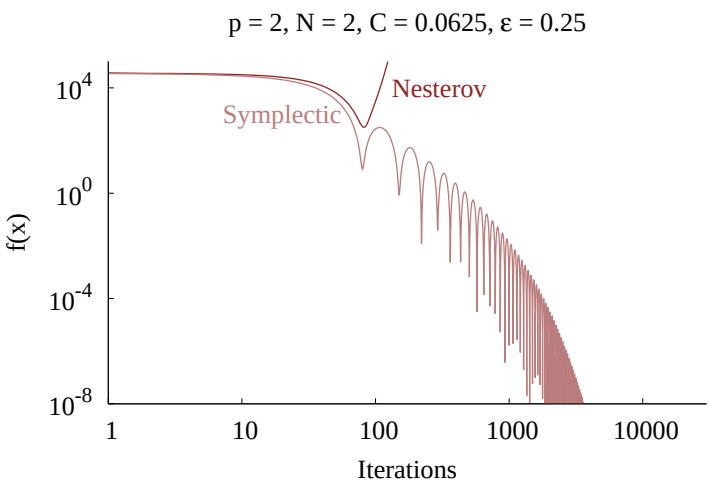
$$\Sigma_{ij} = \rho^{|i-j|},$$

and $\rho = 0.9$. This experiment was carried out in the setting of Eq. (12), for various choices of p, C and N. Representative results are shown in Figure 1(a), which compare the leapfrog integrator with the three-sequence version of Nesterov acceleration from Eq. (15). Here we see that both approaches yield stable, oscillatory dynamics whose asymptotic convergence rate is approximately $\mathcal{O}(k^{-2.95})$. Moreover, as shown in Figure 1(b), the symplectic integrator remains stable when a larger step size is chosen, whereas the three-sequence Nesterov algorithm becomes unstable.

Interestingly, however, while the initial rate is the same, the three-sequence algorithm exhibits exponential convergence near the optimum. This behavior does not hold in general (e.g., Betancourt, Jordan, and Wilson [ibid.] show that it does not hold for quartic functions), but it is nonetheless an interesting feature of the three-sequence method in the case of quadratic objectives. Betancourt, Jordan, and Wilson [ibid.] show



(a)



(b)

Figure 2: (a) By incorporating gradient flow into the leapfrog integration of the Bregman Hamiltonian dynamics we recover the same asymptotic exponential convergence near the minimum of the objective exhibited by the dynamical Nesterov algorithm. (b) These modified Hamiltonian dynamics remain stable even as we increase the step size, allowing for more efficient computation and the advantageous asymptotic behavior.

that it arises from an implicit gradient flow that is a side effect of the three-sequence discretization. Moreover, they note that it is possible to mimic this flow within the symplectic integrator. When this is done the results for the quadratic objective are as shown in Figure 2(a). We see that a symplectic integrator that incorporates gradient flow and the three-sequence integrator yield convergence profiles that are essentially equivalent in this case.

But it is also true that the symplectic approach is topologically more stable than the three-sequence method, a fact which is revealed if one chooses a more aggressive step size. This is exhibited in Figure 2(b), where the symplectic integrator converges while the three-sequence Nesterov method diverges when the step size is increased.

In summary, this section has exhibited a connection between symplectic integration and the acceleration phenomenon in optimization. When the latter is construed as a continuous-time phenomenon, symplectic integration appears to provide an effective and flexible way to obtain discrete-time approximations. Much remains to be done, however, to tighten this link. In particular, we would like to obtain necessary and sufficient conditions for stable integrators that achieve oracle rates, and it is not yet clear what role symplectic geometry will play in uncovering those conditions.

3 Acceleration and the Escape from Saddle Points in Nonconvex Optimization

In this section we turn to *nonconvex* optimization. Although the general nonconvex setting harbors many intractable problems about which little can be said regarding computational or statistical efficiency, it turns out that for a wide range of problems in statistical learning, there is sufficient mathematical structure present in the nonconvex setting that useful mathematical results can be obtained. Indeed, in many cases the ideas and algorithms from convex optimization—suitably modified—can be carried over to the nonconvex setting. In particular, for gradient-based optimization, the same algorithms that perform well in the convex setting also tend to yield favorable performance in the nonconvex setting. In this sense, convex optimization has served as a laboratory for nonconvex optimization, in addition to having many natural applications of its own.

A useful first foothold on nonconvex optimization is obtained by considering the criterion of *first-order stationarity*. Given a differentiable function $f : \mathcal{X} \to \mathbb{R}$, on some well-behaved Euclidean domain $\mathcal{X}$ of dimension d, we define first-order stationary points to be those points $\mathbf{x} \in \mathcal{X}$ where the gradient vanishes: $\|\nabla f(\mathbf{x})\| = 0$. Although first-order stationary points can in general be associated with many kinds of topological singularity, for many statistical learning problems it suffices to consider the categorization into points that are global minima, local minima, local maxima and saddle points. Of these, local maxima are rarely viewed as problematic—simple modifications of gradient descent, such as stochastic perturbation, can suffice to ensure that algorithms do not get stuck at local maxima. Local minima have long been viewed as the core concern in nonconvex optimization for statistical learning problems. Recent work has shown,

however, that in a wide range of nonconvex statistical learning problems, local minima are provably absent, or, in empirical studies, even when local minima are present they do not appear to be discovered by gradient-based algorithms. Such results have been obtained for smooth semidefinite programs (Boumal, Voroninski, and Bandeira [2016]), matrix completion (Ge, J. D. Lee, and Ma [2016]), synchronization and Max-Cut (Bandeira, Boumal, and Voroninski [2016] and Mei, Misiakiewicz, Montanari, and Oliveira [2017]), multi-layer neural networks (Choromanska, Henaff, Mathieu, Arous, and LeCun [2014] and Kawaguchi [2016]) matrix sensing Bhojanapalli, Neyshabur, and Srebro [2016] and robust principal components analysis (Ge, Jin, and Y. Zheng [2017]).

As for global minima, while they are unambiguously the desirable end states for optimization algorithms, when there are multiple global minima it will generally be necessary to impose additional criteria (e.g., statistical) to single out preferable global minima, and to ask that an optimization algorithm respect this preference. We will not discuss these additional criteria here.

It remains to consider saddle points. Naively one might view these as akin to local maxima, in the sense that it is plausible that a simple perturbation could suffice for a gradient-based algorithm to roll down a direction of negative curvature. Such an argument has support from a recent theoretical result: J. D. Lee, Simchowitz, Jordan, and Recht [2016] have shown that under regularity conditions gradient descent will converge asymptotically and almost surely to a (local) minimum and thus avoid saddle points. In particular, a gradient-based algorithm that is initialized at a random point in $\mathcal{X}$ will avoid any and all saddle points in the asymptotic limit. While this result helps to emphasize the strength of gradient descent, it is of limited practical in that it is asymptotic (providing no rate of convergence); moreover, critically, it does not provide any insight into the rate of escape of saddle points as a function of dimension. While under suitable regularity all directions are escape directions for local maxima, it could be that only one direction is an escape direction for a saddle point. The computational burden of finding that direction could be significant; perhaps exponential in dimension. Given that modern statistical learning problems can involve many hundreds of thousands or millions of dimensions, such a burden would be fatal.

We thus focus our discussion on saddle points. To tie the discussion here to the discussion of the previous section, we take a dynamical systems perspective and study the extent to which acceleration (second-order dynamics) is able to improve the rate of escape of saddle points in gradient-based optimization. Intuitively, there is a narrow region around a saddle point in which the flow is principally directed towards the saddle point, and it seems plausible that an accelerated algorithm is able to bypass such a region more effectively than a non-accelerated algorithm. Whether this is actually true has been an open question in the literature.

To understand how dynamics and geometry interact in the neighborhood of saddle points, it is necessary to go beyond first-order stationarity, which lumps saddle points together with local minima, and to impose a condition that excludes saddle points. We define a *second-order stationary point* to be a point $\mathbf{x} \in \mathcal{X}$ such that $\|\nabla f(\mathbf{x})\| = 0$

and $\lambda_{\min}(\nabla^2 f(\mathbf{x})) \geq 0$. This definition includes local minima, but it also allows degenerate saddle points, in which the smallest eigenvalue of the Hessian is zero, and so we also define a *strict saddle point* to be a point $\mathbf{x} \in \mathcal{X}$ for which $\lambda_{\min}(\nabla^2 f(\mathbf{x})) < 0$. These two definitions jointly allow us to separate local minima from most saddle points. In particular, if all saddle points are strict, then an algorithm that converges to a second-order stationary point necessarily converges to a local minimum. (See Ge, Huang, Jin, and Yuan [2015] for further discussion.)

It turns out that these requirements are reasonable in practical applications. Indeed, it it has been shown theoretically that all saddle points are strict in many of the nonconvex problems mentioned earlier, including tensor decomposition, phase retrieval, dictionary learning and matrix completion (Ge, Huang, Jin, and Yuan [2015], J. Sun, Qu, and J. Wright [2016b,a], Bhojanapalli, Neyshabur, and Srebro [2016], and Ge, J. D. Lee, and Ma [2016]). Coupled with the fact (mentioned above) that there is a single global minimum in such problems, we see that an algorithm that converges to a second-order stationary point will actually converge to a global minimum.

To obtain rates of convergence, we need to weaken the definitions of stationarity to allow an algorithm to arrive in a ball of size $\epsilon > 0$ around a stationary point, for varying ϵ. We define an *ϵ-first-order stationary point* as a point $\mathbf{x} \in \mathcal{X}$ such that $\|\nabla f(\mathbf{x})\| \leq \epsilon$. Similarly we define an *ϵ-second-order stationary point* as a point $\mathbf{x} \in \mathcal{X}$ for which $\lambda_{\min}(\nabla^2 f(\mathbf{x})) \geq -\sqrt{\rho\epsilon}$, where ρ is the Hessian Lipschitz constant. (We have followed Nesterov and Polyak [2006] in using a parameterization for the Hessian that is relative to size of the gradient.)

Finally, we need to impose smoothness conditions on f that are commensurate with the goal of finding second-order stationary points. In particular, we require both the gradient and the Hessian to be Lipschitz:

$$\|\nabla f(\mathbf{x}_1) - \nabla f(\mathbf{x}_2)\| \leq \ell \|\mathbf{x}_1 - \mathbf{x}_2\| \tag{17}$$

$$\|\nabla^2 f(\mathbf{x}_1) - \nabla^2 f(\mathbf{x}_2)\| \leq \rho \|\mathbf{x}_1 - \mathbf{x}_2\|, \tag{18}$$

for constants $0 < \ell, \rho < \infty$, and for all $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{X}$.

Before turning to algorithmic issues, let us calibrate our expectations regarding achievable rates of convergence by considering the simpler problem of finding an ϵ-first-order stationary point, under a Lipschitz condition solely on the gradient. Nesterov [1998] has shown that if gradient descent is run with fixed learning rate $\eta = \frac{1}{\ell}$, and the termination condition is $\|\nabla f(\mathbf{x})\| \leq \epsilon$, then the output will be an ϵ-first-order stationary point, and the algorithm will terminate within the following number of iterations:

$$O\left(\frac{\ell(f(\mathbf{x}_0) - f^\star)}{\epsilon^2}\right),$$

where $\mathbf{x}_0$ is the point at which the algorithm is initialized. While the rate here is less favorable than in the case of smooth convex functions—where it is $O(\epsilon)$[1]—the rate

[1] In Section 1 we expressed rates in terms of the achieved ϵ after a given number of iterations; here we use the inverse function, expressing the number of iterations in terms of the accuracy. Expressed in the former way, the rate here is $O(1/\sqrt{k})$.

Algorithm 1 Perturbed Gradient Descent (PGD)

for $k = 0, 1, \ldots,$ **do**
  if $\|\nabla f(\mathbf{x}_k)\| < g$ and *no perturbation in last* $\mathcal{T}$ *steps* **then**
    $\mathbf{x}_k \leftarrow \mathbf{x}_k + \xi_k \qquad \xi_k \sim \mathrm{Unif}(\mathbb{B}_0(r))$
  $\mathbf{x}_{k+1} = \mathbf{x}_k - \eta \nabla f(\mathbf{x}_k)$

retains the essential feature from the convex setting that it is independent of dimension. Recall, however, that saddle points are ϵ-first-order stationary points, and thus this result describes (inter alia) the rate of *approach* to a saddle point. The question that we now turn to is the characterization of the rate of *escape* from a saddle point, where we expect that dimensionality will rear its head.

Turning to algorithmic considerations, we first note that—in contradistinction to the convex case—pure gradient descent will *not* suffice for convergence to a local minimum. Indeed, in the presence of saddle points, the rate of convergence of gradient descent can depend exponentially on dimension (Du, Jin, J. D. Lee, Jordan, Poczos, and A. Singh [2018]). Thus we need to move beyond gradient descent to have a hope of *efficient* escape from saddle points. We could avail ourselves of Hessians, in which case it would be relatively easy to identify directions of escape (as eigenvectors of the Hessian), but as discussed earlier we wish to avoid the use of Hessians on computational grounds. Instead, we focus on gradient descent that is augmented with a stochastic perturbation. Ge, Huang, Jin, and Yuan [2015] and Jin, Ge, Netrapalli, Kakade, and Jordan [2017] studied such an augmentation in which a homogeneous stochastic perturbation (uniform noise in a ball) is added sporadically to the current iterate. Specifically, noise is added when: (1) the norm of the gradient at the current iterate is small, and (2) the most recent such perturbation is at least $\mathcal{T}$ steps in the past, where $\mathcal{T}$ is an algorithmic hyperparameter. We refer to this algorithm as "perturbed gradient descent" (PGD); see Algorithm 1.

We can now state a theorem, proved in Jin, Ge, Netrapalli, Kakade, and Jordan [ibid.], that provides a convergence rate for PGD. Note that PGD has various algorithm hyperparameters, including r (the size of the ball from which the perturbation is drawn), $\mathcal{T}$ (the minimum number of time steps between perturbations), η (the step size), and g (the bound on the norm of the gradient that triggers a perturbation). As shown in Jin, Ge, Netrapalli, Kakade, and Jordan [ibid.], all of these hyperparameters can be specified as explicit functions of the Lipschitz constants ℓ and ρ. The only remaining hyperparameters are a constant quantifying the probability statement in the theorem, and a universal scaling constant.

Theorem 2 (Theorem)**.** *Assume that the function f is ℓ-smooth and ρ-Hessian Lipschitz. Then, with high probability, an iterate $\mathbf{x}_k$ of PGD (Algorithm 1) will be an ϵ-second order stationary point after the following number of iterations k:*

$$O\left(\frac{\ell(f(\mathbf{x}_0) - f^\star)}{\epsilon^2} \log^4\left(\frac{d}{\epsilon^2}\right)\right).$$

We see that the first factor is exactly the same as for ϵ-first-order stationary points with pure gradient descent. The penalty incurred by incorporating the perturbation—and thereby avoiding saddle points—is quite modest—it is only polylogarithmic in the dimension d.

With this convergence result as background, we turn to the main question of this section: Does acceleration aid in the escape from saddle points? In particular, can we improve on the rate in Theorem 2 by incorporating acceleration into the perturbed gradient descent algorithm? We will see that the answer is "yes"; moreover, we will see that a continuous-time dynamical-systems perspective will play a key role in establishing the result.

A major challenge in analyzing accelerated algorithms is that the objective function does not decrease monotonically as is the case for gradient descent. In the convex case, we met this challenge by exploiting the Lagrangian/Hamiltonian formulation to design a Lyapunov function; see Eq. (11). These Lyapunov functions, however, involve the global minimum $\mathbf{x}^\star$, which is unknown to the algorithm. This is not problematic in the convex setting, as terms involving $\mathbf{x}^\star$ can be bounded using convexity; in the nonconvex setting, however, it is fatal.

To overcome this problem in the nonconvex setting, Jin, Netrapalli, and Jordan [2017] developed a Hamiltonian that is appropriate for the analysis of nonconvex problems. Specializing to Euclidean geometry, the function takes the following form:

$$E_t := \frac{1}{2\eta}\|\mathbf{v}_t\|^2 + f(\mathbf{x}_t); \tag{19}$$

a sum of kinetic energy and potential energy terms. Let us consider using this Hamiltonian to analyze the following second-order differential equation:

$$\ddot{\mathbf{x}} + \tilde{\theta}\dot{\mathbf{x}} + \nabla f(\mathbf{x}) = 0. \tag{20}$$

Integrating both sides, we obtain:

$$f(\mathbf{x}(t_2)) + \frac{1}{2}\dot{\mathbf{x}}(t_2)^2 = f(\mathbf{x}(t_1)) + \frac{1}{2}\dot{\mathbf{x}}(t_1)^2 - \tilde{\theta}\int_{t_1}^{t_2} \dot{\mathbf{x}}(t)^2 \mathrm{d}t. \tag{21}$$

The integral shows that the Hamiltonian decreases monotonically with time t, and the decrease is given by the *dissipation* term $\tilde{\theta}\int_{t_1}^{t_2} \dot{\mathbf{x}}(t)^2 \mathrm{d}t$. Note that Eq. (21) holds regardless of the convexity of $f(\cdot)$.

Although the Hamiltonian decreases monotonically in continuous time, Jin, Netrapalli, and Jordan [ibid.] show that this is *not* the case in discrete time (in the nonconvex setting). Thus, once again, the complexity associated with acceleration manifests itself principally in the transition to discrete time. Jin, Netrapalli, and Jordan [ibid.] were able to resolve this problem by isolating a condition under which the change of the Hamiltonian is indeterminate (it may decrease or increase), and show that under the complement of this condition, the Hamiltonian necessarily decreases. Roughly speaking, the condition arises when the function is "too nonconvex." Moreover, this condition can be assayed algorithmically, and the algorithm can be modified to ensure decrease of the Hamiltonian when the condition arises.

Algorithm 2 Perturbed Accelerated Gradient Descent (PAGD)

1: $\mathbf{v}_0 = 0$
2: **for** $k = 0, 1, \ldots,$ **do**
3: **if** $\|\nabla f(\mathbf{x}_k)\| \leq \epsilon$ and *no perturbation in last* $\mathcal{T}$ *steps* **then**
4: $\mathbf{x}_k \leftarrow \mathbf{x}_k + \xi_k \qquad \xi_k \sim \mathrm{Unif}(\mathbb{B}_0(r))$
5: $\mathbf{y}_k = \mathbf{x}_k + \lambda_k \mathbf{v}_k$
6: $\mathbf{x}_{k+1} = \mathbf{y}_k - \eta \nabla f(\mathbf{y}_k)$
7: $\mathbf{v}_{k+1} = \mathbf{x}_{k+1} - \mathbf{x}_k$
8: **if** $f(\mathbf{x}_k) \leq f(\mathbf{y}_k) + \langle \nabla f(\mathbf{y}_k), \mathbf{x}_k - \mathbf{y}_k \rangle - \frac{\gamma}{2}\|\mathbf{x}_k - \mathbf{y}_k\|^2$ **then**
9: $(\mathbf{x}_{k+1}, \mathbf{v}_{k+1}) = \text{Negative-Curvature-Exploitation}(\mathbf{x}_k, \mathbf{v}_k)$

Perturbation	lines 3-4
Standard AGD	lines 5-7
Negative Curvature Exploitation (NCE)	lines 8-9

The overall algorithm, Perturbed Accelerated Gradient Descent (PAGD), is presented in Algorithm 2. Steps 3 and 4 are identical to the PGD algorithm. Steps 5, 6 and 7 replace gradient descent in the latter algorithm with accelerated gradient descent (cf. Eq. (3)). Step 8 measures the "amount of nonconvexity," and if it is too large, makes a call to a function called "Negative-Curvature-Exploitation." This function does one of two things: (1) if the momentum is large, it zeros out the momentum; (2) if the momentum is small, it conducts a local line search along the direction of the momentum. Jin, Netrapalli, and Jordan [2017] prove that this overall algorithm yields monotone decrease in the Hamiltonian. Moreover, they use this result to prove the following theorem regarding the convergence rate of PAGD. (As in the case of PGD, we are not specifying the settings of the various algorithm hyperparameters; we refer to Jin, Netrapalli, and Jordan [ibid.] for these settings. We note, however, that all hyperparameters are functions of the Lipschitz constants ℓ and ρ, with the exception of a constant quantifying the probability statement in the theorem, and a universal scaling constant.)

Theorem 3. *Assume that the function f is ℓ-smooth and ρ-Hessian Lipschitz. Then, with high probability, at least one of iterates, $\mathbf{x}_k$, of PAGD (Algorithm 2) will be an ϵ-second order stationary point after the following number of iterations:*

$$O\left(\frac{\ell^{1/2}\rho^{1/4}(f(\mathbf{x}_0) - f^*)}{\epsilon^{7/4}} \log^6\left(\frac{d}{\epsilon}\right)\right).$$

Comparing this result to Theorem 2, we see that the rate has improved from $1/\epsilon^2$ to $1/\epsilon^{7/4}$. Thus we see that acceleration provably improves upon gradient descent in the nonconvex setting—acceleration hastens the escape from saddle points. We also see that, once again, there is a mild—polylogarithmic—dependence on the dimension d.

The core of the proofs of Theorems 2 and 3 revolves around the study of the local geometry around saddlepoints and its interaction with gradient-descent dynamics. As depicted in Figures 3 and 4, there is a slab-like region in the neighborhood of a saddle

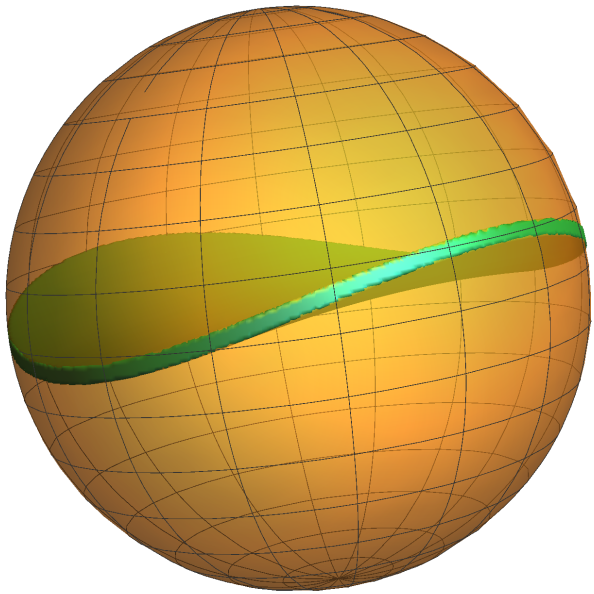


Figure 3: Perturbation ball in three dimensions and "thin pancake" stuck region.

point in which gradient descent will be "stuck"—taking an exponential amount of time to escape. This region is not flat, but instead varies due to the variation of the Hessian in this neighborhood. The Lipschitz assumption gives us control over this variation. To analyze the width of the stuck region, and thus its volume as a fraction of the perturbation ball, Jin, Netrapalli, and Jordan [ibid.] study the rate of escape of a pair of gradient-descent (or accelerated-gradient-descent) sequences that start on the sides of the stuck region. These initial points are a distance r apart along the direction given by the minimum eigenvector of the Hessian, $\nabla^2 f(\mathbf{x})$, at the saddle point. The critical value r for which at least one of the two sequences escapes the stuck region quickly can be computed, and this provides an estimate of the volume of the stuck region. The overall result is that this volume is small compared to that of the perturbation ball, and thus the perturbation is highly likely to cause the optimization algorithm to leave the stuck region (and not return).

4 Underdamped Langevin Diffusion

Our focus thus far has been on dynamical systems which are deterministic, with stochasticity introduced in a limited way—as a perturbation that ensures fast escape from a

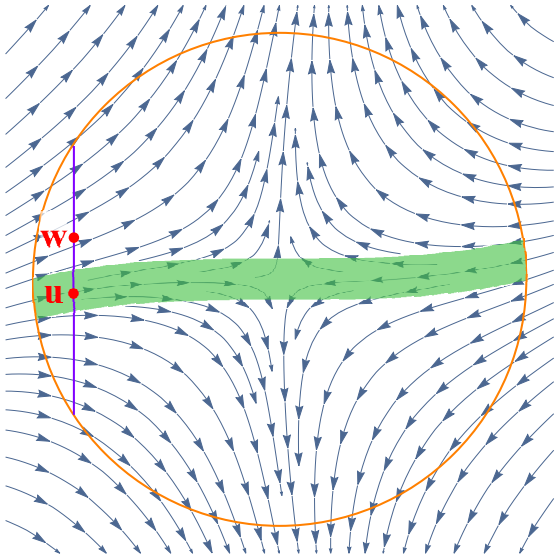


Figure 4: Gradient flow and "narrow band" stuck region in two dimensions.

saddle point. The particular perturbation that we have analyzed—a uniform perturbation in a ball—is sufficient for fast escape, but it is not necessary. Given the success of this simple choice, however, we are motivated to study more thoroughgoing stochastic approaches to our problem. We may wish to investigate, for example, whether a less homogeneous perturbation might suffice. Also, recalling the statistical learning setting that motivates us, even if the optimization problem is formulated as a deterministic one, the underlying data that parameterize this problem are best viewed as random, such that algorithm tractories become stochastic processes, and algorithm outputs become random variables. Thus, taking a stochastic-process point of view opens the door to connecting algorithmic results to inferential results.

We thus turn to a discussion of stochastic dynamics. Given our continuous-time focus, these stochastic dynamics will be expressed as stochastic differential equations. Moreover, we will again be interested in second-order ("momentum") dynamics, and will investigate the extent to which such dynamics can yield improvements over first-order dynamics.

The classical connection between gradient descent and stochastic differential equations is embodied in the *overdamped Langevin diffusion*:

$$d\mathbf{x}_t = -\nabla f(\mathbf{x}_t)dt + \sqrt{2}dB_t,$$

where $\mathbf{x}_t \in \mathbb{R}^d$ and B_t is d-dimensional standard Brownian motion. Under mild regularity conditions, it is known that the invariant distribution of this continuous-time process is proportional to $p^\star(\mathbf{x}) \propto \exp(-(f(\mathbf{x})))$. Thus samples from $p^\star(\mathbf{x})$ can be obtained by solving the diffusion numerically. Such a numerical solution is generally referred to as "Langevin Markov chain Monte Carlo" or "Langevin MCMC."

Asymptotic guarantees for overdamped Langevin MCMC were established in Gelfand and Mitter [1991]. The first explicit proof of non-asymptotic convergence of overdamped Langevin MCMC for log-smooth and strongly log-concave distributions was given by Dalalyan [2017], who showed that discrete, overdamped Langevin diffusion achieves ϵ error, in total variation distance, in $O(d/\epsilon^2)$ steps. Following this, Durmus and Moulines [2016] proved that the same algorithm achieves ϵ error, in 2-Wasserstein distance, in $O(d/\epsilon^2)$ steps.[2]

Our second-order perspective motivates us to consider *underdamped Langevin diffusion*:

$$\begin{aligned} d\mathbf{v}_t &= -\gamma \mathbf{v}_t dt - u\nabla f(\mathbf{x}_t)dt + \sqrt{2\gamma u}dB_t \\ d\mathbf{x}_t &= \mathbf{v}_t dt, \end{aligned} \tag{23}$$

where u and γ are parameters. It can be shown that the invariant distribution of this continuous-time process is proportional to $\exp(-(f(\mathbf{x})+\|\mathbf{v}\|_2^2/2u))$. Thus the marginal distribution of $\mathbf{x}$ is proportional to $\exp(-f(\mathbf{x}))$ and samples from $p^\star(\mathbf{x}) \propto \exp(-f(\mathbf{x}))$ can be obtained by solving the underdamped Langevin diffusion numerically and ignoring the momentum component.

Underdamped Langevin diffusion is interesting because it is analogous to accelerated gradient descent; both are second-order dynamical systems. Moreover, it contains a Hamiltonian component, and its discretization can be viewed as a form of Hamiltonian MCMC. Hamiltonian MCMC has been empirically observed to converge faster to the invariant distribution compared to standard Langevin MCMC (Betancourt, Byrne, Livingstone, and Girolami [2017]).

In recent work, Cheng, Chatterji, Bartlett, and Jordan [2017] have analyzed underdamped Langevin diffusion. They have shown that—in the same setting analyzed by Durmus and Moulines [2016] for overdamped Langevin diffusion—that the underdamped algorithm achieves a convergence rate of $O(\sqrt{d}/\epsilon)$ in 2-Wasserstein distance. This is a significant improvement over the $O(d/\epsilon^2)$ rate of overdamped Langevin diffusion, both in terms of the accuracy parameter ϵ and the dimension d.

5 Discussion

The general topic of gradient-based optimization, and its application to large-scale statistical inference problems, is currently very active. Let us highlight one particular set of questions that appear likely to attract ongoing attention in coming years. Note that optimization methods are used classically in the statistical setting to solve point estimation problems, where the core problem is to output a single point in the configuration

[2] Recall that the Wasserstein distance, $W_2(\mu,\eta)$, between probability measures μ and η is defined as follows:

$$W_2(\mu,\eta) := \left(\inf_{\zeta\in\Gamma(\mu,\eta)} \int \|\mathbf{x}-\mathbf{y}\|_2^2 d\zeta(\mathbf{x},\mathbf{y}) \right)^{1/2}, \tag{22}$$

where ζ ranges over the set of transference plans $\Gamma(\mu,\eta)$.

space that has desirable statistical properties. But the broader problem is to provide in addition an indication of the uncertainty associated with that output, in the form of some summary of a probability distribution. Optimization ideas can be relevant here as well, by considering a configuration space that is a space of probability distributions. Relatedly, one can ask to converge not to a single point, but to a distribution over points. The Hamiltonian approach naturally yields oscillatory solutions, and, as we have seen, some work is required to obtain algorithms that converge to a point. This suggests that the Hamiltonian approach may in fact be easier to employ in the setting of distributional convergence than in the point estimation setting, and thereby provide an algorithmic bridge between point estimation and broader inference problems. Indeed, in Bayesian inference, Hamiltonian formulations (and symplectic integration, in the form of leapfrog integrators) have been successfully employed in the setting of Markov chain Monte Carlo algorithms, where the momentum component of the Hamiltonian (empirically) provides faster mixing. Deeper connections between acceleration and computationally-efficient inference are clearly worth pursuing.

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THREE PUZZLES ON MATHEMATICS, COMPUTATION, AND GAMES

Gil Kalai

Abstract

In this lecture I will talk about three mathematical puzzles involving mathematics and computation that have preoccupied me over the years. The first puzzle is to understand the amazing success of the simplex algorithm for linear programming. The second puzzle is about errors made when votes are counted during elections. The third puzzle is: are quantum computers possible?

1 Introduction

The theory of computing and computer science as a whole are precious resources for mathematicians. They bring up new questions, profound new ideas, and new perspectives on classical mathematical objects, and serve as new areas for applications of mathematics and mathematical reasoning. In my lecture I will talk about three mathematical puzzles involving mathematics and computation (and, at times, other fields) that have preoccupied me over the years. The connection between mathematics and computing is especially strong in my field of combinatorics, and I believe that being able to personally experience the scientific developments described here over the past three decades may give my description some added value. For all three puzzles I will try to describe in some detail both the large picture at hand, and zoom in on topics related to my own work.

Puzzle 1: What can explain the success of the simplex algorithm? Linear programming is the problem of maximizing a linear function ϕ subject to a system of linear inequalities. The set of solutions to the linear inequalities is a convex polyhedron P. The simplex algorithm was developed by George Dantzig. Geometrically it can be described as moving from one vertex to a neighboring vertex of P so as to improve the value of the objective function. The simplex algorithm is one of the most successful mathematical algorithms. The explanation of this success is an applied, vaguely stated problem, that is connected with computers. The problem has strong relations to the study of convex polytopes, which have fascinated mathematicians from ancient times and which served as a starting point for my own research.

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If I were required to choose the single most important mathematical explanation for the success of the simplex algorithm, my choice would point to a theorem about another algorithm. I would choose Khachiyan's 1979 theorem asserting that there is a polynomial-time algorithm for linear programming (or briefly $LP \in \mathbf{P}$). Khachiyan's theorem refers to the ellipsoid method, and the answer is given in the language of computational complexity, a language that did not exist when the question was originally raised.

In Section 2 we will discuss the mathematics of the simplex algorithm, convex polytopes, and related mathematical objects. We will concentrate on the study of the diameter of graphs of polytopes and the discovery of randomized subexponential variants of the simplex algorithm, I'll mention recent advances: the disproof of the Hirsch conjecture by Santos and the connection between linear programming and stochastic games leading to subexponential lower bounds, discovered by Friedmann, Hansen and Zwick, for certain pivot rules.

Puzzle 2: What methods of election are immune to errors in the counting of votes? The second puzzle can be seen in the context of understanding and planning of electoral methods. We all remember the vote recount in Florida in the 2000 US presidential election. Is the American electoral system, based on electoral votes, inherently more susceptible to mistakes than the majority system? And what is the most stable method? Together with Itai Benjamini and Oded Schramm we investigated these and similar problems. We asked the following question: given that there are two candidates, and each voter chooses at random and with equal probability (independently) between them, what is the stability of the outcome, when in the vote-counting process one percent of the votes is counted incorrectly? (The mathematical jargon for these errors is "noise.") We defined a measure of noise sensitivity of electoral methods and found that weighted majority methods are immune to noise; namely, when the probability of error is small, the chances that the election outcome will be affected diminish. We also showed that every stable-to-noise method is "close" (in some mathematical sense) to a weighted majority method. In later work, O'Donnell, Oleszkiewicz, and Mossel showed that the majority system is the most stable to noise among all nondictatorial methods.

Our work was published in 1999, a year before the question appeared in the headlines in the US presidential election, and it did not even deal with the subject of elections. We were interested in understanding the problem of planar percolation, a mathematical model derived from statistical physics. In our article we showed that if we adopt an electoral system based on the model of percolation, this method will be very sensitive to noise. This insight is of no use at all in planning good electoral methods, but it makes it possible to understand interesting phenomena in the study of percolation.

After the US presidential election in 2000 we tried to understand the relevance of our model and the concepts of stability and noise in real-life elections: is the measure for noise stability that we proposed relevant, even though the basic assumption that each voter randomly votes with equal probability for one of the candidates is far from realistic? The attempt to link mathematical models to questions about elections (and, more generally, to social science) is fascinating and complicated, and a pioneer in this

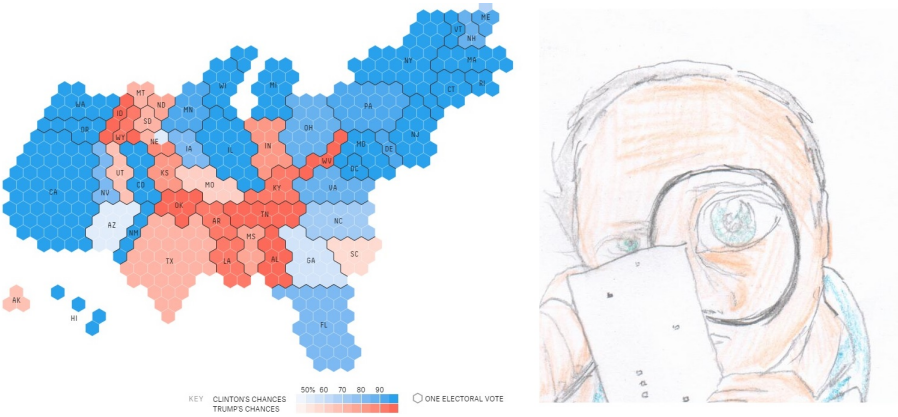

Figure 1: Right: recounts in the 2000 election (drawing: Neta Kalai). Left: Hex-based demonstration in Nate Silver's site

study was the Marquis de Condorcet, a mathematician and philosopher, a democrat, a human rights advocate, and a feminist who lived in France in the 18th century. One of Condorcet's findings, often referred to as Condorcet's paradox, is that when there are three candidates, the majority rule can sometimes lead to cyclic outcomes, and it turns out that the probability for cyclic outcomes depends on the stability to noise of the voting system. In Section 3 we will discuss noise stability and sensitivity, and various connections to elections, percolation, games, and computational complexity.

Puzzle 3: Are quantum computers possible? A quantum computer is a hypothetical physical device that exploits quantum phenomena such as interference and entanglement in order to enhance computing power. The study of quantum computation combines fascinating physics, mathematics, and computer science. In 1994, Peter Shor discovered that quantum computers would make it possible to perform certain computational tasks hundreds of orders of magnitude faster than ordinary computers and, in particular, would break most of today's encryption methods. At that time, the first doubts about the model were raised, namely that quantum systems are of a "noisy" and unstable nature. Peter Shor himself found a key to a possible solution to the problem of "noise": quantum error-correcting codes and quantum fault-tolerance. In the mid-1990s, three groups of researchers showed that "noisy" quantum computers still made it possible to perform all miracles of universal quantum computing, as long as engineers succeeded in lowering the noise level below a certain threshold.

A widespread opinion is that the construction of quantum computers is possible, that the remaining challenge is essentially of an engineering nature, and that such computers will be built in the coming decades. Moreover, people expect to build in the next few years quantum codes of the quality required for quantum fault-tolerance, and to demonstrate the concept of "quantum computational supremacy" on quantum computers with 50 qubits. My position is that it will not be possible to construct quantum codes that are required for quantum computation, nor will it be possible to demonstrate quantum

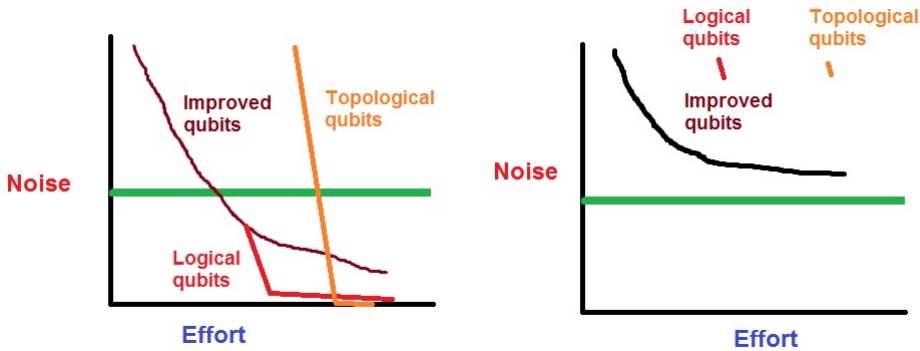

Figure 2: **Left** – It is commonly believed that by putting more effort into creating qubits the noise level can be pushed down to as close to zero as we want. Once the noise level is low enough and crosses the green "threshold" line, quantum error correction allows logical qubits to reduce the noise even further with a small amount of additional effort. Very high-quality topological qubits are also expected. This belief is supported by "Schoelkopf's law," the quantum computing analogue of Moore's law. **Right** – My analysis gives good reasons to expect that we will not be able to reach the threshold line, that all attempts for good quality logical and topological qubits will fail, and that Schoelkopf's law will break down before useful qubits can be created.

computational superiority in other quantum systems. My analysis, based on the same model of noise that led researchers in the 1990s to optimism about quantum computation, points to the need for different analyses on different scales. It shows that small-scale noisy quantum computers (of a few dozen qubits) express such primitive computational power that it will not allow the creation of quantum codes that are required as building blocks for quantum computers on a larger scale.

Near-term plans for "quantum supremacy". By the end of 2017,[1] John Martinis' group is planning to conclude a decisive experiment for demonstrating "quantum supremacy" on a 50-qubit quantum computer. See Boixo, Isakov, Smelyanskiy, Babbush, Ding, Jiang, Bremner, Martinis, and Neven [n.d.]. As they write in the abstract, "A critical question for the field of quantum computing in the near future is whether quantum devices without error correction can perform a well-defined computational task beyond the capabilities of state-of-the-art classical computers, achieving so-called quantum supremacy." The group intends to study "the task of sampling from the output distributions of (pseudo-)random quantum circuits, a natural task for benchmarking quantum computers." The objective of this experiment is to fix a pseudo-random circuit, run it many times starting from a given initial state to create a target state, and then measure the outcome to reach a probability distribution on 0-1 sequences of length 50.

[1]Of course, for such a major scientific project, a delay of a few months and even a couple of years is reasonable.

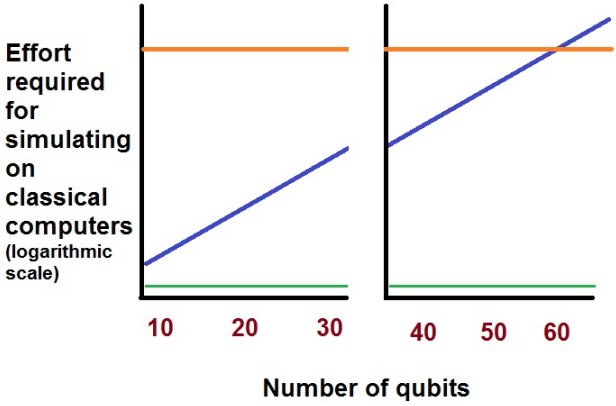

Figure 3: Quantum supremacy via pseudo-random circuits can be tested for 10–30 qubits. The orange line represents the limit for classical computers. Kalai and Kindler [2014] suggest close-to-zero correlation between the experimental results and outcomes based on the noiseless evolution, and further suggests that the experimental results are very easy to simulate (the green line).

The analysis described in Section 4, based on Kalai and Kindler [2014], suggests that the outcome of this experiment will have vanishing correlation with the outcome expected on the "ideal" evolution, and that the experimental outcomes are actually very, very easy to simulate classically. They represent distributions that can be expressed by low-degree polynomials. Testing our alternative can be carried out already with 10–30 qubits (see Fig. 3), and even examined on the 9-qubit experiments described in Neill et al. [n.d.].

The argument for why quantum computers are infeasible is simple.

First, the answer to the question whether quantum devices without error correction can perform a well-defined computational task beyond the capabilities of state-of-the-art classical computers, is negative. The reason is that devices without error correction are computationally very primitive, and primitive-based supremacy is not possible.

Second, the task of creating quantum error-correcting codes is harder than the task of demonstrating quantum supremacy,

Quantum computers are discussed in Section 4, where we first describe the model, then explain the argument for why quantum computers are not feasible, then describe predictions for current and near-future devices, and finally draw some predictions for general noisy quantum systems. The section presents my research since 2005, and it is possible that decisive evidence against my analysis will be announced in a matter of days or a bit later. This is a risk that I and the reader will have to take.

Perspective and resources. For books on linear programming see Matoušek and Gärtner [2007] and Schrijver [1986]. See also Schrijver [2003a,b,c] three-volume book on combinatorial optimization, and a survey article by Todd [2002] on the many facets of linear programming. For books on convex polytopes see Ziegler's book (1995) and Grünbaum's book (1967; 2003). For game theory, let me recommend the books by Maschler, Solan, and Zamir [2013] and Karlin and Peres [2017]. For books on computational complexity, the reader is referred to Goldreich [2008, 2010], Arora and Barak [2009] and Wigderson [2019]. For books on Boolean functions and noise sensitivity see O'Donnell [2014] and Garban and Steif [2015]. The discussion in Section 3 complements my 7ECM survey article "Boolean functions; Fourier, thresholds, and noise." It is also related to Kalai and Safra [2006] survey on threshold phenomena and influence. For quantum information and computation the reader is referred to Nielsen and Chuang [2000]. The discussion in Section 4 follows my Notices AMS paper (Kalai [2016a]) and its expanded version on the ArXiv [Kalai 2016b] (which is also a good source for references). My work has greatly benefited from Internet blog discussions with Aram Harrow, and others, on Regan and Lipton's blog, and my blog, among others.

Remark 1.1. Crucial predictions on quantum computers are going to be tested in the near future, perhaps even in a few months. I hope to post an updated and more detailed version of this paper by the end of 2019.

2 Linear programming, polytopes, and the simplex algorithm

To Micha A. Perles and Victor L. Klee who educated me as a mathematician.

2.1 Linear programming and the simplex algorithm. A linear programming problem is the problem of finding the maximum of a linear functional (called "a linear objective function") ϕ on d variables subject to a system of n inequalities.

Maximize

$c_1x_1 + c_2x_2 + \cdots c_dx_d$

subject to

$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1d}x_d \leq b_1$

$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2d}x_d \leq b_2$

...

$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nd}x_d \leq b_n$

This can be written briefly as: **Maximize** $c^t x$, **subject to** $Ax \leq b$, where vectors are column vectors, $x = (x_1, x_2, \ldots, x_d)$, $b = (b_1, b_2, \ldots, b_n)$, $c = (c_1, c_2, \ldots, c_d)$ and $A = (a_{ij})_{1\leq i\leq n, 1\leq j\leq d}$.

The set of solutions to the inequalities is called the *feasible polyhedron* and the simplex algorithm consists of reaching the optimum by moving from one vertex to a neighboring vertex. The precise rule for this move is called "the pivot rule."

Here is an example where $n = 2d$:

Maximize $x_1 + x_2 + \cdots + x_d$, **subject to:** $0 \leq x_i \leq 1, i = 1, 2, \ldots, d$

In this case, the feasible polyhedron is the d-dimensional cube. Although the number of vertices is exponential, 2^d, for every pivot rule it will take at most d steps to reach the optimal vertex $(1, 1, \ldots, 1)$.

The study of linear programming and its major applications in economics was pioneered by Kantorovich and Koopmans in the early 1940s. In the late 1940's George Dantzig realized the importance of linear programming for planning problems, and introduced the simplex algorithm for solving linear programming problems. [See, Dantzig 1963]. Linear programming and the simplex algorithm are among the most celebrated applications of mathematics. The question can be traced back to an 1827 paper by Fourier. (We will come across Joseph Fourier and John von Neumann in every puzzle.)

2.1.1 Local to global principle. We describe now two basic properties of linear programming.

- If ϕ is bounded from above on P then the maximum of ϕ on P is attained at a face of P; in particular, there is a vertex v for which the maximum is attained. If ϕ is not bounded from above on P then there is an edge of P on which ϕ is not bounded from above.

- A sufficient condition for v to be a vertex of P on which ϕ is maximal is that v is a *local maximum*, namely, $\phi(v) \geq \phi(w)$ for every vertex w that is a neighbor of v.

An *abstract objective function* (AOF) on a polytope P is an ordering of the vertices of P such that every face F of P has a unique local maximum.

Linear programming duality. A very important aspect of linear programming is duality. Linear programming duality associates an LP problem (given as a maximization problem) with d variables and n inequalities with a dual LP problem (given as a minimization problem) with $n - d$ variables and n inequalities with the same solution. Given an LP problem, the simplex algorithm for the dual problem can be seen as a path-following process on vertices of the hyperplane arrangement described by the entire hyperplane arrangement described by the n inequalities. It moves from one dual-feasible vertex to another, where a dual-feasible vertex is the optimal vertex to a subset of the inequalities.

2.2 Overview. *Early empirical experience and expectations.* The performance of the simplex algorithm is extremely good in practice. In the early days of linear programming it was believed that the common pivot rules reach the optimum in a number of steps that is polynomial or perhaps even close to linear in d and n. A related conjecture by Hirsch asserted that for polyhedra defined by n inequalities in d variables, there is always a path of length at most $n - d$ between every two vertices. We review some developments in linear programming and the simplex algorithms, where by "explanations" we refer to theoretical results that give some theoretical support for the excellent behavior of the simplex algorithm, while by "concerns" we refer to results in the opposite direction.

1. *The Klee–Minty example and worst-case behavior (concern 1).* Klee and Minty [1972] found that one of the most common variants of the simplex algorithm is exponential in the worst case. In their example, the feasible polyhedron is combinatorially equivalent to a cube, and all of its vertices are actually visited by the algorithm. Similar results for other pivot rules were subsequently found by several authors.

2. *Klee–Walkup counterexample to the Hirsch Conjecture (concern 2).* Klee and Walkup [1967] found an example of an unbounded polyhedron for which the Hirsch conjecture fails. They additionally showed that also in the bounded case one cannot realize the Hirsch bound by monotone paths. The Hirsch conjecture for polytopes remained open. On the positive side, Barnette and Larman gave an upper bound for the diameter of graphs of d-polytopes with n facets that are exponential in d but linear in n.

3. *$LP \in \mathbf{P}$, via the ellipsoid method (explanation 1).* In 1979 Hačijan (Leonid G. Khachiyan, Леонид Г. Хачиян) proved that $LP \in \mathbf{P}$, namely, that there is a polynomial-time algorithm for linear programming. This was a major open problem ever since the complexity classes P and NP were described in the early 1970s. Khachiyan's proof was based on Yudin, Nemirovski, and Shor's ellipsoid method, which is not practical for LP.

4. *Amazing consequences.* Grötschel, Lovász, and Schrijver [1981, 1993] found many theoretical applications of the ellipsoid method, well beyond its original scope, and found polynomial-time algorithms for several classes of combinatorial optimization problems. In particular, they showed that semi-definite programming, the problem of maximizing a linear objective function on the set of m by m positive definite matrices, is in $\mathbf{P}$.

5. *Interior points methods (explanation 2).* For a few years it seemed like there was a tradeoff between theoretical worst-case behavior and practical behavior. This feeling was shattered with Karmarkar's 1984 interior point method and subsequent theoretical and practical discoveries.

6. *Is there a strongly polynomial algorithm for LP? (concern 3)?* All known polynomial time algorithms for LP require a number of arithmetic operations that is polynomial in d and n and linear in L, the number of bits required to represent the input. Strongly polynomial algorithms are algorithms where the number of arithmetic operations is polynomial in d and n and does not depend on L, and no strongly polynomial algorithm for LP is known.

7. *Average case complexity (explanation 3).* Borgwardt [1982] and Smale [1983] pioneered the study of average-case complexity for linear programming. Borgwardt showed polynomial average-case behavior for a certain model that exhibits rotational symmetry. In 1983, Haimovich and Adler proved that the average length of the shadow boundary path from the bottom vertex to the top, for the regions in an arbitrary arrangement of n-hyperplanes in $\mathbb{R}^d$ is at most d. Adler and Megiddo

[1985], Adler, Karp, and Shamir [1987], and Todd [1986] proved *quadratic* upper bounds for the simplex algorithm for very general random models that exhibit certain sign invariance. All these results are for the shadow boundary rule introduced by Gass and Saaty.

8. *Smoothed complexity (explanation 4).* Spielman and Teng [2004] showed that for the shadow-boundary pivot rule, the average number of pivot steps required for a random Gaussian perturbation of variance σ of an arbitrary LP problem is polynomial in d, n, and σ^{-1}. (The dependence on d is at least d^5.) For many, the Spielman–Teng result provides the best known explanation of the good performance of the simplex algorithm.

9. *LP algorithms in fixed dimensions (explanation 5).* Megiddo [1984] found for a fixed value of d a linear-time algorithm for LP problems with n variables. Subsequent simple randomized algorithms were found by Clarkson [1995b] Clarkson [1995a] Seidel [1991], and Sharir and Welzl [1992]. Sharir and Welzl defined a notion of abstract linear programming problems for which their algorithm applies.

10. *Quasi-polynomial bounds for the diameter (explanation 6).* Kalai [1992b] and Kalai and Kleitman [1992] proved a quasipolynomial upper bound for the diameter of graphs of d-polytopes with n facets.

11. *Sub-exponential pivot rules (explanation 7).* Kalai [1992b] and Matoušek, Sharir and Welzl [1992] proved that there are randomized pivot rules that require in expectation a subexponential number of steps $\exp(K\sqrt{n \log d})$. One of those algorithms is the Sharir–Welzl algorithm.

12. *Subexponential lower bounds for abstract problems (concern 4).* Matoušek [1994] and Matoušek and Szabó [2006] found a subexponential lower bound for the number of steps required by two basic randomized simplex pivot rules, for abstract linear programs.

13. *Santos [2012] found a counterexample to the Hirsch conjecture (concern 5).*

14. *The connection with stochastic games.* Ludwig [1995] showed that the subexponential randomized pivot rule can be applied to the problem posed by Condon [1992] of finding the value of certain stochastic games. For these games this is the best known algorithm.

15. *Subexponential lower bounds for geometric problems (concern 6).* Building on the connection with stochastic games, subexponential lower bounds for genuine LP problems for several randomized pivot rules were discovered by Friedmann, Hansen, and Zwick [2011, 2014].

Most of the developments listed above are on the theoretical side of linear programming research and there are also many other theoretical aspects. Improving the linear algebra aspects of LP algorithms and tailoring the algorithm to specific structural and sparsity features of optimization tasks are both very important undertakings and pose

interesting mathematical challenges. Also of great importance are widening the scope of applications, and choosing the right LP modeling of real-life problems. There is also much theoretical and practical work on special families of LP problems.

2.3 Complexity 1: P, NP, and LP. The complexity of an algorithmic task is the number of steps required by a computer program to perform the task. The complexity is given in terms of the input size, and usually refers to the worst case behavior given the input size. An algorithmic task is in **P** (called "polynomial" or "efficient") if there is a computer program that performs the task in a number of steps that is bounded above by a polynomial function of the input size. (By contrast, an algorithmic task that requires an exponential number of steps in terms of the input size is referred to as "exponential" or "intractable.")

The notion of a nondeterministic algorithm is one of the most important notions in the theory of computation. One way to look at nondeterministic algorithms is to refer to algorithms where some or all steps of the algorithm are chosen by an almighty oracle. Decision problems are algorithmic tasks where the output is either "yes" or "no." A decision problem is in **NP** if when the answer is yes, it admits a nondeterministic algorithm with a polynomial number of steps in terms of the input size. In other words, if for every input for which the answer is "yes," there is an efficient proof demonstrating it, namely, a polynomial-size proof that a polynomial-time algorithm can verify. An algorithmic task A is **NP**-hard if a subroutine for solving A allows solving any problem in **NP** in a polynomial number of steps. An **NP**-complete problem is an **NP**-hard problem in **NP**. The papers by Cook [1971], and Levin [1973] that introduce **P**, **NP**, and **NP**-complete problems, and raising the conjecture that $\mathbf{P} \neq \mathbf{NP}$, and the paper by Karp [1972] that identifies 21 central algorithmic problems as **NP**-complete, are among the scientific highlights of the 20th century.

Graph algorithms play an important role in computational complexity. PERFECT MATCHING, the algorithmic problem of deciding if a given graph G contains a perfect matching, is in **NP** because exhibiting a perfect matching gives an efficient proof that a perfect matching exists. PERFECT MATCHING is in **co-NP** (namely, "not having a perfect matching" is in **NP**) because by a theorem of Tutte, if G does not contain a perfect matching there is a simple efficient way to demonstrate a proof. An algorithm by Edmonds shows that PERFECT MATCHING is in **P**. HAMILTONIAN CYCLE, the problem of deciding if G contains a Hamiltonian cycle, is also in **NP**: exhibiting a Hamiltonian cycle gives an efficient proof of its existence. However, this problem is **NP**-complete.

Remark 2.1. **P**, **NP**, and **co-NP** are three of the lowest computational complexity classes in the *polynomial hierarchy* ***PH***, which is a countable sequence of such classes, and there is a rich theory of complexity classes beyond **PH**. Our understanding of the world of computational complexity depends on a whole array of conjectures: $\mathbf{NP} \neq \mathbf{P}$ is the most famous one. A stronger conjecture asserts that **PH** does not collapse, namely, that there is a strict inclusion between the computational complexity classes defining the polynomial hierarchy. Counting the number of perfect matchings in a graph represents an important complexity class **#P** which is beyond the entire polynomial hierarchy.

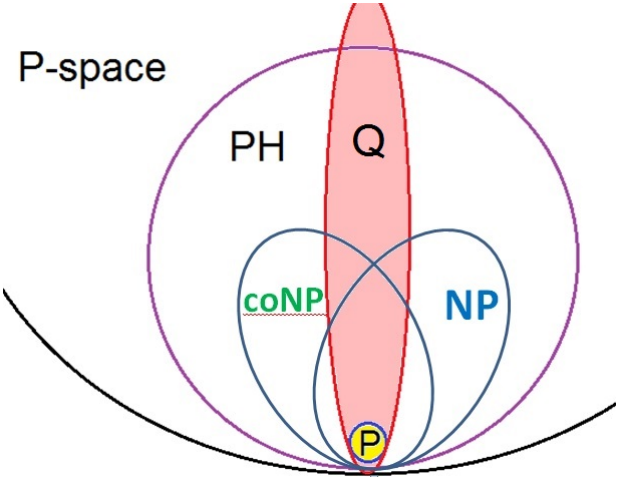

Figure 4: The (conjectured) view of some main computational complexity classes. The red ellipse represents efficient quantum algorithms. (See Section 4.)

2.3.1 The complexity of LP, Khachiyan's theorem, and the quest for strongly polynomial algorithms. It is known that general LP problems can be reduced to the decision problem to decide if a system of inequalities has a solution. It is therefore easy to see that LP is in **NP**. All we need is to identify a solution. The duality of linear programming implies that LP is in **co-NP** (namely, "not having a solution" is in **NP**). For an LP problem, let L be the number of bits required to describe the problem. (Namely, the entries in the matrix A and vectors b and c.)

Theorem 2.2 (Hačijan (Leonid G. Khachiyan, Леонид Г. Хачиян) [1979]). *$LP \in \boldsymbol{P}$. The ellipsoid method requires a number of arithmetic steps that is polynomial in n, d, and L.*[2]

The dependence on L in Khachiyan's theorem is linear and it was met with some surprise. We note that the efficient demonstration that a system of linear inequalities has a feasible solution requires a number of arithmetic operations that is polynomial in d and n but does not depend on L. The same applies to an efficient demonstration that a system of linear inequalities is infeasible. Also the simplex algorithm itself requires a number of arithmetic operations that, while not polynomial in d and n in the worst case, does not depend on L. An outstanding open problem is:

Problem 1. Is there an algorithm for LP that requires a polynomial number in n and d of arithmetic operations that does not depend on L?

Such an algorithm is called a strongly polynomial algorithm, and this problem is one of Smale's "problems for the 21st century." Strongly polynomial algorithms are known for various LP problems. The Edmonds–Karp algorithm (1972) is a strongly polynomial

[2]Khachiyan proved that the number of required arithmetic operations is polynomial in d and n, and linear in L. This bound becomes quadratic in L if we count bit-operations rather than arithmetic operations.

algorithm for the maximal flow problem. Tardos [1986] proved that when we fix the feasible polyhedron (and even only the matrix A used to define the inequalities) there is a strongly polynomial algorithm independent of the objective function (and the vector b).

2.4 Diameter of graphs of polytopes and related objects.

2.4.1 Quasi-polynomial monotone paths to the top. A few important definitions: a d-dimensional polyhedron P is simple if every vertex belongs to d edges (equivalently, to d facets.) A linear objective function ϕ is generic if $\phi(u) \neq \phi(v)$ for two vertices $v \neq u$. The top of P is a vertex for which ϕ attains the maximum or an edge on which ϕ is not bounded. Given a vertex v of P a facet F is active w.r.t. v if $\sup_{x \in F} \phi(x) \geq \phi(v)$.

Theorem 2.3 (Kalai [1992b]). *Let P be a d-dimensional simple polyhedron, let ϕ be a generic linear objective function, and let v be a vertex of P. Suppose that there are n active facets w.r.t. v. Then there is a monotone path of length at most $\leq n^{\log d+1}$ from v to the top.*

Proof: Let $f(d,n)$ denote the maximum value of the minimum length of a monotone path from v to the top. (Here, "the top" refers to either the top vertex or a ray on which ϕ is unbounded.)

Claim: Starting from a vertex v, in $f(d,k)$ steps one can reach either the top or vertices in at least $k+1$ active facets.

Proof: Let S be a set of $n-k$ active facets. Remove the inequalities defined by these facets to obtain a new feasible polyhedron Q. If v is not a vertex anymore than v belongs to some facet in S. If v is still a vertex there is a monotone path of length at most $f(d,k)$ from v to the top. If one of the edges in the path leaves P then it reaches a vertex belonging to a facet in S. Otherwise it reaches the top. Now if a monotone path from v (in P) of length $f(d,k)$ cannot take us to the top, there must be such a path that takes us to a vertex in some facet belonging to every set of $n-k$ facets, and therefore to vertices in at least $k+1$ facets.

Proof of the Theorem: Order the active facets of P according to their top vertex. In $f(d,[n/2])$ steps we can reach from v either the top, or a vertex in the top $[n/2]$ active facets. In $f(d-1,n-1)$ steps we reach the top w of that facet. This leaves us with at most $n/2$ active facets w.r.t. w, and yields

$$f(d,n) \leq 2f(d,[n/2]) + f(d-1,n-1), \tag{2-1}$$

which implies the bound given by the theorem. (In fact, it yields $f(d,n) \leq n \cdot \binom{d+\lceil \log_2 n \rceil}{d}$.)

Remark 2.4. A monotone path can be regarded as a nondeterministic version of the simplex algorithm where the pivot steps are chosen by an oracle.

Remark 2.5. Let me mention a few of the known upper bounds for the diameter of d-polytopes with n facets in some special families of polytopes. The Hirsch bound

was proved for duals of vertex decomoposable spheres [Provan and Billera 1980], transportation polytopes M. L. Balinski [1984], and duals of flag spheres Adiprasito and Benedetti [2014]. Naddef [1989] proved that 0-1 d-polytopes have diameter at most d; Polynomial upper bounds were proved for dual-neighborly polytopes [Kalai 1991], and unimodular polytopes Dyer and Frieze [1994]. Todd [2014] improved the bound of Theorem 2.3 to $(n-d)^{\log d}$ and further small improvements followed.

2.4.2 Reductions, abstractions, and Hähnle's conjecture. Upper bounds for the diameter are attained at *simple* d-polytopes, namely, d-polytopes where every vertex belongs to exactly d facets. A more general question deals with the dual graphs for triangulations of $(d-1)$-spheres with n vertices. All the known upper bounds apply to dual graphs of pure normal $(d-1)$ simplicial complexes. Here "pure" means that all maximal faces have the same dimension and "normal" means that all links of dimension one or more are connected. An even more general framework was proposed by Eisenbrand, Hähnle, Razborov, and Rothvoß [2010].

Problem 2. Consider t pairwise-disjoint nonempty families $F_1, F_2, \ldots, F_t$ of degree d monomials with n variables $(x_1, x_2, \ldots, x_n)$ with the following property: for every $1 \le i < j < k \le t$, if $m_i \in F_i$ and $m_k \in F_k$ then there is a monomial $m_j \in F_j$, such that the greatest common divisor of m_i and m_k divides m_j. How large can t be?[3]

A simple argument shows that the maximum denoted by $g(d,n)$ satisfies relation (2-1).

Conjecture 3 (Hähnle [2010]). $g(d,n) \le d(n-1)+1$.

One example of equality is to let F_k be all monomials $x_{i_1}x_{i_2}\cdots x_{i_d}$ with $i_1+i_2+\cdots+i_d = k-d+1$, $k = d, d+1, \ldots, kd$. Another example of equality is to let F_k be a single monomial of the form $x_i^{d-\ell}x_{i+1}^{\ell}$. ($i = \lfloor k/d \rfloor$ and $\ell = k \pmod d$.)

2.5 Santos' counterexample. The d-step conjecture is a special case of the Hirsch conjecture known to be equivalent to the general case. It asserts that a d-polytope with $2d$ facets has diameter at most d. Santos formulated the following strengthening of the d-step conjecture. **Santos' Spindle (working conjecture):** Let P be a d-polytope with two vertices u and v such that every facet of P contains exactly one of them. (Such a polytope is called a d-spindle.) Then the graph-distance between u and v (called simply the length of the spindle) is at most d. Santos proved

Theorem 2.6 (Santos [2012]).
(i) The spindle conjecture is equivalent to the Hirsch conjecture. More precisely, if there is a d-spindle with n facets and length greater than d then there is a counter-example to the Hirsch conjecture of dimension $n-d$ and with $2n-2d$ facets.

(ii) There is a 5-spindle of length 6.

[3] To see the connection to diameter of polytopes note that a vertex w of a d-polytope P corresponds to a set S of d facets. Now, let F_{k+1} be the family of d-subsets of facets that correspond to vertices in the graph of P at distance k from w.

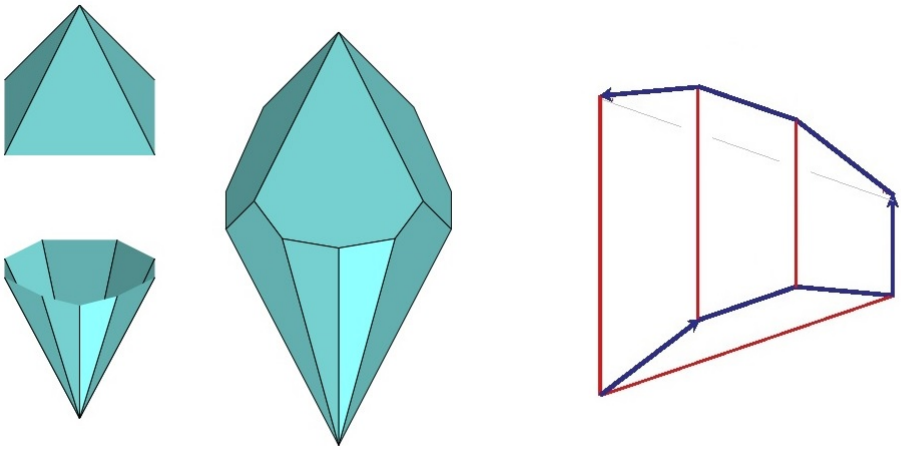

Figure 5: Left: a spindle, right: the Klee–Minty cube

The initial proof of part (2) had 48 facets and 322 vertices, leading to a counterexample in dimension 43 with 86 facets and estimated to have more than a trillion vertices. Matschke, Santos, and Weibel [2015] found an example with only 25 facets leading to a counterexample of the Hirsch conjecture for a 20-polytope with 40 facets and 36,442 vertices. An important lesson from Santos' proof is that although reductions are available to simple polytopes and simplicial objects, studying the problem for general polytopes has an advantage. In the linear programming language this tells us that degenerate problems are important.

Problem 4. Find an abstract setting for the diameter problem for polytopes that will include graphs of general polytopes, dual graphs for normal triangulations, and families of monomials.

2.6 Complexity 2: Randomness in algorithms. One of the most important developments in the theory of computing is the realization that adding an internal randomness mechanism can enhance the performance of algorithms. Two early manifestations of this idea are Monte Carlo methods by Ulam, von Neumann, and Metropolis, and a factoring algorithm by Berlekamp. Since the mid-1970s, and much influenced by Michael Rabin, randomized algorithms have become a central paradigm in computer science. One of the great achievements was the polynomial time randomized algorithms for testing primality by Solovay and Strassen [1977] and Rabin [1980]. Rabin's algorithm was related to an earlier breakthrough – Miller's algorithm for primality (1976), which was polynomial time conditioned on the validity of the generalized Riemann hypothesis. The newly randomized algorithms for testing primality were not only theoretically efficient but also practically excellent! Rabin's paper thus gave "probabilistic proofs" that certain large numbers, like $2^{400} - 593$, are primes, and this was a new kind of a

mathematical proof. (A deterministic polynomial algorithm for primality was achieved by Agrawal, Kayal, and Saxena [2004].) Lovász [1979] offered a randomized efficient algorithm for perfect matching in bipartite graphs: Associate to a bipartite graph G with n vertices on each side, its generic $n \times n$ adjacency matrix A, where a_{ij} is zero if the ith vertex on one side is not adjacent to the jth vertex on the other side, and a_{ij} is a variable otherwise. Note the determinant of A is zero if and only if G has no perfect matching. This can be verified with high probability by replacing a_{ij} with random mod p elements for a large prime p.

We have ample empirical experience and some theoretical support of the fact that pseudo-random number generators are practically sufficient for randomized algorithms. We also have strong theoretical support that weak and imperfect sources of randomness are sufficient for randomized algorithms.

A class of randomized algorithms which are closer to early Monte Carlo algorithms and to randomized algorithms for linear programming, are algorithms based on random walks. Here are two examples: counting the number of perfect matchings for a general graph G is a **#P**-complete problem. Jerrum and Sinclair [1989] and Jerrum, Sinclair, and Vigoda [2001] found an efficient random-walk-based algorithm for estimating the number of perfect matchings up to a multiplicative constant $1 + \epsilon$. Dyer, Frieze, and Kannan [1991] found an efficient algorithm based on random walks to estimate the volume of a convex body in $\mathbb{R}^d$. Both these algorithms rely on the ability to prove a spectral gap (or "expansion") for various Markov chains. Approximate sampling is an important subroutine in the algorithms we have just mentioned and we can regard exact or approximate sampling as an important algorithmic task in its own, as the ability to sample is theoretically and practically important. We mention algorithms by Aldous [1990] and Broder [1989] and Wilson [1996] for sampling spanning trees and by Randall and Wilson [1999] for sampling configurations of the Ising models.

Remark 2.7. The *probabilistic method,* when applied to problems with no mention of probability, led to major developments in combinatorics and several other mathematical disciplines. (See Alon and Spencer [2016].) An interesting question is, to what extent can proofs obtained by the probabilistic method be transformed into efficient randomized algorithms?

2.7 Subexponential randomized simplex algorithms. We start with the following simple observation. Consider the following two sequences. The first sequence is defined by $a_1 = 1$ and $a_{n+1} = a_n + a_{n/2}$, and the second sequence is defined by $b_1 = 1$ and $b_{n+1} = b_n + (b_1 + \cdots + b_n)/n$. Then $a_n = n^{\Theta(\log n)}$, and $b_n = e^{\Theta(\sqrt{n})}$.

Next, we describe two basic randomized algorithms for linear programming.

Random Edge: Choose an improving edge uniformly at random and repeat.

Random Facet: Given a vertex v, choose a facet F containing v uniformly at random. Use the algorithm recursively inside F until reaching its top w, and then repeat. (When $d = 1$, move to the top vertex.)

Random Facet (along with some variants) is the first strongly subexponential algorithm for linear programming, as well as the first subexponential pivot rule for the simplex algorithm.

Theorem 2.8 (Kalai [1992b] and Jiří Matoušek and Welzl [1992]). *Let P be a d-dimensional simple polyhedron, let ϕ be a linear objective function that is not constant on any edge of P, and let v be a vertex of P. Suppose that there are n active facets w.r.t. v. Then* Random Facet *requires an expected number of at most*

$$\exp(K \cdot \sqrt{n \log d}) \tag{2-2}$$

steps from v to the TOP.

Proof: Write $g(d,n)$ for the expected number of pivot steps. The expected number of pivot steps to reach w, the top of the facet chosen first, is bounded above by $g(d-1,n-1)$. With probability $1/d$, w is the ith lowest top among the top vertices of the active facets containing v. This yields

$$g(d,n) \le g(d-1,n-1) + \frac{1}{d-1}\sum_{i=1}^{d-1} g(d,n-i).$$

(Here, we took into account that v itself might be the lowest top.) This recurrence relation leads (with some effort) to equation (2-2). □

Note that the argument applies to abstract objective functions on polyhedra (and, more generally, to abstract LP problems as defined by Sharir–Welzl). The appearance of $\exp(\sqrt{n})$ is related to our observation on the sequence b_n: we get the recurrence relation $G(d+1) = G(d) + (G(1) + G(2) + \cdots + G(d))/d$ for the expected number of steps, $G(d)$, for Random Facet for abstract objective functions in the discrete d-cube. There are few versions of Random Facet that were analyzed (giving slightly worse or slightly better upper bounds). For the best known one see Hansen and Zwick (2015). There are also a few ideas for improved versions: we can talk about a random face rather than a random facet, to randomly walk up and down before setting a new threshold, and to try to learn about the problem and improve the random choices. The powerful results about lower bounds suggest cautious pessimism.

Remark 2.9. Amenta [1994] used Sharir and Welzl's abstract LP problem to settle a Helly-type conjecture of Grünbaum and Motzkin. Halman [2004] considered new large classes of abstract LP problems, found many examples, and also related them to Helly-type theorems.

2.7.1 Lower bounds for abstract problems. As we will see, the hope for better upper bounds for Random Facet and related randomized pivot rules was tempered by formidable examples to the contrary.

Theorem 2.10 (Matoušek [1994]). *There exists an abstract objective function of the d-cube on which* Random Facet *requires on expectation at least* $\exp(C\sqrt{d})$ *steps.*

Matoušek described a large class of AOF's and showed his lower bound to hold in expectation for a randomly chosen AOF. Gärtner proved (1998) that for a geometric AOF in this family, Random Facet requires an expected quadratic time.

Theorem 2.11 (Matoušek and Szabó [2006]). *There exists an AOF on the d-cube on which* Random Edge *requires on expectation of at least* $\exp(Cd^{1/3})$ *steps. [Hansen and Zwick 2016] improved the bound to* $\exp(\sqrt{d \log d})$.*)*

2.8 Games 1: Stochastic games, their complexity, and linear programming.

2.8.1 The complexity of chess and backgammon. Is there a polynomial-time algorithm for chess? Well, if we consider the complexity of chess in terms of the board size then "generalized chess" is very hard. (It is **P-space**-complete.) But if we wish to consider the complexity in terms of the number of all possible positions (which for "generalized chess" is exponential in the board size), given an initial position, it is easy to walk on the tree of positions and determine, in a linear number of steps, the value of the game. (Real life chess is probably intractable, but we note that checkers was solved.)

Now, what about backgammon? This question represents one of the most fundamental open problems in algorithmic game theory. The difference between backgammon and chess is the element of luck: in each position your possible moves are determined by a roll of two dice.

Remark 2.12. Chess and backgammon are games with perfect information and their value is achieved by pure strategies. One of the fundamental insights of game theory is that for zero-sum games with imperfect information, optimal strategies are mixed; namely, they are described by random choices between pure strategies. For mixed strategies, von Neumann's 1928 minmax theorem asserts that a zero-sum game with imperfect information has a value. An optimal strategy for rock-paper-scissors game is to play each strategy with an equal probability of 1/3. An optimal strategy for two-player poker (heads-on poker) is probably much harder to find.

2.8.2 Stochastic games and Ludwig's theorem. A simple stochastic game is a two-player zero-sum game with perfect information, described as follows. We are given one shared token and a directed graph with two sink vertices labeled "1" and "2" which represent winning positions for the two players, respectively. All other vertices have outdegree 2 and are labeled either by the name of a player or as "neutral." In addition, one vertex is the start vertex. Once the token is on a vertex, the player with the vertex labelling moves, and if the vertex is neutral then the move is determined by a toss of a fair coin. Backgammon is roughly a game of this type. (The vertices represent the player whose turn it is to play and the outcome of the two dice, and there are neutral vertices representing the rolls of the dice. The outdegrees are larger than two but this does not make a difference.) This class of games was introduced by Condon in 1992. If there is only one player, the game turns into a one-player game with uncertainty, which is called a Markov decision process. For Markov decision processes, finding the optimal strategy is a linear programming problem.

Theorem 2.13 (Ludwig [1995]). *There is a subexponential algorithm for solving simple stochastic games*

The basic idea of the proof is the following: once the strategy of player 2 is determined the game turns into a Markov decision process and the optimal strategy for player 1 is determined by solving an LP problem. Player one has an optimization problem over the discrete cube whose vertices represent his choices in each vertex labeled

by "1." The crucial observation is that this optimization problem defines an abstract objective function and therefore we can apply Random Facet.

A more general model of stochastic games with imperfect information was introduced by Shapley in 1953. There at each step the two players choose actions independently from a set of possibilities and their choices determine a reward and a probability distribution for the next state of the game.

Problem 5 (Learned from Peter Bro Miltersen)**.**
(i) *Think about backgammon.* Is there a polynomial-time algorithm for finding the value of simple stochastic games?

(ii) Can the problem of finding the sink in a unique sink acyclic orientation of the d-cube be reduced to finding the value of a simple stochastic game?

(iii) (Moving away from zero-sum games.) Is there a polynomial-time algorithm (or at least, subexponential) for finding a Nash equilibrium point for a stochastic two-player game (with perfect information)? What about stochastic games with perfect information with a fixed number of players?

(iv) *Think about two-player poker.* Is there a polynomial-time algorithm (or at least, a subexponential) for finding the value of a stochastic zero-sum game with imperfect information?

Remark 2.14. What is the complexity of finding objects guaranteed by mathematical theorems? Papadimitriou [1994] developed complexity classes and notions of intractability for mathematical methods and tricks! (Finding an efficiently describable object guaranteed by a mathematical theorem cannot be NP-complete Megiddo [1988]. A motivating conjecture that took many years to prove (in a series of remarkable papers) is that Nash equilibria is hard with respect to PPAD, one of the aforementioned Papadimitriou classes.

Problem 6 ([Szabó and Welzl 2001])**.** How does the problem of finding the sink in a unique sink acyclic orientation of the cube, and solving an abstract LP problem, fit into Papadimitriou's classes?

2.9 Lower bounds for geometric LP problems via stochastic games. In this section I discuss the remarkable works of Friedmann, Hansen, and Zwick [2011, 2014]. We talked (having the example of backgammon in mind) about two-player stochastic games with perfect information. (Uri Zwick prefers to think of those as "games with two-and-a-half players" with nature being a nonstrategic player rolling the dice.) The work of Friedmann, Hansen, and Zwick starts by building two-player parity games on which suitable randomized policy-iteration algorithms perform a subexponential number of iterations. Those games are then transformed into one-player Markov decision processes (or $1\frac{1}{2}$-player MDPs in Uri's view) that correspond to concrete linear programs. In their 2014 paper they showed a concrete LP problem, where the feasible polyhedron is combinatorially a product of simplices, on which Random Facet takes an expected number of $\exp(\tilde{\Theta}(d^{1/3}))$ steps, and a variant called Random Bland requires an expected number of $\exp(\tilde{\Theta}(\sqrt{d}))$ steps. The lower bound even applies to linear programming programs that correspond to shortest path problems, (i.e., one-player games, even in Uri's view)

that are very easily solved using other methods (e.g., Dijkstra [1959]). A similar, more involved argument yields an expected $\exp(\tilde{\Theta}(d^{1/4}))$ steps for RANDOM EDGE!

Remark 2.15. Two recent developments: Fearnley and Savani [2015] used the connection between games and LP to show that it is **PSPACE**-complete to find the solution that is computed by the simplex method using Dantzig's pivot rule. Calude, Jain, Khoussainov, Li, and Stephan [2017] achieved a quasi-polynomial algorithm for parity games! So far, it does not seem that the algorithm extends to simple stochastic games or has implications for linear programming.

2.10 Discussion. Is our understanding of the success of the simplex algorithm satisfactory? Are there better practical algorithms for semidefinite and convex programming? Is there a polynomial upper bound for the diameter of graphs of d-polytopes with n facets? (or at least some substantial improvements of known upper bounds)? Is there a strongly polynomial algorithm for LP? Perhaps even a strongly polynomial variant of the simplex algorithm? What is the complexity of finding a sink of an acyclic unique-sink orientation of the discrete cube? Are there other new interesting efficient, or practically good, algorithms for linear programming? What is the complexity of stochastic games? Can a theoretical explanation be found for other practically successful algorithms? (Here, SAT solvers for certain classes of SAT problems, and deep learning algorithms come to mind.) Are there good practical algorithms for estimating the number of matchings in graphs? For computing volumes for high-dimensional polytopes? We also face the ongoing challenge of using linear programming and optimization as a source for deriving further questions and insights into the study of convex polytopes, arrangements of hyperplanes, geometry, and combinatorics.

3 Elections and noise

To Nati Linial and Jeff Kahn who influenced me

3.1 Games 2: Questions about voting games and social welfare.

3.1.1 Cooperative games. A cooperative game (with side payments) is described by a set of n players N, and a payoff function v that associates to every subset S (called a *coalition*) of N a real number $v(S)$. We will assume that $v(\emptyset) = 0$. Cooperative games were introduced by von Neumann and Morgenstern. A game is monotone if $v(T) \geq v(S)$ when $S \subset T$. A voting game is a monotone cooperative game in which $v(S) \in \{0, 1\}$. If $v(S) = 1$ we call S a winning coalition and if $v(S) = 0$ then S is a losing coalition. Voting games represent voting rules for two-candidate elections, the candidates being Anna and Bianca. Anna wins if the set of voters that voted for her is a winning coalition. Important voting rules are the majority rule, where n is odd and the winning coalitions are those with more than $n/2$ voters, and the dictatorship rule, where the winning coalitions are those containing a fixed voter called "the dictator." Voting games are also referred to as monotone Boolean functions.

3.1.2 How to measure power? There are two related measures of power for voting games and both are defined in terms of general cooperative games. The Banzhaf measure of power for player i, $b_i(v)$ (also called the influence of i) is the expected value of $v(S \cup \{i\}) - v(S)$ taken over all coalitions S that do not contain i. The Shapley value of player i is defined as follows: for a random ordering of the players consider the coalition S of players who come before i in the ordering. The Shapley value, $s_i(v)$. is the expectation over all $n!$ orderings of $v(S \cup \{i\}) - v(S)$. (For voting games, the Shapley value is also called the Shapley–Shubik power index.) For voting games, if $v(S) = 0$ and $v(S \cup \{i\}) = 1$, we call voter i *pivotal* with respect to S.

3.1.3 Aggregation of information. For a voting game v and $p, 0 \leq p \leq 1$ denote by $\mu_p(v)$ the probability that a random set S of players is a winning coalition when for every player v the probability that $v \in S$ is p, independently for all players. Condorcet's Jury theorem asserts that when $p > 1/2$, for the sequence v_n of majority games on n players $\lim_{n\to\infty} \mu_p(v_n) = 1$. This property, a direct consequence of the law of large numbers, is referred to as *asymptotically complete aggregation of information*, and we will study it for other voting rules.

A voting game is *strong* (also called *neutral*) if a coalition is winning iff its complement is losing. A voting game is *strongly balanced* if precisely half of the coalitions are winning and it is *balanced* if $0.1 \leq \mu_{1/2}(v) \leq 0.9$. A voting game is *weakly symmetric* if it is invariant under a transitive group of permutations of the voters.

Theorem 3.1 (Friedgut and Kalai [1996] and Kalai [2004]).
(i) Weakly-symmetric balanced voting games lead to asymptotically complete aggregation of information.

(ii) Balanced voting games lead to asymptotically complete aggregation of information iff their maximum Shapley values tend to zero.

3.1.4 Friedgut's Junta theorem. The total influence, $I(v)$, of a balanced voting game is the sum of Banzhaf power indices for all players. (Note that the sum of Shapley values of all players is one.) For the majority rule the total influence is the maximum over all voting games and $I = \theta(\sqrt{n})$. The total influence for dictatorship is one, and this is the minimum for strongly balanced games. A voting game is a C-Junta if there is a a set J, $|J| \leq C$ such that $v(S)$ depends only on $S \cap J$.

Theorem 3.2 (Friedgut's Junta theorem, Friedgut [1998]). *For every $b, \epsilon > 0$, there is $C = C(b, \epsilon)$ with the following property: for every $\epsilon, b > 0$, a voting game v with total influence at most b is ϵ-close to a C-junta g. (Here, ϵ-close means that for all but a fraction ϵ of sets S, $v(S) = g(S)$.)*

3.1.5 Sensitivity to noise. Let $w_1, w_2, \ldots, w_n$ be nonnegative real weights and T be a real number. A weighted majority is defined by $v(S) = 1$ iff $\sum_{i \in S} w_i \geq T$.

Consider a two-candidate election based on a voting game v where each voter votes for one of the two candidates at random, with probability 1/2, and these probabilities are independent. Let S be the set of voters voting for Anna, who wins the election if

$v(S) = 1$. Next consider a scenario where in the vote counting process there is, for every vote, a small probability t that the vote is miscounted, and assume that these mistakes are statistically independent. The set of voters believed to vote for Anna after the counting is T. Define $N_t(v)$ as the probability that $v(T) \neq v(S)$. A family of voting games is called uniformly noise stable if for every $\epsilon > 0$ there exists $t > 0$ such that $N_t(v) < \epsilon$. A sequence v_n of strong voting games is noise sensitive if for every $t > 0$ $\lim_{n\to\infty} N_t(v_n) = 1/2$.

Theorem 3.3 (Benjamini, Kalai, and Schramm [1999]). *For a sequence of balanced voting games v_n each of the following two conditions implies that v_n is noise sensitive:*

(i) The maximum correlation between v_n and a balanced weighted majority game tends to 0.

(ii) $\lim_{n\to\infty} \sum_i b_i^2(v_n) = 0$.

3.1.6 Majority is stablest. Let v_n be the majority voting games with n players. In 1899 Sheppard proved that $\lim_{n\to\infty} N_t(v_n) = \frac{\arccos(1-2t)}{\pi}$.

Theorem 3.4 (Mossel, O'Donnell, and Oleszkiewicz [2010]). *Let v_n be a sequence of games with diminishing maximal Banzhaf power index. Then*

$$N_t(v_n) \geq \frac{\arccos(1-2t)}{\pi} - o(1).$$

3.1.7 The influence of malicious counting errors. Let S be a set of voters. $I_S(v)$ is the probability over sets of voters T that are disjoint from S such that $v(S \cup T) = 1$ and $v(T) = 0$.

Theorem 3.5 (Kahn, Kalai, and Linial [1988]). *For every balanced voting game v:*

(i) There exists a voter k such that

$$b_k(v) \geq C \log n/n.$$

(ii) There exists a set S of $a(n) \cdot n/\log n$ voters, where $a(n)$ tends to infinity with n as slowly as we wish, such that $I_S(v) = 1 - o(1)$.

This result was conjectured by Ben-Or and Linial [1985] who gave a "tribe" example showing that both parts of the theorem are sharp. Ajtai and Linial [1993] found a voting game where no set of $o(n/\log^2(n))$ can influence the outcome of the elections in favor of even one of the candidates.

3.1.8 "It ain't over 'till it's over" theorem. Consider the majority voting game when the number of voters tends to infinity and every voter votes for each candidate with equal probability, independently. There exist (tiny) $\delta > 0$ with the following property: when you count 99% of votes chosen at random, still with probability tending to one, condition on the votes counted, each candidate has a probability larger than δ of winning. We refer to this property of the majority function as the (IAOUIO)-property. Clearly, dictatorship and Juntas do not have the (IAOUIO)-property.

Theorem 3.6 (Mossel, O'Donnell, and Oleszkiewicz [2010]). *Every sequence of voting games with diminishing maximal Banzhaf power index has the (IAOUIO)-property.*

3.1.9 Condorcet's paradox and Arrow's theorem. A generalized social welfare function is a map from n voters' order relations on m alternatives, to a complete anti-symmetric relation for the society, satisfying the following two properties.

(1) If every voter prefers a to b then so does the society. (We do not need to assume that this dependence is monotone.)

(2) Society's preference between a and b depends only on the individual preferences between these candidates.

A social welfare function is a generalized welfare function such that for every n-tuple of order relations of the voters, the society preferences are acyclic ("rational").

Theorem 3.7 (Arrow [1951]). *For three or more alternatives, the only social welfare functions are dictatorial.*

Theorem 3.8 (Kalai [2002], Mossel [2012], and Keller [2012]). *For three or more alternatives the only nearly rational generalized social welfare functions are nearly dictatorial.*

Theorem 3.9 (Mossel, O'Donnell, and Oleszkiewicz [2010]). *The majority gives asymptotically "most rational" social preferences among generalized social welfare functions based on strong voting games with vanishing maximal Banzhaf power.*

A choice function is a rule that, based on individual rankings of the candidates, gives the winner of the election. Manipulation (also called "non-naïve voting" and "strategic voting") is a situation where given the preferences of other voters, a voter may gain by not being truthful about his preferences.

Theorem 3.10 (Gibbard [1973] and Satterthwaite [1975]). *Every nondictatorial choice function is manipulable.*

Theorem 3.11 (Friedgut, Kalai, Keller, and Nisan [2011], Isaksson, Kindeler, and Mossel [2010], and Mossel and Rácz [2015]). *Every nearly nonmanipulable choice function is nearly dictatorial.*

3.1.10 Indeterminacy and chaos. Condorcet's paradox asserts that the majority rule may lead to cyclic outcomes for three candidates. A stronger result was proved by McGarvey [1953]: every asymmetric preference relation on m alternatives is the outcome of majority votes between pairs of alternatives for some individual rational preferences (namely, acyclic preferences) for a large number of voters. This property is referred to as *indeterminacy*. A stronger property is that when the individual order relations are chosen at random, the probability of every asymmetric relation is bounded away from zero. This is called *stochastic indeterminacy*. Finally, *complete chaos* refers to a situation where in the limit all the probabilities for asymmetric preference relations are the same: $2^{-\binom{m}{2}}$.

Theorem 3.12 (Kalai [2004, 2010]).
(i) Generalized social welfare functions based on voting games that aggregate information lead to indeterminacy. In particular this applies when the maximum Shapley value tends to zero.

(ii) Generalized social welfare functions based on voting games where the maximum Banzhaf value tends to zero lead to stochastic indeterminacy.

(iii) Generalized social welfare functions based on noise-sensitive voting games lead to complete chaos.

3.1.11 Discussion. *Original contexts for some of the results.* Voting games are also called monotone Boolean functions and some of the results we discussed were proved in this context. Aggregation of information is also referred to as the sharp threshold phenomenon, which is important in the study of random graphs, percolation theory, and other areas. Theorem 3.5 was studied in the context of distributed computing and the question of collective coin flipping: procedures allowing n agents to reach a random bit. Theorem 3.3 was studied in the context of critical planar percolation. Theorem 3.2 was studied in the context of the combinatorics and probability of Boolean functions. "The majority is stablest" theorem was studied both in the context of hardness of approximation for the MAX CUT problem (see Section 3.5), and in the context of social choice. Arrow's theorem and Theorem 3.10 had immense impact on theoretical economics and political science. There is a large body of literature with extensions and interpretations of Arrow's theorem, and related phenomena were considered by many. Let me mention the books Peleg [1984] and M. Balinski and Laraki [2010] on voting methods that attempt to respond to the challenge posed by Arrow's theorem. Most proofs of the results discussed here go through Fourier analysis of Boolean functions that we discuss in Section 3.2.1.

A little more on cooperative games. I did not tell you yet about the most important solution concept in cooperative game theory (irrelevant to voting games): the core. The core of the game is an assignment of $v(N)$ to the n players so that the members of every coalition S get together at least $v(S)$. Bondareva and Shapley found necessary and sufficient conditions for the core to be nonempty (closely related to linear programming duality). I also did not talk about games without side payments. There, $v(S)$ are sets of vectors that describe the possible payoffs for the player in S if they go together. A famous game with no side payment is Nash's bargaining problem for two players. Now, you are just one step away from one of the deepest and most beautiful results in game theory, Scarf's conditions (1967) for nonemptiness of the core.

But what about real-life elections? The relevance and interpretation of mathematical modeling and results regarding voting rules, games, economics, and social science is a fairly involved matter. It is interesting to examine some notions discussed here in the light of election polls, which are often based on a more detailed model. Nate Silver's detailed forecasts provide a special opportunity. Silver computes the probability of victory for every candidate based on running many noisy simulations, which are in turn based on the outcomes of individual polls. The data in Silver's forecast contain an estimation for the event "recount," which essentially measures noise sensitivity, and it would be interesting to compare noise sensitivity in this more realistic scenario to the simplistic model of i.i.d. voters' behavior. Silver also computes certain power indices based on the probability of pivotality, again, under his model.

But what about real-life elections (2)? Robert Aumann remembers a Hebrew University math department meeting convened to choose two new members from among four very serious candidates. The chairman, a world-class mathematician, asked Aumann for a voting procedure. Aumann referred him to Bezalel Peleg, an expert on social choice and voting methods. The method Peleg suggested was adopted, and two candidates were chosen accordingly. The next day, the chairman met Aumann and complained that a majority of the department opposed the chosen pair, indeed preferred a specific different pair! Aumann replied: "Yes, but there is another pair of candidates that the majority prefers to yours, and yet another pair that the majority prefers to THAT one; and the pair elected is preferred by the majority to that last one! Moreover, there is a theorem that says that such situations cannot be avoided under any voting rule." The chairman was not happy and said dismissively: "Ohh, you guys and your theorems."

3.2 Boolean functions and their Fourier analysis. We start with the discrete cube $\Omega_n = \{-1, 1\}^n$. A Boolean function is a map $f : \Omega_n \to \{-1, 1\}$.

Remark 3.13. A Boolean function represents a family of subsets of $[n] = \{1, 2, \dots, n\}$ (also called a hypergraph), which are central objects in extremal combinatorics. Of course, voting games are monotone Boolean functions. We also note that in agreement with Murphy's law, roughly half of the time it is convenient to consider additive notation, namely, to regard $\{0, 1\}^n$ as the discrete cube and Boolean functions as functions to $\{0, 1\}$. (The translation is $0 \to 1$ and $1 \to -1$.)

3.2.1 Fourier. Every real function $f : \Omega_n \to \mathbb{R}$ can be expressed in terms of the Fourier–Walsh basis (we write here and for the rest of the paper $[n] = \{1, 2, \dots, n\}$):

$$f = \sum \{\hat{f}(S) W_S : S \subset [n]\}, \tag{3-1}$$

where the *Fourier–Walsh function* W_S is simply the monomial $W_S(x_1, x_2, \dots, x_n) = \prod_{i \in S} x_i$.

Note that we have here 2^n functions, one for each subset S of $[n]$. Let μ be the uniform probability measure on Ω_n. The functions W_S form an orthonormal basis of $\mathbb{R}^{\Omega_n}$ with respect to the inner product

$$\langle f, g \rangle = \sum_{x \in \Omega_n} \mu(x) f(x) g(x).$$

The coefficients $\hat{f}(S) = \langle f, W_s \rangle$, $S \subset [n]$, in (3-1) are real numbers, called the *Fourier coefficients* of f. Given a real function f on the discrete cube with Fourier expansion $f = \sum \{\hat{f}(S) W_S : S \subset [n]\}$, the noisy version of f, denoted by $T_\rho(f)$, is defined by $T_\rho(f) = \sum \{\hat{f}(S)(\rho)^{|S|} W_S : S \subset [n]\}$.

3.2.2 Boolean formulas, Boolean circuits, and projections. (Here it is convenient to think about the additive convention.) Formulas and circuits allow us to build complicated Boolean functions from simple ones and they are of crucial importance in computational complexity. Starting with n variables $x_1, x_2, \dots, x_n$, a *literal* is a variable x_i

or its negation $\neg x_i$. Every Boolean function can be written as a formula in conjunctive normal form, namely as AND of ORs of literals. A *circuit* of depth d is defined inductively as follows. A circuit of depth zero is a literal. A circuit of depth one consists of an OR or AND *gate* applied to a set of literals, a circuit of depth k consists of an OR or AND gate applied to the outputs of circuits of depth $k-1$. (We can assume that gates in the odd levels are all OR gates and that the gates of the even levels are all AND gates.) The size of a circuit is the number of gates. Formulas are circuits where we allow the output of a gate to be used as the input of only one other gate. Given a Boolean function $f(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m)$ we can look at its projection $g(x_1, x_2, \ldots, x_n)$ on the first n variables. $g(x_1, x_2, \ldots, x_n) = 1$ if there are values $a_1, a_2, \ldots, a_m)$ (depending on the x_is) such that $f(x_1, x_2, \ldots, x_n, a_1, a_2, \ldots, a_m) = 1$. Monotone formulas and circuits are those where all literals are variables (without negation).

Graph properties. A large important family of examples is obtained as follows. Consider a property P of graphs on m vertices. Let $n = m(m-1)/2$, associate Boolean variables with the n edges of the complete graph K_m, and represent every subgraph of K_m by a vector in Ω_n. The property P is now represented by a Boolean function on Ω_n. We can also start with an arbitrary graph H with n edges and for every property P of subgraphs of H obtain a Boolean function of n variables based on P.

3.3 Noise sensitivity everywhere (but mainly percolation). One thing we learned over the years is that noise sensitivity is (probably) a fairly common phenomenon. This is already indicated by Theorem 3.3. Proving noise sensitivity can be difficult. I will talk in this section about results on the critical planar percolation model, and conclude with a problem by Benjamini and Brieussel. I will not be able to review here many other noise-sensitivity results that justify the name of the section.

3.3.1 Critical planar percolation. The crossing event for planar percolation refers to an n by n square grid and to the event, when every edge is chosen with probability 1/2, that there is a path crossing from the left side to the right side of the square.

Theorem 3.14 (Benjamini, Kalai, and Schramm [1999]). *The crossing event for percolation is sensitive to* $1/o(\log n)$ *noise.*

Theorem 3.15 (Schramm and Steif [2010]). *The crossing event for percolation is sensitive to* $(n^{-c+o(1)})$ *noise, for some* $c > 0$.

Theorem 3.16 (Garban, Pete, and Schramm [2010]). *The crossing event for (hex) percolation is sensitive to* $(n^{-(3/4)+o(1)})$ *noise. The spectral distribution has a scaling limit and it is supported by Cantor-like sets of Hausdorff dimension 3/4.*

Remark 3.17 (Connection to algorithms). The proof of Schramm and Steif is closely related to the model of computation of random decision trees. Decision tree complexity refers to a situation where given a Boolean function we would like to find its value by asking as few questions as possible about specific instances. Random decision trees allow us to add randomization into the choice of the next question. These relations are explored in O'Donnell, Saks, Schramm, and Servedio [2005], and have been very useful in recent works in percolation theory.

Remark 3.18 (Connection to games). Critical planar percolation is closely related to the famous game of Hex. Peres, Schramm, Sheffield, and Wilson [2007] studied random-turn Hex where a coin flip determines the identity of the next player to play. They found a simple but surprising observation that the value of the game when both players play the random-turn game optimally is the same as when both players play randomly. Richman considered auction-based-turn games. Namely, the players bid on who will play the next round. A surprising, very general analysis (Lazarus, Loeb, Propp, Stromquist, and Ullman [1999]) shows that the value of the random-turn game is closely related to that of the auction-based-turn game! Nash famously showed that for ordinary Hex, the first player wins, but his proof gives no clue as to the winning strategy.

3.3.2 Spectral distribution and pivotal distribution. Let f be a monotone Boolean function with n variables. We can associate to f two important probability distributions on subsets of $\{1, 2, \ldots, n\}$. The spectral distribution of f, $\mathcal{S}(f)$ gives a set S a probability $\hat{f}^2(S)$. Given $x \in \Omega_n$ the ith variable is *pivotal* if when we flip the value of x_i the value of f is flipped as well. The pivotality distribution $\mathcal{P}(f)$ gives to a set S the probability that S is the set of pivotal variables. It is known that the first two moments of $\mathcal{S}$ and $\mathcal{P}$ agree.

Problem 7. Find further connections between $\mathcal{S}(f)$ and $\mathcal{P}(f)$ for all Boolean functions and for specific classes of Boolean functions.

Conjecture 8. Let f represent the crossing event in planar percolation. Show that $H(\mathcal{S}(f)) = O(I(f))$ and $H(\mathcal{P}(f)) = O(I(f))$. (Here H is the entropy function.)

The first inequality is a special case of the entropy-influence conjecture of Friedgut and Kalai [1996] which applies to general Boolean functions. The second inequality is not so general. We note that if f is in **P** the pivotal distribution can be efficiently sampled. The spectral distribution can be efficiently sampled on a quantum computer (Section 4.2.2).

3.3.3 First-passage percolation. Consider an infinite planar grid where every edge is assigned a length: 1 with probability 1/2 and 2 with probability 1/2 (independently). This model of a random metric on the planar grid is called first-passage percolation. An old question is to understand what is the variance $V(n)$ of the distance D from $(0, 0)$ to $(n, 0)$. Now, let M be the median value of D and consider the Boolean function f describing the event "$D \geq M$." Is f noise sensitive?

Benjamini, Kalai, and Schramm [2003] showed, more or less, that f is sensitive to a logarithmic level of noise, and concluded that $V(n) = O(n/\log n)$. To show that f is sensitive to a noise level of n^{δ} for $\delta > 0$ would imply that $V(n) = O(n^{1-c})$. A very interesting question is whether methods used for critical planar percolation for obtaining stronger noise sensitivity results can also be applied here.

3.3.4 A beautiful problem by Benjamini and Brieussel. Consider an n-step simple random walk (SRW) X_n on a Cayley graph of a finitely generated infinite group Γ. Refresh independently each step with probability ϵ, to get Y_n from X_n. Are there groups

for which at time n the positions X_n and Y_n are asymptotically independent? That is, does the l_1 (total variation) distance between the chain (X_n, Y_n) and two independent copies (X_n', X_n'') go to 0, as $n \to \infty$?

Note that on the line $\mathbb{Z}$, they are uniformly correlated, and therefore also on any group with a nontrivial homomorphism to $\mathbb{R}$, or on any group that has a finite index subgroup with a nontrivial homomorphism to $\mathbb{R}$. On the free group and for any non-Liouville group, X_n and Y_n are correlated as well, but for a different reason: both X_n and Y_n have a nontrivial correlation with X_1. Itai Benjamini and Jeremie Brieussel conjecture that these are the only ways not to be noise sensitive. That is, if a Cayley graph is Liouville and the group does not have a finite index subgroup with a homomorphism to the reals, then the Cayley graph is noise sensitive for the simple random walk. In particular, the Grigorchuk group is noise sensitive for the simple random walk!

3.4 Boolean complexity, Fourier, and noise.

3.4.1 P $\neq$ NP – circuit version. The **P** $\neq$ **NP**-conjecture (in a slightly stronger form) asserts that the Boolean function described by the graph property of containing a Hamiltonian cycle cannot be described by a polynomial size circuit. Equivalently, the circuit form of the **NP** $\neq$ **P**-conjecture asserts that there are Boolean functions that can be described by polynomial-size nondeterministic circuits, namely as the projection to n variables of a polynomial-size circuit, but cannot be described by polynomial-size circuits. A Boolean function f is in **co-NP** if $-f$ is in **NP**.

Remark 3.19. Projection to n variables of a Boolean function in **co-NP** is believed to enlarge the family of functions even further. The resulting class is denoted by Π_P^2 and the class of functions $-f$ when $f \in \Pi_P^2$ is denoted by Σ_P^2. By repeating the process of negating and projecting we reach a whole hierarchy of complexity classes, **PH**, called the polynomial hierarchy.

3.4.2 Well below P. The class **NC** describes Boolean functions that can be expressed by polynomial-size polylogarithmical-depth Boolean circuits. This class (among others) is used to model the notion of parallel computing. Considerably below **NC**, the class $\mathbf{AC}^0$ describes Boolean functions that can be expressed by bounded-depth polynomial-size circuits, where we allow AND and OR gates to apply to more than two inputs. A celebrated result in computational complexity asserts that majority and parity do not belong to $\mathbf{AC}^0$. However, the noise stability of majority implies that majority can be well approximated by functions in $\mathbf{AC}^0$. $\mathbf{mAC}^0$ is the class of functions described by bounded-depth polynomial-size monotone circuits. We note that functions in $\mathbf{AC}^0$ are already very complex mathematical objects.

A monotone threshold circuit is a circuit built from gates that are are weighted majority functions (without negations). A general threshold circuit is a circuit built from gates that are threshold linear functions, i.e. we allow negative weights. $\mathbf{TC}^0$ ($\mathbf{mTC}^0$) is the class of functions described by bounded-depth polynomial-size (monotone) threshold circuits.

3.4.3 Some conjectures on noise sensitivity and bounded-depth monotone threshold circuits.

Conjecture 9 (Benjamini, Kalai, and Schramm [1999]).
(i) Let f be a Boolean function described by a monotone threshold circuit of size M and depth D. Then f is stable to $(1/t)$-noise where $t = (\log M)^{100D}$.

(ii) Let f be a monotone Boolean function described by a threshold circuit of size M and depth D. Then f is stable to $(1/t)$-noise where $t = (\log M)^{100D}$.

The constant 100 in the exponent is, of course, negotiable. In fact, replacing $100D$ with any function of D will be sufficient for most applications. The best we can hope for is that the conjectures are true if t behaves like $t = (\log M)^{D-1}$. Part (i) is plausible but looks very difficult. Part (ii) is quite reckless and may well be false. (See, however, Problem 11 below.) Note that the two parts differ "only" in the location of the word "monotone."

There are many Boolean functions that are very noise sensitive. A simple example is the recursive majority on threes, denoted by RM3 and defined as follows. Suppose that $n = 3^m$. Divide the variables into three equal parts. Compute the RM3 separately for each of these parts and apply majority to the three outcomes. Conjecture 9 would have the following corollaries (C1)–(C4). Part (i) implies: (C1) – RM3 is not in $\mathbf{mTC}^0$, and even C2 – RM3 cannot be approximated by a function in $\mathbf{mTC}^0$. Yao [1989] and Håstad and Goldmann [1991] proved a variant of (C1) and these results motivated our conjecture. (C2) already seems well beyond reach. Part (ii) implies: (C3) – RM3 is not in $\mathbf{TC}^0$ and (C4) – RM3 cannot be approximated by a function in $\mathbf{TC}^0$. (We can replace RM3 with other noise-sensitive properties like the crossing event in planar percolation.)

3.4.4 Bounded depth Boolean circuits and the reverse Håstad conjecture. For a monotone Boolean function f on Ω_n, a Fourier description of the total influence is $I(f) = \sum \hat{f}^2(S)|S|$, and we can take this expression as the definition of $I(f)$ for nonmonotone functions as well. The following theorem describes some connections between functions in $\mathbf{AC}^0$, influence and Fourier. The second and third items are based on Håstad's switching lemma.

Theorem 3.20.
(i) Boppana [n.d.]: If f is a (monotone) Boolean function that can be described by a depth-D size-M monotone Boolean circuit then $I(f) \le C(\log M)^{D-1}$.

(ii) (Håstad [1986] and Boppana [1997]) If f is a function that can be described by a depth-D size-M Boolean circuit then $I(f) \le C(\log M)^{D-1}$.

(iii) Linial, Mansour, and Nisan [1993]; improved by Håstad [2001]: If f is a function that can be described by a depth-D size-M monotone Boolean circuit then $\{\sum \hat{f}^2(S) : |S| = t\}$ decays exponentially with t when $t > C(\log M)^{D-1}$.

We conjecture that functions with low influence can be approximated by low-depth small-size circuits. A function g ϵ-approximates a function f if $\mathbb{E}(f-g)^2 \le \epsilon$.

Conjecture 10 (Benjamini, Kalai, and Schramm [1999]). For some absolute constant C the following holds. A Boolean function f can be 0.01-approximated by a circuit of depth d of size M where $(\log M)^{Cd} \le I(f)$.

3.4.5 Positive vs. monotone. We stated plausible but hard conjectures on functions in $\mathbf{mTC}^0$ and reckless and perhaps wrong conjectures on monotone functions in $\mathbf{TC}^0$. But we cannot present a single example of a monotone function in $\mathbf{TC}^0$ that is not in $\mathbf{mTC}^0$. To separate the conjectures we need monotone functions in $\mathbf{TC}^0$ that cannot even be approximated in $\mathbf{mTC}^0$. Ajtai and Gurevich [1987] proved that there are monotone functions in $\mathbf{AC}^0$ that are not in $\mathbf{mAC}^0$.

Problem 11.
(i) Are there monotone functions in $\mathbf{AC}^0$ that cannot be approximated by functions in $\mathbf{mAC}^0$?

(ii) Are there monotone functions in $\mathbf{TC}^0$ that are not in $\mathbf{mTC}^0$?

(iii) Are there monotone functions in $\mathbf{TC}^0$ that cannot be approximated by functions in $\mathbf{mTC}^0$?

3.5 A taste of PCP, hardness of approximation, and Max Cut. A vertex cover of a graph G is a set of vertices such that every edge contains a vertex in the set. Vertex Cover is the algorithmic problem of finding such a set of vertices of minimum size. Famously, this problem is an **NP**-complete problem; in fact, it is one of the problems in Karp's original list. A matching in a graph is a set of edges such that every vertex is included in at most one edge. Given a graph G there is an easy efficient algorithm for finding a maximal matching. Finding a maximal matching with r edges with respect to inclusion, gives us at the same time a vertex cover of size $2r$ and a guarantee that the minimum size of a vertex cover is at least r. A very natural question is to find an efficient algorithm for a better approximation. There is by now good evidence that this might not be possible. It is known to derive Khot and Regev [2003] from Khot's unique game conjecture Khot [2002].

A cut in a graph is a partition of the vertices into two sets. The Max Cut problem is the problem of finding a cut with the maximum number of edges between the parts. Also this problem is **NP**-complete, and in Karp's list. The famous Goemans–Williamson algorithm based on semidefinite programming achieves α-approximation for max cut where $\alpha_{GM} = .878567$. Is there an efficient algorithm for a better approximation? There is by now good evidence that this might not be possible.

3.5.1 Unique games, the unique game conjecture, and the PCP theorem. We have a connected graph and we want to color it with colors from a set Σ. For every edge e we are given an orientation of the edge and a permutation π_e on Σ. In a good coloring of the edge, if the tail is colored c then the head must be colored $\pi_e(c)$. It is easy to check efficiently if a global good coloring exists since coloring one vertex forces the coloring of all the others.

Given ϵ, δ, the unique game problem is for a graph G, a set of colors Σ, and a permutation constraint for each edge, to decide algorithmically between two scenarios (when we are promised that one of them holds):

(i) There is no coloring for which more than a fraction ϵ of the edges are colored good.

(ii) There is a coloring for which at least a fraction $1 - \delta$ of the edges are colored good.

The unique game conjecture asserts that for every $\epsilon > 0$ and $\delta > 0$ it is NP-hard to decide between these two scenarios.

If one does not insist on the constraints being permutations and instead allows them to be of general form, then the above holds, and it is called the PCP Theorem – one of the most celebrated theorems in the theory of computation.

Remark 3.21. A useful way to describe the situation (which also reflects the historical path leading to it) is in terms of a three-player game in which there are two "provers" and a verifier. A verifier is trying to decide which of the two cases he is in, and can communicate with two all powerful (noncommunicating) provers. To do so, the verifier samples an edge, and sends one endpoint to each prover. Upon receiving their answers, the verifier checks that the two colors satisfy the constraint. The provers need to convince the verifier that a coloring exists by giving consistent answers to simultaneous questions drawn at random.

3.5.2 The theorem of Khot, Kindler, Mossel, and O'Donnell.

Theorem 3.22 (Khot, Kindler, Mossel, and O'Donnell [2007]). *Let $\beta > \alpha_{GM}$ be a constant. Then an efficient β-approximation algorithm for* Max Cut *implies an efficient algorithm for unique games.*

The reduction relies on the "majority is stablest" theorem (Theorem 3.4), which was posed by Khot, Kindler, Mossel, and O'Donnell as a conjecture and later proved by Mossel, O'Donnell, and Oleszkiewicz. This result belongs to the theory of hardness of approximation and probabilistically checkable proofs (PCPs), which is among the most important areas developed in computational complexity in the past three decades. For quite a few problems in Karp's original list of NP-complete problems (and many other problems added to the list), there is good evidence that the best efficient approximation is achieved by a known relatively simple algorithm. For a large class of problems it is even known [Raghavendra 2008] (based on hardness of the unique game problem) that the best algorithm is either a very simple combinatorial algorithm (like that for Vertex Cover), or a more sophisticated application of semidefinite programming (like that for Max Cut). I will give a quick and very fragmented taste of three ingredients of the proof of Theorem 3.22.

The noisy graph of the cube. The proof of the hardness of max cut relative to unique games is based on the weighted graph whose vertices are the vertices of the discrete cube, all pairs are edges, and the weight of an edge between two vertices of distance k is $(1-p)^k p^{n-k}$. It turns out that in order to analyze the reduction, it suffices to study the structure of good cuts in this very special graph.

The least efficient error-correcting codes. Error-correcting codes have, for many decades, been among the most celebrated applications of mathematics, with a huge impact on technology. They also play a prominent role in theoretical computer science and in PCP theory. The particular code needed for max cut is the following: encode a number k between 1 to n (and thus $\log n$ bits) by a Boolean function: a dictatorship where the kth variable is the dictator!

Testing dictatorship. An important ingredient of a PCP proof is "property testing," i.e., testing by looking at a bounded number of values if a Boolean function satisfies a certain property, or is very far from satisfying it. In our case we would like to test (with a high probability of success) if a Boolean function is very far from dictatorship, or has substantial correlation with it. The test is the following: choose x at random, let $y = N_\epsilon(-x)$. Test if $f(x) = -f(y)$. For the majority function the probability that majority passes the test is roughly $\arccos(\epsilon - 1)$; the "majority is stablest" theorem implies that anything that is more stable has a large correlation with a dictator.

3.5.3 Discussion: integrality gap and polytope integrality gap. Given a graph G and nonnegative weights on its vertices, the weighted version of VERTEX COVER is the algorithmic problem of finding a set of vertices of minimum weight that covers all edges.

Minimize $w_1x_1 + w_2x_2 + \cdots + w_nx_n$ where $x = (x_1, x_2, \ldots, x_n)$ is a 0-1 vector,
subject to: $x_i + x_j \geq 1$ for every edge $\{i, j\}$.

Of course, this more general problem is also **NP**-complete. The linear programming relaxation allows x_is to be real and belong to the interval [0,1]. The *integrality gap* for general vertex cover problems is 2 and given the solution to the linear programming problem you can just consider the set of vertices i so that $x_i \geq 1/2$. This will be a cover and the ratio between this cover and the optimal one is at most 2. The integrality gap for the standard relaxation of max cut is $\log n$. The integrality gap is an important part of the picture in PCP theory. I conclude with a beautiful problem that I learned from Anna Karlin.

Consider the integrality gap (called the *polytope integrality gap*) between the covering problem and the linear programming relaxation when the graph G is fixed. In greater generality, consider a general covering problem of maximizing $c^t x$ subject to $Ax \leq b$ where A is an integral matrix of nonnegative integers. Next, considered the integrality gap between 0-1 solutions and real solutions in $[0, 1]$ when A and b are fixed (thus the feasible polyhedron is fixed, and hence the name "polytope integrality gap") and only c (the objective function) varies. The problem is if, for VERTEX COVER for every graph G and every vector of weights, there is an efficient algorithm achieving the polytope integrality gap. The same question can be asked about a polytope integrality gap of arbitrary covering problems.

4 The quantum computer challenge

To Robert Aumann, Maya Bar-Hillel, Dror Bar-Nathan, Brendan McKay, and Ilya Rips who trained me as an applied mathematician.

4.1 Quantum computers and noise. Recall that the basic memory component in classical computing is a "bit," which can be in two states, "0" or "1." A computer, as modeled by a Boolean circuit, has n bits and it can perform certain logical operations on them. The NOT gate, acting on a single bit, and the AND gate, acting on two bits, suffice for *universal* classical computing. This means that a computation based on another collection of logical gates, each acting on a bounded number of bits, can be replaced by

a computation based only on NOT and AND. Classical circuits equipped with random bits lead to randomized algorithms, which, as mentioned before, are both practically useful and theoretically important. Quantum computers allow the creation of probability distributions that are well beyond the reach of classical computers with access to random bits.

4.1.1 Quantum circuits. A qubit is a piece of quantum memory. The state of a qubit can be described by a unit vector in a two-dimensional complex Hilbert space H. For example, a basis for H can correspond to two energy levels of the hydrogen atom, or to horizontal and vertical polarizations of a photon. Quantum mechanics allows the qubit to be in a *superposition* of the basis vectors, described by an arbitrary unit vector in H. The memory of a quantum computer ("quantum circuit") consists of n qubits. Let H_k be the two-dimensional Hilbert space associated with the kth qubit. The state of the entire memory of n qubits is described by a unit vector in the tensor product $H_1 \otimes H_2 \otimes \cdots \otimes H_n$. We can put one or two qubits through *gates* representing unitary transformations acting on the corresponding two- or four-dimensional Hilbert spaces, and as for classical computers, there is a small list of gates sufficient for universal quantum computing. At the end of the computation process, the state of the entire computer can be *measured*, giving a probability distribution on 0–1 vectors of length n.

A few words on the connection between the mathematical model of quantum circuits and quantum physics: in quantum physics, states and their evolutions (the way they change in time) are governed by the Schrödinger equation. A solution of the Schrödinger equation can be described as a unitary process on a Hilbert space and quantum computing processes of the kind we just described form a large class of such quantum evolutions.

Remark 4.1. Several universal classes of quantum gates are described in Nielsen and Chuang [2000, Ch. 4.5]. The gates for the IBM quantum computer are eight very basic one-qubit gates, and the 2-qubit CNOT gate according to a certain fixed directed graph. This is a universal system and in fact, an over complete one.

4.1.2 Noise and fault-tolerant computation. The main concern regarding the feasibility of quantum computers has always been that quantum systems are inherently noisy: we cannot accurately control them, and we cannot accurately describe them. The concern regarding noise in quantum systems as a major obstacle to quantum computers was put forward in the mid-90s by Landauer [1995], Unruh [1995], and others.

What is noise? As we said already, solutions of the Schrödinger equation ("quantum evolutions") can be regarded as unitary processes on Hilbert spaces. Mathematically speaking, the study of noisy quantum systems is the study of *pairs* of Hilbert spaces (H, H'), $H \subset H'$, and a unitary process on the larger Hilbert space H'. Noise refers to the general effect of neglecting degrees of freedom, namely, approximating the process on a large Hilbert space by a process on the small Hilbert space. For controlled quantum systems and, in particular, quantum computers, H represents the controlled part of the system, and the large unitary process on H' represents, in addition to an "intended" controlled evolution on H, also the uncontrolled effects of the *environment*. The study

of noise is relevant, not only to controlled quantum systems, but also to many other aspects of quantum physics.

A second, mathematically equivalent way to view noisy states and noisy evolutions, is to stay with the original Hilbert space H, but to consider a mathematically larger class of states and operations. In this view, the state of a noisy qubit is described as a classical probability distribution on unit vectors of the associated Hilbert spaces. Such states are referred to as *mixed states*.

It is convenient to think about the following simple form of noise, called *depolarizing noise*: in every computer cycle a qubit is not affected with probability $1 - p$, and, with probability p, it turns into the *maximal entropy mixed state*, i.e., the average of all unit vectors in the associated Hilbert space.

Remark 4.2. It is useful to distinguish between the model error rate, which is p in the above example, and the effective error rate, which is the probability that a qubit is corrupted at a computation step, conditioned on it having survived up to this step. The effective error rate depends not only on the model error rate but also on the computation sequence. When the computation is nontrivial (for example, for pseudo-random circuits) the effective error rate grows linearly with the number of qubits.[4] This is a familiar fact that is taken into account by the threshold theorem described below.

To overcome noise, a theory of quantum fault-tolerant computation based on quantum error-correcting codes was developed. Fault-tolerant computation refers to computation in the presence of errors. The basic idea is to represent (or "encode") a single (logical) qubit by a large number of physical qubits, so as to ensure that the computation is robust even if some of these physical qubits are faulty.

Theorem 4.3 (Threshold theorem – informal statement [Aharonov and Ben-Or 1999; Kitaev 1997; Knill, Laflamme, and Zurek 1998])**.** *When the level of noise is below a certain positive threshold ρ, noisy quantum computers allow universal quantum computation.*

Theorem 4.3 shows that once high-quality quantum circuits are built for roughly 100–500 qubits then it will be possible in principle to use quantum error-correction codes to amplify this achievement for building quantum computers with an unlimited number of qubits. The interpretation of this result took for granted that quantum computers with a few dozen qubits are feasible, and this is incorrect.

Let A be the maximal number of qubits for which a reliable quantum circuit can be engineered. Let B be the number of qubits required for good quantum error-correcting codes needed for quantum fault-tolerance. B is in the range of 100–1000 qubits.

The optimistic scenario: $A > B$.

The pessimistic scenario: $B > A$.

As we will see, there are good theoretical reasons for the pessimistic scenario (even for $B \gg A$) as well as interesting consequences from it. We emphasize that both scenarios are compatible with quantum mechanics.

[4] We refer to error rate in terms of qubit errors, which are relevant to quantum error-correction. There is no difference between model error rate and effective error rate for rate based on trace distance.

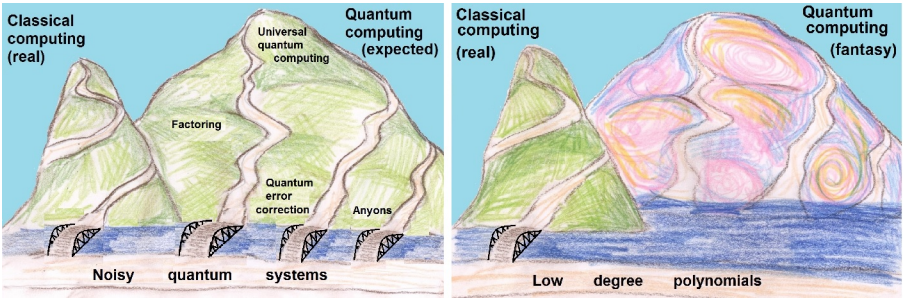

Figure 6: **Two scenarios: Left** – Quantum fault-tolerance mechanisms, via quantum error-correction, allow robust quantum information and computationally superior quantum computation. **Right** – Noisy quantum evolutions, described by low-degree polynomials, allow, via the mechanisms of averaging/repetition, robust *classical* information and computation, but do not allow reaching the starting points for quantum supremacy and quantum fault-tolerance. Drawing by Neta Kalai.

4.2 Complexity 6: quantum computational supremacy.

4.2.1 Factoring is in BQP. Recall that computational complexity is the theory of *efficient computations*, where "efficient" is an asymptotic notion referring to situations where the number of computation steps ("time") is at most a polynomial in the number of input bits. We already discussed the complexity classes **P** and **NP**, and let us (abuse notation and) allow classical randomization to be added to all the complexity classes we discuss. There are important intermediate problems between **P** and **NP**. Factoring – the task of factoring an n-digit integer to its prime decomposition is not known to be in **P**, as the best algorithms are exponential in the cube root of the number of digits. Factoring is in **NP**, hence it is unlikely that factoring is **NP**-complete. Practically, Factoring is hard and this is the basis for most of the current cryptosystems.

The class of decision problems that quantum computers can efficiently solve is denoted by **BQP**. Shor's algorithm shows that quantum computers can factor n-digit integers efficiently – in $\sim n^2$ steps! Quantum computers are not known to be able to solve **NP**-complete problems efficiently, and there are good reasons to think that they cannot do so. However, quantum computers can efficiently perform certain computational tasks beyond **NP** and even beyond **PH**.

The paper by Shor [1999] presenting an efficient factoring algorithm for quantum computers is among the scientific highlights of the 20th century, with an immense impact on several theoretical and experimental areas of physics.

4.2.2 Fourier sampling, boson sampling, and other quantum sampling. As mentioned in Section 2, exact and approximate sampling are important algorithmic tasks on their own and as subroutines for other tasks. Quantum computers enable remarkable new forms of sampling. Quantum computers would allow the creation of probability distributions that are beyond the reach of classical computers with access to random

bits. Let QUANTUM SAMPLING denote the class of distributions that quantum computers can efficiently sample. An important class of such distributions is FOURIER SAMPLING. Start with a Boolean function f. (We can think of $f(x)$ as the winner in HEX.) If f is in **P** then we can classically sample $f(x)$ for a random $x \in \Omega_n$. With quantum computers we can do more. A crucial ability of quantum computers is to prepare a state $2^{-n/2} \cdot \sum f(x)|x>$, which is a superposition of all 2^n vectors weighted by the value of f. Next a quantum computer can easily take the Fourier transform of f and thus sample exactly a subset S according to $\hat{f}^2(S)$. This ability of quantum computers goes back essentially to Simon [1997], and is crucial for Shor's factoring algorithm.

Another important example is BOSON SAMPLING, which refers to a class of probability distributions (that quantum computers can efficiently create) representing a collection of noninteracting bosons. BOSON SAMPLING was introduced by Troyansky and Tishby [1996] and was intensively studied by 2013, who offered it as a quick path for experimentally showing that quantum supremacy is a real phenomenon. Given an n by n matrix A, let $per(A)$ denote the permanent of A. Let M be a complex $n \times m$ matrix, $m \geq n$, with orthonormal rows. Consider all $\binom{m+n-1}{n}$ sub-multisets S of n columns (namely, allow columns to repeat), and for every sub-multiset S consider the corresponding $n \times n$ submatrix A (with column i repeating r_i times). BOSON SAMPLING is the algorithmic task of sampling these multisets S according to $|per(A)|^2/(r_1!r_2!\cdots r_n!)$.

4.2.3 Hierarchy collapse theorems. Starting with Terhal and DiVincenzo [2004] there has been a series of works showing that it is very unreasonable to expect a classical computer to perform quantum sampling even regarding distributions that express very limited quantum computing. (Of course quantum computers can perform these sampling tasks.)

Theorem 4.4 (Terhal and DiVincenzo [ibid.], Aaronson and Arkhipov [2013], Bremner, Jozsa, and Shepherd [2011])**.** *If a classical computer can exactly sample according to either*

(i) General QUANTUM SAMPLING,

(ii) BOSON SAMPLING,

(iii) FOURIER SAMPLING, or

(iv) Probability distributions obtained by bounded-depth polynomial-size quantum circuits,

then the polynomial hierarchy collapses.

The proof is by showing that if a classical computer that allows any of the sampling tasks listed above, is equipped with an **NP** oracle, then it is able to efficiently perform **#P-complete** computations. This implies that the class **#P** that includes **PH** already collapses to the third level in the polynomial hierarchy.

4.3 Computation and physics 1.

4.3.1 Variants of the Church–Turing thesis. The famous Church–Turing thesis (CTT) asserts that everything computable is computable by a Turing machine. Although initially this was a thesis about computability, there were early attempts to relate it to

physics, namely, to assert that physical devices obey the CTT. The efficient (or strong) Church–Turing thesis (ECTT) in the context of feasible computations by physical devices was considered early on by Wolfram [1985], Pitowsky [1990] and others. It asserts that only efficient computations by a Turing machine are feasible physical computation. Quantum computers violate the ECTT. The pessimistic scenario brings us back to the ECTT, and, in addition, it proposes an even stronger limitation for "purely quantum processes" (suggested from Kalai and Kindler [2014]).

> Unitary evolutions that can be well approximated by physical devices can be approximated by low-degree polynomials, and are efficiently learnable.

The following **NPBS**-principle (no primitive-based supremacy) seems largely applicable in the interface between practice and theory in the theory of computing.

> Devices that express (asymptotically) primitive (low-level) computational power cannot be engineered or programmed to achieve superior computational tasks.

4.3.2 What even quantum computers cannot achieve and the modeling of locality. The model of quantum computers already suggests important limitations on what local quantum systems can compute.

- *Random unitary operations on large Hilbert spaces.* A quantum computer with n qubits cannot reach a random unitary state since reaching such a state requires an exponential number of computer cycles. (Note also that since an ϵ-net of states for n-qubits quantum computer requires a set of size doubly exponential in n, most states are beyond the reach of a quantum computer.)

- *Reaching ground states for complex quantum systems.* A quantum computer is unlikely to be able to reach the ground state of a quantum system (that admits an efficient description). As a matter of fact, reaching a ground state is **NP**-complete even for classical systems and for quantum computing the relevant complexity class is an even larger **QMA**.

These limitations are based on the model of quantum computers (and the second also on $\mathbf{NP} \neq \mathbf{P}$) and thus do not formally follow from the basic framework of quantum mechanics (for all we know). They do follow from a principle of "locality" asserting that quantum evolutions express interactions between a small number of physical elements. This principle is modeled by quantum computers, and indeed a crucial issue in the debate on quantum computers is what is the correct modeling of local quantum systems. Let me mention three possibilities.

(A) The model of quantum circuits is the correct model for local quantum evolutions. Quantum computers are possible, the difficulties are matters of engineering, and quantum computational supremacy is amply manifested in quantum physics.

(B) The model of noisy quantum circuits is the correct model for local quantum evolutions. In view of the threshold theorem, quantum computers are possible and the remaining difficulties are matters of engineering.

(C) The model of noisy quantum circuits is the correct model for local quantum evolutions, and further analysis suggests that the threshold in the threshold theorem cannot be reached. Quantum circuits with noise above the threshold is the correct modeling of local quantum systems. Quantum computational supremacy is an artifact of incorrect modeling of locality.

Computational complexity insights (and some common sense) can assist us in deciding between these possibilities. While each of them has its own difficulties, in my view the third one is correct.

4.3.3 Feynman's motivation for quantum computing.

Conjecture 12 (Feynman [1981/82] motivation for quantum computation)**.** High energy physics computations, especially computations in QED (quantum electrodynamics) and QCD (quantum chromodynamics), can be carried out efficiently by quantum computers.

This question touches on the important mathematical question of giving rigorous mathematical foundations for QED and QCD computations. Efficient quantum computation for them will be an important (while indirect) step toward putting these theories on rigorous mathematical grounds. Jordan, Lee, and Preskill [2014] found an efficient algorithm for certain computations in (ϕ^4) quantum field theory for cases where a rigorous mathematical framework is available.

4.4 The low-scale analysis: Why quantum computers cannot work.

4.4.1 Noisy systems of noninteracting photons.

Theorem 4.5 (Kalai and Kindler [2014])**.** *When the noise level is constant, Boson Sampling distributions are well approximated by their low-degree Fourier–Hermite expansion. Consequently, noisy Boson Sampling can be approximated by bounded-depth polynomial-size circuits.*

It is reasonable to assume that for all proposed implementations of Boson Sampling the noise level is at least a constant and, therefore, an experimental realization of Boson Sampling represents, asymptotically, bounded-depth computation. In fact noisy Boson Sampling belongs to a computational class **LDP** (approximately sampling distributions described by bounded-degree polynomials) which is well below $\mathbf{AC}^0$. The next theorem shows that implementation of Boson Sampling will actually require pushing down the noise level to below $1/n$.

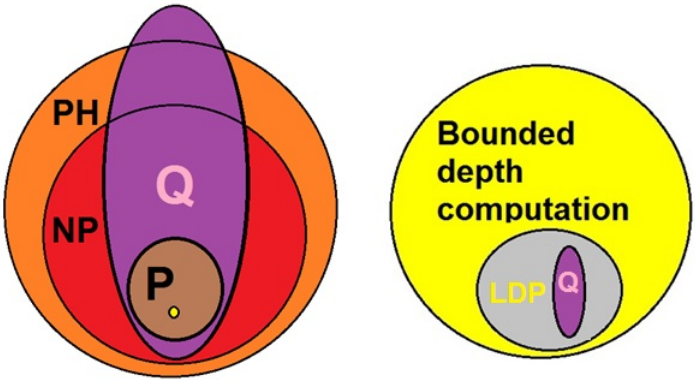

Figure 7: Quantum computers offer mind-boggling computational superiority (left), but in the small scale, noisy quantum circuits are computationally very weak, unlikely to allow quantum codes needed for quantum computers (right).

Theorem 4.6 (Kalai and Kindler [2014]). *When the noise level is $\omega(1/n)$, and $m \gg n^2$, Boson Sampling is very sensitive to noise with a vanishing correlation between the noisy distribution and the ideal distribution.*[5]

4.4.2 Noisy quantum circuits.

Conjecture 13.
(i) The insights for noisy Boson Sampling apply to all versions of realistic forms of noise.

(ii) These insights extend further to quantum circuits and other quantum devices in the small scale.

(iii) These insights extend even further to quantum devices, including microscopic processes, that do not use quantum error-correction.

(iv) This rules out quantum computational supremacy and the needed quantum error-correcting codes.

The first item seems a quite reasonable extension of Theorem 4.5. In fact, the argument applies with small changes to a physical modeling of mode-mismatch noise (when bosons are not fully indistinguishable). Each item represents quite a leap from the previous one. The last item expresses the idea that superior computation cannot be manifested by primitive asymptotic computational power. Theorem 4.5 put noisy Boson Sampling in a very low-level class, **LDP**, even well below $\mathbf{AC}^0$. It is not logically impossible but still quite implausible that such a primitive computing device will manifest superior computing power for 50 bosons.

Remark 4.7. *Why robust classical information and computation is possible and ubiquitous.* The ability to approximate low-degree polynomials still supports robust *classical*

[5]The condition $m \gg n^2$ can probably be removed by a more detailed analysis.

information. This is related to our second puzzle. The majority function allows for very robust bits based on a large number of noisy bits and admits excellent low-degree approximations. Both encoding (by some repetition procedure) and decoding (by majority or a variation of majority) that are needed for robust classical information are supported by low-degree polynomials.

4.5 Predictions regarding intermediate goals and near-term experiments.

- *Demonstrating quantum supremacy.* A demonstration of quantum computing supremacy, namely, crossing the line where classical simulation is possible, requires, e.g., building of pseudo-random quantum circuits of 50–70 qubits. As we mentioned in the Introduction, this idea can be partially tested already for quantum circuits with 10–30 qubits, and there are plans for a decisive demonstration on 50 qubits in the near future. Quantum supremacy could be demonstrated via implementation of Boson Sampling and in various other ways. Theorems 4.5 and 4.6 and the **NPBS** principle suggest that all these attempts will fail.

- *Robust quantum qubits via quantum error-correction.* The central goal toward quantum computers is to build logical qubits based on quantum error-correction, and a major effort is being made to demonstrate a distance-5 surface code that requires 100 or so qubits. It is now commonly agreed that this task is harder than "simply" demonstrating quantum computational supremacy. Therefore, the **NPBS** principle suggests that these attempts will fail as well.

- *Good-quality individual qubits and gates (and anyonic qubits).* The quality of individual qubits and gates is the major factor in the quality of quantum circuits built from them. The quantum computing analogue of Moore's law, known as "Schoelkopf's law," asserts that roughly every three years, quantum decoherence can be delayed by a factor of ten. The analysis leading to the first two items suggests that Schoelkopf's law will be broken before reaching the quality needed for quantum supremacy and quantum fault-tolerance. This is an indirect argument, but more directly, the microscopic process leading to the qubits (for all we know) also represents low level complexity power. This last argument also casts doubt on any hopes of reaching robust quantum qubits via anyons.

4.6 Computation and physics 2: noisy quantum systems above the noise threshold. Noisy quantum systems under the pessimistic scenario, namely, under the assumption that the noise level is above the fault-tolerance threshold are quite interesting! There are two basic premises for modeling noisy quantum evolutions under the pessimistic scenario. The first is that the modeling is implicit; namely, it is given in terms of conditions that the noisy process must satisfy, rather than a direct description for the noise. The second premise is that there are systematic relations between the (effective) noise and the entire quantum evolution and also between the target state and the noise.

4.6.1 Correlated errors and error synchronization. The following prediction regarding noisy entangled pairs of qubits is perhaps the simplest prediction on noisy quantum circuits under the pessimistic scenario. Entanglement is a name for quantum correlation, and it is an important feature of quantum physics and a crucial ingredient of quantum computation. A *cat state* of the form $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$ represents the simplest (and strongest) form of entanglement between two qubits.

Prediction 1: Two-qubit behavior. For any implementation of quantum circuits, cat states are subject to qubit errors with substantial positive correlation.

Error synchronization refers to a substantial probability that a large number of qubits, well beyond the average rate of noise, are corrupted. This is a very rare phenomenon for the model noise and, when quantum fault-tolerance is in place, error synchronization is an extremely rare event also for the effective noise.

Prediction 2: Error synchronization. For pseudo-random circuits, highly synchronized errors will necessarily occur.

Remark 4.8. Both predictions 1 and 2 can already be tested via the quantum computers of Google, IBM, and others. (It will be interesting to test prediction 1 even on gated qubits, where it is not in conflict with the threshold theorem, but may still be relevant to the required threshold constant.)

4.6.2 Modeling general noisy quantum systems.

Prediction 3: Bounded-degree approximations, and effective learnability. Unitary evolutions that can be approximated by noisy quantum circuits (and other devices) are approximated by low-degree polynomials and are efficiently learnable.

Prediction 4: Rate. For a noisy quantum system a lower bound for the rate of noise in a time interval is a measure of noncommutativity for the projections in the algebra of unitary operators in that interval.

Prediction 5: Convoluted time smoothing. Quantum evolutions are subject to noise with a substantial correlation with time-smoothed evolutions.

Time-smoothed evolutions form an interesting restricted class of noisy quantum evolutions aimed at modeling evolutions under the pessimistic scenario when quantum fault-tolerance is unavailable to suppress noise propagation. The basic example of time-smoothing is the following: start with an ideal quantum evolution given by a sequence of T unitary operators, where U_t denotes the unitary operator for the t-th step, $t = 1, 2, \ldots T$. For $s < t$ we denote $U_{s,t} = \prod_{i=s}^{t-1} U_i$ and let $U_{s,s} = I$ and $U_{t,s} = U_{s,t}^{-1}$. The next step is to add noise in a completely standard way: consider a noise operation E_t for the tth step. We can think about the case where the unitary evolution is a quantum-computing process and E_t represents a depolarizing noise at a fixed rate acting independently on the qubits. And, finally, replace E_t with a new noise operation E'_t defined as the average[6]

$$E'_t = \frac{1}{T} \cdot \sum_{s=1}^{T} U_{s,t} E_s U_{s,t}^{-1}. \tag{4-1}$$

[6]The need for smoothing into the future as well as into the past follows from Kalai and Kuperberg [2015].

Predictions 1–5 are implicit and describe systematic relations between the (effective) noise and the evolution. We expect that time-smoothing will suppress high terms for some Fourier-like expansion,[7] thus relating Predictions 3 and 5. Prediction 4 resembles the picture drawn by Polterovich [2014] of the "unsharpness principle" in symplectic geometry, quantization, and quantum noise.

Remark 4.9. It is reasonable to assume that time-dependent quantum evolutions are inherently noisy since time dependency indicates interaction with an environment. Two caveats: famously, time-dependent evolutions can be simulated by time-independent evolutions, but we can further assume that in such cases the noise lower bounds will transfer. Second, in the context of noise, the environment of, say, an electron refers also to its internal structure.

Remark 4.10. Physical processes are not close to unitary evolutions and there are systematic classical effects (namely, robust effects of interactions with a large "environment"). We certainly cannot model everything with low-degree polynomials. On the other hand, it is unlikely that natural physical evolutions express the full power of **P**. It will be interesting to understand the complexity of various realistic physical evolutions , and to identify larger relevant classes within **P**, especially classes for which efficient learnability is possible.

Remark 4.11. Let me list, for more details see [Kalai 2016b], a few features of noisy quantum evolutions and states "above the threshold." (a) *Symmetry.* Noisy quantum states and evolutions are subject to noise that respects their symmetries. (b) *Entropy lower bounds.* Within a symmetry class of quantum states/evolutions (or for classes of states defined in a different way), there is an absolute positive lower bound for entropy. (c) *Geometry.* Quantum states and evolutions reveal some information on the geometry of (all) their physical realizations. (d) *Fluctuation.* Fluctuations in the rate of noise for interacting N-element systems (even in cases where interactions are weak and unintended) scale like N and not like $\sqrt{N}$. (e) *Time.* The difficulty in implementing a local quantum computing process is not invariant under reversing time.

4.7 Our computational world. The emerging picture from our analysis is that the basic computational power of quantum devices is very limited: unitary evolutions described by noisy local quantum devices are confined to low-degree polynomials. It is classical information and computation that emerge via noise-stable encoding and decoding processes that enable the wealth of computation witnessed in nature. This picture offers many challenges, in its mathematical, physical, and computational aspects, and these can serve as a poor man's replacement for quantum supremacy dreams.

5 Conclusion

We have talked about three fascinating puzzles on mathematics and computation, telling a story that involves pure and applied mathematics, theoretical computer science, games of various kinds, physics, and social sciences. The connection and tension between

[7] Pauli expansion seems appropriate for the case of quantum circuits; see Montanaro and Osborne [2010].

the pure and the applied, between models and reality, and the wide spectrum between foundations and engineering is common to all three puzzles. We find great expectations, surprises, mistakes, disappointments, and controversies at the heart of our endeavor, while seeking truth and understanding in our logical, physical, and human reality. In our sweet professional lives, being wrong while pursuing dreams unfounded in reality is sometimes of value, and second only to being right.

Appendix: abstract objective functions and telling a polytope from its graph

The linear programming local-to-global principle has very nice connections and applications to the combinatorial theory of convex polytopes. Abstract linear objective functions (and Sharir–Welzl's abstract linear programming problems) are related to the notion of shellability. We will bring here one such application: Kalai [1988] – a beautiful proof that I found for the following theorem of Blind and Mani-Levitska [1987] conjectured by Micha A. Perles:

Theorem 5.1. *The combinatorial structure of a simple polytope P is determined by its graph.*

We recall that a d-polytope P is simple if every vertex belongs to exactly d edges. Thus the graph of P is a d-regular graph. For a simple polytope, every set of r edges containing a vertex v determines an r-face of P. Faces of simple polytopes are simple. Consider an ordering $\prec$ of the vertices of a simple d-polytope P. For a nonempty face F we say that a vertex v of F is a local maximum in F if v is larger w.r.t. the ordering $\prec$ than all its neighboring vertices in F. Recall that an abstract objective function (AOF) of a simple d-polytope is an ordering that satisfies the basic property of linear objective functions: every nonempty face F of P has a unique local maximum vertex.

If P is a simple d-polytope and $\prec$ is a linear ordering of the vertices we define the degree of a vertex v w.r.t. the ordering as the number of adjacent vertices to v that are smaller than v w.r.t. $\prec$. Thus, the degree of a vertex is a nonnegative number between 0 and d. Let $h_k^{\prec}$ be the number of vertices of degree k. Finally, let $F(P)$ be the total number of nonempty faces of P.

Claim 1: $\sum_{r=0}^{d} 2^k h_k^{\prec} \quad \geq \quad F(P)$, and the equality holds if and only if the ordering $\prec$ is an AOF.

Proof: Count pairs (F, v) where F is a nonempty face of P (of any dimension) and v is a vertex that is local maximum in F w.r.t. the ordering $\prec$. On the one hand, every vertex v of degree k contributes precisely 2^k pairs (F, v) corresponding to all subsets of edges from v leading to smaller vertices w.r.t. $\prec$. Therefore the number of pairs is precisely $\sum_{r=0}^{d} 2^k h_k^{\prec}$. On the other hand, the number of such pairs is at least $F(P)$ (every face has at least one local maximum) and it is equal to $F(P)$ iff every face has exactly one local maximum, i.e, if the ordering is an AOF.

Claim 2: A connected k-regular subgraph H of $G(P)$ is the graph of a k-face, if and only if there is an AOF in which all vertices in H are smaller than all vertices not in H.

Proof: If H is the graph of a k-face F of P then consider a linear objective function ψ that attains its minimum precisely at the points in F. (By definition for every nontrivial face such a linear objective function exists.) Now perturb ψ a little to get a generic linear objective function ϕ in which all vertices of H have smaller values than all other vertices.

On the other hand, if there is an AOF $\prec$ in which all vertices in H are smaller than all vertices not in H, consider the vertex v of H that is the largest w.r.t. $\prec$. There is a k-face F of P determined by the k-edges in H adjacent to v and v is a local maximum in this face. Since the ordering is an AOF, v must be larger than all vertices of F and hence the vertices of F are contained in H and the graph of F is a subgraph of H. But the only k-regular subgraph of a connected k-regular graph is the graph itself and therefore H is the graph of F.

Proof of Theorem 5.1: Claim 1 allows us to determine just from the graph all the orderings that are AOF's. Using this, claim 2 allows to determine which sets of vertices form the vertices of some k-dimensional face. □

The proof gives a poor algorithm and it was an interesting problem to find better algorithms. This is an example where seeking an efficient algorithm was not motivated by questions from computer science but rather a natural aspect of our mathematical understanding. Friedman [2009] found a remarkable LP-based polynomial-time algorithm to tell a simple polytope from its graph. Another important open problem is to extend the theorem to dual graphs of arbitrary triangulations of $(d-1)$-dimensional spheres. This is related to deep connections between polytopes and spheres and various areas in commutative algebra and algebraic geometry, pioneered by Richard Stanley.

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KNOTS, THREE-MANIFOLDS AND INSTANTONS

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Abstract

Low-dimensional topology is the study of manifolds and cell complexes in dimensions four and below. Input from geometry and analysis has been central to progress in this field over the past four decades, and this article will focus on one aspect of these developments in particular, namely the use of Yang–Mills theory, or gauge theory. These techniques were pioneered by Simon Donaldson in his work on 4-manifolds, but the past ten years have seen new applications of gauge theory, and new interactions with more recent threads in the subject, particularly in 3-dimensional topology.

This is a field where many mathematical techniques have found applications, and sometimes a theorem has two or more independent proofs, drawing on more than one of these techniques. We will focus primarily on some questions and results where gauge theory plays a special role.

1 Representations of fundamental groups

1.1 Knot groups and their representations. Knots have long fascinated mathematicians. In topology, they provide blueprints for the construction of manifolds of dimension three and four. For this exposition, a *knot* is a smoothly embedded circle in 3-space, and a *link* is a disjoint union of knots. The simplest examples, the trefoil knot and the Hopf link, are shown in Figure 1, alongside the trivial round circle, the "unknot".

Knot theory is a subject with many aspects, but one place to start is with the *knot group*, defined as the fundamental group of the complement of a knot $K \subset \mathbb{R}^3$. We will write it as $\pi(K)$. For the unknot, $\pi(K)$ is easily identified as $\mathbb{Z}$. One of the basic tools of 3-dimensional topology is Dehn's Lemma, proved by Papakyriakopoulos in 1957, which provides a converse:

Theorem 1.1 (Papakyriakopoulos [1957]). *If the knot group $\pi(K)$ is $\mathbb{Z}$, then K is the unknot.*

It is a consequence of Alexander duality that the *abelianization* of $\pi(K)$ is $\mathbb{Z}$ for any knot. (This is the first homology of the complement.) So we may restate the result above as saying that the unknot is characterized by having abelian fundamental group.

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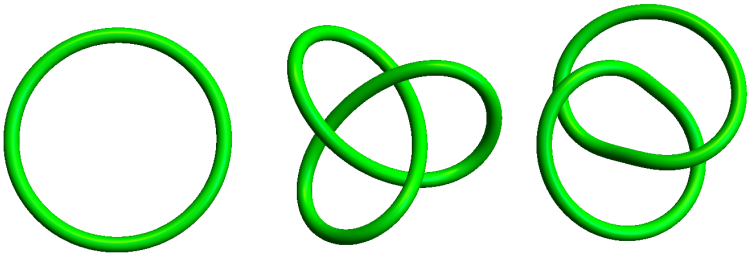


Figure 1: The unknot, trefoil and Hopf link.

In particular, if we are able to find a homomorphism $\pi(K) \to G$ with non-abelian image in any target group G, then K must be genuinely knotted. We begin our account of more modern results with the following theorem.

Theorem 1.2 (Kronheimer and Mrowka [2010]). *If K is a non-trivial knot, then there exists a homomorphism,*

$$\rho : \pi(K) \to SO(3),$$

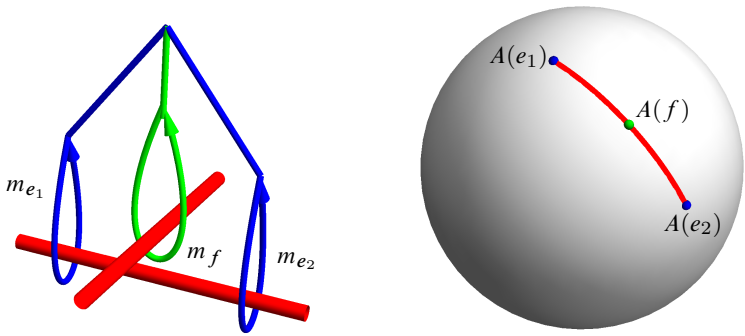
with non-abelian image, from the knot group to the 3-dimensional rotation group.

There are two aspects to why this result is interesting. First, representations of the knot group in particular types of target groups are a central part of the subject: the case that G is dihedral leads to the "Fox colorings" Fox [1962], and the more general case of a two-step solvable group is captured by the Alexander polynomial and related invariants. But there are non-trivial knots with no Fox colorings and trivial Alexander polynomial. It is known that $\pi(K)$ is always a residually finite group, so there are always non-trivial homomorphisms to finite groups; but it is perhaps surprising that the very smallest simple Lie group is a target for all non-trivial knots.

Secondly, the theorem is of interest for the techniques that are involved in its proof, some of which we will describe later. A rich collection of tools from gauge theory are needed, and these are coupled with more classical tools from 3-dimensional topology, namely the theory of incompressible surfaces and decomposition theory, organized in Gabai's theory of sutured manifolds Gabai [1983].

1.2 Orbifolds from knots. The knot group $\pi(K)$ has a distinguished conjugacy class, namely the class of the *meridional elements*. A meridional element m is one represented by a small loop running once around a circle linking K. If we take a planar diagram of a knot (a generic projection of K into $\mathbb{R}^2$), and take our basepoint for the fundamental group to lie above the plane, then there is a distinguished meridional element m_e for each *arc* e of the diagram (a path running from one undercrossing to the next). The elements m_e generate the knot group and satisfy a relation at each crossing, the *Wirtinger relations* Fox [1962]. (See Figure 2.)

Theorem 1.2 can be refined to say that ρ can be chosen so that $\rho(m)$ has order 2 in $SO(3)$, for one (and hence all) meridional elements. This refinement can be helpfully



(a) (b)

Figure 2: (A) The *Wirtinger relation*, $m_f m_{e_2} = m_{e_1} m_f$ holds in the fundamental group of the complement. (B) The corresponding points on the sphere lie on a geodesic arc. The reflection about the green axis interchanges the two blue points.

reinterpreted in terms of the fundamental group of an *orbifold*. Recall that an orbifold is a space locally modeled on the quotient of a manifold by a finite group, and that its *singular* set is the locus of points which have non-trivial stabilizer in the local models. Given a knot or link K in a 3-manifold Y, one can equip Y with the structure of an orbifold whose non-trivial stabilizers are all $\mathbb{Z}/2$ and whose singular set is K. Let us write $\mathrm{Orb}(Y, K)$ for this orbifold. The *orbifold fundamental group* in this situation can be described as the fundamental group of the complement of the singular set with relations

$$m^2 = 1$$

imposed, for all meridional elements. Thus the refinement we seek can be stated:

Theorem 1.3 (Kronheimer and Mrowka [2010]). *If K is a non-trivial knot in S^3, and $O = \mathrm{Orb}(S^3, K)$ is the corresponding orbifold with $\mathbb{Z}/2$ stabilizers, then there exists a homomorphism from the orbifold fundamental group,*

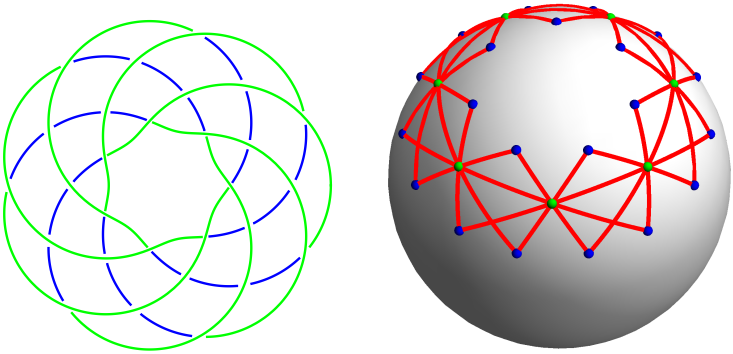
$$\rho : \pi_1(O) \to SO(3),$$

with non-abelian image.

Using the Wirtinger presentation described above, this result can be given a concrete interpretation. An element of order 2 in $SO(3)$ is a 180° rotation about an axis A in $\mathbb{R}^3$, an these are therefore parametrized by the points A of $\mathbb{RP}^2$. So if we are given a diagram of K, then a representation $\rho : \pi(K) \to SO(3)$ which sends meridians to elements of order 2 can be described by giving a point $A(e)$ in $\mathbb{RP}^2$ for each arc e, satisfying a collection of constraints coming from the Wirtinger relations at the crossings. So the concrete version of Theorem 1.3 is the following.

Theorem 1.4. *Given any diagram of a non-trivial knot K, we can find a non-trivial assignment $e \mapsto A(e)$,*

$$\{\textit{arcs of the diagram}\} \to \mathbb{RP}^2,$$



(a) (5,7)-torus knot diagram

(b) Axes of one representation

Figure 3: The green arcs in the diagram on the left contain all the over-crossings. The blue arcs consist only of under-crossings. In the right-hand picture, the green vertices are axes $A(e)$ corresponding to green arcs e in the knot diagram. The blue vertices correspond to blue arcs.

so that the following condition holds: whenever e_1, e_2, f are arcs meeting at a crossing, with f being the overcrossing arc (see Figure 2), the point $A(e_2)$ is the reflection of $A(e_1)$ in the point $A(f)$.

The last condition in the theorem means that $A(e_1)$, $A(f)$, $A(e_2)$ are equally spaced along a geodesic, see Figure 2. The case that all the $A(e)$ are equal is the trivial case, and corresponds to the abelian representation. Dihedral representations arise when the points $A(e)$ lie at the vertices of a regular polygon on $\mathbb{RP}^1$. A configuration corresponding to a non-dihedral representation of the $(5,7)$-torus knot is illustrated in Figure 3.

1.3 Three-manifolds and *SO*(3) . Having considered a knot or link in $\mathbb{R}^3$ and an associated orbifold, we consider next a closed 3-manifold Y and its fundamental group $\pi_1(Y)$. From the solution of the Poincaré conjecture Lott [2007], we know that $\pi_1(Y)$ is non-trivial if Y is not the 3-sphere. Motivated by the discussion of knot groups in the previous section, one might ask:

Question 1.5. *Let Y be a closed 3-manifold with non-trivial fundamental group. Does there exist a non-trivial homomorphism $\rho : \pi_1(Y) \to SO(3)$?*

It is not known whether the answer is yes in general. Stated this way, the interesting case for this question is when Y is a *homology 3-sphere*, i.e. a 3-manifold with the same (trivial) homology as S^3. (If Y has non-trivial homology, then $\pi_1(Y)$ has a cyclic group as a quotient, and there will always be a representation in $SO(3)$ with cyclic image.) For homology 3-spheres, an affirmative answer to the question is known when Y has non-zero Rohlin invariant Akbulut and McCarthy [1990], when Y is obtained by Dehn surgery on a knot in S^3 Kronheimer and Mrowka [2004a, 2010], or when Y carries a taut foliation Kronheimer and Mrowka [2004b]. See also Baldwin and Sivek [2016].

There is an interesting variant of this question, for 3-manifolds with non-trivial homology. A representation $\rho : \pi_1(Y) \to SO(3)$ defines a flat vector bundle on Y with fiber $\mathbb{R}^3$, and such a vector bundle has a second Stiefel–Whitney class w_2. Thus the representations ρ can be grouped by this class,

$$w_2(\rho) \in H^2(Y; \mathbb{Z}/2).$$

which is the obstruction to lifting ρ to the double cover $SU(2) \to SO(3)$. The following result completely describes the classes which arise as $w_2(\rho)$.

Theorem 1.6 (Kronheimer and Mrowka [2010]). *Let Y be a closed, oriented 3-manifold and let $\omega \in H^2(Y; \mathbb{Z}/2)$ be given. Suppose that for every embedded 2-sphere S in Y, the pairing $\omega \cdot [S]$ is zero mod 2. Then there exists a homomorphism $\rho : \pi_1(Y) \to SO(3)$ with $w_2(\rho) = \omega$.*

Remark 1.7. The restriction on ω is also necessary as well as sufficient, because a flat vector bundle on a 2-sphere is trivial and must therefore have trivial Stiefel–Whitney class on the sphere.

Remark 1.8. The condition on $\omega \cdot [S]$ is automatically satisfied if Y is *irreducible*, i.e. if every 2-sphere in Y bounds a ball. To prove the theorem it is enough to consider only irreducible 3-manifolds.

Remark 1.9. If ρ has cyclic image, then $w_2(\rho)$ has a lift to a *torsion* class in the integer homology, $H^2(Y)$. The case that there is no such lift is the case that $w_2(\rho)$ has non-trivial image in $\mathrm{Hom}(H_2(Y), \mathbb{Z}/2)$, and in this case ρ must be non-cyclic. The case that ω has non-trivial image in $\mathrm{Hom}(H_2(Y), \mathbb{Z}/2)$ is the difficult case for the theorem.

An interesting special case of this theorem is the case that Y is the mapping torus of a diffeomorphism, $h : \Sigma_g \to \Sigma_g$, of a surface of genus g. The conjugacy classes of representations $\pi_1(\Sigma_g) \to SO(3)$ with non-zero w_2 are parametrized by an orbifold $M(\Sigma_g)$ of dimension $6g - 6$, and the diffeomorphism h gives rise to a map $h^* : M(\Sigma_g) \to M(\Sigma_g)$ by pull-back. It is then a consequence of the above theorem that the diffeomorphism h^* has fixed points in $M(\Sigma_g)$. In this form, the result was proved independently and with different methods by Smith [2012].

1.4 Spatial graphs. A spatial graph is a graph (tamely) embedded as a topological space in $\mathbb{R}^3$. We will be interested here in finite, trivalent graphs (also called cubic graphs). Thus we are generalizing classical knots and links by allowing vertices of valence 3. We allow that the set of vertices may be empty, so knots and links are included as a special case, regarded as vertexless graphs. There is a significant literature on spatial graphs: see for example Conway and Gordon [1983].

As with knots and links, we write $\pi(K)$ for the fundamental group of the complement of a spatial graph $K \subset \mathbb{R}^3$. For each edge e of K, there is corresponding distinguished conjugacy class of meridional curves m_e, obtained from the small circles linking e. Following the same lines as before, we wish to study representations of $\pi(K)$ in $SO(3)$, with the constraint that the meridional elements map to elements of order 2. Representations of this sort are parametrized by a topological space, the *representation variety*,

$$\mathcal{R}(K) = \{\rho : \pi(K) \to SO(3) \mid \rho(m_e) \text{ has order 2 for all edges } e\} \tag{1}$$

As with knots and links, this representation variety for a spatial graph can be interpreted as a space of representations for the fundamental group of an orbifold. Given a trivalent graph K in a 3-manifold Y, we may construct a 3-dimensional orbifold $\mathrm{Orb}(Y, K)$ whose underlying topological space is Y, whose singular set is K, and whose local stabilizer groups are $\mathbb{Z}/2$ at the interior points of edges of K. At vertices of K where three edges meet, the local model for the orbifold is the quotient of the 3-ball by the Klein four-group, V_4. In this way, $\mathcal{R}(K)$ becomes the space of homomorphisms from the orbifold fundamental group,

$$\rho : \pi_1(\mathrm{Orb}(S^3, K)) \to SO(3),$$

with the additional property that ρ is injective on each of the non-trivial local stabilizer groups.

The Klein 4-group V_4 is contained in $SO(3)$ as the subgroup of diagonal matrices, and representations $\rho : \pi(K) \to SO(3)$ with image in V_4 play a special and already subtle role. Since V_4 is abelian, a representation into the Klein four-group factors through the abelianization of $\pi(K)$, namely the homology $H_1(S^3 \setminus K))$, so we are considering homomorphisms

$$\tau : H_1(S^3 \setminus K) \to V_4$$

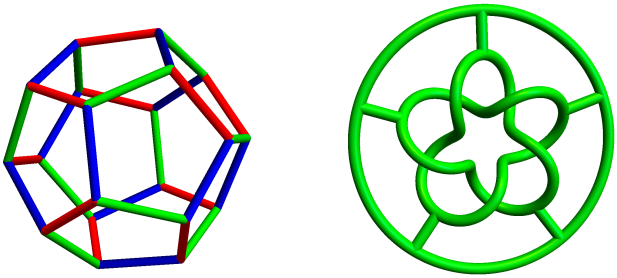
which map meridional elements to elements of order 2. If we write the elements of V_4 as $\{1, A, B, C\}$, then τ assigns one of three "colors" $\{A, B, C\}$ to each edge e, and this coloring must satisfy the constraint that, at a vertex, the colors of the three incident edges are all different (because the sum of the corresponding elements of H_1 is zero). Such a 3-coloring of the edges of a trivalent graph is a called a *Tait coloring*. So we have:

Proposition 1.10. *For a trivalent spatial graph K, the representations $\rho \in \mathcal{R}(K)$ whose image is contained in the Klein 4-group V_4 are in one-to-one correspondence with Tait colorings of K.*

Notice in particular that this set depends only on the abstract graph K, independent of the embedding. This is a reflection of the fact that the homology group $H_1(S^3 \setminus K)$ is isomorphic to $H^1(K)$ by Lefschetz duality.

The question of whether a cubic graph admits a Tait coloring is difficult. Indeed Tait [1884] observed that the four color theorem Appel and Haken [1989] and Robertson, Sanders, Seymour, and Thomas [1997] could be reframed as a question about the existence of Tait colorings as follows. A planar map determines a graph, giving the borders of the countries. A map is called proper if no country has a border with itself, in which case the graph of the borders is *bridgeless*; that is, no edge (or "bridge") can be removed making the graph disconnected. It is also elementary to see that it suffices to verify the four color theorem for planar maps whose border graph is trivalent. Tait's observation is that the four-colorability of the regions of the map is equivalent to the existence of a Tait coloring of the border graph. So the four color theorem is equivalent to the statement that every bridgeless trivalent graph admits a Tait coloring.

While the methods of gauge theory at the time of writing have not given a proof the four color theorem, one can prove some suggestive results. To state the main result we



(a) (b)

Figure 4: The dodecahedral graph admits a Tait coloring, while the Petersen graph, the simplest snark, has none.

observe that for spatial graphs there is a natural extension of being bridgeless. A *spatial bridge* is an edge of a spatial graph K for which the meridional loop is contractible in the complement of the graph. Equivalently, it is an edge e for which we can find a sphere S for which $K \cap S$ is a single point of e, with transverse intersection. Note that the existence of such a spatial bridge implies that $\mathcal{R}(K)$ is empty. The converse is the following non-trivial theorem.

Theorem 1.11 (Kronheimer and Mrowka [n.d.]). *] For any trivalent graph $K \subset \mathbb{R}^3$ without a spatial bridge, the representation variety $\mathcal{R}(K)$ is non-empty.*

We conclude this section with some remarks to put this result in context. There is an action of $SO(3)$ on $\mathcal{R}(K)$ by conjugacy. Representation with image V_4 are characterized by the fact their stabilizer is exactly V_4 under this action. For a graph with at least one vertex, the possible groups that can arise as stabilizers are V_4, $\mathbb{Z}/2$ and the trivial group. One can show (see Kronheimer and Mrowka [ibid.]) that, for planar graphs, if there is a representation with non-trivial stabilizer then there is also one with V_4 stabilizer, and hence a Tait coloring of K. Theorem 1.11 is agnostic regarding the possible stabilizers of the representation that it guarantees. There are planar graphs with only V_4 representations, as well as many with both V_4 representations and irreducible representations (the simplest being the 1-skeleton of the dodecahedron.)

Bridgeless trivalent graphs with no Tait colorings are called snarks. The simplest one is the Petersen graph, shown in a spatial embedding in Figure 4.

Trivially, the four color theorem says that there are no planar snarks. For any spatial embedding of a snark, Theorem 1.11 guarantees the existence of a representation $\pi(K) \to SO(3)$.

Theorem 1.11 also says that even a graph with a bridge, when embedded in a spatially bridgeless manner, will have a nontrivial representation. An example is shown Figure 5. On the left, the "handcuffs" are shown embedded in $\mathbb{R}^3$ with a spatial bridge, and the representation variety $\mathcal{R}(K_1)$ is empty. On the right, the same abstract graph

Figure 5: The standard handcuffs and the tangled handcuffs.

is shown with a more interesting embedding. The representation variety $\mathcal{R}(K_2)$ in this case consists of the $SO(3)$ orbit of a single representation ρ whose image in $SO(3)$ is the symmetry group of a cube.

2 Background on instantons and four-manifolds

The theorems discussed above are proved by means of more general results based on non-vanishing theorems for Floer's instanton homology for 3-manifolds, first introduced in Floer [1988]. Before introducing the instanton homology groups, we discuss their natural historical precursor, the invariants of smooth 4-manifolds developed by Simon Donaldson in the 1980's Donaldson [1990].

2.1 Instanton moduli spaces. The story begins with the work of Donaldson and his use of gauge theory in 4-dimensional topology. On an oriented Riemannian manifold X of dimension $2n$, the Hodge $*$-operator maps n-forms to n-forms,

$$* : \Omega^n(X) \to \Omega^n(X)$$

and satisfies $*^2 = (-1)^n$. When n is even, this gives rise to a decomposition into the ± 1 eigenspaces, the *self-dual* and *anti-self-dual* n-forms,

$$\Omega^n(X) = \Omega^n_+(X) \oplus \Omega^n_-(X).$$

The case of dimension 4 and $n = 2$ plays a special role, because if $E \to X$ is a vector bundle and A is a connection in E, then the *curvature* of the connection is a 2-form with values in the endomorphisms of E:

$$F_A \in \Omega^2\big(X; \mathrm{End}(E)\big).$$

Only in dimension 4, therefore, we can decompose the curvature into its self-dual and anti-self-dual parts, $F_A^+ + F_A^-$ and we can consider the *anti-self-dual Yang–Mills equations*,

$$F_A^+ = 0.$$

The solutions are the *anti-self-dual* connections A, sometimes called *instantons* on X.

We shall first consider the case that the structure group G for the bundle is $SU(N)$, so that E is a rank-N bundle with a hermitian metric and trivialized determinant. Our connections A will be $SU(N)$ connections: they will respect the trivialization. The

isomorphism classes of pairs (E, A) consisting of an $SU(N)$ bundle E with an anti-self-dual connection A are parametrized by a *moduli space* $M_N(X)$ (which depends also on the Riemannian metric). When X is closed and connected, the bundles E themselves are classified by a single integer k, the second Chern number, or instanton number,

$$k = c_2(E)[X].$$

We therefore have a decomposition

$$M_N(X) = \bigcup_k M_{N,k}(X).$$

Each $M_{N,k}(X)$ is finite-dimensional, and for generic choice of Riemannian metric it will be a smooth manifold, except at *reducible* solutions: i.e. those where A preserves some orthogonal decomposition of E. An index calculation yields a formula for the dimension of $M_{N,k}$,

$$\dim M_{N,k} = 4Nk - (N^2 - 1)(\chi + \sigma)/2, \tag{2}$$

in which χ and σ are the signature an Euler number of X. The quantity $(\chi + \sigma)/2$ is an integer, which can also be written as

$$b_+^2 - b^1 + 1$$

where b^i is the rank of $H^i(X)$ and b_+^2 is the dimension of a maximal positive-definite subspace for the quadratic form on $H^2(X; \mathbb{R})$ defined by the cup-square.

The space $M_{N,k}(X)$ will usually be non-compact, because there may be sequences of solutions (E_n, A_n) in which the point-wise norm of the curvature, $|F_{A_n}|$, diverges near finitely many points in X, a "bubbling" phenomenon analyzed by Uhlenbeck [1982].

Having associated to each closed Riemannian 4-manifold an infinite sequence of new spaces, one is led to ask whether the moduli spaces $M_{N,k}(X)$ are non-empty. Do the anti-self-dual Yang–Mills equations have solutions? This question was answered in the affirmative by Taubes [1982, 1984], who constructed solutions on general 4-manifolds X using a grafting technique to transfer standard solutions from flat $\mathbb{R}^4$. Taubes' results tell us in particular that $M_{2,k}(X)$ is non-empty for all $k \geq k_0$, where the value of k_0 depends only on the topology of X. The resulting solutions have curvature concentrated near points in X, the same situation that is allowed in Uhlenbeck's work.

2.2 Donaldson's polynomial invariants. Although they may be non-compact, Donaldson showed that the moduli spaces $M_{N,k}(X)$ have sufficient compactness properties as a consequence of Uhlenbeck's theorems that they may (under mild conditions) be regarded as possessing a *fundamental class* $[M_{N,k}(X)]$ in the homology of the ambient space in which they sit, namely the space $\mathcal{B}_{N,k}(X)$ which parametrizes *all* isomorphism classes of $SU(N)$ connections with instanton number k.

To elaborate on this, the space $\mathcal{B}_{N,k}(X)$ (or more relevantly, the open subspace $\mathcal{B}^*_{N,k}(X)$ of irreducible connections) has a well-understood topology, and the "fundamental class" $[M_{N,k}(X)]$ gives rise to a rich collection of invariants, the *Donaldson*

invariants of X. For these to be defined, it is important that the moduli space be contained in the irreducible part, $\mathcal{B}^*_{N,k}(X)$, and this will be true for $N = 2$ and for generic choice of Riemannian metric, as long as $b^2_+(X) \geq 1$ and $k > 0$. (We discuss this point again in the next subsection.) Furthermore, if $b^2_+(X) \geq 2$, then the fundamental class of the moduli space in $\mathcal{B}^*_{2,k}(X)$ is independent of the choice of metric, and so can be regarded as an *invariant* of the underlying smooth 4-manifold X.

These invariants, defined originally using the $N = 2$ moduli spaces, are usually referred to as Donaldson's *polynomial invariants*. In the $N = 2$ case, the rational cohomology of $\mathcal{B}^*_{2,k}$ contains a polynomial algebra Donaldson [1990]. More specifically, let us introduce the symmetric algebra

$$\mathbb{A}(X) = \mathrm{Sym}(H_{even}(X;\mathbb{Q}))$$

graded so that $H_r(X;\mathbb{Q})$ lies in $\mathbb{A}_{4-r}(X)$. Then there is an injection, for each k,

$$\mu : \mathbb{A}_d(X) \to H^d(\mathcal{B}_{2,k}(X);\mathbb{Q}).$$

The polynomial invariants defined by the moduli spaces $M_{2,k}(X)$ are linear maps

$$q_{X,k} : \mathbb{A}_{d(k)}(X) \to \mathbb{Q}$$

where $d(k)$ is the dimension of the moduli space (2). If we accept that $M_{2,k}(X)$ carries a fundamental class in homology, then we can regard the definition as:

$$q_{X,k}(z) = \langle \mu(z), [M_{2,k}(X)] \rangle.$$

The definition can be generalized in various way, in particular by considering $N > 2$. We can omit k from the notation by taking the sum,

$$q_X = \bigoplus_k q_{X,k} : \mathbb{A}(X) \to \mathbb{Q}. \tag{3}$$

There is an understanding that q_X is zero on $\mathbb{A}_d(X)$ for integers d not of the form $d(k)$.

The invariants q_X of smooth 4-manifolds, together with some closely-related invariants Donaldson [1987], were the first tools which were able to show that the the diffeomorphism type of a simply-connected compact 4-manifold is not determined by its cohomology ring alone.

2.3 A generalization, $U(N)$ bundles. In the discussion above, the bundle E had structure group $SU(N)$. We now consider $U(N)$ bundles with non-trivial determinant instead. A $U(N)$ connection A can be described locally as the sum of an $SU(N)$ connection and a $U(1)$ connection. More invariantly, and globally, A is determined by

- a $PU(N)$ connection A^o; and
- a connection $\mathrm{tr}(A)$ in the determinant line bundle, the top exterior power $\Lambda^N E$.

Whether there are anti-self-dual connections on the line bundle $\Lambda^N E$ is determined by Hodge theory on X, and usually they will not exist if the line bundle is non-trivial. The appropriate set-up is to ask only that A^o is anti-self-dual. More specifically, we *fix* a line bundle $W \to X$ and the data we seek is:

- a $U(N)$ bundle $E \to X$ with $c_1(E) = c_1(W)$;
- a chosen isomorphism $\iota : \Lambda^N E \to L$;
- an anti-self-dual $PU(N)$ connection A' in the associated $PU(N)$ bundle of E.

The resulting moduli space of solutions (E, ι, A') depends on the Riemannian manifold X and the choice of the class $w = c_1(W)$ in $H^2(X;\mathbb{Z})$. The topology of E is determined by w and the the *instanton number*, defined now as an appropriately normalized Pontryagin number of the associated $PU(N)$ bundle. With a standard normalization, the instanton number k is not an integer but satisfies a congruence

$$k = -\left(\frac{N-1}{2N}\right) w \cdot w \pmod{\mathbb{Z}}.$$

We write $M_{N,k}(X)^w$ for this moduli space of anti-self-dual connections with instanton number k. It leads to polynomial invariants,

$$q^w_{X,k} : \mathbb{A}_{d(k)}(X) \to \mathbb{Q},$$

generalizing the $q_{X,k}$ (defined using the $N = 2$ moduli spaces) and we can combine these again as

$$q^w_X = \bigoplus_k q^w_{X,k} : \mathbb{A}(X) \to \mathbb{Q}.$$

This extra generality is introduced not so much for its own sake, but because it serves to avoid the difficulty that was mentioned in our discussion of the polynomial invariants above. The difficulty is the possible presence of *reducible* connections A in E. We will say that E or w is *admissible* if there is an integer homology class σ in $H_2(X)$ such that

$$(w \cdot \sigma) \text{ is prime to } N. \tag{4}$$

The relevance of admissibility is in the following result.

Proposition 2.1. *If E is admissible and $b^2_+(X) > 0$, then the moduli space of anti-self-dual connections contains no reducible solutions, for a generic metric on X. The same is true for a generic path of metrics if $b^2_+(X) > 1$.*

Remark 2.2. A prototype which captures part of this is the more elementary statement, that if E is a $U(N)$ bundle on a closed, oriented 2-manifold and the degree of E is prime to N, then the associated $PU(N)$ bundle E' admits no reducible flat connections. In this way, reducible solutions can be avoided, and the invariants q^w_X can be generalized to higher-rank bundles with admissible w.

Remark 2.3. In the case $N = 2$, the group $PU(2)$ is $SO(3)$ and elements of the moduli space $M_{2,k}(X)^w$ give rise to anti-self-dual $SO(3)$ connections with second Stiefel–Whitney class w mod 2. There is a distinction between the two setups however. The automorphisms of the pair (E, ι) are the bundle automorphisms of E that have determinant 1 on each fiber. If $H^1(X;\mathbb{Z}/2)$ is non-zero, then not every automorphism of the associated $PU(2)$ bundle E' lifts to a determinant-1 automorphism of E. The moduli space of anti-self-dual $SO(3)$ or $PU(2)$ connections is the quotient of $M_{2,k}(X)^w$ by an action of the finite group $H^1(X;\mathbb{Z}/2)$.

2.4 Non-vanishing for the polynomial invariants. While Taubes' results inform us that $M_{N,k}(X)$ is non-empty for large enough k, one can now ask a different question whose answer reflects the non-triviality of the moduli space in a different way: one can ask whether the Donaldson invariants of X are non-zero; or equivalently, is the fundamental class $[M_{N,k}(X)]$ non-zero? Donaldson proved the following non-vanishing theorem for the polynomial invariants q_X arising from the $SU(2)$ instanton moduli spaces.

Theorem 2.4 (Donaldson [1990]). *If X is the smooth 4-manifold underlying a simply-connected complex projective algebraic surface, then the polynomial invariants $q_{X,k}$ are non-zero for all sufficiently large k. In particular, $q_{X,k}(h^{d/2})$ is non-zero, where h is the hyperplane class and $d = d(k)$ is the dimension of the moduli space.*

This result moves us from the simple non-emptiness of a moduli space to non-triviality in homology. Donaldson's proof uses the fact that, for the Kähler metrics adapted to the complex-algebraic structure, the moduli spaces of instantons can be identified with moduli spaces of stable holomorphic bundles, which are quasi-projective varieties. The non-vanishing eventually derives from the positivity of intersections in complex geometry.

The theorem generalizes to $U(2)$ bundles and the corresponding invariants $q^w_{X,k}$ for non-zero w. The authors believe that, using later results and constructions from Gieseker and Li [1996], O'Grady [1996] and Kronheimer [2005], the restriction to $N = 2$ can be dropped, and that the hypothesis that X is simply-connected is also unnecessary.

2.5 Non-vanishing for symplectic 4-manifolds. The non-vanishing theorem for algebraic surfaces was proved when Donaldson's invariants were first introduced in Donaldson [1990]. A class of 4-manifolds that is in many ways closely related are the *symplectic* 4-manifolds, i.e. those which carry a closed 2-form ω for which $\omega \wedge \omega$ is a volume form. Although the original proof of Theorem 2.4 does not extend to the symplectic case, the more general theorem does hold:

Theorem 2.5. *The non-vanishing statement of Theorem 2.4 continues to hold for the larger class of symplectic 4-manifolds, with the role of the hyperplane class h now played by the de Rham class $[\omega]$ of the symplectic form.*

Part of the history of this result is as follows. A non-vanishing theorem was proved by Taubes [1994] for the Seiberg–Witten invariants of symplectic 4-manifolds, and the above theorem should then follow from Witten's conjecture Witten [1994] relating the Donaldson invariants to the Seiberg–Witten invariants. A weakened version of Witten's conjecture has been proved by Feehan and Leness [2015], building on ideas of Pydstrigach and Tyurin, and this work can be used to deduce Theorem 2.5 for a large class of symplectic 4-manifolds. The general version of Theorem 2.5 was later proved more cleanly, and without use of the Seiberg–Witten invariants: an argument was outlined in Kronheimer and Mrowka [2010] and a variant is given in Sivek [2015]. These later proofs make use of another theorem of Donaldson, on the existence of Lefschetz pencils for symplectic 4-manifolds Donaldson [1999].

3 Instanton homology for 3-manifolds

3.1 Formalities, and non-vanishing. The instanton homology groups of an oriented 3-manifold Y arise naturally when one seeks to understand the Donaldson invariants of a 4-manifold X which is decomposed as a union of two manifolds with common boundary Y:

$$\begin{aligned} X &= X_+ \cup_Y X_- \\ \partial X_+ &= \quad Y \\ \partial X_- &= -Y. \end{aligned} \tag{5}$$

(We use $-Y$ to denote Y equipped with the opposite orientation.) There are several variants of Floer's construction depending on, among other choices, the gauge group and the coefficient ring, and some variants are applicable only to certain 3-manifolds (such as homology spheres) or only allow certain bundles.

Floer's first construction worked only for homology 3-spheres and structure group $SU(2)$. To each oriented connected homology 3-sphere Y it gave a finitely-generated abelian group $I(Y)$. The simplest property of this invariant is that the instanton homologies of Y and $-Y$ are related as the homology and cohomology of a complex, so that in particular there is a perfect pairing

$$I(Y) \otimes I(-Y) \to \mathbb{Z}. \tag{6}$$

If we work with rational coefficients, as we often will, then these are dual vector spaces.

When a connected, oriented 4-manifold X is decomposed as in (5) where Y is a homology sphere and $b_+^2(X_\pm) > 0$ then the Donaldson invariant q_X (3) can be expressed in terms of the *relative invariants* of the two pieces $X_\pm$. These relative invariants take the form of linear maps

$$\begin{aligned} q_{X_+} &: \mathbb{A}(X_+) \to I(Y) \\ q_{X_-} &: \mathbb{A}(X_-) \to I(-Y) \end{aligned}$$

and the Donaldson invariant of the closed manifold X is expressed using the pairing (6) as

$$q_X(z) = \langle q_{X_+}(z_+), q_{X_-}(z_-) \rangle, \tag{7}$$

where $z = i_+(z_+) i_i(z_-)$ and $i_\pm : \mathbb{A}(X_\pm) \to \mathbb{A}(X)$ arise from the inclusion maps. This pairing formula has a straightforward corollary:

Proposition 3.1. *If $q_X \neq 0$ for some 4-manifold X, and a homology 3-sphere Y can be placed into X in such a way that X is decomposed into two pieces, each with $b_+^2 > 0$, then it must be that $I(Y) \otimes \mathbb{Q}$ is non-zero.*

Remark 3.2. Unlike ordinary homology, the instanton homology groups are not $\mathbb{Z}$-graded. The version $I(Y)$ discussed here has a cyclic grading by $\mathbb{Z}/8$. Some versions we will encounter later have not grading at all, as they arise as the homology $\ker(d)/\mathrm{im}(d)$ for a differential d on an ungraded abelian group rather than a chain complex.

3.2 Sketch of the construction. The basic idea for instanton Floer homology Floer [1988] can be motivated by thinking of solutions to the anti-self-duality equations on a closed 4-manifold X, decomposed as above, but with a Riemannian metric containing long cylinder $[-L, L] \times Y$. By means of a gauge transformation on this cylinder we can assume that a connection A in $E \to [-L, L] \times Y$ is pulled back from path of connections $B(t)$, for $t \in [-L, L]$, in $E \to Y$. The anti-self-duality equation for A becomes the equation

$$\frac{\partial B}{\partial t} + *F_B. \tag{8}$$

In particular, translationally invariant solutions to the ASD equation (i.e. solutions with $\frac{\partial B}{\partial t} = 0$) are flat connections B on Y, so that $F_B = 0$. A key observation in Floer [ibid.] is that the above equation for a path $B(t)$ is formally the downward gradient flow for a functional – the *Chern–Simons functional* – on a space of connections on the 3-manifold. To see this, consider for simplicity a trivial bundle on a 3-manifold Y. We write a connection B as sum of the connection Γ coming from a trivialization and a 1-form with values in the Lie algebra $\mathfrak{su}(N)$:

$$B = \Gamma + b, \text{ where } b \in \Omega^1(Y) \otimes \mathfrak{su}(N).$$

In this form, the Chern–Simons function is given by

$$CS(B) = -\frac{1}{2}\int_Y \operatorname{tr}(b \wedge db + \frac{1}{3} b \wedge b \wedge b).$$

The first variation of CS is given by

$$\begin{aligned} \frac{d}{dt} CS(B + t\beta)|_{t=0} &= -\int_Y \operatorname{tr}(\beta \wedge (db + \frac{1}{2} b \wedge b)) \\ &= -\int_Y \operatorname{tr}(\beta \wedge F_B), \end{aligned}$$

so the stationary points are flat connections, $F_B = 0$. If Y is given a Riemannian metric, then the standard inner product on $su(N)$-valued one forms can be written

$$\langle \alpha, \beta \rangle = -\int_Y \operatorname{tr}(\alpha \wedge *\beta).$$

With respect to this inner product, the gradient of CS at the connection B is $*F_B$, which verifies that the equation (8) is indeed the downward gradient flow.

Floer's construction applies the ideas of Morse theory to the Chern–Simons functional. In the case of a finite-dimensional compact manifold B carrying a Morse function f, the ordinary homology of B can be computed as the *Morse homology* of f. This is the homology of a complex whose generators correspond to critical points of f and whose boundary map records intersection numbers between ascending a descending submanifolds. After a generic perturbation, these intersection numbers can be interpreted as counting gradient flow lines between critical points of adjacent index.

When this framework from finite dimensions is applied to the Chern–Simons functional on the space of gauge-equivalence classes of connections on Y, the relation with ordinary homology is lost. The "instanton homology" which results is something new. It is the homology of a complex whose generators are critical points of the (perturbed) Chern–Simons functional and whose boundary map counts gradient flow lines. In line with the discussion above, the critical points are flat $SU(N)$ connections on Y (corresponding to representations of $\pi_1(Y)$ in $SU(N)$) and the boundary map counts solutions of the anti-self-duality equations on the cylinder.

Floer's original construction ignores the trivial connection, and the assumption that Y is a homology sphere is used in a crucial manner to prove that $I(Y)$ is independent of the choice of metric on Y and the perturbation. This restriction on Y also means that the non-trivial representations of $\pi_1(Y)$ in $SU(2)$ are all irreducible. It follows then from the construction of $I(Y)$ as a Morse homology that if there are no irreducible representations, i.e. no critical points for CS used in the construction of $I(Y)$, then $I(Y)$ is trivial. So Proposition 3.1 has the following straightforward corollary:

Corollary 3.3. *If $q_X \neq 0$ for some 4-manifold X, and a homology 3-sphere Y can be placed into X in such a way that X is decomposed into two pieces, each with $b^2_+ > 0$, then there is a non-trivial homomorphism $\rho : \pi_1(Y) \to SU(2)$.*

In particular (from the case that Y is S^3), if a 4-manifold admits a connected sum decomposition $X = X_1 \# X_2$ where $b^2_+(X_i) > 0$, then $q_X = 0$, which is an earlier vanishing theorem due to Donaldson.

As a tool to prove existence of non-trivial homomorphisms, the Corollary is useful, but it is not a completely general tool. For example, it is not known which homology 3-spheres can be embedded in complex algebraic surfaces or in symplectic 4-manifolds. (See Theorem 2.4 and Theorem 2.5.)

In a positive direction, it is shown Y. Eliashberg [2004], Etnyre [2004], and Kronheimer and Mrowka [2004b] that 3-manifolds carrying a *taut foliation* can always be embedded in a symplectic 4-manifold. The argument combines deep and difficult work of Y. M. Eliashberg and Thurston [1998] and Giroux [2002]. One can deduce:

Corollary 3.4. *If the homology 3-sphere Y admits a taut foliation, then there is a non-trivial homomorphism $\rho : \pi_1(Y) \to SU(2)$.*

As indicated earlier (Question 1.5), the general case of a 3-manifold with non-zero fundamental group remains open.

3.3 Using $U(N)$ bundles. The instanton homology we have just described is the first version which Floer defined. Rather than work with homology spheres where the unique reducible connection can be excluded, Floer observed that there is an alternative setup for 3-manifolds with $b_1(Y) \neq 0$ where reducible flat connections can be avoided entirely. One works with 3-manifolds that carry an $SO(3)$ bundle whose the second Stiefel–Whitney class $\omega \in H^2(Y; \mathbb{Z}/2)$ has non-zero evaluation on some integral homology class. Almost the same, as we did for Donaldson's invariants in section 3.3, we may fix a line bundle $W \to Y$ and work with triples (E, ι, A'), where E a $U(2)$ bundle,

ι is an isomorphism $\Lambda^2 E \to W$, and A' is a connection in the associated bundle $PU(2)$. We define *admissibility* for W – or equivalently for its first Chern class $w = c_1(W)$ – just as in the 4-dimensional situation, equation (4). In the admissible case there are no reducible flat connections A'. (See Remark 2.2.)

We arrive at instanton homology groups $I^w(Y)$, labeled by admissible classes w. A pairing formula similar to equation 7 holds in this context. Suppose again that X is decomposed along Y as in (5), and suppose now that v is a class in $H^2(X;\mathbb{Z})$ whose restriction, w, to Y is also admissible. Then we have relative invariants,

$$q^{v_+}_{X_+} : \mathbb{A}(X_+) \to I^w(Y)$$
$$q^{v_-}_{X_-} : \mathbb{A}(X_-) \to I^w(-Y)$$

and a pairing formula,

$$q^v_X(z) = \langle q^{v_+}_{X_+}, q^{v_-}_{X_-} \rangle, \tag{9}$$

where $z = i_+(z_+) i_-(z_-)$ as before.

Along the same lines as Proposition 3.1, we now have:

Proposition 3.5. *Let Y be given, and let w be an admissible class on Y. Suppose that Y can be embedded as a separating hypersurface in X in such a way that the class w extends to a class $v \in H^2(X;\mathbb{Z})$, and suppose that the Donaldson invariant q^v_X is non-zero. Then the instanton homology group $I^w(Y) \otimes \mathbb{Q}$ is non-zero.*

As in the case of a homology 3-sphere (see Corollary 3.4), one can deduce that if Y admits a taut foliation, then $I^w(Y)$ is non-zero for any admissible w, and there exists a representation $\rho : \pi_1(Y) \to SO(3)$ with $w_2(\rho) = w \bmod 2$. Unlike the case of homology 3-spheres however, irreducible 3-manifolds with non-zero Betti number *all* carry taut foliations, by a deep existence result due to Gabai [1983]. So we have:

Corollary 3.6. *If Y is irreducible with $b_1 \neq 0$ and w is any admissible class, then $I^w(Y) \otimes \mathbb{Q}$ is non-zero.*

In this way we arrive at an existence result for representations that is sufficiently general to deduce the necessary and sufficient condition, Theorem 1.6.

4 Sutured manifolds

In Kronheimer and Mrowka [2010], the authors found a much more efficient proof of Corollary 3.6 and Theorem 1.6. The original arguments outlined above used the existence of a taut foliation, which had been proved by Gabai [1983] using his theory of sutured manifolds. The later argument in Kronheimer and Mrowka [2010] uses Gabai's sutured manifold theory more directly. This strategy is inspired in part by the construction of sutured Heegaard Floer homology by Juhász [2008] and its precursors in the work of Ni [2007] and Ghiggini [2008]. On the gauge theory side, the non-vanishing theorem for Donaldson invariants and the difficult proof of the relation between Donaldson and Seiberg–Witten invariants is replaced by work of Muñoz [1999] in computing of the instanton homology groups of $S^1 \times \Sigma_g$. We will explain how some of this works, beginning with Gabai's work.

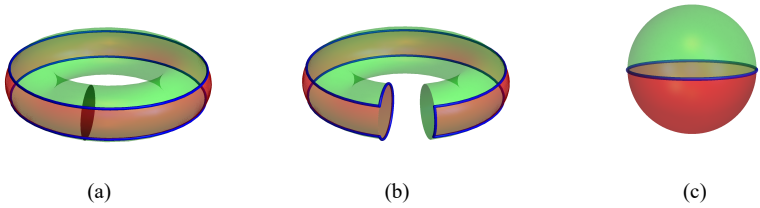


Figure 6: An example of sutured manifold decomposition: a sutured solid torus is decomposed along an embedded disk. The new sutured manifold (center) is isomorphic to a standard sutured ball (right). Red and green indicate R_+ and R_-, while blue curves are sutures.

4.1 Sutured manifold decompositions. An important idea in 3-manifold topology going back to Haken [1961] and Waldhausen [1968] and further developed by Gabai is that of surface decomposition: cutting a 3-manifold along a surface may result in a simpler 3-manifold. In order to organize a sequence of surface decompositions of manifolds with boundary, Gabai defined a notion of *sutured manifold*. This is an oriented 3-manifold with boundary, Y, together with a decomposition of its boundary into two parts,

$$\partial Y = R_+ \cup R_-,$$

intersecting along a union of simple closed curves $\gamma \subset \partial Y$. These simple closed curves are the *sutures*. The simplest example is a 3-ball, with its boundary divided into upper and lower hemispheres, meeting at the equator (Figure 6c). We can always orient γ by first orienting R_+ as the boundary of Y and then orienting γ as the boundary of R_+.

A *decomposing surface* S for a sutured manifold (Y, γ) is an oriented embedded surface $S \subset Y$ with $\partial S \subset \partial Y$. It is required that S and ∂Y meet transversally, so that ∂S a union of simply closed curves in ∂Y; and each of these is required to either meet the sutures γ transversally, or to coincide with a component of γ as an oriented 1-manifold. If a component of ∂S is disjoint from γ, then it is required that this circle does not bound a disk in $R_\pm$ nor a disk in S. Given such a decomposing surface, one obtains a new sutured manifold Y' by cutting Y open along S and smoothing the corners. The new decomposition of $\partial Y'$ as $R'_+ \cup R'_-$ is defined by setting

$$R'_+ = R_+ \cup S_+ \qquad R'_- = R_- \cup S_-$$

where S_+ is the copy of S in $\partial Y'$ picked out by the oriented normal to $S \subset Y$. The process of forming (Y', γ') from (Y, γ) in this way is called a *sutured manifold decomposition*. (See Figure 6.)

The following is a slightly special case of one of Gabai's central results about sutured manifolds.

Theorem 4.1 (Gabai [1983]). *Let Y be a closed irreducible 3-manifold, regarded as a sutured manifold without boundary. Suppose that the Betti number $b_1(Y)$ is non-zero. Then we can find a sequence of sutured manifolds, starting with Y, each obtained from*

the previous one by sutured manifold decomposition, and ending with a disjoint union of 3-balls with one equatorial suture each:

$$(10) \qquad (Y, \varnothing) = (Y^0, \gamma^0) \rightsquigarrow (Y^1, \gamma^1) \rightsquigarrow \cdots \rightsquigarrow (Y^k, \gamma^k) = \coprod_1^m (B^3, \text{equator}).$$

Furthermore, for the first decomposition in the sequence, the decomposing surface $S \subset Y$ can be chosen to be any connected genus-minimizing surface: a surface that achieves the minimum genus among all oriented surfaces in the same homology class. Conversely, the genus-minimizing property for the first cut is a necessary condition for the existence of such a decomposition ending with standard 3-balls.

This result provides a broad framework for proving existence results for structures on an irreducible 3-manifold Y, by starting with existence (of whatever structure) on the trivial (Y^k, γ^k), and working back up to Y. In Gabai's work this framework is used to prove the existence of taut foliations, and in our context it can be used to prove that the instanton homology $I^w(Y)$ is non-zero (Corollary 3.6). What needs to be done is:

1. extend the definition of instanton homology $I^w(Y)$ to the case of sutured manifolds;

2. show that the rank of the instanton homology of sutured manifolds is monotone decreasing in any sequence of decompositions such as (10);

3. show that the instanton homology has non-zero rank for the disjoint union of sutured balls.

Once one has item (1) of the above three, the remaining pieces fall into place quite easily. The non-trivial ingredient needed for the definition in (1) is what we turn to next.

4.2 Munoz' computation of $I^w(S^1 \times \Sigma)$ and its consequences. In Muñoz [1999] Munoz gave a description of the instanton homology $I^w(S^1 \times \Sigma)$, where Σ is a surface of genus $g \geq 1$ and w is the 2-dimensional cohomology class Poincaré dual to $S^1 \times \{\text{point}\}$.

In general, ordinary homology classes h in a 3-manifold Y give rise to operators $\hat{h}$ on $I^w(Y)$. This can be seen as arising naturally from the formalism of Donaldson's invariants, as follows. We working with coefficients in a field and consider (roughly speaking) the relative Donaldson invariants of the 4-manifold $Z = [-1, 1] \times Y$ as defining a linear map

$$\mathbb{A}(Z) \to I^w(\partial Z) \otimes \mathbb{Q},$$

or equivalently

$$\mathbb{A}(Y) \to I^w(-Y) \otimes I^w(Y) \otimes \mathbb{Q}.$$

The algebra $\mathbb{A}(Y)$ contains $H_*(Y)$ as a linear subspace, and $I^w(-Y)$ is dual to $I^w(Y)$, so we obtain the operators we seek:

$$\hat{h} : I^w(Y) \otimes \mathbb{Q} \to I^w(Y) \otimes \mathbb{Q}, \qquad h \in H_*(Y).$$

If we have two homology classes h_1 and h_2 then the corresponding operators commute in the graded sense.

Munoz' work can be used to determine the spectrum, and indeed joint spectrum, of the operators coming from $H_*(S^1 \times \Sigma)$. Let $s \in H_2(S^1 \times \Sigma)$ be the homology class of (point) $\times\, \Sigma$ and $y \in H_0(S^1 \times \Sigma)$ be the homology class of a point.

Theorem 4.2. *Then the simultaneous eigenvalues of the action of $\hat{s}$ and $\hat{y}$ on $I^w(S^1 \times \Sigma_g) \otimes \mathbb{C}$ are the pairs of complex numbers $\big(i^m(2k),\ (-1)^m 2\big)$ for all the integers k in the range $0 \le k \le g-1$ and all $m = 0, 1, 2, 3$. Here i denotes $\sqrt{-1}$.*

Furthermore the generalized eigenspace corresponding to $(2g-2, 2)$ is one-dimensional hence simple.

As a corollary of this one can deduce an result for a general 3-manifold Y. If we have a 2-dimensional homology class s represented by a connected surface S in Y, with genus $g > 1$, and a point y thought of as a 0-dimensional homology class, then the following holds.

Theorem 4.3. *For any admissible class $w \in H^2(Y)$ with $w \cdot s = 1$, the simultaneous eigenvalues of the action of $\hat{s}$ and $\hat{y}$ on $I^w(Y) \otimes \mathbb{C}$ are* ***contained*** *in the pairs that arise in the case of the product manifold $S^1 \times S$. That is, they are pairs of complex numbers*

$$\big(i^m(2k),\ (-1)^m 2\big)$$

where k is in the range $0 \le k \le g-1$.

We return now to a *sutured* manifold (Y, γ), which we shall suppose satisfies the condition that Juhász [2008] calls *balanced*, namely we require that $\chi(R_+) = \chi(R_-)$, that no component of Y is a closed 3-manifold, and that every component of ∂Y contains a suture. The first of these conditions holds automatically for the sutured manifolds (Y^i, γ^i) in Theorem 4.1, and one can arrange that the other two mild conditions hold from Y^2 onwards. For such balanced sutured manifold one can form (not uniquely) a *closure* $\bar{Y}$ as follows.

Choose an oriented connected surface T whose boundary admits an orientation-reversing diffeomorphism $\partial T \to \gamma$. Extend this diffeomorphism to a diffeomorphism ϕ of $[-1, 1] \times \partial T$ with a tubular neighborhood (in ∂Y) of γ. Then form the new 3-manifold with boundary

$$\tilde{Y} = Y \cup_\phi [-1, 1] \times \partial T.$$

Note that our assumptions imply that the boundary of $\tilde{Y}$ has two connected components $\bar{R}_\pm$ formed from $R_\pm$ and $\pm 1 \times T$. The balanced assumption implies that $\bar{R}_+$ $\bar{R}_-$ have the same Euler characteristic, and since these are connected these surfaces are diffeomorphic. Choosing a diffeomorphism ψ we construct an closed, connected 3-manifold

$$\bar{Y} = \tilde{Y}|_\psi$$

where the two boundary components are glued together using ψ. The original sutured manifold Y can be obtained from the closed manifold $\bar{Y}$ by a sequence of two sutured manifold decompositions, decomposing first along $\bar{R}$ and then along the annuli $[-1, 1] \times \partial T$.

Formed in this way, the closure $\bar{Y}$ contains a distinguished non-separating connected surface $\bar{R}$ carrying a homology class $r \in H_2(\bar{Y})$. Let w be an admissible class with $w \cdot r = 1$, and consider the application of Theorem 4.3 to the operators

$$\hat{r}, \hat{y} : I^w(\bar{Y}) \otimes \mathbb{C} \to I^w(\bar{Y}) \otimes \mathbb{C}.$$

According to the theorem, the integers that arise in the spectrum of $\hat{r}$ are bounded above by $2g - 2$. We make the following definition:

Definition 4.4. The *sutured instanton homology* of the sutured manifold (Y, γ), written $SHI(Y, \gamma)$ is defined to be the simultaneous eigenspace for the pair $(2g - 2, 2)$ for the operators $(\hat{r}, \hat{y})$ on $I^w(\bar{Y}) \otimes \mathbb{Q}$ for any closure $\bar{Y}$.

In showing that this definition is good (i.e. is independent of the choice of closure) an important role is played by last clause of Theorem 4.2. Note in particular that it tells us that the dimension of $SHI(Y, \gamma)$ is 1 in the case that the sutured manifold is $(B^3, \text{equator})$ or a union of such, for in this case we can take the closure to be $S^1 \times \bar{R}$.

Returning to the three-step plan (1)–(3) from the end of section 4.1, we see that what remains for a proof of the non-vanishing theorem, Corollary 3.6, is item (2) there. That is, one must show that if (Y, γ) is decomposed along S to obtain (Y', γ'), then $SHI(Y', \gamma')$ has rank no larger than the rank of $SHI(Y, \gamma)$. The idea of the proof here is to construct a particular closure $\bar{Y}$ for Y so that S becomes a closed surface $\bar{S}$, and consider the operators $\hat{s}$ that it gives rise to on $SHI(Y, \gamma)$. One then seeks to identify $SHI(Y', \gamma')$ with an eigenspace of the operator $\hat{s}$, thus exhibiting it as a subspace of $SHI(Y, \gamma)$.

Note that in nearly all cases, this line of proof gives a considerable strengthening of the non-vanishing theorem, Corollary 3.6. As a very simple example:

Corollary 4.5. *Let Y be an irreducible 3-manifold containing a non-separating connected surface S of genus at least 2 which is genus-minimizing in its homology class s. Then $I^w(Y)$ has rank at least 4, for every admissible w with $w \cdot s = 1$.*

Proof. Consider the operators $(\hat{s}, \hat{y})$ again. The proof of non-vanishing shows that the simultaneous eigenspace for the pair $(2g - 2, 2)$ for $\hat{s}$ is non-zero. For formal reasons, the eigenspaces of the pair $(i^r(2g - 2), (-1)^r 2)$ are all of the same dimension, and if $2g - 2$ is non-zero then these four pairs are distinct. □

4.3 Sutured manifolds and knots. As Juhasz observed in Juhász [2008], one can use the sutured manifold formalism to define an instanton homology for knots. For simplicity, let us consider a classical knot K in S^3, and let Y be the "knot complement": the manifold with torus boundary obtained by removing from S^3 an open tubular neighborhood of K. Let γ be the union of two disjoint meridional curves on ∂Y, with opposite orientations. In this way we associate a sutured manifold (Y, γ) to $K \subset S^3$, and the sutured instanton homology of (Y, γ) is an invariant, which we can call the knot instanton homology $KHI(K)$.

To understand this knot invariant better, one can describe an explicit closure $\bar{Y}$ in this case, using an annulus for the auxiliary surface T. To describe this closure, consider

the 3-torus T^3 as $S^1 \times T^2$, and let $\lambda \subset T^3$ be the circle $S^1 \times \{\text{point}\}$. Remove an open neighborhood of λ and glue the resulting torus boundary to the boundary of knot complement Y. The gluing should be chosen so that the longitudinal curves in ∂Y are glued to the meridional curves of λ and vice versa. In this closed 3-manifold $\bar{Y}$, the distinguished surface $\bar{R} \subset \bar{Y}$ is a standard torus disjoint from Y. Note if we are given a Seifert surface for K (an oriented surface with boundary a longitude of K), then we obtain a closed surface $S \subset \bar{Y}$ as the union of the Seifert surface and the punctured torus $T^2 \setminus \{\text{point}\}$.

Following Juhász [ibid.], one can show that the invariant *KHI* can be used to detect the unknot:

Proposition 4.6. *For a classical knot K, the dimension of $KHI(K)$ is greater than or equal to 1, with equality if and only if K is the unknot.*

Proof. Unwrapping the definitions we see that $KHI(K)$ is the simultaneous eigenspace for the eigenvalues $(0, 2)$ of the operators $(\hat{r}, \hat{y})$. Since $\hat{r}$ has genus 1, this is simply the eigenspace of $\hat{y}$ for the eigenvalue 2. In the case of the unknot, $\bar{Y}$ is a 3-torus, the instanton homology $I^w(\bar{Y})$ has rank 2, and the $+2$ and -2 eigenspaces of $\hat{y}$ are both 1-dimensional. For a non-trivial knot, the dimensions are larger, by the argument of Corollary 4.5, for there is a genus-minimizing surface $S \subset Y$ of genus at least 2, obtained from a Seifert surface for K as described above. □

Since instanton homology is defined ultimately in terms of flat connections, one can use the above proposition to deduce that, if K is a non-trivial knot, then the $SU(2)$ or $SO(3)$ representations of the knot group $\pi(K)$ is strictly larger than the case of the unknot. In this way, one can derive Theorem 1.2 from the first section.

Remark 4.7. In the Heegaard Floer homology setting, Juhasz's construction recovers the simplest version of the Heegaard knot homology of Ozsváth and Szabó [2004] and Rasmussen [2003]. On the basis of the few existing calculations in the instanton case, one can conjecture that the rank of $SHI(K)$ is equal to the rank of the Heegaard knot homology group.

Remark 4.8. The generalized eigenspaces of the operator defined by S give a direct sum decomposition of $KHI(K)$. There is also a $\mathbb{Z}/2$ grading, so it makes sense to compute the Euler characteristic of the summands. In this way one recovers the coefficients of the Alexander polynomial, just as one does in the case of the Heegaard knot homology.

Remark 4.9. There is work of Daemi and Xie [2017] on generalizing the sutured instanton homology *SHI* by using the gauge groups $SU(N)$. The essential step is in establishing an appropriate replacement for the results of Munoz.

5 Instanton homology for knots, links and spatial graphs

We have explained above that Theorem 1.6, which asserts the existence of a non-abelian representation of the fundamental group of a 3-manifold in $SO(3)$, can be seen as a corollary of a non-vanishing theorem for the instanton Floer homology group $I^w(Y)$ (Corollary 3.6). The non-vanishing theorem can be proved using sutured manifold decompositions, as outlined in section 4; and from these results about sutured manifolds,

we deduced results about representations of $\pi(K)$ for knots and links such as Proposition 4.6 via the knot homology group $KHI(K)$.

There is a more direct approach to defining an instanton homology group for knots, which we will outline next. Although the invariant defined by this approach turns out to be isomorphic to $KHI(K)$, the alternative approach plays dividends in its extra flexibility.

5.1 Instanton homology for orbifolds. Our strategy is to define an instanton Floer homology group for a certain class or orbifolds, and the class we have in mind are closed, oriented 3-dimensional orbifolds O whose singular set is a knot or link and whose nontrivial local stabilizers are all $\mathbb{Z}/2$. Thus our class includes the orbifolds that appear in the statement Theorem 1.3 in the introduction. The material here is drawn from Kronheimer and Mrowka [2012].

There is no particular difficulty in studying bundles, connections and the anti-self-duality equations on orbifolds. To avoid reducible connections we need to work again with $U(2)$ bundles with fixed determinant, as in subsection 3.3. At points of the singular set, where the stabilizer is $\mathbb{Z}/2$, the local model will be the quotient of a smooth $U(2)$ bundle over the ball, and we ask that the action on the fiber of the bundle be by the element

$$\tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The first Chern class of such an orbifold bundle E is a class dual to a relative 1-cycle w (thought of geometrically as a 1-manifold in the smooth part of O with possible endpoints on the singular part).

As long as w is admissible one can define an instanton homology group $I^w(O)$ much as before. The critical points of the Chern–Simons functional are flat $PU(2)$ connections on the complement of the singular set whose monodromy around the meridional links has order 2 and whose Stiefel–Whitney class is w mod 2.

Recall that, for a classical link K in S^3, we write $\mathrm{Orb}(S^3, K)$ for the orbifold whose singular set is K. A first Chern class w on $\mathrm{Orb}(S^3, K)$ will be admissible if it is dual to an arc joining two components of K, because such a w evaluates to 1 on an oriented surface separating the components. To achieve admissibility in general, we adopt the following device. Given classical knot or link K, we form a new link by taking the union of K with a small meridional loop L, linking K at chosen point $x \in K$. We take w to be the admissible class dual to an arc joining the new loop L to K. We may then define:

$$I^\natural(K) = I^w(K \cup L). \tag{11}$$

If K has more than one component, then the choice of the point x may be material, but we still omit x from the notation.

To understand the definition a little, observe that when K is an unknot, the union $K \cup L$ is a Hopf link. It is not hard to verify in this case that there is exactly one critical point of the Chern–Simons functional on the corresponding orbifold with the correct determinant w. This flat connection corresponds to the Klein 4-group representation of

the fundamental group of the complement, $\pi(K \cup L) = \mathbb{Z} \oplus \mathbb{Z}$. This unique critical point is the generator for $I^\natural(K)$. Having just one generator, $I^\natural(K)$ is infinite cyclic.

Although the definition is different, the same techniques of cut-and-paste topology that are used in the construction of sutured instanton Floer homology can be used to show that the two approaches yield the same result, for knots:

Proposition 5.1. *For a knot K, the homology groups $KHI(K)$ and $I^\natural(K)$ are isomorphic.*

As a trivial corollary, the orbifold version also detects knottedness:

Corollary 5.2. *For a knot K in S^3, the rank of $I^\natural(K)$ is at least 1, with equality if and only K is the unknot.*

5.2 Khovanov homology. An advantage of the orbifold approach to the definition of $I^\natural(K)$ is that it allows a straightforward approach to functoriality. A *cobordism* between classical links K_0 and K_1 is an embedded surface Σ in $[0,1] \times S^3$, meeting the boundary transversely in K_0 and K_1 at the two ends. Without any requirement of orientability, such a cobordism gives rise to a homomorphism $I^\natural(K_0) \to I^\natural(K_1)$.

This functoriality is the starting point in making an unexpected connection between this instanton homology and a knot homology group from a quite different stable, namely the *Khovanov homology* groups introduced in Khovanov [2000]. The Khovanov homology $Kh(K)$ for a classical knot or link is a "categorification" of the Jones polynomial. It has a definition which is entirely algebraic, and eventually elementary, but $Kh(K)$ and its generalizations have turned out to have deep connections with geometry, in several directions. In our particular context, we have the following result:

Theorem 5.3 (Kronheimer and Mrowka [2012])**.** *For a classical knot or link K, there is a spectral sequence whose E_2 page is the (reduced variant of) the Khovanov homology of K and which abuts to the orbifold instanton homology, $I^\natural(K)$.*

Like $I^\natural(K)$, the reduced Khovanov homology has rank 1 if K is the unknot. From Corollary 5.2 we and the existence of the spectral sequence, we therefore obtain:

Corollary 5.4. *For a knot K in S^3, the rank of the reduced Khovanov homology is at least 1, with equality if and only K is the unknot.*

It is an open question whether the Jones polynomial itself is an unknot-detector. Although many geometric techniques can be used to characterize the unknot algorithmically (starting with Haken's work in Haken [1961]), the above corollary stands somewhat apart, because of the origins of Khovanov homology in quantum algebra and representation theory.

An interesting avenue to pursue is to replace $U(2)$ in the orbifold setup with $U(N)$ and to explore the relationship to generalizations such as Khovanov–Rozansky homology Khovanov and Rozansky [2008]. See Daemi [2014] and Xie [2016].

5.3 Instanton homology for spatial graphs. We return to the material of section 1.4, to consider a trivalent spatial graph $K \subset S^3$. As we did for knots and links, we can

apply instanton Floer homology to the associated orbifold $O = \mathrm{Orb}(S^3, K)$. Allowing trivalent vertices in K leads to new issues, related in particular to the possibility of the Uhlenbeck bubbling phenomenon occurring at orbifold points corresponding to vertices of K. In order to have a well-defined instanton homology, it turns out to be necessary to use a ring of coefficients of characteristic 2.

Following this line, the authors defined in Kronheimer and Mrowka [n.d.] an invariant of trivalent spatial graphs K which takes the form of a $\mathbb{Z}/2$ vector space $J^\sharp(K)$. This variant of instanton homology arises from a Chern–Simons functional whose set of critical points can be identified with the space of $SO(3)$ representations $\mathcal{R}(K)$ considered at (1). (In particular, this is essentially an $SO(3)$ gauge theory, not the type of $U(2)$ gauge theory used in the definition of $I^w(Y)$ before.)

Once again, by reducing the question to one about the instanton homology of a sutured manifold (essentially the complement of K) one can prove a non-vanishing theorem for graphs that are spatially bridgeless in the sense of section 1.4:

Theorem 5.5. *If $K \subset \mathbb{R}^3$ is a spatially bridgeless trivalent graph, then the instanton homology group $J^\sharp(K)$ is non-zero.*

An immediate corollary is that the space of representations $\mathcal{R}(K)$ is non-empty, which is the statement of Theorem 1.11 in the introduction.

As mentioned in section 1.4, the space of $SO(3)$ representations $\mathcal{R}(K)$ contains the set of representations, $\pi(K) \to V_4$, into the Klein 4-group, which are in one-to-one correspondence with Tait colorings of K. It is difficult to compute $J^\sharp(K)$, but an examination of the simplest examples prompts this question.

Question 5.6. *For a spatial trivalent graph K that is* planar *(that is, embedded in a plane $\mathbb{R}^2$ in $\mathbb{R}^3$), is it the case that the dimension of $J^\sharp(K)$ is equal to the number of Tait colorings?*

It is known that, if the answer is no, then a minimal counterexample can have no bigons, triangles or squares Kronheimer and Mrowka [ibid.]. Various equivalent forms of the question are given in Kronheimer and Mrowka [2016] and Kronheimer and Mrowka [2017]. It is also known Kronheimer and Mrowka [ibid.] that the number of Tait colorings is a lower bound for the dimension of $J^\sharp(K)$. Because of the connection between Tait coloring the edges and four-coloring the regions of a planar trivalent graph (see section 1.4 again), an affirmative answer to the question would provide a new proof that every planar map can be four-colored.

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SHTUKAS FOR REDUCTIVE GROUPS AND LANGLANDS CORRESPONDENCE FOR FUNCTION FIELDS

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Abstract

This text gives an introduction to the Langlands correspondence for function fields and in particular to some recent works in this subject.

We begin with a short historical account (all notions used below are recalled in the text).

The Langlands correspondence Langlands [1970] is a conjecture of utmost importance, concerning global fields, i.e. number fields and function fields. Many excellent surveys are available, for example Gelbart [1984], Bump [1997], Bump, Cogdell, de Shalit, Gaitsgory, Kowalski, and Kudla [2003], Taylor [2004], Frenkel [2007], and Arthur [2014]. The Langlands correspondence belongs to a huge system of conjectures (Langlands functoriality, Grothendieck's vision of motives, special values of L-functions, Ramanujan–Petersson conjecture, generalized Riemann hypothesis). This system has a remarkable deepness and logical coherence and many cases of these conjectures have already been established. Moreover the Langlands correspondence over function fields admits a geometrization, the "geometric Langlands program", which is related to conformal field theory in Theoretical Physics.

Let G be a connected reductive group over a global field F. For the sake of simplicity we assume G is split.

The Langlands correspondence relates two fundamental objects, of very different nature, whose definition will be recalled later,

- the automorphic forms for G,
- the global Langlands parameters , i.e. the conjugacy classes of morphisms from the Galois group $\mathrm{Gal}(\overline{F}/F)$ to the Langlands dual group $\widehat{G}(\overline{\mathbb{Q}_\ell})$.

For $G = GL_1$ we have $\widehat{G} = GL_1$ and this is class field theory, which describes the abelianization of $\mathrm{Gal}(\overline{F}/F)$ (one particular case of it for $\mathbb{Q}$ is the law of quadratic reciprocity, which dates back to Euler, Legendre and Gauss).

Now we restrict ourselves to the case of function fields.

In the case where $G = GL_r$ (with $r \geq 2$) the Langlands correspondence (in both directions) was proven by Drinfeld [1980, 1987b, 1988, 1987a] for $r = 2$ and by L. Lafforgue [2002] for arbitrary r. In fact they show the "automorphic to Galois" direction

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by using the cohomology of stacks of shtukas and the Arthur–Selberg trace formula, and deduce from it the reverse direction by using the inverse theorems of Weil, Piatetski–Shapiro and Cogdell (see Cogdell and Piatetski-Shapiro [1994] which are specific to the case of GL_r) as well as Grothendieck's functional equation and Laumon's product formula Laumon [1987b] (which are specific to the case of function fields). Other works using the Arthur–Selberg trace formula for stacks of shtukas are, in chronological order, Laumon, Rapoport, and Stuhler [1993] and Laumon [1997], Châu [1999, 2006] and Châu and Tuấn [2008], Tuấn [2007, 2009], Lau [2004, 2007], Kazhdan, Varshavsky [2004b], Badulescu and Roche [2017].

In V. Lafforgue [2012] we show the "automorphic to Galois" direction of the Langlands correspondence for all reductive groups over function fields. More precisely we construct a *canonical* decomposition of the vector space of cuspidal automorphic forms, indexed by global Langlands parameters. This decomposition is obtained by the spectral decomposition associated to the action on this vector space of a commutative algebra $\mathcal{B}$ of "excursion operators" such that each character of $\mathcal{B}$ determines a unique global Langlands parameter. Unlike previous works, our method is independent on the Arthur–Selberg trace formula. We use the following two ingredients:

- the classifying stacks of shtukas, introduced by Drinfeld for GL_r Drinfeld [1980, 1987b] and generalized to all reductive groups and arbitrary number of "legs" by Varshavsky [2004b] (shtukas with several legs were also considered in Lau [2004] and Châu [2006]),
- the geometric Satake equivalence, due to Lusztig, Drinfeld, Ginzburg and Mirkovic–Vilonen (see Beilinson and Drinfeld [1999] and Mirković and Vilonen [2007]) (it is a fundamental ingredient of the geometric Langlands program, whose idea comes from the fusion of particles in conformal field theory).

In the last sections we discuss recent works related to the Langlands program over function fields, notably on the independence on ℓ and on the geometric Langlands program. We cannot discuss the works about number fields because there are too many and it is not possible to quote them in this short text. Let us only mention that, in his lectures at this conference, Peter Scholze will explain local analogues of shtukas over $\mathbb{Q}_p$.

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1 Preliminaries

1.1 Basic notions in algebraic geometry. Let k be a field. The ring of functions on the n-dimensional affine space $\mathbb{A}^n$ over k is the ring $k[x_1, ..., x_n]$ of polynomials in

n variables. For any ideal I, the quotient $A = k[x_1, ..., x_n]/I$ is the ring of functions on the closed subscheme of $\mathbb{A}^n$ defined by the equations in I and we obtain in this way all affine schemes (of finite type) over k. An affine scheme over k is denoted by $\mathrm{Spec}(A)$ when A is the k-algebra of functions on it. It is equipped with the Zariski topology (generated by open subschemes of the form $f \neq 0$ for $f \in A$). It is called a variety when A has no non zero nilpotent element. General schemes and varieties are obtained by gluing. The projective space $\mathbb{P}^n$ over k is the quotient of $\mathbb{A}^{n+1} \setminus \{0\}$ by homotheties and can be obtained by gluing $n+1$ copies of $\mathbb{A}^n$ (which are the quotients of $\{(x_0, ..., x_n), x_i \neq 0\}$, for $i = 0, ..., n$). Closed subschemes (resp. varieties) of $\mathbb{P}^n$ are called projective schemes (resp varieties) over k. Schemes over k have a dimension and a curve is a variety purely of dimension 1.

1.2 Global fields. A number field is a finite extension of $\mathbb{Q}$, i.e. a field generated over $\mathbb{Q}$ by some roots of a polynomial with coefficients in $\mathbb{Q}$.

A function field F is the field of rational functions on an irreducible curve X over a finite field $\mathbb{F}_q$.

We recall that if q is a prime number, $\mathbb{F}_q = \mathbb{Z}/q\mathbb{Z}$. In general q is a power of a prime number and all finite fields of cardinal q are isomorphic to each other (although non canonically), hence the notation $\mathbb{F}_q$.

The simplest example of a function field is $F = \mathbb{F}_q(t)$, namely the field of rational functions on the affine line $X = \mathbb{A}^1$. Every function field is a finite extension of such a field $\mathbb{F}_q(t)$.

Given a function field F there exists a unique smooth projective and geometrically irreducible curve X over a finite field $\mathbb{F}_q$ whose field of rational functions is F: indeed for any irreducible curve over $\mathbb{F}_q$ we obtain a smooth projective curve with the same field of rational functions by resolving the singularities and adding the points at infinity. For example $F = \mathbb{F}_q(t)$ is the field of rational functions of the projective line $X = \mathbb{P}^1$ over $\mathbb{F}_q$ (we have added to $\mathbb{A}^1$ the point at infinity).

For the rest of the text we fix a smooth projective and geometrically irreducible curve X over $\mathbb{F}_q$. We denote by F the field of functions of X (but F may also denote a general global field, as in the next subsection).

1.3 Places of global fields and local fields. A place v of a global field F is a non trivial multiplicative norm $F \to \mathbb{R}_{\geq 0}$, up to equivalence (where the equivalence relation identifies $\|.\|$ and $\|.\|^s$ for any $s > 0$). The completion F_v of the global field F for this norm is called a local field. It is a locally compact field and the inclusion $F \subset F_v$ determines v. Therefore a place is "a way to complete a global field into a local field".

For any local field there is a canonical normalization of its norm given by the action on its Haar measure. For any non zero element of a global field the product of the normalized norms at all places is equal to 1.

For example the places of $\mathbb{Q}$ are

- the archimedean place, where the completion is $\mathbb{R}$ (with normalized norm equal to the usual absolute value),

- for every prime number p, the place p where the completion is $\mathbb{Q}_p$ (the normalized norm in $\mathbb{Q}_p$ of a number $r \in \mathbb{Q}^\times$ is equal to $p^{-n_p(r)}$, where $n_p(r) \in \mathbb{Z}$ is the exponent of p in the decomposition of r as the product of a sign and powers of the prime numbers).

Thus the local fields obtained by completion of $\mathbb{Q}$ are $\mathbb{Q}_p$, for all prime numbers p, and $\mathbb{R}$. A place v is said to be archimedean if F_v is equal to $\mathbb{R}$ or $\mathbb{C}$. These places are in finite number for number fields and are absent for function fields. For each non archimedean place v we denote by $\mathcal{O}_v$ the ring of integers of F_v, consisting of elements of norm ≤ 1. For example it is $\mathbb{Z}_p$ if $F_v = \mathbb{Q}_p$.

In the case of function fields, where we denote by F the field of functions of X, the places are exactly the closed points of X (defined as the maximal ideals). The closed points are in bijection with the orbits under $\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ on $X(\overline{\mathbb{F}_q})$ (Galois groups are recalled below). For every closed point v of X, we denote by $n_v : F^\times \to \mathbb{Z}$ the valuation which associates to a rational function other than 0 its vanishing order at v. We can see $\mathcal{O}_v$ as the $\mathbb{F}_q$-algebra of functions on the formal neighborhood around v in X and F_v as the $\mathbb{F}_q$-algebra of functions on the punctured formal neighborhood. We denote by $\kappa(v)$ the residue field of $\mathcal{O}_v$; it is a finite extension of $\mathbb{F}_q$, whose degree is denoted by $\deg(v)$, therefore it is a finite field with $q^{\deg(v)}$ elements. The normalized norm on F associated to v sends $a \in F^\times$ to $q^{-\deg(v)n_v(a)}$.

In the example where $X = \mathbb{P}^1 = \mathbb{A}^1 \cup \infty$, the unitary irreducible polynomials in $\mathbb{F}_q[t]$ (which is the ring of functions on $\mathbb{A}^1$) play a role analoguous to that of the prime numbers in $\mathbb{Z}$: the places of $\mathbb{P}^1$ are

- the place ∞, at which the completion is $\mathbb{F}_q((t^{-1}))$,

- the places associated to unitary irreducible polynomials in $\mathbb{F}_q[t]$ (the degree of such a place is simply the degree of the polynomial). For example the unitary irreducible polynomial t corresponds to the point $0 \in \mathbb{A}^1$ and the completion at this place is $\mathbb{F}_q((t))$.

We recall that the local field $\mathbb{F}_q((t))$ consists of Laurent series, i.e. sums $\sum_{n\in\mathbb{Z}} a_n t^n$ with $a_n \in \mathbb{F}_q$ and $a_n = 0$ for n negative enough.

1.4 Galois groups. If k is a field, we denote by $\overline{k}$ an algebraic closure of k. It is generated over k by the roots of all polynomials with coefficients in k. The separable closure $k^{\mathrm{sep}} \subset \overline{k}$ consists of the elements whose minimal polynomial over k has a non zero derivative. We denote by $\mathrm{Gal}(\overline{k}/k) = \mathrm{Gal}(k^{\mathrm{sep}}/k)$ the group of automorphisms of $\overline{k}$ (or equivalently of k^{sep}) which act by the identity on k. It is a profinite group, i.e. a projective limit of finite groups: an element of $\mathrm{Gal}(\overline{k}/k)$ is the same as a family, indexed by the finite Galois extensions $k' \subset \overline{k}$ of k, of elements $\theta_{k'} \in \mathrm{Gal}(k'/k)$, so that if $k'' \supset k'$, $\theta_{k''}\big|_{k'} = \theta_{k'}$. We recall that $k' \subset \overline{k}$ is said to be a finite Galois extension of k if it is a finite dimensional k-vector subspace of k^{sep} and is stable under the action of $\mathrm{Gal}(\overline{k}/k) = \mathrm{Gal}(k^{\mathrm{sep}}/k)$ (and then $\mathrm{Gal}(k'/k)$ is a finite group of cardinal equal to the dimension of k' over k).

A simple example is given by finite fields: $\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ is equal to the profinite completion $\widehat{\mathbb{Z}}$ of $\mathbb{Z}$ in such a way that $1 \in \widehat{\mathbb{Z}}$ is the Frobenius generator $x \mapsto x^q$ (which is an automorphism of $\overline{\mathbb{F}_q}$ equal to identity on $\mathbb{F}_q$).

We recall that for any $\mathbb{F}_q$-algebra, $x \mapsto x^q$ is a morphism of $\mathbb{F}_q$-algebras, in particular $(x+y)^q = x^q + y^q$. For any scheme S over $\mathbb{F}_q$ we denote by $\mathrm{Frob}_S : S \to S$ the morphism acting on functions by $\mathrm{Frob}_S^*(f) = f^q$.

We come back to the function field F of X. Our main object of interest is the Galois group $\Gamma = \mathrm{Gal}(\overline{F}/F) = \mathrm{Gal}(F^{\mathrm{sep}}/F)$.

By the point of view of Grothendieck developed in SGA1, we have an equivalence between

- the category of finite sets A endowed with a continuous action of Γ
- the category of finite separable F-algebras

where the functor from the first category to the second one maps A to the finite separable F-algebra $((F^{\mathrm{sep}})^A)^\Gamma$ (here $(F^{\mathrm{sep}})^A$ is the direct sum of copies of F^{sep} indexed by A and Γ acts on each copy and permutes them at the same time). We write $\eta = \mathrm{Spec}(F)$ and $\overline{\eta} = \mathrm{Spec}(\overline{F})$. Then, for any dense open $U \subset X$, Γ has a profinite quotient $\pi_1(U, \overline{\eta})$ such that a continuous action of Γ on a finite set A factors through $\pi_1(U, \overline{\eta})$ if and only if $\mathrm{Spec}(((F^{\mathrm{sep}})^A)^\Gamma)$ extends (uniquely) to an étale covering of U. We will not explain the notion of étale morphism in general and just say that a morphism between smooth varieties over a field is étale if and only if its differential is everywhere invertible. Thus we have an equivalence between

- the category of finite sets A endowed with a continuous action of $\pi_1(U, \overline{\eta})$
- the category of finite étale coverings of U.

For any place v the choice of an embedding $\overline{F} \subset \overline{F_v}$ provides an inclusion $\mathrm{Gal}(\overline{F_v}/F_v) \subset \mathrm{Gal}(\overline{F}/F)$ (well defined up to conjugation). We denote by $\mathrm{Frob}_v \in \mathrm{Gal}(\overline{F}/F)$ the image of any element of $\mathrm{Gal}(\overline{F}_v/F_v)$ lifting the Frobenius generator Frobenius $x \mapsto x^{q^{\deg(v)}}$ in $\mathrm{Gal}(\overline{\kappa(v)}/\kappa(v)) = \widehat{\mathbb{Z}}$. When U is open dense in X as above and v is a place in U, the image of Frob_v in $\pi_1(U, \overline{\eta})$ is well defined up to conjugation.

1.5 A lemma of Drinfeld [1980]. Let $U \subset X$ open dense as above. For any $i \in I$ we denote by Frob_i the "partial Frobenius" morphism $U^I \to U^I$ which sends $(x_j)_{j \in I}$ to $(x'_j)_{j \in I}$ with $x'_i = \mathrm{Frob}_U(x_i)$ and $x'_j = x_j$ for $j \neq i$. For any scheme T and any morphism $T \to U^I$, we say that a morphism $a : T \to T$ is "above" Frob_i if the square

$$\begin{tikzcd} T \arrow[r, "a"] \arrow[d] & T \arrow[d] \\ U^I \arrow[r, "\mathrm{Frob}_i"] & U^I \end{tikzcd}$$

is commutative.

Lemma 1.1. *We have an equivalence of categories between*

- *the category of finite sets A endowed with a continuous action of $(\pi_1(U,\overline{\eta}))^I$,*
- *the category of finite étale coverings T of U^I, equipped with partial Frobenius morphisms, i.e. morphisms $F_{\{i\}}$ above* Frob_i*, commuting with each other, and whose composition is* Frob_T.

The functor from the first category to the second one is the following: if the action of $(\pi_1(U,\overline{\eta}))^I$ on A factorizes through $\prod_{i\in I}\mathrm{Gal}(U_i/U)$ where for each i, U_i is a finite étale Galois covering of U (and $\mathrm{Gal}(U_i/U)$ is its automorphism group), then the image by the functor is $(\prod_{i\in I} U_i)\times_{\prod_{i\in I}\mathrm{Gal}(U_i/U)} A$, equipped with the partial Frobenius morphisms $F_{\{i\}}$ given by $\left(\mathrm{Frob}_{U_i}\times\prod_{j\neq i}\mathrm{Id}_{U_j}\right)\times\mathrm{Id}_A$.

1.6 Split connected reductive groups and bundles. We denote by $\mathbb{G}_m = GL_1$ the multiplicative group. A split torus over a field k is an algebraic group T which is isomorphic to $\mathbb{G}_m^r$ for some r.

A connected reductive group over a field k is a connected, smooth, affine algebraic group whose extension to $\overline{k}$ has a trivial unipotent radical (i.e. any normal, smooth, connected, unipotent subgroup scheme of it is trivial). A connected reductive group G over k is said to be split if it has a split maximal torus T. Then (after choosing a Borel subgroup containing T) the lattices $\mathrm{Hom}(\mathbb{G}_m,T)$ and $\mathrm{Hom}(T,\mathbb{G}_m)$ are called the coweight and weight lattices of G. The split connected reductive groups over a field k are exactly the quotients by central finite subgroup schemes of products of $\mathbb{G}_m$, simply-connected split groups in the four series SL_{n+1}, $Spin_{2n+1}$, Sp_{2n}, $Spin_{2n}$, and five simply-connected split exceptional groups.

Let G be a split connected reductive group over a field k, and X a scheme over k. Then a G-bundle over X is a morphism $Y\to X$, together which an action of G on the fibers which is simply transitive. A GL_r-bundle $\mathcal{E}$ gives rise to the vector bundle of rank r equal to $\mathcal{E}\times_{GL_r}\mathbb{A}^r$ and the notions are equivalent.

2 Reminder on automorphic forms

For the moment we take $G = GL_r$. When the global field is $\mathbb{Q}$, an automorphic form (without level at finite places) is a function on the quotient $GL_r(\mathbb{Z})\backslash GL_r(\mathbb{R})$ (the best known example is the particular case of modular functions, for which $r = 2$). This quotient classifies the free $\mathbb{Z}$-modules (or, equivalently, projective $\mathbb{Z}$-modules) M of rank r equipped with a trivialization $M\otimes_{\mathbb{Z}}\mathbb{R} = \mathbb{R}^r$ (i.e. an embedding of M as a lattice in $\mathbb{R}^r$). Indeed if we choose a basis of M over $\mathbb{Z}$ its embedding in $\mathbb{R}^r$ is given by a matrix in $GL_r(\mathbb{R})$ and the change of the basis of M gives the quotient by $GL_r(\mathbb{Z})$.

Now we come back to our function field F. To explain the analogy with $\mathbb{Q}$ we choose a place v of X (of degree 1 to simplify) playing the role of the archimedean place of $\mathbb{Q}$ (but this choice is not natural and will be forgotten in five lines). An analogue of a projective $\mathbb{Z}$-module M of rank r equipped with a trivialization $M\otimes_{\mathbb{Z}}\mathbb{R} = \mathbb{R}^r$ is a vector bundle of rank r over X equipped with a trivialization on the formal neighborhood around v. Now we forget the trivialization on the formal neighborhood around v (because we do not want to introduce a level at v) and then we forget the choice of v.

Thus an automorphic form (without level at any place) for GL_r is a function on the set $\mathrm{Bun}_{GL_r}(\mathbb{F}_q)$ of isomorphism classes of vector bundles of rank r over X.

Now we consider the case of a general group G. From now on we denote by G a connected reductive group over F, assumed to be split for simplicity. An automorphic form (without level) for G is a function on the set $\mathrm{Bun}_G(\mathbb{F}_q)$ of isomorphism classes of G-bundles over X.

Remark 2.1. This remark can be skipped. In fact the G-bundles over X have finite automorphism groups. Therefore it is more natural to consider $\mathrm{Bun}_G(\mathbb{F}_q)$ as a groupoid, i.e. a category where all arrows are invertible. It is the groupoid of points over $\mathbb{F}_q$ of the Artin stack Bun_G over $\mathbb{F}_q$ whose groupoid of S-points (with S a scheme over $\mathbb{F}_q$) classifies the G-bundles over $X \times S$. We refer to Laumon and Moret-Bailly [2000] for the notion of Artin stack and we say only that examples of Artin stacks are given by the quotients of algebraic varieties by algebraic groups. In Artin stacks the automorphism groups of the points are algebraic groups (for example in the case of a quotient they are the stabilizers of the action). The Quot construction of Grothendieck implies that Bun_G is an Artin stack (locally it is even the quotient of a smooth algebraic variety by a smooth algebraic group). The automorphisms groups of points in the groupoid $\mathrm{Bun}_G(\mathbb{F}_q)$ are finite, in fact they are the points over $\mathbb{F}_q$ of automorphisms groups of points in Bun_G, which are algebraic groups of finite type.

It is convenient to impose a condition relative to the center Z of G. From now on we fix a subgroup Ξ of finite index in $\mathrm{Bun}_Z(\mathbb{F}_q)$ (for example the trivial subgroup if Z is finite) and we consider functions on $\mathrm{Bun}_G(\mathbb{F}_q)/\Xi$. However, except when G is a torus, $\mathrm{Bun}_G(\mathbb{F}_q)/\Xi$ is still infinite. To obtain vector spaces of finite dimension we now restrict ourselves to cuspidal automorphic forms.

For any field $E \supset \mathbb{Q}$, we denote by $C_c^{\mathrm{cusp}}(\mathrm{Bun}_G(\mathbb{F}_q)/\Xi, E)$ the E-vector space of finite dimension consisting of *cuspidal* functions on $\mathrm{Bun}_G(\mathbb{F}_q)/\Xi$. It is defined as the intersection of the kernel of all "constant term" morphisms $C_c(\mathrm{Bun}_G(\mathbb{F}_q)/\Xi, E) \to C(\mathrm{Bun}_M(\mathbb{F}_q)/\Xi, E)$ (which are given by the correspondence

$$\mathrm{Bun}_G(\mathbb{F}_q) \leftarrow \mathrm{Bun}_P(\mathbb{F}_q) \to \mathrm{Bun}_M(\mathbb{F}_q)$$

and involve only finite sums), for all proper parabolic subgroups P of G with associated Levi quotient M (defined as the quotient of P by its unipotent radical). For readers who do not know these notions, we recall that in the case of GL_r a parabolic subgroup P is conjugated to a subgroup of upper block triangular matrices and that the associated Levi quotient M is isomorphic to the group of block diagonal matrices. It is legitimate in the Langlands correspondence to restrict oneself to cuspidal automorphic forms because all automorphic forms for G can be understood from cuspidal automorphic forms for G and for the Levi quotients of its parabolic subgroups.

Let ℓ be a prime number not dividing q. To simplify the notations we assume that $\mathbb{Q}_\ell$ contains a square root of q (otherwise replace $\mathbb{Q}_\ell$ everywhere by a finite extension containing a square root of q). For Galois representations we have to work with coefficients in $\mathbb{Q}_\ell$ and $\overline{\mathbb{Q}_\ell}$, and not $\mathbb{Q}$, $\overline{\mathbb{Q}}$ and even $\mathbb{C}$ (to which $\overline{\mathbb{Q}_\ell}$ is isomorphic algebraically but not topologically) because the Galois representations which are continuous with coefficients in $\mathbb{C}$ always have a finite image (unlike those with coefficients in $\overline{\mathbb{Q}_\ell}$) and are not

enough to match automorphic forms in the Langlands correspondence. Therefore, even if the notion of cuspidal automorphic form is (in our case of function fields) algebraic, to study the Langlands correspondence we will consider cuspidal automorphic forms with coefficients in $E = \mathbb{Q}_\ell$ or $\overline{\mathbb{Q}_\ell}$.

3 Class field theory for function fields

It was developped by Rosenlicht and Lang (see Serre [1975]). Here we consider only the unramified case.

Let Pic be the relative Picard scheme of X over $\mathbb{F}_q$, whose definition is that, for any scheme S over $\mathbb{F}_q$, $\mathrm{Pic}(S)$ (the set of morphisms $S \to \mathrm{Pic}$) classifies the isomorphism classes $[\mathcal{E}]$ of line bundles $\mathcal{E}$ on $X \times S$ (a line bundle is a vector bundle of rank 1, so it is the same as a $\mathbb{G}_m$-bundle). The relation with Bun_{GL_1} is that Bun_{GL_1} can be identified with the quotient of Pic by the trivial action of $\mathbb{G}_m$.

Let Pic^0 be the neutral component of Pic, i.e. the kernel of the degree morphism $\mathrm{Pic} \to \mathbb{Z}$. It is an abelian variety over $\mathbb{F}_q$, also called the jacobian of X.

Class field theory states (in the unramified case to which we restrict ourselves in this text) that there is a canonical isomorphism

$$\big(\pi_1(X, \overline{\eta})\big)^{\mathrm{ab}} \times_{\widehat{\mathbb{Z}}} \mathbb{Z} \xrightarrow{\sim} \mathrm{Pic}(\mathbb{F}_q) \tag{3-1}$$

characterized by the fact that for any place v of X, it sends Frob_v to $[\mathcal{O}(v)]$, where $\mathcal{O}(v)$ is the line bundle on X whose sections are the functions on X with a possible pole of order ≤ 1 at v.

The isomorphism (3-1) implies that for any $a \in \mathrm{Pic}(\mathbb{F}_q)$ of non zero degree we can associate to any (multiplicative) character χ of the finite abelian group $\mathrm{Pic}(\mathbb{F}_q)/a^{\mathbb{Z}}$ (with values in any field, e.g. $\overline{\mathbb{Q}_\ell}$ for ℓ prime to q) a character $\sigma(\chi)$ of $\pi_1(X, \overline{\eta})$. We now give a geometric construction of $\sigma(\chi)$, which is in fact the key step in the proof of the isomorphism (3-1).

The Lang isogeny $L : \mathrm{Pic} \to \mathrm{Pic}^0$ is such that, for any scheme S over $\mathbb{F}_q$ and every line bundle $\mathcal{E}$ on $X \times S$, $[\mathcal{E}] \in \mathrm{Pic}(S)$ is sent by L to $[\mathcal{E}^{-1} \otimes (\mathrm{Frob}_S \times \mathrm{Id}_X)^*(\mathcal{E})] \in \mathrm{Pic}^0(S)$. We note that $[(\mathrm{Frob}_S \times \mathrm{Id}_X)^*(\mathcal{E})] \in \mathrm{Pic}(S)$ is the image by $\mathrm{Frob}_{\mathrm{Pic}}$ of $[\mathcal{E}] \in \mathrm{Pic}(S)$. The Lang isogeny is surjective and its kernel is $\mathrm{Pic}(\mathbb{F}_q)$. For any finite set I and any family $(n_i)_{i\in I} \in \mathbb{Z}^I$ satisfying $\sum_{i\in I} n_i = 0$, we consider the Abel–Jacobi morphism $AJ : X^I \to \mathrm{Pic}^0$ sending $(x_i)_{i\in I}$ to the line bundle $\mathcal{O}(\sum_{i\in I} n_i x_i)$. We form the fiber product

$$\begin{tikzcd} \mathrm{Cht}_{I,(n_i)_{i\in I}} \arrow[r] \arrow[d, "\mathfrak{p}"] \arrow[dr, phantom, "\square"] & \mathrm{Pic} \arrow[d, "L"] \\ X^I \arrow[r, "AJ"'] & \mathrm{Pic}^0 \end{tikzcd}$$

and see that $\mathfrak{p}$ is a Galois covering of X^I with Galois group $\mathrm{Pic}(\mathbb{F}_q)$. Thus, up to an automorphism group ${\mathbb{F}_q}^\times$ which we neglect, for any scheme S over $\mathbb{F}_q$, $\mathrm{Cht}_{I,(n_i)_{i\in I}}(S)$ classifies

- morphisms $x_i : S \to X$

- a line bundle $\mathcal{E}$ on $X \times S$
- an isomorphism $\mathcal{E}^{-1} \otimes (\mathrm{Frob}_S \times \mathrm{Id}_X)^*(\mathcal{E}) \simeq \mathcal{O}(\sum_{i \in I} n_i x_i)$.

Moreover $\mathrm{Cht}_{I,(n_i)_{i\in I}}$ is equipped with partial Frobenius morphisms $F_{\{i\}}$ sending $\mathcal{E}$ to $\mathcal{E} \otimes \mathcal{O}(n_i x_i)$. The morphism $F_{\{i\}}$ is above $\mathrm{Frob}_i : X^I \to X^I$, because

$$(\mathrm{Frob}_S \times \mathrm{Id}_X)^*(\mathcal{O}(x_i)) = \mathcal{O}(\mathrm{Frob}_S(x_i))$$

Taking the quotient by $a^{\mathbb{Z}}$ we obtain a *finite* Galois covering

$$\begin{array}{c} \mathrm{Cht}_{I,(n_i)_{i\in I}}/a^{\mathbb{Z}} \\ \downarrow \mathfrak{p} \\ X^I \end{array}$$

with Galois group $\mathrm{Pic}(\mathbb{F}_q)/a^{\mathbb{Z}}$ and equipped with the partial Frobenius morphisms $F_{\{i\}}$. Then Drinfeld's lemma gives rise to a morphism $\alpha_{I,(n_i)_{i\in I}} : \pi_1(X, \overline{\eta})^I \to \mathrm{Pic}(\mathbb{F}_q)/a^{\mathbb{Z}}$. The character $\sigma(\chi)$ of $\pi_1(X, \overline{\eta})$ is characterized by the fact that for any I and $(n_i)_{i\in I}$ with sum 0, $\chi \circ \alpha_{I,(n_i)_{i\in I}} = \boxtimes_{i\in I} \sigma(\chi)^{n_i}$ and this gives in fact a construction of $\sigma(\chi)$.

4 The Langlands correspondence for split tori

Split tori are isomorphic to $\mathbb{G}_m^r$, so there is nothing more than in the case of $\mathbb{G}_m = GL_1$ explained in the previous section. Nevertheless the isomorphism of a split torus with $\mathbb{G}_m^r$ is not canonical (because the automorphism group of $\mathbb{G}_m^r$ is non trivial, equal to $GL_r(\mathbb{Z})$). Let T be a split torus over F. To obtain a canonical correspondence we introduce the Langlands dual group $\widehat{T}$, defined as the split torus over $\mathbb{Q}_\ell$ whose weights are the coweights of T and reciprocally. In other words the lattice $\Lambda = \mathrm{Hom}(\widehat{T}, \mathbb{G}_m)$ is equal to $\mathrm{Hom}(\mathbb{G}_m, T)$. Then the Langlands correspondence gives a bijection $\chi \mapsto \sigma(\chi)$ between

- characters $\mathrm{Bun}_T(\mathbb{F}_q) \to \overline{\mathbb{Q}_\ell}^\times$ with finite image
- continuous morphisms $\pi_1(X, \overline{\eta}) \to \widehat{T}(\overline{\mathbb{Q}_\ell})$ with finite image

characterized by the fact that for any place v of X and any $\lambda \in \Lambda$ the image of $\sigma(\chi)(\mathrm{Frob}_v)$ by $\widehat{T}(\overline{\mathbb{Q}_\ell}) \xrightarrow{\lambda} \overline{\mathbb{Q}_\ell}^\times$ is equal to the image of $\mathcal{O}(v)$ by

$$\mathrm{Pic}(\mathbb{F}_q) \xrightarrow{\lambda} \mathrm{Bun}_T(\mathbb{F}_q) \xrightarrow{\chi} \overline{\mathbb{Q}_\ell}^\times$$

(this condition is the particular case for tori of the condition of "compatibility with the Satake isomorphism" which we will consider later for all reductive groups).

The construction of $\sigma(\chi)$ works as in the previous section, except that $a^{\mathbb{Z}}$ has to be replaced by a subgroup Ξ of $\mathrm{Bun}_T(\mathbb{F}_q)$ of finite index which is included in the kernel of χ, and we now have to use schemes of T-shtukas, defined using T-bundles instead of line bundles.

5 Reminder on the dual group

Let G be a split reductive group over F. We denote by $\widehat{G}$ the Langlands dual group of G. It is the split reductive group over $\mathbb{Q}_\ell$ characterized by the fact that its roots and weights are the coroots and coweights of G, and reciprocally. Here are some examples:

G	$\widehat{G}$
GL_n	GL_n
SL_n	PGL_n
SO_{2n+1}	Sp_{2n}
Sp_{2n}	SO_{2n+1}
SO_{2n}	SO_{2n}

and if G is one of the five exceptional groups, $\widehat{G}$ is of the same type. Also the dual of a product of groups is the product of the dual groups.

Definition 5.1. A global Langlands parameter is a conjugacy class of morphisms $\sigma : \mathrm{Gal}(\overline{F}/F) \to \widehat{G}(\overline{\mathbb{Q}_\ell})$ factorizing through $\pi_1(U, \overline{\eta})$ for some open dense $U \subset X$, defined over a finite extension of $\mathbb{Q}_\ell$, continuous and semisimple.

We say that σ is semisimple if for any parabolic subgroup containing its image there exists an associated Levi subgroup containing it. Since $\overline{\mathbb{Q}_\ell}$ has characteristic 0 this means equivalently that the Zariski closure of its image is reductive, Serre [2005].

We now define the Hecke operators (the spherical ones, also called unramified, i.e. without level). They are similar to the Laplace operators on graphs.

Let v be a place of X. If $\mathcal{G}$ and $\mathcal{G}'$ are two G-bundles over X we say that $(\mathcal{G}', \phi)$ is a modification of $\mathcal{G}$ at v if ϕ is an isomorphism between the restrictions of $\mathcal{G}$ and $\mathcal{G}'$ to $X \setminus v$. Then the relative position is a dominant coweight λ of G (in the case where $G = GL_r$ it is the r-uple of elementary divisors). Let λ be a dominant coweight of G. We get the Hecke correspondence

$$\begin{tikzcd} & \mathcal{H}_{v,\lambda} \arrow[dl, "h^{\leftarrow}"'] \arrow[dr, "h^{\rightarrow}"] & \\ \mathrm{Bun}_G(\mathbb{F}_q) & & \mathrm{Bun}_G(\mathbb{F}_q) \end{tikzcd}$$

where $\mathcal{H}_{v,\lambda}$ is the groupoid classifying modifications $(\mathcal{G}, \mathcal{G}', \phi)$ at v with relative position λ and $h^{\leftarrow}$ and $h^{\rightarrow}$ send this object to $\mathcal{G}'$ and $\mathcal{G}$. Then the Hecke operator acts on functions by pullback by $h^{\leftarrow}$ followed by pushforward (i.e. sums in the fibers) by $h^{\rightarrow}$. In other words

$$\begin{aligned} T_{\lambda,v} : C_c^{\mathrm{cusp}}(\mathrm{Bun}_G(\mathbb{F}_q)/\Xi, \mathbb{Q}_\ell) &\to C_c^{\mathrm{cusp}}(\mathrm{Bun}_G(\mathbb{F}_q)/\Xi, \mathbb{Q}_\ell) \\ f &\mapsto \Big[\mathcal{G} \mapsto \sum_{(\mathcal{G}',\phi)} f(\mathcal{G}')\Big] \end{aligned}$$

where the finite sum is taken over all the modifications $(\mathcal{G}', \phi)$ of $\mathcal{G}$ at v with relative position λ.

These operators form an abstract commutative algebra $\mathcal{H}_v$, the so-called spherical (or unramified) Hecke algebra at v, and this algebra acts on $C_c^{\mathrm{cusp}}(\mathrm{Bun}_G(\mathbb{F}_q)/\Xi, \mathbb{Q}_\ell)$. This algebra $\mathcal{H}_v$ is equal to $C_c(G(\mathcal{O}_v)\backslash G(F_v)/G(\mathcal{O}_v), \mathbb{Q}_\ell)$ and it is possible to write its action with the help of adèles. The actions of these algebras $\mathcal{H}_v$ for different v commute with each other.

The Satake isomorphism Satake [1963] and Cartier [1979] can be viewed Gross [1998] as a canonical isomorphism

$$[V] \mapsto T_{V,v}$$

from the Grothendieck ring of representations of $\widehat{G}$ (with coefficients in $\mathbb{Q}_\ell$) to the unramifed Hecke algebra $\mathcal{H}_v$, namely we have $T_{V\oplus V',v} = T_{V,v} + T_{V',v}$ and $T_{V\otimes V',v} = T_{V,v}T_{V',v}$. If V is an irreducible representation of $\widehat{G}$, $T_{V,v}$ is a combination of the $T_{\lambda,v}$ for λ a weight of V.

6 Presentation of the main result of V. Lafforgue [2012]

We now explain the construction in V. Lafforgue [ibid.] of a canonical decomposition of $\overline{\mathbb{Q}_\ell}$-vector spaces

$$C_c^{\mathrm{cusp}}(\mathrm{Bun}_G(\mathbb{F}_q)/\Xi, \overline{\mathbb{Q}_\ell}) = \bigoplus_\sigma \mathfrak{S}_\sigma \tag{6-1}$$

where the direct sum is taken over global Langlands parameters $\sigma : \pi_1(X, \overline{\eta}) \to \widehat{G}(\overline{\mathbb{Q}_\ell})$. This decomposition is respected by and compatible with the action of Hecke operators. In fact we construct a commutative $\mathbb{Q}_\ell$-algebra

$$\mathcal{B} \subset \mathrm{End}(C_c^{\mathrm{cusp}}(\mathrm{Bun}_G(\mathbb{F}_q)/\Xi, \mathbb{Q}_\ell))$$

containing the image of $\mathcal{H}_v$ for all places v and such that each character ν of $\mathcal{B}$ with values in $\overline{\mathbb{Q}_\ell}$ corresponds in a unique way to a global Langlands parameter σ.

Since $\mathcal{B}$ is commutative we deduce a canonical spectral decomposition

$$C_c^{\mathrm{cusp}}(\mathrm{Bun}_G(\mathbb{F}_q)/\Xi, \overline{\mathbb{Q}_\ell}) = \bigoplus_\nu \mathfrak{S}_\nu$$

where the direct sum is taken over characters ν of $\mathcal{B}$ with values in $\overline{\mathbb{Q}_\ell}$ and $\mathfrak{S}_\nu$ is the generalized eigenspace associated to ν. By associating to each ν a global Langlands parameter σ we deduce the decomposition (6-1) we want to construct. We show in V. Lafforgue [ibid.] that any σ obtained in this way factorizes through $\pi_1(X, \overline{\eta})$, and that the decomposition (6-1) is compatible with the Satake isomorphism at every place v of X, in the sense that for every representation V of $\widehat{G}$, $T_{V,v}$ acts on $\mathfrak{S}_\sigma$ by multiplication by $\mathrm{Tr}_V(\sigma(\mathrm{Frob}_v))$.

The elements of $\mathcal{B}$ are constructed with the help of the ℓ-adic cohomology of stacks of shtukas and are called *excursion operators*.

In the case of GL_r, since every semisimple linear representation is determined up to conjugation by its character and since the Frobenius elements Frob_v are dense in

$\mathrm{Gal}(\overline{F}/F)$ by the Chebotarev theorem, the decomposition (6-1) is uniquely determined by its compatibility with the Satake isomorphism.

On the contrary, for some groups G other than GL_r, according to Blasius [1994] and Lapid [1999] it may happen that different global Langlands parameters correspond to the same characters of $\mathcal{H}_v$ for every place v. This comes from the fact that it is possible to find finite groups Γ and couples of morphisms $\sigma, \sigma' : \Gamma \to \widehat{G}(\overline{\mathbb{Q}_\ell})$ such that σ and σ' are not conjugated but that for any $\gamma \in \Gamma$, $\sigma(\gamma)$ and $\sigma'(\gamma)$ are conjugated Larsen [1994, 1996].

Thus for a general group G, the algebra $\mathcal{B}$ of excursion operators may not be generated by the Hecke algebras $\mathcal{H}_v$ for all places v and the compatibility of the decomposition (6-1) with Hecke operators may not characterize it in a unique way. Therefore we wait for the construction of the excursion operators (done in section 8) before we write the precise statement of our main result, which will be theorem 8.4.

7 The stacks of shtukas and their ℓ-adic cohomology

The ℓ-adic cohomology of a variety (over any algebraically closed field of characteristic $\neq \ell$) is very similar to the Betti cohomology of a complex variety, but it has coefficients in $\mathbb{Q}_\ell$ (instead of $\mathbb{Q}$ for the Betti cohomology). For its definition Grothendieck introduced the notions of site and topos, which provide an extraordinary generalization of the usual notions of topological space and sheaf of sets on it.

To a topological space X we can associate the category whose

- objects are the open subsets $U \subset X$
- arrows $U \to V$ are the inclusions $U \subset V$

and we have the notion of a covering of an open subset by a family of open subsets. A site is an abstract category with a notion of covering of an object by a family of arrows targeting to it, with some natural axioms. A topos is the category of sheaves of sets on a site (a sheaf of sets $\mathcal{F}$ on a site is a contravariant functor of "sections of $\mathcal{F}$" from the category of the site to the category of sets, satisfying, for each covering, a gluing axiom). Different sites may give the same topos.

To define the étale cohomology of an algebraic variety X we consider the étale site

- whose objects are the étale morphisms

$$\begin{tikzcd} U \arrow[d] \\ X \end{tikzcd}$$

- whose arrows are given by commutative triangles of étale morphisms,

$$\begin{tikzcd} U \arrow[rr] \arrow[dr] & & V \arrow[dl] \\ & X & \end{tikzcd}$$

- with the obvious notion of covering.

The étale cohomology is defined with coefficients in $\mathbb{Z}/\ell^n\mathbb{Z}$, whence $\mathbb{Z}_\ell$ by passing to the limit, and $\mathbb{Q}_\ell$ by inverting ℓ.

The stacks of shtukas, introduced by Drinfeld, play a role analoguous to Shimura varieties over number fields. But they exist in a much greater generality. Indeed, while the Shimura varieties are defined over the spectrum of the ring of integers of a number field and are associated to *a minuscule coweight* of the dual group, the stacks of shtukas exist over *arbitrary powers* of the curve X, and can be associated to *arbitrary coweights*, as we will see now. One simple reason for this difference between function fields and number fields is the following: in the example of the product of two copies, the product $X \times X$ is taken over $\mathbb{F}_q$ whereas nobody knows what the product $\operatorname{Spec}\mathbb{Z} \times \operatorname{Spec}\mathbb{Z}$ should be, and over what to take it.

Let I be a finite set and $W = \boxtimes_{i\in I} W_i$ be an irreducible $\mathbb{Q}_\ell$-linear representation of $\widehat{G}^I$ (in other words each W_i is an irreducible representation of $\widehat{G}$).

We define $\mathrm{Cht}_{I,W}$ as the reduced Deligne–Mumford stack over X^I whose points over a scheme S over $\mathbb{F}_q$ classify shtukas, i.e.

- points $(x_i)_{i\in I} : S \to X^I$, called the legs of the shtuka ("les pattes du chtouca" in French),
- a G-bundle $\mathcal{G}$ over $X \times S$,
- an isomorphism

$$\phi : \mathcal{G}\big|_{(X\times S)\smallsetminus(\bigcup_{i\in I}\Gamma_{x_i})} \xrightarrow{\sim} (\mathrm{Id}_X \times \mathrm{Frob}_S)^*(\mathcal{G})\big|_{(X\times S)\smallsetminus(\bigcup_{i\in I}\Gamma_{x_i})}$$

  (where Γ_{x_i} denotes the graph of x_i), such that

  (7-1) the relative position at x_i of the modification ϕ is bounded by the dominant coweight of G corresponding to the dominant weight of W_i.

The notion of Deligne–Mumford stack is in algebraic geometry what corresponds to the topological notion of orbifold. Every quotient of an algebraic variety by a finite étale group scheme is a Deligne–Mumford stack and in fact $\mathrm{Cht}_{I,W}$ is locally of this form.

Remark 7.1. Compared to the notion of Artin stacks mentioned in remark 2.1, a Deligne–Mumford stack is a particular case where the automorphism groups of geometric points are finite groups (instead of algebraic groups).

Remark 7.2. In the case of GL_1, *resp.* split tori, we had defined schemes of shtukas. With the above definition, the stacks of shtukas are the quotients of these schemes by the trivial action of ${\mathbb{F}_q}^\times$, *resp.* $T(\mathbb{F}_q)$.

We denote by $H_{I,W}$ the $\mathbb{Q}_\ell$-vector space equal to the "Hecke-finite" subspace of the ℓ-adic intersection cohomology with compact support, in middle degree, of the fiber of $\mathrm{Cht}_{I,W}/\Xi$ over a generic geometric point of X^I (or, in fact equivalently, over a generic geometric point of the diagonal $X \subset X^I$). To give an idea of intersection cohomology,

let us say that for a smooth variety it is the same as the ℓ-adic cohomology and that for (possibly singular) projective varieties it is Poincaré self-dual. An element of this ℓ-adic intersection cohomology is said to be Hecke-finite if it belongs to a sub-$\mathbb{Z}_\ell$-module of finite type stable by all Hecke operators $T_{\lambda,v}$ (or equivalently by all Hecke operators $T_{V,v}$). Hecke-finiteness is a technical condition but Cong Xue has proven Xue [2017] that $H_{I,W}$ can equivalently be defined by a cuspidality condition (defined using stacks of shtukas for parabolic subgroups of G and their Levi quotients) and that it has finite dimension over $\mathbb{Q}_\ell$.

Drinfeld has constructed "partial Frobenius morphisms" between stacks of shtukas. To define them we need a small generalization of the stacks of shtukas where we require a factorization of ϕ as a composition of several modifications. Let $(I_1, ..., I_k)$ be an ordered partition of I. An example is the coarse partition (I) and in fact the stack $\mathrm{Cht}_{I,W}$ previously defined is equal to $\mathrm{Cht}^{(I)}_{I,W}$ in the following definition.

Definition 7.3. We define $\mathrm{Cht}^{(I_1,...,I_k)}_{I,W}$ as the *reduced* Deligne–Mumford stack whose points over a scheme S over $\mathbb{F}_q$ classify

$$\big((x_i)_{i\in I}, \mathcal{G}_0 \xrightarrow{\phi_1} \mathcal{G}_1 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_{k-1}} \mathcal{G}_{k-1} \xrightarrow{\phi_k} (\mathrm{Id}_X \times \mathrm{Frob}_S)^*(\mathcal{G}_0)\big) \tag{7-2}$$

with

- $x_i \in (X \smallsetminus N)(S)$ for $i \in I$,
- for $i \in \{0, ..., k-1\}$, $\mathcal{G}_i$ is a G-bundle over $X \times S$ and we write $\mathcal{G}_k = (\mathrm{Id}_X \times \mathrm{Frob}_S)^*(\mathcal{G}_0)$ to prepare the next item,
- for $j \in \{1, ..., k\}$

$$\phi_j : \mathcal{G}_{j-1}\big|_{(X\times S)\smallsetminus(\bigcup_{i\in I_j}\Gamma_{x_i})} \xrightarrow{\sim} \mathcal{G}_j\big|_{(X\times S)\smallsetminus(\bigcup_{i\in I_j}\Gamma_{x_i})}$$

  is an isomorphism such that the relative position of $\mathcal{G}_{j-1}$ with respect to $\mathcal{G}_j$ at x_i (for $i \in I_j$) is bounded by the dominant coweight of G corresponding to the dominant weight of W_i.

We can show that the obvious morphism $\mathrm{Cht}^{(I_1,...,I_k)}_{I,W} \to \mathrm{Cht}_{I,W}$ (which forgets the intermediate modifications $\mathcal{G}_1, ..., \mathcal{G}_{k-1}$) gives an isomorphism at the level of intersection cohomology. The interest of $\mathrm{Cht}^{(I_1,...,I_k)}_{I,W}$ is that we have the partial Frobenius morphism $\mathrm{Frob}_{I_1} : \mathrm{Cht}^{(I_1,...,I_k)}_{I,W} \to \mathrm{Cht}^{(I_2,...,I_k,I_1)}_{I,W}$ which sends (7-2) to

$$\begin{aligned}\big((x'_i)_{i\in I}, \mathcal{G}_1 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_{k-1}} \mathcal{G}_{k-1} \xrightarrow{\phi_k} (\mathrm{Id}_X \times \mathrm{Frob}_S)^*(\mathcal{G}_0)\\ \xrightarrow{(\mathrm{Id}_X \times \mathrm{Frob}_S)^*(\phi_1)} (\mathrm{Id}_X \times \mathrm{Frob}_S)^*(\mathcal{G}_1)\big)\end{aligned}$$

where $x'_i = \mathrm{Frob}(x_i)$ if $i \in I_1$ and $x'_i = x_i$ otherwise. Taking I_1 to be a singleton we get the action on $H_{I,W}$ of the partial Frobenius morphisms. Thanks to an extra work (using the Hecke-finiteness condition and Eichler–Shimura relations), we are able in

V. Lafforgue [2012] to apply Drinfeld's lemma, and this endows the $\mathbb{Q}_\ell$-vector space $H_{I,W}$ with a continuous action of $\mathrm{Gal}(\overline{F}/F)^I$.

For $I = \varnothing$ and $W = \mathbf{1}$ (the trivial representation), we have

$$H_{\varnothing,\mathbf{1}} = C_c^{\mathrm{cusp}}(\mathrm{Bun}_G(\mathbb{F}_q)/\Xi, \mathbb{Q}_\ell). \tag{7-3}$$

Indeed the S-points over $\mathrm{Cht}_{\varnothing,\mathbf{1}}$ classify the G-bundles $\mathcal{G}$ over $X \times S$, equipped with an isomorphism

$$\phi : \mathcal{G} \xrightarrow{\sim} (\mathrm{Id}_X \times \mathrm{Frob}_S)^*(\mathcal{G}).$$

If we see $\mathcal{G}$ as a S-point of Bun_G, $(\mathrm{Id}_X \times \mathrm{Frob}_S)^*(\mathcal{G})$ is its image by $\mathrm{Frob}_{\mathrm{Bun}_G}$. Therefore $\mathrm{Cht}_{\varnothing,\mathbf{1}}$ classifies the fixed points of $\mathrm{Frob}_{\mathrm{Bun}_G}$ and it is discrete (i.e. of dimension 0) and equal to $\mathrm{Bun}_G(\mathbb{F}_q)$. Therefore the ℓ-adic cohomology of $\mathrm{Cht}_{\varnothing,\mathbf{1}}/\Xi$ is equal to $C_c(\mathrm{Bun}_G(\mathbb{F}_q)/\Xi, \mathbb{Q}_\ell)$ and in this particular case it is easy to see that Hecke-finiteness is equivalent to cuspidality, so that (7-3) holds true.

Up to now we defined a vector space $H_{I,W}$ for every isomorphism class of irreducible representation $W = \boxtimes_{i\in I} W_i$ of $\widehat{G}^I$. A construction based on the geometric Satake equivalence enables to

a) define $H_{I,W}$ functorialy in W

b) understand the fusion of legs

as explained in the next proposition.

Proposition 7.4. *a) For every finite set I,*

$$W \mapsto H_{I,W}, \quad u \mapsto \mathcal{H}(u)$$

is a $\mathbb{Q}_\ell$-linear functor from the category of finite dimensional representations of $\widehat{G}^I$ to the category of finite dimensional and continuous representations of $\mathrm{Gal}(\overline{F}/F)^I$.

This means that for every morphism $u : W \to W'$ of representations of $\widehat{G}^I$, we have a morphism $\mathcal{H}(u) : H_{I,W} \to H_{I,W'}$ of representations of $\mathrm{Gal}(\overline{F}/F)^I$.

b) For each map $\zeta : I \to J$ between finite sets, we have an isomorphism

$$\chi_\zeta : H_{I,W} \xrightarrow{\sim} H_{J,W^\zeta}$$

which is

- *functorial in W, where W is a representation of $\widehat{G}^I$ and W^ζ denotes the representation of $\widehat{G}^J$ on W obtained by composition with the diagonal morphism*

$$\widehat{G}^J \to \widehat{G}^I, (g_j)_{j\in J} \mapsto (g_{\zeta(i)})_{i\in I}$$

- $\mathrm{Gal}(\overline{F}/F)^J$*-equivariant, where* $\mathrm{Gal}(\overline{F}/F)^J$ *acts on the LHS by the diagonal morphism*

$$\mathrm{Gal}(\overline{F}/F)^J \to \mathrm{Gal}(\overline{F}/F)^I, \ (\gamma_j)_{j\in J} \mapsto (\gamma_{\zeta(i)})_{i\in I},$$

- *and compatible with composition, i.e. for every* $I \xrightarrow{\zeta} J \xrightarrow{\eta} K$ *we have* $\chi_{\eta\circ\zeta} = \chi_\eta \circ \chi_\zeta$.

The statement b) is a bit complicated, here is a basic example of it. For every finite set I we write $\zeta_I : I \to \{0\}$ the tautological map (where $\{0\}$ is an arbitrary choice of notation for a singleton). If W_1 and W_2 are two representations of $\widehat{G}$, the statement of b) provides a canonical isomorphism

$$\chi_{\zeta_{\{1,2\}}} : H_{\{1,2\},W_1\boxtimes W_2} \xrightarrow{\sim} H_{\{0\},W_1\otimes W_2} \tag{7-4}$$

associated to $\zeta_{\{1,2\}} : \{1,2\} \to \{0\}$. We stress the difference between $W_1 \boxtimes W_2$ which is a representation of $(\widehat{G})^2$ and $W_1 \otimes W_2$ which is a representation of $\widehat{G}$.

Another example of b) is the isomorphism on the left in

$$H_{\{0\},\mathbf{1}} \xrightarrow[\sim]{\chi_{\zeta_\varnothing}^{-1}} H_{\varnothing,\mathbf{1}} \overset{(7\text{-}3)}{=} C_c^{\mathrm{cusp}}(\mathrm{Bun}_G(\mathbb{F}_q)/\Xi, \mathbb{Q}_\ell) \tag{7-5}$$

which is associated to $\zeta_\varnothing : \varnothing \to \{0\}$ (the idea of the isomorphism $\chi_{\zeta_\varnothing}$ is that $H_{\varnothing,\mathbf{1}}$ *resp.* $H_{\{0\},\mathbf{1}}$ is the cohomology of the stack of shtukas without legs, *resp.* with a inactive leg, and that they are equal). Thanks to (7-5) we are reduced to construct a decomposition

$$H_{\{0\},\mathbf{1}} \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}_\ell} = \bigoplus_\sigma \mathfrak{S}_\sigma. \tag{7-6}$$

Idea of the proof of proposition 7.4. We denote by Cht_I the inductive limit of Deligne–Mumford stacks over X^I, defined as $\mathrm{Cht}_{I,W}$ above, but without the condition (7-1) on the relative position. In other words, and with an extra letter $\mathcal{G}'$ to prepare the next definition, the points of Cht_I over a scheme S over $\mathbb{F}_q$ classify

- points $(x_i)_{i\in I} : S \to X^I$,
- two G-bundles $\mathcal{G}$ and $\mathcal{G}'$ over $X \times S$,
- a modification ϕ at the x_i, i.e. an isomorphism
$$\phi : \mathcal{G}\big|_{(X\times S)\smallsetminus(\bigcup_{i\in I}\Gamma_{x_i})} \xrightarrow{\sim} \mathcal{G}'\big|_{(X\times S)\smallsetminus(\bigcup_{i\in I}\Gamma_{x_i})}$$
- an isomorphism $\theta : \mathcal{G}' \xrightarrow{\sim} (\mathrm{Id}_X \times \mathrm{Frob}_S)^*(\mathcal{G})$.

We introduce the "prestack" $\mathfrak{M}_I$ of "modifications on the formal neighborhood of the x_i", whose points over a scheme S over $\mathbb{F}_q$ classify

- points $(x_i)_{i\in I} : S \to X^I$,
- two G-bundles $\mathcal{G}$ and $\mathcal{G}'$ on the formal completion $\widehat{X \times S}$ of $X \times S$ in the neighborhood of the union of the graphs Γ_{x_i},
- a modification ϕ at the x_i, i.e. an isomorphism
$$\phi : \mathcal{G}\big|_{(\widehat{X\times S})\smallsetminus(\bigcup_{i\in I}\Gamma_{x_i})} \xrightarrow{\sim} \mathcal{G}'\big|_{(\widehat{X\times S})\smallsetminus(\bigcup_{i\in I}\Gamma_{x_i})}.$$

The expert reader will notice that for any morphism $S \to X^I$, $\mathfrak{M}_I \times_{X^I} S$ is the quotient of the affine grassmannian of Beilinson–Drinfeld over S by $\Gamma(\widehat{X \times S}, G)$. We have a formally smooth morphism $\epsilon_I : \mathrm{Cht}_I \to \mathfrak{M}_I$ given by restricting $\mathcal{G}$ and $\mathcal{G}'$ to the formal neighborhood of the graphs of the x_i and forgetting θ.

The geometric Satake equivalence, due to Lusztig, Drinfeld, Ginzburg and Mirković–Vilonen Beilinson and Drinfeld [1999] and Mirković and Vilonen [2007], is a fundamental statement which *constructs* $\widehat{G}$ from G and is the cornerstone of the geometric Langlands program. It is a canonical equivalence of tensor categories between

- the category of perverse sheaves on the fiber of $\mathfrak{M}_{\{0\}}$ above any point of X (where $\{0\}$ is an arbitrary notation for a singleton)
- the tensor category of representations of $\widehat{G}$.

For the non expert reader we recall that perverse sheaves, introduced in Beĭlinson, Bernstein, and Deligne [1982], behave like ordinary sheaves and have, in spite of their name, very good properties. An example is given by intersection cohomology sheaves of closed (possibly singular) subvarieties, whose total cohomology is the intersection cohomology of this subvarieties.

The tensor structure on the first category above is obtained by "fusion of legs", thanks to the fact that $\mathfrak{M}_{\{1,2\}}$ is equal to $\mathfrak{M}_{\{0\}} \times \mathfrak{M}_{\{0\}}$ outside the diagonal of X^2 and to $\mathfrak{M}_{\{0\}}$ on the diagonal. The first category is tannakian and $\widehat{G}$ is defined as the group of automorphisms of a natural fiber functor.

This equivalence gives, for every representation W of $\widehat{G}^I$, a perverse sheave $\mathcal{S}_{I,W}$ on $\mathfrak{M}_I$, with the following properties:

- $\mathcal{S}_{I,W}$ is functorial in W,
- for every surjective map $I \to J$, $\mathcal{S}_{J,W^\xi}$ is canonically isomorphic to the restriction of $\mathcal{S}_{I,W}$ to $\mathfrak{M}_I \times_{X^I} X^J \simeq \mathfrak{M}_J$, where $X^J \to X^I$ is the diagonal morphism,
- for every irreducible representation W, $\mathcal{S}_{I,W}$ is the intersection cohomology sheaf of the closed substack of $\mathfrak{M}_I$ defined by the condition (7-1) on the relative position of the modification ϕ at the x_i.

Then we define $H_{I,W}$ as the "Hecke-finite" subspace of the cohomology with compact support of $\epsilon_I^*(\mathcal{S}_{I,W})$ on the fiber of Cht_I/Ξ over a geometric generic point of X^I (or, in fact equivalently, over a geometric generic point of the diagonal $X \subset X^I$). The first two properties above imply a) and b) of the proposition. The third one and the smoothness of ϵ_I ensure that, for W irreducible, $\epsilon_I^*(\mathcal{S}_{I,W})$ is the intersection cohomology sheaf of $\mathrm{Cht}_{I,W}$ and therefore the new definition of $H_{I,W}$ generalizes the first one using the intersection cohomology of $\mathrm{Cht}_{I,W}$.

8 Excursion operators and the main theorem of V. Lafforgue [2012]

Let I be a finite set. Let $(\gamma_i)_{i\in I} \in \mathrm{Gal}(\overline{F}/F)^I$. Let W be a representation of $\widehat{G}^I$ and $x \in W$ and $\xi \in W^*$ be invariant by the diagonal action of $\widehat{G}$. We define the

endomorphism $S_{I,W,x,\xi,(\gamma_i)_{i\in I}}$ of (7-5) as the composition

$$H_{\{0\},\mathbf{1}} \xrightarrow{\mathcal{H}(x)} H_{\{0\},W^{\zeta_I}} \xrightarrow[\sim]{\chi_{\zeta_I}^{-1}} H_{I,W} \xrightarrow{(\gamma_i)_{i\in I}} H_{I,W} \xrightarrow[\sim]{\chi_{\zeta_I}} H_{\{0\},W^{\zeta_I}} \xrightarrow{\mathcal{H}(\xi)} H_{\{0\},\mathbf{1}} \tag{8-1}$$

where $\mathbf{1}$ denotes the trivial representation of $\widehat{G}$, and $x : \mathbf{1} \to W^{\zeta_I}$ and $\xi : W^{\zeta_I} \to \mathbf{1}$ are considered as morphisms of representations of $\widehat{G}$ (we recall that $\zeta_I : I \to \{0\}$ is the obvious map and that W^{ζ_I} is simply the vector space W equipped with the diagonal action of $\widehat{G}$).

Paraphrasing (8-1) this operator is the composition

- of a creation operator associated to x, whose effect is to create legs at the same (generic) point of the curve,
- of a Galois action, which moves the legs on the curve independently from each other, then brings them back to the same (generic) point of the curve,
- of an annihilation operator associated to ξ.

It is called an "excursion operator" because it moves the legs on the curve (this is what makes it non trivial).

To W, x, ξ we associate the matrix coefficient f defined by

$$f((g_i)_{i\in I}) = \langle \xi, (g_i)_{i\in I} \cdot x\rangle. \tag{8-2}$$

We see that f is a function on $\widehat{G}^I$ invariant by left and right translations by the diagonal $\widehat{G}$. In other words $f \in \mathcal{O}(\widehat{G}\backslash\widehat{G}^I/\widehat{G})$, where $\widehat{G}\backslash\widehat{G}^I/\widehat{G}$ denotes the coarse quotient, defined as the spectrum of the algebra of functions f as above. Unlike the stacky quotients considered before, the coarse quotients are schemes and therefore forget the automorphism groups of points.

For every function $f \in \mathcal{O}(\widehat{G}\backslash\widehat{G}^I/\widehat{G})$ we can find W, x, ξ such that (8-2) holds. We show easily that $S_{I,W,x,\xi,(\gamma_i)_{i\in I}}$ does not depend on the choice of W, x, ξ satisfying (8-2), and therefore we denote it by $S_{I,f,(\gamma_i)_{i\in I}}$.

The conjectures of Arthur and Kottwitz on multiplicities in vector spaces of automorphic forms and in the cohomologies of Shimura varieties Arthur [1989] and Kottwitz [1990] give, by extrapolation to stacks of shtukas, the following heuristics.

Remark 8.1. Heuristically we conjecture that for every global Langlands parameter σ there exists a $\overline{\mathbb{Q}_\ell}$-linear representation $\mathfrak{A}_\sigma$ of its centralizer $S_\sigma \subset \widehat{G}$ (factorizing through $S_\sigma/Z_{\widehat{G}}$), so that we have a $\mathrm{Gal}(\overline{F}/F)^I$-equivariant isomorphism

$$H_{I,W} \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}_\ell} \overset{?}{=} \bigoplus_\sigma \left(\mathfrak{A}_\sigma \otimes_{\overline{\mathbb{Q}_\ell}} W_{\sigma^I}\right)^{S_\sigma} \tag{8-3}$$

where W_{σ^I} is the $\overline{\mathbb{Q}_\ell}$-linear representation of $\mathrm{Gal}(\overline{F}/F)^I$ obtained by composition of the representation W of $\widehat{G}^I$ with the morphism $\sigma^I : \mathrm{Gal}(\overline{F}/F)^I \to \widehat{G}(\overline{\mathbb{Q}_\ell})^I$, and S_σ acts diagonally. We conjecture that (8-3) is functorial in W, compatible to χ_ξ and that it is equal to the decomposition (7-6) when $W = \mathbf{1}$ (so that $\mathfrak{S}_\sigma = (\mathfrak{A}_\sigma)^{S_\sigma}$).

In the heuristics (8-3) the endomorphism $S_{I,f,(\gamma_i)_{i\in I}} = S_{I,W,x,\xi,(\gamma_i)_{i\in I}}$ of

$$H_{\{0\},\mathbf{1}} \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}_\ell} \xrightarrow[\sim]{\chi_{\zeta_\varnothing}^{-1}} H_{\varnothing,\mathbf{1}} \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}_\ell} \overset{?}{=} \bigoplus_\sigma (\mathfrak{A}_\sigma)^{S_\sigma}$$

acts on $(\mathfrak{A}_\sigma)^{S_\sigma}$ by the composition

$$(\mathfrak{A}_\sigma)^{S_\sigma} \xrightarrow[\sim]{\chi_{\zeta_\varnothing}} (\mathfrak{A}_\sigma \otimes \mathbf{1})^{S_\sigma} \xrightarrow{\mathrm{Id}_{\mathfrak{A}_\sigma} \otimes x} (\mathfrak{A}_\sigma \otimes W_{\sigma^I})^{S_\sigma} \xrightarrow{(\sigma(\gamma_i))_{i\in I}} (\mathfrak{A}_\sigma \otimes W_{\sigma^I})^{S_\sigma}$$
$$\xrightarrow{\mathrm{Id}_{\mathfrak{A}_\sigma} \otimes \xi} (\mathfrak{A}_\sigma \otimes \mathbf{1})^{S_\sigma} \xrightarrow[\sim]{\chi_{\zeta_\varnothing}^{-1}} (\mathfrak{A}_\sigma)^{S_\sigma}$$

i.e. by the scalar

$$\langle \xi, (\sigma(\gamma_i))_{i\in I} \cdot x \rangle = f\big((\sigma(\gamma_i))_{i\in I}\big).$$

In other words we should have

$$\mathfrak{S}_\sigma \overset{?}{=} \text{eigenspace of the } S_{I,f,(\gamma_i)_{i\in I}} \text{ with the eigenvalues } f\big((\sigma(\gamma_i))_{i\in I}\big). \tag{8-4}$$

The heuristics (8-4) clearly indicates the path to follow. We show in V. Lafforgue [2012] that the $S_{I,f,(\gamma_i)_{i\in I}}$ generate a commutative $\mathbb{Q}_\ell$-algebra $\mathfrak{B}$ and satisfy some properties implying the following proposition.

Proposition 8.2. *For each character ν of $\mathfrak{B}$ with values in $\overline{\mathbb{Q}_\ell}$ there exists a unique global Langlands parameter σ such that for all I, f and $(\gamma_i)_{i\in I}$, we have*

$$\nu(S_{I,f,(\gamma_i)_{i\in I}}) = f((\sigma(\gamma_i))_{i\in I}). \tag{8-5}$$

The unicity of σ in the previous proposition comes from the fact that, for any integer n, taking $I = \{0, ..., n\}$, the coarse quotient $\widehat{G}\backslash\widehat{G}^I/\widehat{G}$ identifies with the coarse quotient of $(\widehat{G})^n$ by diagonal conjugation by $\widehat{G}$, and therefore, for any $(\gamma_1, ..., \gamma_n) \in (\mathrm{Gal}(\overline{F}/F))^n$, (8-5) applied to $(\gamma_i)_{i\in I} = (1, \gamma_1, ..., \gamma_n)$ determines $(\sigma(\gamma_1), ..., \sigma(\gamma_n))$ up to conjugation and semisimplification (thanks to Richardson [1988]). The existence and continuity of σ are justified thanks to relations and topological properties satisfied by the excursion operators.

Since $\mathfrak{B}$ is commutative we obtain a canonical spectral decomposition $H_{\{0\},\mathbf{1}} \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}_\ell} = \bigoplus_\nu \mathfrak{S}_\nu$ where the direct sum is taken over characters ν of $\mathfrak{B}$ with values in $\overline{\mathbb{Q}_\ell}$. Associating to each ν a unique global Langlands parameter σ as in the previous proposition, we deduce the decomposition (7-6) we wanted to construct. We do not know if $\mathfrak{B}$ is reduced.

Moreover the unramified Hecke operators are particular cases of excursion operators: for every place v and for every irreducible representation V of $\widehat{G}$ with character χ_V, the unramified Hecke operator $T_{V,v}$ is equal to the excursion operator $S_{\{1,2\},f,(\mathrm{Frob}_v,1)}$ where $f \in \mathcal{O}(\widehat{G}\backslash(\widehat{G})^2/\widehat{G})$ is given by $f(g_1, g_2) = \chi_V(g_1 g_2^{-1})$, and Frob_v is a Frobenius element at v. This is proven in V. Lafforgue [2012] by a geometric argument (essentially a computation of the intersection of algebraic cycles in a stack of shtukas). It implies the compatibility of the decomposition (7-6) with the Satake isomorphism at all places.

Remark 8.3. By the Chebotarev density theorem, the subalgebra of $\mathcal{B}$ generated by all the Hecke algebras $\mathcal{H}_v$ is equal to the subalgebra generated by the excursion operators with $\sharp I = 2$. The remarks at the end of section 6 show that in general it is necessary to consider excursion operators with $\sharp I > 2$ to generate the whole algebra $\mathcal{B}$.

Finally we can state the main theorem.

Theorem 8.4. *We have a canonical decomposition of $\overline{\mathbb{Q}_\ell}$-vector spaces*

$$C_c^{\mathrm{cusp}}(\mathrm{Bun}_G(\mathbb{F}_q)/\Xi, \overline{\mathbb{Q}_\ell}) = \bigoplus_\sigma \mathfrak{H}_\sigma, \tag{8-6}$$

where the direct sum in the RHS is indexed by global Langlands parameters , i.e. $\widehat{G}(\overline{\mathbb{Q}_\ell})$-conjugacy classes of morphisms $\sigma : \mathrm{Gal}(\overline{F}/F) \to \widehat{G}(\overline{\mathbb{Q}_\ell})$ factorizing through $\pi_1(X, \overline{\eta})$, defined over a finite extension of $\mathbb{Q}_\ell$, continuous and semisimple.

This decomposition is uniquely determined by the following property : $\mathfrak{H}_\sigma$ is equal to the generalized eigenspace associated to the character ν of $\mathcal{B}$ defined by

$$\nu(S_{I,f,(\gamma_i)_{i\in I}}) = f((\sigma(\gamma_i))_{i\in I}. \tag{8-7}$$

This decomposition is respected by the Hecke operators and is compatible with the Satake isomorphism at all places v of X.

Everything is still true with a level (a finite subscheme N of X). We denote by $\mathrm{Bun}_{G,N}(\mathbb{F}_q)$ the set of isomorphism classes of G-bundles over X trivialized on N. Then we have a canonical decomposition

$$C_c^{\mathrm{cusp}}(\mathrm{Bun}_{G,N}(\mathbb{F}_q)/\Xi, \overline{\mathbb{Q}_\ell}) = \bigoplus_\sigma \mathfrak{H}_\sigma, \tag{8-8}$$

where the direct sum is taken over global Langlands parameters $\sigma : \pi_1(X \smallsetminus N, \overline{\eta}) \to \widehat{G}(\overline{\mathbb{Q}_\ell})$. This decomposition is respected by all Hecke operators and compatible with the Satake isomorphism at all places of $X \smallsetminus N$. If G is split we have, by Thắng [2011],

$$\mathrm{Bun}_{G,N}(\mathbb{F}_q) = G(F)\backslash G(\mathbb{A})/K_N \tag{8-9}$$

(where $\mathbb{A}$ is the ring of adèles, $\mathbb{O}$ is the ring of integral adèles, $\mathcal{O}_N$ the ring of functions on N and $K_N = \mathrm{Ker}(G(\mathbb{O}) \to G(\mathcal{O}_N))$). When G is non necessarily split the RHS of (8-9) must be replaced by a direct sum, indexed by the finite group $\ker^1(F, G)$, of adelic quotients for inner forms of G and in the definition of global Langlands parameters we must replace $\widehat{G}$ by the L-group (see Borel [1979] for L-groups).

We have a statement similar to theorem 8.4 with coefficients in $\overline{\mathbb{F}_\ell}$ instead of $\overline{\mathbb{Q}_\ell}$.

We can also consider the case of metaplectic groups thanks to the metaplectic variant of the geometric Satake equivalence due to Finkelberg and Lysenko [2010], Lysenko [2014], and Gaitsgory and Lysenko [2016].

Remark 8.5. Drinfeld gave an idea to prove something like the heuristics (8-3) but it is a bit difficult to formulate the result. Let Reg be the left regular representation of $\widehat{G}$ with coefficients in $\mathbb{Q}_\ell$ (considered as an inductive limit of finite dimensional representations). We can endow the $\mathbb{Q}_\ell$-vector space $H_{\{0\},\mathrm{Reg}}$ (of infinite dimension in general) with

a) a structure of module over the algebra of functions on the "affine space $\mathcal{S}$ of morphisms $\sigma : \mathrm{Gal}(\overline{F}/F) \to \widehat{G}$ with coefficients in $\mathbb{Q}_\ell$-algebras",

b) an algebraic action of $\widehat{G}$ (coming from the right action of $\widehat{G}$ on Reg) which is compatible with conjugation by $\widehat{G}$ on $\mathcal{S}$.

The space $\mathcal{S}$ is not rigorously defined and the rigorous definition of structure a) is the following. For any finite dimensional $\mathbb{Q}_\ell$-linear representation V of $\widehat{G}$, with underlying vector space $\underline{V}$, $H_{\{0\},\mathrm{Reg}} \otimes \underline{V}$ is equipped with an action of $\mathrm{Gal}(\overline{F}/F)$, making it an inductive limit of finite dimensional continuous representations of $\mathrm{Gal}(\overline{F}/F)$, as follows. We have a $\widehat{G}$-equivariant isomorphism

$$\begin{aligned} \theta : \mathrm{Reg} \otimes \underline{V} &\simeq \mathrm{Reg} \otimes V \\ f \otimes x &\mapsto [g \mapsto f(g) g.x] \end{aligned}$$

where $\widehat{G}$ acts diagonally on the RHS, and to give a meaning to the formula the RHS is identified with the vector space of algebraic functions $\widehat{G} \to V$. Therefore we have an isomorphism

$$H_{\{0\},\mathrm{Reg}} \otimes \underline{V} = H_{\{0\},\mathrm{Reg}\otimes\underline{V}} \overset{\theta}{\rightsquigarrow} H_{\{0\},\mathrm{Reg}\otimes V} \simeq H_{\{0,1\},\mathrm{Reg}\boxtimes V}$$

where the first equality is tautological (since $\underline{V}$ is just a vector space) and the last isomorphism is the inverse of the fusion isomorphism $\chi_{\xi_{\{0,1\}}}$ of (7-4). Then the action of $\mathrm{Gal}(\overline{F}/F)$ on the LHS is defined as the action of $\mathrm{Gal}(\overline{F}/F)$ on the RHS corresponding to the leg 1. If V_1 and V_2 are two representations of $\widehat{G}$, the two actions of $\mathrm{Gal}(\overline{F}/F)$ on $H_{\{0\},\mathrm{Reg}} \otimes \underline{V_1} \otimes \underline{V_2}$ associated to the actions of $\widehat{G}$ on V_1 and V_2 commute with each other and the diagonal action of $\mathrm{Gal}(\overline{F}/F)$ is the action associated to the diagonal action of $\widehat{G}$ on $V_1 \otimes V_2$. This gives a structure as we want in a) because if V is as above, $x \in V$, $\xi \in V^*$, f is the function on $\widehat{G}$ defined as the matrix coefficient $f(g) = \langle \xi, g.x \rangle$, and $\gamma \in \mathrm{Gal}(\overline{F}/F)$ then we say that $F_{f,\gamma} : \sigma \mapsto f(\sigma(\gamma))$, considered as a "function on $\mathcal{S}$", acts on $H_{\{0\},\mathrm{Reg}}$ by the composition

$$H_{\{0\},\mathrm{Reg}} \xrightarrow{\mathrm{Id}\otimes x} H_{\{0\},\mathrm{Reg}} \otimes \underline{V} \xrightarrow{\gamma} H_{\{0\},\mathrm{Reg}} \otimes \underline{V} \xrightarrow{\mathrm{Id}\otimes\xi} H_{\{0\},\mathrm{Reg}}.$$

Any function f on $\widehat{G}$ can be written as such a matrix coefficient, and the functions $F_{f,\gamma}$ when f and γ vary are supposed to "generate topologically all functions on $\mathcal{S}$". The property above with V_1 and V_2 implies relations among the $F_{f,\gamma}$, namely that $F_{f,\gamma_1\gamma_2} = \sum_\alpha F_{f_1^\alpha,\gamma_1} F_{f_2^\alpha,\gamma_2}$ if the image of f by comultiplication is $\sum_\alpha f_1^\alpha \otimes f_2^\alpha$. In Xinwen [2017] Xinwen Zhu gives an equivalent construction of the structure a). Structures a) and b) are compatible in the following sense: the conjugation $gF_{f,\gamma}g^{-1}$ of the action of $F_{f,\gamma}$ on $H_{\{0\},\mathrm{Reg}}$ by the algebraic action of $g \in \widehat{G}$ is equal to the action of $F_{f^g,\gamma}$ where $f^g(h) = f(g^{-1}hg)$.

The structures a) and b) give rise to a "$\mathcal{O}$-module on the stack $\mathcal{S}/\widehat{G}$ of global Langlands parameters" (such that the vector space of its "global sections on $\mathcal{S}$" is $H_{\{0\},\mathrm{Reg}}$). For any morphism $\sigma : \mathrm{Gal}(\overline{F}/F) \to \widehat{G}(\overline{\mathbb{Q}_\ell})$, we want to define $\mathfrak{A}_\sigma$ as the fiber of this

$\mathcal{O}$-module at σ (considered as a "$\overline{\mathbb{Q}_\ell}$-point of $\mathcal{S}$ whose automorphism group in the stack $\mathcal{S}/\widehat{G}$ is S_σ"). Rigorously we define $\mathfrak{A}_\sigma$ as the biggest quotient of $H_{\{0\},\mathrm{Reg}} \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}_\ell}$ on which any function $F_{f,\gamma}$ as above acts by multiplication by the scalar $f(\sigma(\gamma))$, and S_σ acts on $\mathfrak{A}_\sigma$. If the heuristics (8-3) is true it is the same as $\mathfrak{A}_\sigma$ from the heuristics. When σ is elliptic (i.e. when $S_\sigma/Z_{\widehat{G}}$ is finite), σ is "isolated in $\mathcal{S}$" in the sense that it cannot be deformed (among continuous morphisms whose composition with the abelianization of $\widehat{G}$ is of fixed finite order) and, as noticed by Xinwen Zhu, heuristics (8-3) is true when we restrict on both sides to the parts lying over σ. In general due to deformation of some non elliptic σ there could a priori be nilpotents, and for example we don't know how to prove that $\mathcal{B}$ is reduced so we don't know how to prove the heuristics (8-3).

We can see the heuristics (8-3), and the structures a) and b) above, as an illustration of the general idea that, in a spectral decomposition, when the points of the spectrum naturally have automorphism groups, the multiplicities should be associated to representations of these groups. By contrast the algebra $\mathcal{B}$ of excursion operators gives the spectral decomposition with respect to the coarse quotient associated to $\mathcal{S}/\widehat{G}$, where we forget the automorphism groups S_σ.

Remark 8.6. The previous remark makes sense although $\mathcal{S}$ was not defined. To define a space like $\mathcal{S}$ rigorously it may be necessary to consider continuous morphisms $\sigma : \mathrm{Gal}(\overline{F}/F) \to \widehat{G}$ with coefficients in $\mathbb{Z}_\ell$-algebras where ℓ is nilpotent (such σ have finite image), and $\mathcal{S}$ would be an ind-scheme over $\mathrm{Spf}\,\mathbb{Z}_\ell$. Then to define structure a) we would need to consider Reg with coefficients in $\mathbb{Z}_\ell$, and, for any representation W of $\widehat{G}^I$ with coefficients in $\mathbb{Z}_\ell$, to construct $H_{I,W}$ as a $\mathbb{Z}_\ell$-module.

9 Local aspects: joint work with Alain Genestier

In Genestier and V. Lafforgue [2017], Alain Genestier and I construct the local parameterization up to semisimplification and the local-global compatibility.

Let G be a reductive group over a local field K of equal characteristics. We recall that the Bernstein center of $G(K)$ is defined, in two equivalent ways, as

- the center of the category of smooth representations of $G(K)$,
- the algebra of central distributions on $G(K)$ acting as multipliers on the algebra of locally constant functions with compact support.

On every $\overline{\mathbb{Q}_\ell}$-linear irreducible smooth representation of $G(K)$, the Bernstein center acts by a character.

The main result of Genestier and V. Lafforgue [ibid.] associates to any character ν of the Bernstein center of $G(K)$ with values in $\overline{\mathbb{Q}_\ell}$ a local Langlands parameter $\sigma_K(\nu)$ up to semisimplification , i.e. (assuming G split to simplify) a conjugacy class of morphisms $\mathrm{Weil}(\overline{K}/K) \to \widehat{G}(\overline{\mathbb{Q}_\ell})$ defined over a finite extension of $\mathbb{Q}_\ell$, continuous and semisimple.

We show in Genestier and V. Lafforgue [ibid.] the local-global compatibility up to semisimplication, whose statement is the following. Let X be a smooth projective and geometrically irreducible curve over $\mathbb{F}_q$ and let N be a level. Then if σ is a global

Langlands parameter and if $\pi = \bigotimes \pi_v$ is an irreducible representation of $G(\mathbb{A})$ such that π^{K_N} is non zero and appears in $\mathfrak{S}_\sigma$ in the decomposition (8-8), then for every place v de X we have equality between

- the semisimplification of the restriction of σ to $\mathrm{Gal}(\overline{F_v}/F_v) \subset \mathrm{Gal}(\overline{F}/F)$,
- the semisimple local parameter $\sigma_K(\nu)$ where ν is the character of the Bernstein center by which it acts on the irreducible smooth representation π_v of $G(K)$.

We use nearby cycles on arbitrary bases (Deligne, Laumon, Gabber, Illusie, Orgogozo), which are defined on oriented products of toposes and whose properties are proven in Orgogozo [2006] (see also Illusie [2006] for an excellent survey). Technically we show that if all the γ_i are in $\mathrm{Gal}(\overline{F_v}/F_v) \subset \mathrm{Gal}(\overline{F}/F)$ then the global excursion operator $S_{I,f,(\gamma_i)_{i\in I}} \in \mathrm{End}\left(C_c^{\mathrm{cusp}}(\mathrm{Bun}_{G,N}(\mathbb{F}_q)/\Xi, \mathbb{Q}_\ell)\right)$ acts by multiplication by an element $\mathfrak{z}_{I,f,(\gamma_i)_{i\in I}}$ of the ℓ-adic completion of the Bernstein center of $G(F_v)$ which depends only on the local data at v. We construct $\mathfrak{z}_{I,f,(\gamma_i)_{i\in I}}$ using stacks of "restricted shtukas", which are analogues of truncated Barsotti–Tate groups.

Remark 9.1. In the case of GL_r the local correspondence was known by Laumon, Rapoport, and Stuhler [1993] and the local-global compatibility (without semisimplification) was proven in L. Lafforgue [2002]. Badulescu and Henniart explained us that in general we cannot hope more that the local-global compatibility up to semisimplication.

10 Independence on ℓ

Grothendieck motives (over a given field) form a $\overline{\mathbb{Q}}$-linear category and unify the ℓ-adic cohomologies (of varieties over this field) for different ℓ: a motive is "a factor in a universal cohomology of a variety". We consider here motives over F. We conjecture that the decomposition

$$C_c^{\mathrm{cusp}}(\mathrm{Bun}_G(\mathbb{F}_q)/\Xi, \overline{\mathbb{Q}_\ell}) = \bigoplus_\sigma \mathfrak{S}_\sigma$$

we have constructed is defined over $\overline{\mathbb{Q}}$ (instead of $\overline{\mathbb{Q}_\ell}$), indexed by motivic Langlands parameters σ, and independent on ℓ. This conjecture seems out of reach for the moment.

The notion of motivic Langlands parameter is clear if we admit the standard conjectures. A motivic Langlands parameter defined over $\overline{\mathbb{Q}}$ would give rise to a "compatible" family of morphisms $\sigma_{\ell,\iota} : \mathrm{Gal}(\overline{F}/F) \to \widehat{G}(\overline{\mathbb{Q}_\ell})$ for any ℓ not dividing q and any embedding $\iota : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_\ell}$. When $G = GL_r$, the condition of compatibility is straightforward (the traces of the Frobenius elements should belong to $\overline{\mathbb{Q}}$ and be the same for all ℓ and ι) and the fact that any irreducible representation (with determinant of finite order) for some ℓ belongs to such a family (and has therefore "compagnons" for other ℓ and ι) was proven as a consequence of the Langlands correspondence in L. Lafforgue [ibid.]. It was generalized in the two following independent directions

- Abe [2013] used the crystalline cohomology of stacks of shtukas to construct crystalline compagnons,

- when F is replaced by the field of rational functions of any smooth variety over $\mathbb{F}_q$, Deligne proved that the traces of Frobenius elements belong to a finite extension of $\mathbb{Q}$ and Drinfeld constructed these compatible families Deligne [2012], Drinfeld [2012], and Esnault and Kerz [2012].

For a general reductive group G the notion of compatible family is subtle (because the obvious condition on the conjugacy classes of the Frobenius elements is not sufficient). In Drinfeld [2018] Drinfeld gave the right conditions to define compatible families and proved that any continuous semisimple morphism $\mathrm{Gal}(\overline{F}/F) \to \widehat{G}(\overline{\mathbb{Q}_\ell})$ factorizing through $\pi_1(U, \overline{\eta})$ for some open dense $U \subset X$ (and such that the Zariski closure of its image is semisimple) belongs to a unique compatible family.

11 Conjectures on Arthur parameters

We hope that all global Langlands parameters σ which appear in this decomposition come from elliptic Arthur parameters, i.e. conjugacy classes of continuous semisimple morphisms $\mathrm{Gal}(\overline{F}/F) \times SL_2(\overline{\mathbb{Q}_\ell}) \to \widehat{G}(\overline{\mathbb{Q}_\ell})$ whose centralizer is finite modulo the center of $\widehat{G}$. This SL_2 should be related to the Lefschetz SL_2 acting on the intersection cohomology of compactifications of stacks of shtukas. We even hope a parameterization of the vector space of discrete automorphic forms (and not only cuspidal ones) indexed by elliptic Arthur parameters.

Moreover we expect that generic cuspidal automorphic forms appear exactly in $\mathfrak{S}_\sigma$ such that σ is elliptic as a Langlands parameter (i.e. that it comes from an elliptic Arthur parameter with trivial SL_2 action). This would imply the Ramanujan–Petersson conjecture (an archimedean estimate on Hecke eigenvalues).

By Drinfeld and Kedlaya [2016] the conjectures above would also imply p-adic estimates on Hecke eigenvalues which would sharper than those in V. Lafforgue [2011].

12 Recent works on the Langlands program for function fields in relation with shtukas

In Böckle, Harris, Khare, and Thorne [2016] G. Böckle, M. Harris, C. Khare, and J. Thorne apply the results explained in this text together with Taylor–Wiles methods to prove (in the split and everywhere unramified situation) the potential automorphy of all Langlands parameters with Zariski-dense image. Thus they prove a weak form of the "Galois to automorphic" direction.

In Yun and Zhang [2017] Zhiwei Yun and Wei Zhang proved analogues of the Gross–Zagier formula, namely equality between the intersection numbers of some algebraic cycles in stacks of shtukas and special values of derivatives of L-functions (of arbitrary order equal to the number of legs).

In Xiao and Zhu [2017] Liang Xiao and Xinwen Zhu construct algebraic cycles in special fibers of Shimura varieties. Their construction was inspired by the case of the stacks of shtukas and is already new in this case (it gives a conceptual setting for the Eichler–Shimura relations used in V. Lafforgue [2012]).

13 Geometric Langlands program

The results explained above are based on the geometric Satake equivalence Mirković and Vilonen [2007], and are inspired by the factorization structures studied by Beilinson and Drinfeld [2004]. The geometric Langlands program was pioneered by Drinfeld [1983] and Laumon [1987a], and then developed itself in two variants, which we will discuss in turn.

13.1 Geometric Langlands program for ℓ-adic sheaves. Let X be a smooth projective curve over an algebraically closed field of characteristic different from ℓ.

For any representation W of $\widehat{G}^I$ the Hecke functor

$$\phi_{I,W} : D_c^b(\mathrm{Bun}_G, \overline{\mathbb{Q}_\ell}) \to D_c^b(\mathrm{Bun}_G \times X^I, \overline{\mathbb{Q}_\ell})$$

is given by

$$\phi_{I,W}(\mathfrak{F}) = q_{1,!}\big(q_0^*(\mathfrak{F}) \otimes \mathfrak{F}_{I,W}\big)$$

where $\mathrm{Bun}_G \xleftarrow{q_0} \mathrm{Hecke}_I \xrightarrow{q_1} \mathrm{Bun}_G \times X^I$ is the Hecke correspondence classifying modifications of a G-bundle at the x_i, and $\mathfrak{F}_{I,W}$ is defined as the inverse image of $\mathcal{S}_{I,W}$ by the natural formally smooth morphism $\mathrm{Hecke}_I \to \mathfrak{M}_I$.

Let σ be a $\widehat{G}$-local system on X. Then $\mathfrak{F} \in D_c^b(\mathrm{Bun}_G, \overline{\mathbb{Q}_\ell})$ is said to be an eigensheaf for σ if we have, for any finite set I and any representation W of $(\widehat{G})^I$, an isomorphism $\phi_{I,W}(\mathfrak{F}) \xrightarrow{\sim} \mathfrak{F} \boxtimes W_\sigma$, functorial in W and compatible to exterior tensor products and fusion. The conjecture of the geometric Langlands program claims the existence of an σ-eigensheaf $\mathfrak{F}$ (it should also satisfy a Whittaker normalization condition which in particular prevents it to be 0). For $G = GL_r$ this conjecture was proven by Frenkel, Gaitsgory, and Vilonen [2002] and Gaitsgory [2004]

When X, Bun_G, σ and $\mathfrak{F}$ are defined over $\mathbb{F}_q$ (instead of $\overline{\mathbb{F}_q}$), a construction of Varshavsky [2004a] produces subspaces of cohomology classes in the stacks of shtukas and this allows to show that the function given by the trace of Frobenius on $\mathfrak{F}$ belongs to the factor $\mathfrak{H}_\sigma$ of decomposition (8-6), as explained in section 15 of V. Lafforgue [2012].

The ℓ-adic setting is truly a geometrization of automorphic forms over function fields, and many constructions were geometrized: Braverman and Gaitsgory [2002] geometrized Eisenstein series, and Lysenko geometrized in particular Rankin–Selberg integrals Lysenko [2002], theta correspondences Lysenko [2006, 2011] and V. Lafforgue and Lysenko [2013], and several constructions for metaplectic groups Lysenko [2015a, 2017].

13.2 Geometric Langlands program for D-modules. Now let X be a smooth projective curve over an algebraically closed field of characteristic 0. A feature of the setting of D-modules is that one can upgrade the statement of Langlands correspondence to a conjecture about an equivalence between categories on the geometric and spectral sides, respectively. See Gaitsgory [2015a] for a precise statement of the conjecture and Gaitsgory [2017] for a survey of recent progress. Such an equivalence can in principle make sense due to the fact that Galois representations into $\widehat{G}$, instead of being taken

individually, now form an algebraic stack $\mathrm{LocSys}_{\widehat{G}}$ classifying $\widehat{G}$-local systems, i.e. $\widehat{G}$-bundles with connection (by contrast one does not have such an algebraic stack in the ℓ-adic setting).

On the geometric side, one considers the derived category of D-modules on Bun_G, or rather a stable ∞-category enhancing it. It is denoted $\text{D-mod}(\mathrm{Bun}_G)$ and is defined and studied in Drinfeld and Gaitsgory [2015]. The category on the spectral side is a certain modification of $\mathrm{QCoh}(\mathrm{LocSys}_{\widehat{G}})$, the (derived or rather ∞-) category of quasi-coherent sheaves on the stack $\mathrm{LocSys}_{\widehat{G}}$. The modification in question is denoted $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\widehat{G}})$, and it has to do with the fact that $\mathrm{LocSys}_{\widehat{G}}$ is not smooth, but rather *quasi-smooth* (a.k.a. derived locally complete intersection). The difference between $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\widehat{G}})$ and $\mathrm{QCoh}(\mathrm{LocSys}_{\widehat{G}})$ is measured by *singular support* of coherent sheaves, a theory developed in Arinkin and Gaitsgory [2015]. The introduction of Nilp in Arinkin and Gaitsgory [ibid.] was motivated by the case of $\mathbb{P}^1$ V. Lafforgue [n.d.] and the study of the singular support of the geometric Eisenstein series. In terms of Langlands correspondence, this singular support may also be seen as accounting for *Arthur parameters*. More precisely the singularities of $\mathrm{LocSys}_{\widehat{G}}$ are controlled by a stack $\mathrm{Sing}(\mathrm{LocSys}_{\widehat{G}})$ over $\mathrm{LocSys}_{\widehat{G}}$ whose fiber over a point σ is the H^{-1} of the cotangent complex at σ, equal to $H^2_{dR}(X, \widehat{\mathfrak{g}}_\sigma)^* \simeq H^0_{dR}(X, \widehat{\mathfrak{g}}^*_\sigma) \simeq H^0_{dR}(X, \widehat{\mathfrak{g}}_\sigma)$ where the first isomorphism is Poincaré duality and the second depends on the choice of a non-degenerate ad-invariant symmetric bilinear form on $\widehat{\mathfrak{g}}$. Therefore $\mathrm{Sing}(\mathrm{LocSys}_{\widehat{G}})$ is identified to the stack classifying (σ, A), with $\sigma \in \mathrm{LocSys}_{\widehat{G}}$ and A an horizontal section of the local system $\widehat{\mathfrak{g}}_\sigma$ associated to σ with the adjoint representation of $\widehat{G}$. Then Nilp is the cone of $\mathrm{Sing}(\mathrm{LocSys}_{\widehat{G}})$ defined by the condition that A is nilpotent. By the Jacobson–Morozov theorem, any such A is the nilpotent element associated to a morphism of SL_2 to the centralizer of σ in $\widehat{G}$, i.e. it comes from an Arthur parameter. The singular support of a coherent sheaf on $\mathrm{LocSys}_{\widehat{G}}$ is a closed substack in $\mathrm{Sing}(\mathrm{LocSys}_{\widehat{G}})$. The category $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\widehat{G}})$ (compared to $\mathrm{QCoh}(\mathrm{LocSys}_{\widehat{G}})$) corresponds to the condition that the singular support of coherent sheaves has to be included in Nilp (compared to the zero section where $A = 0$). The main conjecture is that there is an equivalence of categories

$$\text{D-mod}(\mathrm{Bun}_G) \simeq \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\widehat{G}}). \tag{13-1}$$

Something weaker is known: by Gaitsgory [2015a], $\text{D-mod}(\mathrm{Bun}_G)$ "lives" over $\mathrm{LocSys}_{\widehat{G}}$ in the sense that $\mathrm{QCoh}(\mathrm{LocSys}_{\widehat{G}})$, viewed as a monoidal category, acts naturally on $\text{D-mod}(\mathrm{Bun}_G)$. Note that $\mathrm{QCoh}(\mathrm{LocSys}_{\widehat{G}})$ acts on $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\widehat{G}})$ (one can tensor a coherent complex by a perfect one and obtain a new coherent complex) and the conjectured equivalence (13-1) should be compatible with the actions of $\mathrm{QCoh}(\mathrm{LocSys}_{\widehat{G}})$ on both sides.

Theorem 8.4 (refined in remark 8.5) can be considered as an arithmetic analogue of the fact that $\text{D-mod}(\mathrm{Bun}_G)$ "lives" over $\mathrm{LocSys}_{\widehat{G}}$ (curiously, due to the lack of an ℓ-adic analogue of $\mathrm{LocSys}_{\widehat{G}}$, that result does not have an analogue in the ℓ-adic geometric Langlands program, even if the vanishing conjecture proven by Gaitsgory [2004] goes in this direction). And the fact that Arthur multiplicities formula is still unproven in general is parallel to the fact that the equivalence (13-1) is still unproven.

When $G = T$ is a torus, there is no difference between $\mathrm{QCoh}(\mathrm{LocSys}_{\widehat{T}})$ and $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\widehat{T}})$. In this case, the desired equivalence

$$\mathrm{QCoh}(\mathrm{LocSys}_{\widehat{T}}) \simeq \text{D-mod}(\mathrm{Bun}_T)$$

is a theorem, due to Laumon [1996].

The formulation of the geometric Langlands correspondence as an equivalence of categories (13-1), and even more the proofs of the results, rely on substantial developments in the technology, most of which had to do with the incorporation of the tools of higher category theory and higher algebra, developed by Lurie [2009, n.d.]. Some of the key constructions use categories of D-modules and quasi-coherent sheaves on algebro-geometric objects more general than algebraic stacks (a typical example is the moduli space of G-bundles on X equipped with a reduction to a subgroup *at the generic point of* X).

13.3 Work of Gaitsgory and Lurie on Weil's conjecture on Tamagawa numbers over function fields. In Gaitsgory and Lurie [n.d.(a),(b)] (see also Gaitsgory [2015b]) Gaitsgory and Lurie compute the cohomology with coefficients in $\mathbb{Z}_\ell$ of the stack Bun_G when X is any smooth projective curve over an algebraically closed field of characteristic other than ℓ, and G is a smooth affine group scheme over X with connected fibers, whose generic fiber is semisimple and simply connected. They use in particular a remarkable geometric ingredient, belonging to the same framework of factorization structures Beilinson and Drinfeld [2004] (which comes from conformal field theory) as the geometric Satake equivalence. The Ran space of X is, loosely speaking, the prestack classifying non-empty finite subsets Z of X. The affine grassmannian Gr_{Ran} is the prestack over the Ran space classifying such a Z, a G-bundle $\mathcal{G}$ on X, and a trivialization α of $\mathcal{G}$ on $X \setminus Z$. Then the remarkable geometric ingredient is that the obvious morphism $\mathrm{Gr}_{Ran} \to \mathrm{Bun}_G, (Z, \mathcal{G}, \alpha) \mapsto \mathcal{G}$ has contractible fibers in some sense and gives an isomorphism on homology. Note that when $k = \mathbb{C}$ and G is constant on the curve, their formula implies the well-known Atiyah–Bott formula for the cohomology of Bun_G, whose usual proof is by analytic means.

Now assume that the curve X is over $\mathbb{F}_q$. By the Grothendieck–Lefschetz trace formula their computation of the cohomology of Bun_G over $\overline{\mathbb{F}_q}$ gives a formula for $|\mathrm{Bun}_G(\mathbb{F}_q)|$, the number of $\mathbb{F}_q$-points on the stack Bun_G. Note that since Bun_G is a stack, each isomorphism class y of points is weighted by $\frac{1}{\mathrm{Aut}_y(\mathbb{F}_q)}$, where Aut_y is the algebraic group of automorphisms of y, and $\mathrm{Aut}_y(\mathbb{F}_q)$ is the finite group of its $\mathbb{F}_q$-points. Although the set of isomorphism classes y of points is infinite, the weighted sum converges. Gaitsgory and Lurie easily reinterpret $|\mathrm{Bun}_G(\mathbb{F}_q)|$ as the volume (with respect to some measure) of the quotient $G(\mathbb{A})/G(F)$ (where F is the function field of X and $\mathbb{A}$ is its ring of adèles) and prove in this way, in the case of function fields, a formula for the volume of $G(\mathbb{A})/G(F)$ as a product of local factors at all places. This formula, called the Tamagawa number formula, had been conjectured by Weil for any global field F.

Over number fields Bun_G does not make sense, only the conjecture of Weil on the Tamagawa number formula remains and it had been proven by Kottwitz after earlier

works of Langlands and Lai by completely different methods (residues of Eisenstein series and trace formulas).

14 Homage to Alexandre Grothendieck (1928–2014)

Modern algebraic geometry was built by Grothendieck, together with his students, in the realm of categories: functorial definition of schemes and stacks, Quot construction for Bun_G, tannakian formalism, topos, étale cohomology, motives. His vision of topos and motives already had tremendous consequences and others are certainly yet to come. He also had a strong influence outside of his school, as testified by the rise of higher categories and the work of Beilinson, Drinfeld, Gaitsgory, Kontsevich, Lurie, Voevodsky (who, sadly, passed away recently) and many others. He changed not only mathematics, but also the way we think about it.

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CONFORMALLY INVARIANT LOOP MEASURES

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Abstract

There have been incredible progress in the last twenty years in the rigorous analysis of planar statistical mechanics models whose limits are conformally invariant. This paper will not try to survey all the recent advances. Instead, it will discuss some recent results about particular conformally invariant measures on loops and paths.

1 Introduction

One of the main goals in statistical physics is to understand macroscopic behavior of a system given the interactions which are mainly microscopic but may exhibit long range correlations. Such models often depend on a parameter and at a critical value of the parameter the collective interaction switches from being microscopic to macroscopic. Critical phenomena is the study of such systems at or near this critical value.

There is a wide class of models (percolation, self-avoiding walk, Ising and Potts model, loop-erased random walk and spanning trees,...) whose behavior is very dependent on the spatial dimension. There exists a critical dimension above which the behavior is relatively simple (although it is not always trivial to prove this is true!), but below the critical dimension there is "non mean-field" behavior with nontrivial critical exponents for long-range correlations and fractal structures arising.

It was first predicted by Belavin, Polyakov, and Zamolodchikov [1984b,a] that the continuum limit of critical fields in two dimensions would exhibit some kind of conformal invariance. This idea along with the related Coulomb gas techniques allowed for a number of nonrigorous predictions of critical exponents, see, e.g., Cardy [1992, 1996], Di Francesco, Mathieu, and Sénéchal [1997], Nienhuis [1982, 1984], and Saleur and Duplantier [1987]. These exact exponents agreed with simulations so even though the theoretical arguments were far from being mathematically rigorous, it seemed clear that they were giving correct predictions and hence there should be mathematical structures and theorems to make precise and prove these predictions.

Major breakthroughs in the rigorous theory happened in around the turn of the twenty-first century. Probably the most important is Schramm's creation of what it now called

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the Schramm–Loewner evolution (SLE). This combined with ideas of Werner and myself on the Brownian intersection exponent opened up the understanding of the continuum limit for curves and interfaces of fields. On the discrete side, Kenyon used conformal invariance to prove the exact value of the dimension of the loop-erased walk and Smirnov proved "Cardy's formula" for the crossing probabilities of critical percolation on the triangular lattice.

These works were just a start to what may be called a major subfield studying critical behavior of two-dimensional systems. This has included two Fields medals Newman [2007] and Kesten [2010], two other plenary talks Schramm [2007] and Le Gall [2014], at least four previous invited talks Smirnov [2006], Lawler [2002], and Duplantier [2014a] plus a number of other invited talks somewhat related, and it has been a part of the work of at least four invited speakers in this conference.

Given the explosive nature of the field, I will not try to give an overview. I have decided to give a personal perspective and to focus on several loop measures and related models, loop-erased random walk (related to uniform spanning trees) and the Gaussian free field. I start by introducing one of the main characters, discrete loop measures, and show how they are related to some well known objects, spanning trees and determinant of the Laplacian. It also generates one of the random fractals, the loop-erased random walk, and we then discuss what it means to take a scaling limit. This leads to a review of two of the main players in the field: the conformally invariant Brownian loop measure and the Schramm–Loewner evolution (SLE). I discuss a number of properties of SLE and focus on the most recent part to finish the characterization, the natural or fractal parametrization of the curve.

The Gaussian free field in two dimensions is the next topic. I start with the discrete field and show recent results that construct the field using the loop measure with some extra randomness. I then introduce the continuous free field which has become the centerpiece of much of the work in conformally invariant systems. Here I only give a quick introduction. In respect for my advisor, Ed Nelson, I have decided to phrase this section in terms of nonstandard analysis Nelson [1987]. While I am not sure this will add to the mathematical development of the field, I have a hope that it will be a pedagogical tool in the future to explain the relationship between the discrete and the continuous. Here I use it to help define "Liouville quantum gravity" which is the exponential of the Gaussian field.

The next part of the paper concerns the second type of loop, SLE type loops. I again give a discrete introduction focusing on the loop-erased random walk and spanning tree model. I then discuss recent constructions of such loops — either as part of conformal loop ensembles (CLE) or directly from the definition of SLE. The first construction uses the Brownian soup directly and the second construction is modeled on the definition of the Brownian loop measure and makes use of the natural parametrization.

I finish my discussing some recent results that combine the ideas in this survey, the convergence of the loop-erased walk to SLE_2 in the natural parametrization. This requires a combinatorial estimate involving a signed loop measure and reduces the discrete problem to a calculation for the Brownian loop measure. Then, it is shown how this relates to the the natural parametrization of SLE.

2 Loop measures and spanning trees

We start with a simple definition. Given a countable set $\mathcal{X}$ and a function $p : \mathcal{X} \times \mathcal{X} \to [0, \infty)$, we define a *(rooted) loop* $l = [l_0, l_1, \dots, l_n]$ to be a finite sequence with $l_j \in \mathcal{X}$ and $l_0 = l_n$. An important case is when p is the transition matrix for a Markov or subMarkov chain. We let $|l| = n$ denote the number of vertices in the loop; if $|l| = 0$, the loop is *trivial* and $p(l) = 1$; if $|l| > 0$, $p(l) = p(l_0, l_1)\, p(l_1, l_2) \cdots p(l_{n-1}, l_n)$. Our loop measures will always be on nontrivial loops.

An *(unrooted)* loop ℓ is an equivalence class of rooted loops under the equivalence

$$[l_0, l_1, \dots, l_m] \sim [l_1, \dots, l_m, l_1] \sim [l_2, l_3, \dots, l_m, l_1, l_2] \sim \cdots .$$

In other words, an unrooted loop is a rooted loop that has retained its orientation but has forgotten its starting point. We can write $p(\ell)$ since $p(l)$ is the same for all representatives. The number of representatives of an unrooted loop divides n but can be strictly less than n; for example, the unrooted loop generated by $[x, y, x, y, x]$ has only two representatives. The loop measure on rooted loops is given by

$$\tilde{m}(l) = \tilde{m}_p(l) = \frac{p(l)}{|l|},$$

and the loop measure on the unrooted loops is the induced measure

$$m(\ell) = m_p(\ell) = \sum_{l \in \ell} \tilde{m}(l) = \frac{s(\ell)\, p(\ell)}{|\ell|},$$

where $s(\ell)$ denotes the number of rooted representatives of ℓ. In this generality, there is no need to restrict to positive values of p; indeed it can be complex-valued or even matrix-valued.

As an example, we will assume that $\mathcal{X}$ is a finite, connected, (undirected) graph and that p gives the transition probabilities for simple random walk on the graph. In other words, $p(x, y) = 1/d_x$ if x is adjacent to y where d_x is the degree of x. Wilson [1996] found the following algorithm for choosing a spanning tree from the uniform distribution over all spanning trees. Let us write $\mathcal{X} = \{x_0, x_1, \dots, x_n\}$ where we have chosen an arbitrary ordering of the vertices. We choose a spanning tree as follows thinking of x_0 as the root vertex:

- Start a random walk at x_1 and stop it when it reaches x_0 and erase the loops chronologically from the path. Add these edges to the tree.
- Recursively, choose the vertex of smallest index that has not been added to the tree; start a random walk there and stop it when it reaches a vertex in the tree; erase loops and add those edges to the tree.

We continue until we have a spanning tree. A straightforward analysis of the algorithm (see Lawler and Limic [2010, Chapter 9]) shows that the probability that a particular tree $\mathcal{T}$ is chosen is

$$\left[\prod_{j=1}^{n} p(y_j, \hat{y}_j)\right] F(A) = \left[\prod_{y \in \mathcal{X} \setminus \{x_0\}} d_y\right]^{-1} F(A), \qquad F(A) := \prod_{j=1}^{n} G_{A_j}(y_j, y_j).$$

Here $\{y_1, \dots, y_n\}$ is a permutation of $A := \{x_1, \dots, x_n\}$ (determined by $\mathcal{T}$); $\hat{y}_j$ is the vertex adjacent to y_j in $\mathcal{T}$ on the path to x_0; $A_j = A \setminus \{y_1, \dots, y_{j-1}\}$; and G_{A_j} denotes the usual random walk Green's function for the walk killed upon leaving A_j. The term in brackets is clearly independent of the permutation. While it is not obvious that our definition for $F(A)$ does not depend on the ordering of the vertices, it indeed does not. One can check this as a simple exercise in Markov chain theory but it is more illuminating to write it in one of two order independent ways:

- $F(A) = 1/\det \Delta$ where $\Delta = G^{-1} = (I - P)$ is the (negative of the random walk) Laplacian considered as a matrix indexed by A_1.
- If $m = m_p$,

$$F(A) = \exp\left\{\sum_{\ell \subset A} m(\ell)\right\}, \tag{1}$$

The surprising fact is that Wilson's algorithm gives equal probability to each spanning tree; moreover, since we know what this probability, is we can conclude that the number of spanning trees is

$$\left[\prod_{y \in \mathcal{X} \setminus \{x_0\}} d_y\right] F(A)^{-1} = \left[\prod_{y \in \mathcal{X} \setminus \{x_0\}} d_y\right] \det \Delta.$$

This looks even nicer if we use the graph Laplacian Δ_g (the degree matrix minus the adjacency matrix) in which case the right-hand side becomes just $\det \Delta_g$. This is far from being a new result — it was proved by Kirchhoff in the nineteenth century.

The fact that the quantity in (1) is a determinant can be seen if we write it in terms of the *rooted* loop measure and use a well known identity,

$$\exp\left\{\sum_{l \subset A_1} \hat{m}(l)\right\} = \exp\left\{\sum_{n=1}^{\infty} \frac{1}{n} \operatorname{Tr}(P^n)\right\} = \det[I - P].$$

The great utility of the loop measure comes from its description in terms of *unrooted* loops; indeed, the proof of Wilson's algorithm uses the fact that one sample from a "soup" of unrooted loops in any order.

Although this can be done in generality, we will be focusing on a special case. Suppose that A is a finite, simply connected subset of the integer lattice $\mathbb{Z}^2$ containing the origin so that $\partial A = \{x \in \mathbb{Z}^2 : \operatorname{dist}(x, A) = 1\}$. The usual simple random walk measure gives $p(x, y) = 1/4$ if $|x - y| = 1$. Suppose we take a simple random walk starting at the origin, stop it when it reaches ∂A, and then erase loops to give a self-avoiding path η. This gives a probability measure on self-avoiding walks (SAW) starting at the origin ending and ∂A. By Wilson's algorithm, it is the probability that the unique path from the origin to ∂A in the uniform spanning tree of the graph $\mathcal{X} = A \cup \{\partial A\}$ is η (here, ∂A is considered as a single point — this is called the wired spanning tree). In this case, the number of spanning trees is $4^{\#(A)}\, F(A)^{-1}$.

Associated to loop measures are *loop soups*. This is a colorful term for a Poissonian realization from m. Let $\mathcal{L}_A$ denote the set of unrooted loops in the set A. At each time t, the soup $\mathcal{C}_t(A)$ is a multiset from $\mathcal{L}_A$ where loop ℓ appears N_t^ℓ times. It is defined by saying that $\{N_t^\ell : \ell \in \mathcal{L}_A\}$ are independent Poisson processes with parameter $m(\ell)$.

There are various ways to describe the probability distribution for loop-erased random walk from the origin to ∂A without making reference to loop erasure. One nice one is as a *Laplacian random walk*. Suppose the path starts as $\eta = [\eta_0 = 0, \eta_1, \dots, \eta_k]$. Then the probabilities for the next step are given by weighting by the solution of the Dirichlet problem (for the discrete Laplacian) in $A \setminus \eta$ with boundary value 0 on η and 1 on ∂A. In other words, loop-erased random walk is Laplacian growth where the growth only occurs at the tip.

Suppose we observe the loop-erased walk η. Can we recover (with added randomness) the simple random walk that produced η? The answer is yes, and the way to do it is by taking a realization of the loop soup $\mathcal{C}_1$ at time $t = 1$. We then use η to "explore" the loop soup. We start at the origin and view all loops in $\mathcal{C}_1$ that intersect the origin. We turn these into rooted loops by choosing the origin as the root (if the origin is visited several times choose randomly) and then add all the loops to the path in the order they appeared in the soup. At this point we have not observed the soup in $A \setminus \{0\}$. We take our next step η_1 and observe the loops in $A \setminus \{0\}$ that intersect η_1, and continue. A short combinatorial argument Lawler and Limic [2010, Chapter 9] shows that the distribution of the path at the end is that of a usual simple random walk. Note that the order in which we discover loops in the soup depends on the choice of η, and for a particular η we only observe the loops that intersect η.

The probability that a particular η is chosen for the loop-erased walk in A is $4^{-|\eta|}\, F_\eta(A)$, where $\log F_\eta(A)$ denotes the loop measure of loops in A that intersect η. What happens if we "perturb" the domain, say, consider $\tilde{A} \subset A$? How does this change the probability of seeing a certain η? This probability is zero if $\eta \cap (A \setminus \tilde{A})$ is nonempty, but otherwise it is $4^{-|\eta|}\, F_\eta(\tilde{A})$. In other words, the Radon–Nikodým derivative is given in terms of the loops in the larger domain that are lost when shrinking:

$$\frac{F_\eta(\tilde{A})}{F_\eta(A)} = \exp\left\{ - \sum_{\ell \subset A, \ell \cap (A \setminus A') \neq \varnothing, \ell \cap \eta \neq \varnothing} m(\ell) \right\}. \tag{2}$$

In continuous models for paths and loops, one of the key quantities to consider is the effect of perturbation of a domain on the measure of a particular path.

One observable of a loop soup is $n_x = n_x(t)$, the number of times that a vertex x is visited by time t. In the case $t = 1$ and the loop-erased walk, n_x is a geometric random variable,

$$\mathbf{P}\{n_x = k\} = q^k\,(1-q), \qquad q = \frac{1}{G_A(x,x)}.$$

More generally, the distribution of $n_x(t)$ is negative binomial,

$$\mathbf{P}\{n_x(t) = k\} = \binom{k+t-1}{k} q^k\,(1-q)^t.$$

One can convert to continuous times by adding independent exponential waiting times. For $t = 1$, given n_x, we define $L_x(1)$ to be the sum of $n_x + 1$ independent exponential random variables with parameter 1. Then a standard computation shows that $L_x(t)$ has an exponential distribution with parameter q. For other t, we can choose $L_x(t)$ to be the sum of $n_x(t)$ independent exponential random variables and a Gamma random variable with parameters t and 1. In particular, if $t = 1/2$, then $L_x(t)$ is the sum of $n_x(1/2)$ exponentials plus a random variable with the same distribution as $Z^2/2$ where Z is a standard normal.

As we discuss below, $\{L_x(1)/2 : x \in \mathcal{X}\}$ has the same distribution as $\{|Z_x|^2 : x \in \mathcal{X}\}$ where $\{Z_x = X_x + iY_x : x \in \mathcal{X}\}$ is a complex Gaussian field with independent real and imaginary parts each having covariance matrix G. Equivalently, $\{L_x(1/2)\}$ has the distribution of $\{X_x^2\}$.

3 Scaling limits

In two dimensions, conformally invariant objects are obtained as scaling limits of discrete models. Both the loop soup and the loop-erased walks have limits that we describe here. To each finite, connected subset A of $\mathbb{Z}^2 = \mathbb{Z} + i\mathbb{Z}$ we associate a domain in $\mathbb{C}$,

$$D_A = \operatorname{int}\left[\bigcup_{z \in A}(z + \mathcal{S})\right], \quad \mathcal{S} = \{x + iy \in \mathbb{C} : |x|, |y| \le 1/2\}.$$

Conversely, if D is a simply connected domain in $\mathbb{C}$ containing the origin and $1/n$ is a lattice spacing, we let A_n be the connected component containing the origin of all w with $w + \mathcal{S} \subset nD$. Then we set $D_n = n^{-1} D_{A_n}$ as a lattice approximation of the domain D. If $z \in n^{-1}\mathbb{Z}^2$, we write $\mathcal{S}_z = z + n^{-1}\mathcal{S}$.

As an example, let D be the square $\{x + iy : |x|, |y| < 1\}$ so that $A_n = \{x + iy \in \mathbb{Z}^2 : |x|, |y| < n\}$. Let $z_n = -n, w_n = n$. Then in the limit we get the square $D = \{x + iy \in \mathbb{C} : |x|, |y| < 1\}$ with the boundary points $z = -1, w = +1$. On A_n we have the LERW from z_n to w_n; to be more precise, we consider all simple random walk paths starting at z_n, ending at w_n, otherwise staying in A_n, and erase the loops chronologically. We also have the random walk loop measure. Here we consider the scaling limits of both.

We start with the loop measure which does not depend on the boundary points z, w. To each rooted loop $l = [l_0, \dots, l_{2k}]$ in A_n we associate the scaled loop $l^{(n)}(t), 0 \le t \le k/n^2$,

$$l^{(n)}(j/n^2) = n^{-1} l_{2j}, \quad 0 \le j \le k, \tag{3}$$

extended to other t by linear interpolation. We give $l^{(n)}$ the same measure as l; in other words, the loop measure on D_n is the same as the loop measure on A_n except that the paths are scaled, As $n \to \infty$, the total mass of the measure goes to ∞. The Brownian scaling in (3) uses the relationship $dt = (dx)^2$ where 2 is the fractal dimension of Brownian paths. It is well known that the probability that a two-dimensional simple random walk is at its starting point after $2k$ steps is asymptotic to $(\pi k)^{-1}$, and hence the

measure of loops $l^{(n)}$ rooted at a particular point $\zeta \in n^{-1}\,\mathbb{Z}^2$ of time duration $t = k/n^2$ is asymptotic to

$$\frac{1}{\pi k}\,\frac{1}{2k} = \frac{1}{2\pi t^2}\,dt\,\mathrm{Area}(\mathcal{S}_\zeta) \quad dt = \frac{1}{n^2}. \tag{4}$$

The scaling limit of this can now be determined Lawler and Trujillo Ferreras [2007] and is described in the next section.

The definition of the LERW from z to w starts with the usual random walk measure from z to w; to be more precise, every nearest neighbor walk $\omega = [\omega_0, \dots, \omega_k]$ with $\omega_0 = -n, \omega_k = n$ and $\{\omega_1, \dots, \omega_{k-1}\} \in A_n$ gets measure 4^{-k}. The total mass of this measure is the probability that a simple random walk starting at $-n$ immediately enters A_n and then exits at n. It can be shown (using, for example, the "gambler's ruin" estimate for random walk or in this case by explicit computation as a finite Fourier series) that the total mass is asymptotic to $c\,n^{-2}$ for a (computable) constant c. If we erase loops from the paths we get a measure on self-avoiding paths with the same total mass. As before, we can consider this as the measure on self-avoiding paths $\eta = [\eta_0 = -n, \dots, \eta_k = n]$ with $\eta_1, \dots, \eta_{k-1} \in A_n$, that gives measure $4^{-k}\,F_\eta(A_n)$, to each such η. The total mass can be written as the value of the *partition function*

$$Z_{A_n}(\beta; -n, n) := \sum_{\eta: -n \longrightarrow n, \eta \subset A_n} e^{-\beta|\eta|}\,F_\eta(A_n),$$

evaluated at the *critical* $\beta_c = \log 4$. The mass can also be described in terms of spanning trees. A wired spanning tree of A_n is a spanning tree of the graph $A_n \cup \{\partial A_n\}$ where all the points in the boundary have been identified to a single point. Using Wilson's algorithm rooted at the boundary with $x_1 = -n + 1$, we can see that the total mass is ($1/4$ times) the probability that the uniform spanning tree contains a path starting at $-n + 1$ and reaching the boundary along the edge adjacent to n.

In order to scale the paths, we need to know the fractal dimension d which can be defined roughly by saying the that typical LERW crossing A_n has on the order of n^d steps. Using the Brownian scaling as the model, we associate to each LERW $\eta = [\eta_0, \dots, \eta_k]$ connecting $-n$ to n in A_n, the scaled path of time duration $k\,n^{-d}$,

$$\eta^{(n)}(j/n^d) = n^{-1}\,\eta_j, \quad 0 \le j \le k. \tag{5}$$

When taking the limit, we first multiply the measure by n^2 so that the limit has finite total mass, and then we take a limit of the paths above. The limit is be a version of the Schramm–Loewner evolution (SLE).

4 Brownian loop measure and soup in $\mathbb{C}$

Let D be a bounded, simply connected domain in $\mathbb{C}$. For every n, let $A_n \subset \mathbb{Z}^2$ and $D_n = n^{-1}\,A_n$ be defined as in the previous section. The *Brownian loop measure* is the scaling limit as $n \to \infty$ of the random walk loop measure on A_n. It can be constructed directly Lawler and Werner [2004], and, indeed, it was defined before the discrete loop measures.

A *rooted loop* is a curve $\gamma : [0, t_\gamma] \to \mathbb{C}$ with $\gamma(0) = \gamma(t_\gamma)$ where $t_\gamma \in (0, \infty)$. An *(unrooted) loop* is an equivalence class of rooted loops under the equivalence $\gamma \sim \gamma^r$ where $\gamma^r(t) = \gamma(t + r)$ and $t + r$ is interpreted modulo t_γ. We will define the loop measure first for rooted loops and then use this to define the measure for unrooted loops. It is useful to view a rooted loop γ as a triple $(z, t_\gamma, \hat{\gamma})$ where z is the root, t_γ is the time duration, and $\hat{\gamma}$ is a loop of time duration 1 rooted at the origin. The bijection is given using Brownian scaling

$$\gamma(t) = z + t_\gamma^{1/2}\, \hat{\gamma}(t/t_\gamma), \quad 0 \le t \le t_\gamma.$$

The rooted loop measure on $\mathbb{C}$ can be given by

$$(\text{Area}) \times \frac{dt}{2\pi t^2} \times (\text{Brownian bridge}),$$

Here Brownian bridge refers to the probability measure associated to (appropriately defined) Brownian motion starting at 0 conditioned to return to 0 at time one. The middle term can be written as $t^{-1}\,(2\pi t)^{-1}$. The factor $(2\pi t)^{-1}$ is the density of Brownian motion at time t evaluated at the origin and the t^{-1} is the analogue of the $|l|^{-1}$ term from the discrete loop measure. This is the natural continuum analogue of (4). To give the measure on a domain D one restricts this measure to loops in D. This is an infinite measure even for bounded D because the measure of small loops blows up. This measure induces a measure on unrooted loops which we call the *Brownian loop measure*. Poissonian realizations of the Brownian loop measure are called Brownian *loop soups*.

The Brownian loop measure satisfies two important properties. The first is immediate but still very important.

- **Restriction property**. If $D' \subset D$ then the loop measure on D' is the same as the loop measure on D restricted to loops that lie in D'.

The second is particular to two dimensions and is a starting point for analysis of conformally invariant processes.

- **Conformal invariance** Lawler and Werner [2004]. If $f : D \to f(D)$ is a conformal transformation, then the image of the loop measure on D is the loop measure on $f(D)$.

Let us explain the last statement. The Brownian (heat equation) scaling can be written intuitively as $dx = \sqrt{dt}$. This is the scaling for a path with fractal dimension 2. (We will consider paths of fractal dimension d for which $dx = (dt)^{1/d}$.) The parametrization for the Brownian motion is a "natural" two-dimensional parametrization $|dB_t| \approx t^{1/2}$. As shown first by Lévy [1940], complex Brownian motion is conformally invariant provided that one changes the parametrization to respect the fractal dimension. In our case, if γ is a loop in D of time duration t_γ, we define a loop $f \circ \gamma$ in $f(D)$ by

$$t_{f\circ\gamma} = \int_0^{t_\gamma} |f'(\gamma(s))|^2\, ds,$$

and $f \circ \gamma(r) = f[\gamma(\sigma(r))]$ where

$$\int_0^{\sigma(r)} |f'(\gamma(s))|^2 \, ds = r.$$

If μ_D denotes the Brownian loop measure on *unrooted loops* and $f \circ \mu_D$ is defined by

$$f \circ \mu_D(V) = \mu_D\{\gamma : f \circ \gamma \in V\},$$

then $f \circ \mu_D = \mu_{f(D)}$. This result requires no topological assumptions on the domain D.

The definition of the loop measure is not very conducive to calculation. When computing measures of sets it is often useful to use a decomposition of into *Brownian (boundary) bubbles*. This focuses on a particular rooted representative of an unrooted loop. For example, if $\mu = \mu_{\mathbb{C}}$ is the measure on the entire plane, then we can write

$$\mu_{\mathbb{C}} = \int_{\mathbb{C}} \mu^{\text{bub}}_{\mathbb{H}+iy}(x+iy) \, dx \, dy.$$

This is a decomposition focusing on the (unique) point on the loop of smallest imaginary part. Here $\mu^{\text{bub}}_{\mathbb{H}}(0)$ is a σ-finite measure on loops rooted at the origin and otherwise staying in the upper half plane $\mathbb{H}$. It can be defined in a number of equivalent ways by taking limits. More generally, if $f : D \to f(D)$ is a conformal transformation, $z \in \partial D$, and z and $f(z)$ are analytic boundary points, we have the conformal covariance rule

$$f \circ \mu^{\text{bub}}_D(z) = |f'(z)|^2 \, \mu^{\text{bub}}_{f(D)}(z).$$

Another useful way to write μ_C is by focusing on the point of greatest magnitude

$$\mu_{\mathbb{C}} = \int_{\mathbb{C}} \mu^{\text{bub}}_{|z|\mathbb{D}}(z) \, dA(z), \tag{6}$$

where $\mathbb{D}$ is the unit disk.

5 Measures on self-avoiding curves

The loop-erased random walk is one of many lattice models for which scaling limits are expected to exist. Many of them are parts of more complicated fields, for example, loop-erased random walks arise as macroscopic paths in scaling limits of uniform spanning trees. Suppose D is a bounded, simply connected subdomain of $\mathbb{C}$ containing the origin which for ease we will assume has an analytic boundary. Let z, w be distinct points on the boundary. We will consider measures on (continuous) curves $\gamma : (0, t_\gamma) \to D$ with $\gamma(0-) = z, \gamma(t_\gamma+) = w$ (we sometimes allow $t_\gamma = \infty$). Much of the work of the last eighteen years has built on work of Oded Schramm [2000] to understand the possible limits under the assumption that the limit is conformallly invariant or covariant. We will consider (positive) measures $\mu_D(z, w)$ on such curves; if $\|\mu_D(z, w)\| < \infty$, we write $\mu^{\#}_D(z, w)$ for the corresponding probability measure obtained by normalization. The goal is to find families of measures indexed by D, z, w which have the following properties.

- **Fractal dimension**. The measure is supported on simple curves of Hausdorff dimension $d \in [1,2]$ with the appropriate fractal parametrization. If $f : D \to f(D)$ is a conformal transformation, then $f \circ \gamma$ is defined by $f \circ \gamma(t) = f[\gamma(\sigma_t)]$ where
$$\int_0^{\sigma_t} |f'(s)|^d \, ds = t.$$
Note that Brownian curves satisfy this with $d = 2$ but they are not simple. Although we start with an assumption of simplicity of the curves, it turns out that many important examples gives curves that are not simple. However, they will be "non-crossing". For the moment we restrict to the simple case.

- **Conformal covariance**. If $f : D \to f(D)$ is a conformal transformation, then
$$f \circ \mu_D(z,w) = |f'(z)|^b \, |f'(w)|^b \, \mu_{f(D)}(f(z), f(w)),$$
where b is a scaling exponent. In particular the probability measures are conformally *invariant*:
$$f \circ \mu_D^{\#}(z,w) = \mu_{f(D)}^{\#}(f(z), f(w)),$$
and $\mu_D^{\#}(z,w)$ can be defined even for nonsmooth boundaries.

- **Reversibility**. The measure $\mu_D(w,z)$ is obtained from $\mu_D(z,w)$ by reversing the paths.

- **Boundary perturbation or generalized restriction**. Suppose $D' \subset D$ and the domains agree in neighborhoods of z, w. Then $\mu_{D'}(z,w)$ is mutually absolutely continuous with the measure given by $\mu_D(z,w)$ *restricted* to curves with $\gamma(0,t_\gamma) \subset D'$. If $\Phi_{D,D'}$ denotes the Radon–Nikodým derivative, then it is a conformal invariant,
$$\Phi_{D,D'}(\gamma) = \Phi_{f(D),f(D')}(f \circ \gamma).$$

- **Domain Markov property**. Suppose an initial segment $\tilde{\gamma}$ of γ ending at $z' \in D$ is observed. Then in the probability measure $\mu_D^{\#}(z,w)$, the conditional distribution of the remainder of the curve given $\tilde{\gamma}$ is given by $\mu_{D\setminus\tilde{\gamma}}(z',w)$. By reversibility, we should be able to also grow ends of the curve from w.

The big breakthrough by Schramm [2000] described in our notation is as follows. Let us restrict to simply connected domains D, consider only the probability measures $\mu_D^{\#}(z,w)$ (which do not require boundary smoothness), and assume conformal invariance and the domain Markov property. Finally, consider the curves γ and $f \circ \gamma$ only *modulo reparametrization*. Then there is only a one parameter family of curves that are candidates for this. This is now called the *(chordal) Schramm–Loewner evolution (SLE)* with parameter $\kappa > 0$. It describes the curve $\gamma[0,t_\gamma]$ in terms of the collection of conformal maps $g_t : D \setminus \gamma[0,t] \to D$ with $g_t(\gamma(t)) = z$, $g_t(w) = w$. The evolution of g_t is described with a Loewner equation driven by a Brownian input. The parameter κ gives the variance of the Brownian motion. It takes some work to understand the curve $\gamma[0,t]$ from the maps g_t but we now know a lot.

- The measure is supported on simple curves for $\kappa \le 4$; it is supported on plane-filling curves for $\kappa \ge 8$; and for $4 < \kappa < 8$, it is supported on self-intersecting but "non-crossing" curves that are not plane filling. Rohde and Schramm [2005]

- For $\kappa < 8$, the measure is supported on curves of Hausdorff dimension

$$d = 1 + \frac{\kappa}{8}.$$

  In particular, for each $1 < d < 2$, there is a unique family of curves. Rohde and Schramm [2005] and Beffara [2008]

- The measure is reversible for $\kappa < 8$ Miller and Sheffield [2016a] and Zhan [2008].

The relationship with the Brownian loop measure comes in the boundary perturbation rule. We define the conformal invariant: if D is a domain and K, K' are disjoint, relative closed, subsets, then $\Lambda_D(K, K') = \exp\{m(\mathfrak{L})\}$, where $\mathfrak{L} = \mathfrak{L}_D(K, K')$ is the Brownian measure of loops in D that intersect both K and K'. Using mainly the work in Lawler, Schramm, and Werner [2003], for $d \le 4$ we can define the measure $\mu_D(z, w)$ such that the following is true.

- The total mass of $\mu_D(z, w)$ is $H_D(z, w)^b$ where $b = \frac{6-\kappa}{2\kappa}$, and $H_D(z, w)$ denotes the boundary Poisson kernel (the normal derivative in each component of the Green's function) normalized so that $H_{\mathbb{H}}(0, 1) = 1$.

- If $D' \subset D$ as above and $\gamma(0, t_\gamma) \subset D'$, the Radon–Nikodým derivative is given by

$$\frac{d\mu_{D'}(z, w)}{d\mu_D(z, w)}(\gamma) = \Lambda_D(\gamma, D \setminus D')^{\mathbf{c}/2} = \exp\left\{\frac{\mathbf{c}}{2}\, m[\mathfrak{L}_D(\gamma, D \setminus D')]\right\}, \tag{7}$$

  where $\mathbf{c}$ is the *central charge* given by

$$\mathbf{c} = \frac{(3\kappa - 8)(6 - \kappa)}{2\kappa}.$$

  This is the same central charge that is a fundamental parameter in conformal field theories. In statistical physics, it also can be described in terms of infinitesimal changes of the "stress energy tensor". Here we see it as measuring the effect on the path measure of infinitesimal changes to the ambient domain.

The scaling limit of LERW is SLE_2. (This was predicted in Schramm [2000] and proved, at least for the related radial case, in Lawler, Schramm, and Werner [2004] for curves modulo reparametrization.) Here we see $\mathbf{c} = -2$; indeed (7) is the scaling limit of the relation (2). This answers all of the questions above except for giving the path the correct parametrization; we discuss this in Section 11.

The theory of chordal SLE in simply connected domains can be derived from the assumptions of conformal invariance and domain Markov property of the probability measures $\mu_D^{\#}(z, w)$. In fact, the role of the partition function and the Brownian loop measure was found by studying the unique one-parameter family of measures in simple

connected domains. There is a another way of looking at this that is important. Let us consider the case of the upper half plane and boundary points 0 and infinity. The fundamental observation of Schramm was the following. Suppose that we consider the case of the upper half plane $\mathbb{H}$ and boundary points 0 and infinity. One version of "Loewner chains" (which were developed by Loewner to understand the Bieberbach conjecture) states that if γ is a simple curve from 0 to ∞; $H_t = \mathbb{H} \setminus \gamma[0,t]$, and $g_t : H_t \to \mathbb{H}$ is the unique conformal transformation satisfying

$$g_t(z) = z + o(1), \quad z \to \infty,$$

then *with an appropriate parametrization of* γ, there is a continuous function real-valued function U_t such that

$$\partial g_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z.$$

Schramm noted that conformal invariance and the domain Markov property implied that U_t is a driftless Brownian motion, and hence $U_t = \sqrt{\kappa}\, W_t$, where W_t is a standard Brownian motion and κ is the parameter. Setting $a = 2/\kappa$ and doing a linear time change, we get

$$\partial g_t(z) = \frac{a}{g_t(z) - U_t}, \quad g_0(z) = z,$$

where $U_t = -B_t$ is a standard Brownian motion. If $Z_t(z) = g_t(z) - U_t$, we can write this as a Bessel equation,

$$dZ_t(z) = \frac{a\, dt}{Z_t(z)} + dB_t.$$

For $\kappa \le 4$, this gives a measure on simple curves and the following is true. Suppose $D = \mathbb{H} \setminus K$ is a simply connected subdomain where K is compact, not containing the origin. There are three equivalent ways to find $\mu_D^{\#}(0,\infty)$.

- Use a conformal transformation to map $\mathbb{H}$ onto D fixing 0 and ∞ and use conformal invariance.

- Give the Radon–Nikodým derivative of the two probability measures. Assuming $\gamma \cap K = \varnothing$, the value is

$$\Phi_D'(0)^{-b} \exp\left\{\frac{\mathbf{c}}{2}\, m[\mathcal{L}_D(\gamma, D \setminus D')]\right\}.$$

  where $\Phi_D : D \to \mathbb{H}$ with $\Phi_D(0) = 0, \Phi_D(\infty) = \infty, \Phi_D'(\infty) = 1$. Equivalently, we can define the measure $\mu_D(0,\infty)$ with total mass (partition function)

$$\Phi_D'(0)^b := \mathbf{E}\left[\exp\left\{\frac{\mathbf{c}}{2}\, m[\mathcal{L}_D(\gamma, D \setminus D')]\right\}\right],$$

  and this satisfies the generalized restriction property.

- Give the Radon–Nikodým derivative on the probability space on which the Brownian motion is defined. One standard way to construct "adapted" absolutely continuous measures to a Brownian motion is to give a drift,

$$dY_t = R_t\, dt + dB_t,$$

where R_t is measurable with respect to the process at time t. The process in D is SLE in all of $\mathbb{H}$ "weighted by" or "tilted" by the partition function. The precise meaning of this is given by the Girsanov theorem. Let M_t be the partition function of the remaining domain seen at time t; more precisely, M_t is a (local) martingale of the form

$$M_t = C_t \, \Phi'_{g_t(D)-U_t}(0)^b,$$

where C_t is a differentiable process that can be considered as a continuous normalization to be a probability measure. Then if we change the probability measure to weight by M_t, then

$$dB_t = R_t \, dt + d\tilde{B}_t, \qquad R_t = b \, [\log \Phi'_{g_t(D)-U_t}(0)]'$$

where $\tilde{B}_t$ is a standard Brownian motion in the new measure. An equivalent way to specify the process in D is to give the drift term R_t which is the logarithmic derivative of the partition function.

One of the main reasons that the conformal Markov property (domain Markov property and conformal invariance) determine SLE up to a single parameter, is the fact that a domain obtained from slitting a simply connected domain from the boundary is still simply connected and hence conformally equivalent to the original domain. This is not true for nonsimply connected domains and more general Riemann surfaces. Extending SLE to more general domains requires making more assumptions than just the conformal Markov property. One possibility is to use the Brownian loop measure and use the generalized restriction measure to define the measure $\mu_D(z, w)$ for other domains. This will satisfy the conformal Markov property but is not the unique process. Another idea is to find the partition functions $\Phi_D(z, w)$; if one does (perhaps as limits of some discrete model) and can prove sufficient smoothness, then one can define the process in terms of the logarithmic derivative. Finding the correct partition function for a scaling limit of a model is a big step to understanding the behavior.

6 Natural parametrization

The use of the Loewner differential equation to study SLE requires parametrization by some form of capacity, that is, by the size of the set seen by a Brownian motion starting away from the set. For example, if we consider SLE from 0 to infinity in the upper half plane $\mathbb{H}$ with corresponding maps g_t, then the parametrization is such that for $z \in \mathbb{H}$, $t \mapsto g_t(z)$ is continuously differentiable.

The scaling limit of discrete curves such as in (5) should also give a parametrization of the curves. It turns out that not only is this not the same measure, it and the capacity parametrization are *singular* with respect to each other. However, one can use the properties of SLE to find this new parametrization which is sometimes called *natural parametrization*. It is the fractal d-dimensional analogue of parametrization by arc length. Brownian paths have the natural 2-dimensional parametrization.

Given a path γ, the natural parametrization would be defined so that the "d-dimensional length" of $\gamma[0, t]$ is t. One well-known d-dimensional measure is Hausdorff

measure defined (at least up to a constant) by

$$\mathcal{H}^d(V) = \lim_{\epsilon \downarrow 0} \inf \sum_{j=1}^{\infty} [\operatorname{diam} U_j]^d,$$

where the infimum is over all covers of V by sets of diameter at most ϵ. The Hausdorff dimension of V is the unique d at which $\mathcal{H}^d(V)$ jumps from ∞ to 0; the value at d can be anything. For random sets of Hausdorff dimension d, typically we have $\mathcal{H}^d(V) = 0$. Roughly speaking, this is because we can take covers by sets of any size less than or equal to ϵ, and given a realization of the random set, the optimal cover takes advantage of this freedom. There are refinements of Hausdorff measure using gauge functions, and the optimal gauge function is well understood for some processes such as Brownian motion. However, for processes with very strong dependence on the immediate past such as SLE, determining a correct gauge correction is difficult and open.

To parameterize SLE paths it is more useful to take a naïve approach and try to cover by balls of radius ϵ; this is much closer to the approximation by a lattice since one has a fixed lattice size. A similar idea is the *d-dimensional Minkowski content* which for subsets V of $\mathbb{C} = \mathbb{R}^2$ is given by

$$\operatorname{Cont}_d(V) = \lim_{\epsilon \downarrow 0} \epsilon^{d-2} \operatorname{Area}\{z : \operatorname{dist}(z, V) \le \epsilon\}, \tag{8}$$

provided that this limit exists. Lawler and Rezaei [2015] were able to show that this limit exists and is nontrivial for $SLE_\kappa, \kappa < 8$ (for $\kappa \ge 8$ the curve is plane filling and the natural parametrization should be parametrization by area). In particular, the curve γ can be reparametrized such that for each s, $\operatorname{Cont}_d[\gamma[0,s]) = s$.

Proving the existence of this limit starts with hoping that it exists and seeing what this would imply. Consider SLE from z to w in a domain D and let $\zeta \in D$. The (chordal SLE) Green's function $G_D(\zeta; z, w)$ is the normalized probability that the SLE path goes through ζ, more precisely,

$$G_D(\zeta; z, w) = \lim_{\epsilon \downarrow 0} \epsilon^{d-2}\, \mathbf{P}\{\operatorname{dist}(\zeta, \gamma) \le \epsilon\}.$$

Establishment of the limit on the right-hand side is essentially the same as showing that for fixed $0 < \rho < 1$ as $\epsilon \to 0$,

$$\mathbf{P}\{\operatorname{dist}(\zeta, \gamma) \le \rho\epsilon \mid \operatorname{dist}(\zeta, \gamma) \le \epsilon\} \sim \rho^{2-d}.$$

This requires understanding the distribution of the tip of γ when it first gets within ϵ of ζ.

For simply connected D, it was noted in Rohde and Schramm [2005] that if such a function existed, then $M_t := G_{D \setminus \gamma_t}(\zeta; \gamma(t), w)$ would have to be a local martingale and an Itô's formula calculation gave $d = 1 + \frac{\kappa}{8}$ and

$$G_D(\zeta; z, w) = r_D(\zeta)^{d-2}\, S_D(\zeta; z, w)^{\frac{8}{\kappa}-1}, \tag{9}$$

where $r_D(\zeta)$ denotes conformal radius and $S_D(\zeta; z, w)$ denotes the sine of the (conformally invariant) argument of ζ with respect to z, w. Here $\gamma_t = \gamma[0,t]$. Having made

the observation, we can use the local martingale given by the left-hand side of (9) and the Girsanov theorem to understand the local behavior of the path as it gets near ζ. To establish the Minkowski content, one needs to improve this to a "two-point" estimate,

$$\mathbf{P}\{\mathrm{dist}(\zeta,\gamma) \le \rho\epsilon, \mathrm{dist}(\zeta',\gamma) \le \rho\epsilon \mid \mathrm{dist}(\zeta,\gamma) \le \epsilon, \mathrm{dist}(\zeta',\gamma) \le \epsilon\} \sim \rho^{2(2-d)}.$$

The natural parametrization satisfies a kind of Markovian property. Suppose D is a bounded domain, z, w are distinct boundary points, and $\gamma(t)$ is an SLE_κ path from z to w in D. Let $\Theta_t = \mathrm{Cont}_D(\gamma_t)$. Then

$$\mathbf{E}[\Theta_\infty] = \int_D G_D(\zeta; z, w)\, dA(\zeta).$$

$$\mathbf{E}\left[\Theta_\infty \mid \gamma_t\right] = \Theta_t + \Psi_t, \quad \Psi_t := \int_{D_t} G_{D_t}(\zeta; \gamma(t), w)\, dA(\zeta) \tag{10}$$

where $D_t = D \setminus \gamma_t$. Since $\mathbf{E}\left[\Theta_\infty \mid \gamma_t\right]$ is a martingale, we can characterize Θ_t as the unique increasing process such that $\Psi_t + \Theta_t$ is a martingale (Doob–Meyer decomposition). The first construction Lawler and Sheffield [2011] and Lawler and Zhou [2013] of the natural parametrization used this characterization and it is important in the proof of the discrete parametrization of LERW to the natural parametrization of SLE_2.

Another way of viewing a "d-dimensional" parametrization is in terms of the Hölder exponent. Under the natural parametrization, the SLE_κ curves are Hölder continuous for all $\alpha < 1/d$ Zhan [2017a].

7 Gaussian field

Maybe the most fundamental random field is the *Gaussian (free) field*, that is, variables $\{Z_x : x \in \mathcal{X}\}$ indexed by a set which can be finite, countable, or uncountable, such that each finite dimensional distribution is multivariate Gaussian. The distribution is determined by the means and the covariances and we say it is centered if the means are zero. A relationship between random paths and Gaussian fields has been known for a while, (see, e.g., Brydges, Fröhlich, and Spencer [1982], Dynkin [1984], and Symanzik [1969]) but what we describe here relating to loop measures is more recent due to Le Jan [2010, 2011] and Lupu [2016].

We started with a discrete-time, discrete-space loop measure and then described the Brownian loop measure which is continuous-time, continuous space. We will also consider continuous-time, discrete space. There are two ways to get a loop measure with continuous times on a discrete space. Le Jan's approach is to use a definition analogous to Brownian loop measure by having paths from a continuous time Markov chain. The other is to start with discrete time loops and then add waiting times. Both approaches have advantages; we will use the latter approach here. Suppose we have a finite set $\mathcal{X}$ and a real-valued symmetric function q on edges; for ease, we will assume $q(x,x) = 0$ although the definitions here can be adapted to allow for self-edges. Such a weight gives a measure on paths by multiplying the weight of the edges and hence also gives a weight

on loops. We will assume that this weight is actually a measure

$$\sum_{\omega} |q(\omega)| < \infty.$$

where the sum is over all finite length paths in A. In particular the Green's function can be written as

$$G(x,y) = [I-Q]^{-1}_{x,y} = \sum_{\omega: x \to y} q(\omega).$$

A particular case is when q are the transition probabilities for a subMarkov chain. There corresponds a weight on rooted loops $m(l) = m_q(l) = q(l)/|l|$ and the corresponding measure on unrooted loops. The centered Gaussian field $\{Z_x : x \in \mathcal{X}\}$ with covariance matrix G is the random vector whose Radon–Nikodým derivative with respect to independent, standard Gaussians is

$$\sqrt{\det(I-Q)}\, \exp\left\{\sum_e q_e\, Z_e\right\}$$

where the sum is over all undirected edges $e = \{x, y\}$ and $Z_e = Z_x Z_y$. If we consider the random field $\bar{T} = \{T_x = Z_x^2/2; x \in \mathcal{X}\}$, then the density of $\bar{T}$ can be written as

$$g(\bar{t})\, \sqrt{\det(I-Q)}\, \mathbf{E}\left[\exp\left\{\sum_e 2\, J_e\, q_e\, \sqrt{t_e}\right\}\right]$$

where $g(\bar{t})$ is the density for independent $\chi_1^2/2$ random variables; the sum is over all edges $e = \{x, y\}$, $t_e = t_x t_y$, $J_e = J_x J_y$; and $\{J_x\}$ are independent with $\mathbf{P}\{J_x = \pm 1\} = 1/2$.

To get a realization of $\bar{T}$ we can proceed as follows.

- Start with a realization of $\bar{T}$ for independent standard normals, that is, $\{t'_x\}$ independent with $\chi_1^2/2$ distributions.
- Take a realization of the discrete loop soup giving local times $\{n_x\}$.
- Replace each n_x with the sum of n_x independent exponentials with rate 1 and add this to $\{t'_x\}$ to get $\{t_x\}$.

There are several ways to verify it; in Lawler [2017], motivated by Lupu and Werner [2016], it was done in a way to also get the joint distribution with the distribution on currents, that is, functions $\bar{k}$ on edges with the property that each vertex has an an even number of edges coming out of it. We start with a realization of the loop soup from m with intensity $1/2$. If $q \geq 0$, this induces a probability distribution on currents $\bar{k}$. Given $\bar{k}$, we also get the local times n_x on vertices (n_x is $1/2$ times the number of edges in $\bar{k}$ that intersect x). A little work shows that the probability that the current $\{k_e : e \in \mathcal{E}\}$ with corresponding local times $\{n_x : x \in \mathcal{X}\}$ is chosen is

$$\sqrt{\det(I-Q)}\left[\prod_{x\in\mathcal{X}} \frac{\Gamma(n_x + \frac{1}{2})}{\sqrt{\pi}}\right]\left[\prod_{e\in\mathcal{E}} \frac{\theta_e^{k_e}}{k_e!}\right], \quad \theta_e = 2q_e.$$

This formula works for nonpositive integrable weights if we interpret the Poisson process as the measure on loops. With this the validity of the claim above is straightforward to verify.

The loop soup and some extra randomness give the square of the field Z_x^2, but that is insufficient for determining the signs. For positive weights, the signs of the field can be chosen so that the clusters formed by the loop soup are all of the same sign. However, some more randomness is needed. Suppose $V \subset \mathcal{X}$ and let G_V, G_{V^c} be the corresponding Green's functions for the random walks restricted to those sets. Then the density of the random field on $\mathcal{X}$ with respect to independent fields on V and $\mathcal{X} \setminus V$ with covariance G_V, G_{V^c} is given by

$$\exp\left\{-\frac{1}{2}\sum_{\ell\in\mathcal{L}^*} m(\ell)\right\} \exp\left\{\sum_{e\in\mathcal{E}^*} q_e\, Z_e\right\},$$

where $\mathcal{E}^*$ denote the set of edges with one vertex in V and one in V^c, and $\mathcal{L}^*$ denotes the set of loops that intersect both V and V^c. This gives a way, first proposed by Lupu, to put the signs on the field.

- Take a Poissonian realization of the loop measure giving the local times n_x and then choose continuous local times giving t_x.
- Open each edge e that has been traversed by a loop in the soup.
- Independently, open each edge with probability $1 - \exp\{-2q_e\sqrt{t_e}\}$.
- Give each Z_x in a connected cluster the same sign using independent fair coins in each cluster.

This formulation has started with a measure on discrete time loops and added some waiting times. The original construction started with a continuous time loop measure that can be derived from the discrete time measure by adding exponential waiting times of mean one which is the same as $\chi_2^2/2$. In this formulation, one also needs to have some "trivial loops" that do not move and whose time duration have a $\chi_1^2/2$ distribution. Another nice variation due to Lupu [2016] only uses a loop soup on a slightly different graph (metric graph or cable system) and some random coin flips. In this case the connected clusters for the loop soup are exactly the clusters for which one choose random coin flips.

We note that when n_x is large, it makes little difference whether we use n_x or t_x. Indeed,

$$t_x = n_x + Y_x + O(1),$$

where Y_x is an independent, mean zero, random variable with variance n_x and

$$\sqrt{2t_x} = \sqrt{2n_x} + \frac{Y_x}{\sqrt{2n_x}} + O(n_x^{-1/2}),$$

where $\{Y_x\}$ are conditionally independent given $\{n_x\}$ with mean zero and variance n_x.

8 Continuous Gaussian field and quantum gravity

As fundamental a conformal invariant as Brownian motion in $\mathbb{C}$ is the (continuous) Gaussian free field. One way that it can be obtained is as a limit of the Gaussian field from the previous section. Let D be a simply connected bounded subdomain of $\mathbb{C}$ containing the origin. Let g_D denote the Green's function for the Laplacian: if $\mathbb{D}$ is the unit disk and $f : D \to \mathbb{D}$ is a conformal transformation with $f(z) = 0$, then $g_D(z, w) = -\log|f(w)|$. Let N be a large integer; in the spirit of nonstandard analysis Nelson [1987] we can consider $N \simeq \infty$. We will use the notation from Section 3 with lattice spacing $1/N$ and corresponding sets $D_N \subset D$ and $A_N \subset \mathbb{Z}^2 \cap ND$.

Let $G_N = G_{A_N}$ denote the Green's function for the usual random walk killed upon leaving A_N, and let g_N denote the function $g_N(z, w) = (\pi/2)\, G_N(zN, wN)$. For standard $z, w \in D$ with $z \neq w$,

$$g_N(z, w) \simeq g_D(z, w)$$

(precise error estimates can be given, see, e.g., Kozdron and Lawler [2005], but we will not discuss them). The Gaussian field on A_N, $\{Z_z : z \in A_N\}$ can be viewed as a piecewise constant function $\phi_N(z) = \sqrt{\pi/2}\, Z_{wN}$ for $z \in \mathcal{S}_w$ (we do not need to worry about the values on the boundaries of the squares $\mathcal{S}_w$). In the terminology of nonstandard analysis, the Gaussian free field ϕ on D with Dirichlet boundary conditions is the "standard part" of ϕ_N.

The macroscopic object ϕ is a little tricky because it is not defined pointwise. For example, $\phi_N(0)$ is a normal random variable of variance (up to an infinitesimal)

$$g_N(0, 0) = \log N - \log|f'(0)| + c_0$$

and hence the standard part is not defined. One way to get well-defined quantities is to take averages. For example, if f is a standard L^2 function on D, then we can define $\phi(f)$ to be the standard part of

$$\int_{D_N} f(z)\, \phi_N(z)\, dA(z) = \sqrt{\frac{\pi}{2}}\, N^{-2} \sum_{z \in A_N} f(z/N)\, Z_z,$$

which is a centered normal random variable with variance

$$\int_D \int_D f(z)\, f(w)\, g_D(z, w)\, dA(z)\, dA(w).$$

This observation gives one way to construct the continuous field rigorously — as the centered Gaussian process indexed by functions f with

$$\mathbf{E}\,[\phi_f\, \phi_h] = \int_D \int_D f(z)\, h(w)\, g_D(z, w)\, dA(z)\, dA(w).$$

We will retain the discrete picture to consider what is sometimes called Liouville quantum gravity. This is a fancy term for the exponential $e^{\gamma\phi}$ where γ is a constant that we will choose to be nonnegative. To make sense of this, consider the field $\psi_N(z) =$

$N^{-\gamma^2/2}\exp\{\gamma\phi_N(z)\}$. We choose this normalization so that the expected value is of order one,

$$\mathbf{E}\left[\psi_N(z)\right] = N^{-\gamma^2/2}\,\mathbf{E}\left[\exp\{\gamma\phi_N(z)\}\right] = N^{-\gamma^2/2}\exp\left\{\frac{\gamma^2\,\mathrm{Var}(\phi_N)}{2}\right\} \asymp 1.$$

We let μ_N be the random measure on D_N whose Radon–Nikodým derivative with respect to area is ψ_N, that is, $\mu_N[\mathcal{S}_z] = N^{-2}\,\psi_n(z)$. Let μ be the standard part of this measure. It is not so clear whether this make sense. By construction we can see that μ_N is a random measure on D such that $\mathbf{E}[\mu_N(D)] \asymp 1$. However, it does not follow from this calculation that the "typical" value of $\mu_N(D)$ is of order 1; it is possible that the typical value is infinitesimal. Whether or not this is true depends on the value γ.

We can do a "back of the envelope" calculation to find the critical value. Suppose Φ is a normal random variable with mean zero, variance $\sigma^2 = \log N + O(1)$, and $Y = e^{\gamma\Phi}$. If we tilt the distribution by Y, that is consider the random variable Φ under the measure tilted by $e^{-\gamma^2/2}\,e^{\gamma\Phi}$, the induced distribution on Φ is that of a normal random variable with mean $\gamma\sigma^2$ and variance σ^2. The original probability of getting a value as large as $\gamma\sigma^2$ is of order $e^{-\gamma\sigma^2/2}$. That is, the typical value of $\psi_N(z)$ in the tilted measure is of order $N^{\gamma^2/2}$ and the probability (in the original measure) of such of value is of order $N^{-\gamma^2/2}$. Since there are of order N^2 points, we see that critical value is $\gamma = 2$; if $\gamma < 2$, then we would expect that the measure μ would be supported on a set of $N^{2-\frac{\gamma^2}{2}}$ squares, that is, on a set of "fractal dimension" $2 - \frac{\gamma^2}{2}$.

We will now use Liouville quantum gravity to reparametrize a curve. Let us consider the loop-erased random walk which has dimension $d = 5/4$. Then we can reparametrize the curve as in (5) and get a curve whose macroscopic time duration is finite and positive. The number of points visited by a typical path is comparable to N^d and the amount of time it spends on each of these points is $N^{-d} = [\mathrm{area}(\mathcal{S}_z)]^{-d/2}$. Suppose an independent realization of the Liouville quantum gravity is given. Then we can also reparametrize our curve so that the amount of time spent on square $\mathcal{S}_z$ is $[\sqrt{\mu_N(\mathcal{S}_z)}]^{-\alpha}$. Here we can view α as the "quantum fractal dimension" of the path chosen so that

$$\sum_z [\sqrt{\mu_N(\mathcal{S}_z)}]^{-\alpha} \asymp 1.$$

If a random set of dimension d is chosen independently of the Gaussian field, then the expected value of the left-hand side is comparable to

$$N^d\,\mathbf{E}\left[(\mu_N(\mathcal{S}_z))^{-\alpha/2}\right]$$

where z is a typical interior point for which we see that

$$\mathbf{E}\left[(\mu_N(\mathcal{S}_z))^{-\alpha/2}\right] = N^{-\alpha(1+\frac{\gamma^2}{4})}\,\mathbf{E}\left[\exp\{\alpha\gamma Z_{Nz}/2\}\right] \asymp N^{-\alpha(1+\frac{\gamma^2}{4})+\frac{\alpha^2\gamma^2}{8}}.$$

This gives the KPZ relation

$$d = \alpha\left(1+\frac{\gamma^2}{4}\right) - \frac{\alpha^2\gamma^2}{8}, \tag{11}$$

which is often written in terms of the scaling exponents x, Δ defined by $d = 2-2x, \alpha = 2 - 2\Delta$,

$$x = \left(1 - \frac{\gamma^2}{4}\right) \Delta + \frac{\gamma^2}{4} \Delta^2.$$

As in the case of the loop measure, for each $\kappa \le 4$, there is a corresponding value of γ. In this case γ is chosen so that the quantum fractal dimension α_0 of the SLE_κ path is 1. Using (11) we can see which γ to choose for each κ.

- If $\gamma^2 = \kappa$, then the quantum fractal dimension of an independent set of Euclidean fractal dimension $1 + \frac{\kappa}{8}$ is one.

In the case of the loop-erased random walk, we choose $\gamma = \sqrt{2}$, and then we have a one-dimensional parametrization of the d-dimensional curve. For $\kappa' > 4$, a similar association is appropriate; indeed, the outer boundary of $SLE_{\kappa'}$ curves are locally like SLE_κ curves with $\kappa\,\kappa' = 16$. These values of κ, κ' share the same central charge.

One of the most exciting recent developments in conformally invariant systems has been the work of Scott Sheffield, Jason Miller, Bertrand Duplantier, and others in understanding the random geometry and surfaces produced by taking independent realizations of the Gaussian free field (and hence of the quantum gravity) and realizations of SLE_κ or $SLE_{\kappa'}$ curves and loops. I am not going to try to explain this work for two reasons: it would take too much space to give even a reasonable description and I do not feel I have sufficient expertise to do it justice. I suggest the paper Duplantier [2014b] whose abstract starts with the inviting sentence "There is a simple way to "glue together" a coupled pair of continuum random trees (CRTs) to produce a topological sphere", but then is followed by a very technical paper of over 200 pages! Another major breakthrough by Miller and Sheffield [2015] is making rigorous the relation between the $\gamma^2 = 8/3$ ($\mathbf{c} = 0$) case and combinatorial models for random graphs and the Brownian map Le Gall [2014].

9 Random simple loops

A rooted self-avoiding loop (rSAL) is a path $[l_0, l_1, \dots, l_{2n}]$ with $l_0 = l_{2n}$ and all other vertices distinct. We will call ℓ an (unrooted) self-avoiding loop (SAL) if it is an equivalence class of rooted self-avoiding loops as before. For self-avoiding loops, there are exactly $2n$ rooted loops associated to an SAL. We have retained the orientation of the loop. A self-avoiding polygon (SAP) is an equivalence class of SAL where we ignore the orientation; to each SAP of length $2n > 2$, there are 2 SALs and $4n$ rSALs.

When studying SALs or SAPs in D_N, we can either consider loops in the (scaled) lattice or the dual lattice. We note that SAPs on the dual lattice are in one-to-one correspondence with finite simply connected subsets of $\mathbb{Z}^2$ where the correspondence is given by the boundary. For finite (not necessarily simply) connected subsets we can fill in the bounded components of the complement (giving the hull of the set) and then take the outer boundary. Of course, this is not a bijection since the outer boundary of a set is the same as the outer boundary of its hull. We will be studying measures on SAPS or SALs with an emphasis on the macroscopic (that is, noninfinitesimal) diameter. Some

of these measures will be infinite because they give large measure to small loops, but the measure on large loops is bounded.

We start by considering a simple to define measure on loops using the random walk measure similar to one in Kozdron and Lawler [2007]. We will define it as a measure on SALs, but one could equally consider it as a measure on SAPs (being careful of factors of 2 since the relation between SALs and SAPs is 2-to-1). There are two variants of the measure, depending on whether the loops lie on the lattice or the dual lattice. In either case we will be considering the random walk loop measure on the original lattice. Recall that if η is a loop in the lattice, then

$$F_\eta(A) = \exp\left\{ - \sum_{\ell \subset A, \ell \cap \eta \neq \varnothing} m(\ell) \right\}.$$

Here $\ell \cap \eta \neq \varnothing$ means that the loops share a vertex. If η is a loop in the dual lattice, we define $F_\eta(A)$ in the same way but in this case $\ell \subset A$ means that the edges of ℓ are parts of boundaries of squares centered at $z \in A$, and $\ell \cap \eta \neq \varnothing$ means that ℓ includes a vertex adjacent to η. Our simple candidate for a measure is to give each η measure

$$m_A(\eta) = e^{-\beta|\eta|}\, F_\eta(A)^{-\mathbf{c}/2}, \tag{12}$$

where $\beta = \beta_c$ is a critical value and $\mathbf{c}$ denotes the central charge. Part of the conjecture is a form of *hyperscaling*, which can be stated roughly that at the critical value of β, the total measure of loops of diameter at least 1 contained in D is of order 1. The conjecture is that many of the interesting measures on loops are absolutely continuous with respect to this measure but that there may be domain corrections that will depend on the particular model studied.

One way to compensate, which will turn out to be natural at least in the case $\mathbf{c} = -2$, is to include an extra term

$$\hat{m}_A(\eta) = e^{-\beta|\eta|}\, \left[H_A(\eta, \partial A)\, F_\eta(A)\right]^{-\mathbf{c}/2},$$

where $H_A(\eta, \partial A)$ denotes an "excursion measure" term,

$$H_A(\eta, \partial A) = \sum_{x \in \eta} \mathrm{Es}_{\eta,A}(x) = \frac{1}{4} \sum_{x \in \eta} \sum_{|y-x|=1} [1 - g(y)].$$

Here $g(x) = g_{\eta,A}(x)$ is the probability that a simple random walk starting at x reaches η before A (so that $g \equiv 1$ on η), and $\mathrm{Es}_{\eta_A}(x) = -\Delta g$ is the probability that a simple random walk starting at η reaches ∂A before returning to η. If the scaled walk η is of diameter 1 and is not too close to the boundary, then $H_A(\eta, \partial A) \asymp 1$. Indeed, $(2/\pi)\, H_A(\eta, \partial A) \sim r^{-1}$ where r is chosen so that annular region between η and ∂D_A is conformally equivalent to $\{1 < |x| < e^r\}$. In particular, the continuum limit is a conformal invariant at least for transformations of the annular region. In this case one can show similarly to Field and Lawler [2013] that the limit

$$\lim_{A \uparrow \mathbb{Z}^2} H_A(\eta, \partial A)\, F_\eta(A)$$

exists and is nontrivial.

9.1 $\mathbf{c} = 0$: Self-avoiding polygons. The case $\mathbf{c} = 0$ where $\hat{m}_A(\eta)$ depends only on $|\eta|$ is a version of one of the big open questions in the intersection of probability, combinatorics, and statistical physics. The value e^{β} is called the connective constant and its value is not known (although it is known on the honeycomb lattice Duminil-Copin and Smirnov [2012]). However, its continuum limit is perhaps the easiest to construct because it satisfies the restriction property: the value $\hat{m}_A(\eta)$ does not change if A changes, provided that $\eta \subset A$.

A very similar measure can be constructed from the random walk loop measure. To each unrooted loop we can associate its outer boundary. To be more precise, the set of vertices visited by an unrooted loop is a connected set and this set can become a simply connected A by filling in the finite holes. The outer boundary is the simple loop in the dual lattice given by ∂D_A. Mandelbrot [1982] made the remarkable heuristic observation that the outer boundary of these loops looked like self-avoiding walks. The random walk loop measure therefore generates a measure on SAPs on the dual lattice (one could also specify or choose a random orientation to get a measure on SALs). For the continuous limit, Brownian motion, this was proved, first in Lawler, Schramm, and Werner [2003] where it was shown that locally the paths are the same as $SLE_{8/3}$ paths. A direct construction of $SLE_{8/3}$ loops without topological constraints on a domain was done by Werner [2008].

9.2 $\mathbf{c} = -2$: Loop-erased loops. We will call a SAW ξ a *near-SAL* if ξ has an odd number of steps and ends distance 1 from the starting position. In other words, ξ can be turned into a SAL by adding the edge connecting the initial and terminal vertices. For each SAL η with $2n$ steps, there exist $2n$ near-SALs ξ (each with $2n-1$ steps) that produce η. For each $\xi = [x = \xi_0, \xi_1, \dots, y = \xi_k]$ in A the quantity $4^{-k} F_\xi(A)$ represents the expected number of times that one views ξ if one starts a random walk at x, erases loops as they appear chronologically, and stops the walk when it leaves A. Equivalently,

$$4^{-k} F_\xi(A) = \sum_{\omega : x \to y, LE(\omega) = \eta} 4^{-|\omega|},$$

where the sum is over all ordinary (not necessarily self-avoiding) random walks in A from x to y whose loop-erasure is ξ. In analogy with the case of the loop measure, if we give each near-SAL measure

$$\frac{1}{4(|\xi|+1)} 4^{-k} F_\xi(A), \tag{13}$$

then the induced measure on SALs is m_A.

Using Beneš, Lawler, and Viklund [2016], one can see that the expected number of times that the loop-erasing process starting at x (not too close to the boundary) produces a near-SAL with diameter greater than 1 is comparable to $N^{-3/4}$, and the typical number of steps of such a near-SAL is of order $N^{5/4}$. Hence the total mass of the measure in (13) for near-SALs rooted at x of diameter greater than one is comparable to N^{-2}. Summing over the $O(N^2)$ points, see that the measure μ_A of macroscopic loops is comparable to one. For macroscopic loops that are not too close to ∂A we also get that $H_{\partial A}(\eta, A)$ is comparable to one.

The measure $\hat{m}$ arises naturally in the study of uniform spanning trees. If A is a finite subset of $\mathbb{Z}^2$ with n elements, then a *wired spanning tree* is a spanning tree of the graph of $n+1$ vertices obtained by identifying all the boundary points as a single vertex we can call ∂A. Using Wilson's algorithm with ∂A as the root, we can see that the number of wired spanning trees is $4^n \det[I - Q_A] = 4^n/F(A)$, where Q_A is the matrix indexed by A with $Q_A(x,y) = 1/4$ if x, y are nearest neighbors and equals zero otherwise.

As an extension if the boundary is partitioned into two sets ∂_1 and ∂_2 and we wire ∂_1, ∂_2 separately, giving a graph of $n+2$ vertices, then the number of spanning trees is

$$\frac{4^{n+1}}{F(A)\, H_{\partial A}(\partial_1, \partial_2)},$$

where

$$H_{\partial A}(\partial_1, \partial_2) = \sum_{x \in \partial_1} \mathrm{Es}_{\partial_1}(x) = \sum_{x \in \partial_2} \mathrm{Es}_{\partial_2}(x)$$

and $\mathrm{Es}_{\partial_j}(x)$ is the probability that a simple random walk starting at x reaches ∂_{3-j} before returning to ∂_j. Indeed, this is what is output from Wilson's algorithm if one makes ∂_1 the root and ∂_2 the initial vertex from which loop-erased random walks are chosen. In the case of an annular region with ∂_1, ∂_2 being the components of the boundary we call it a *(wired) crossing spanning tree*.

If η is an SAL of length n surrounding the origin, we say that a spanning tree includes η if all but one of the edges of η are included in the tree (it is impossible for all the edges of η to be included). We claim that the probability that a uniform spanning tree contains η is $\hat{m}(\eta)$. Indeed, if we partition the vertices on A into η, A_η^-, A_η^+ where A_η^- is the connected component of $A \setminus \eta$ containing the origin, then we can choose a tree $\mathcal{T}$ including η as follows:

- Choose any crossing spanning tree $\mathcal{T}^+$ of A_η^+ from η to ∂A in A_η and add those edges to $\mathcal{T}$.
- Choose any wired spanning tree $\mathcal{T}^-$ in A_η^-, and add those edges to $\mathcal{T}$.

Given $(\mathcal{T}^+, \mathcal{T}^-)$, $\mathcal{T}$ is determined as follows.

- There is a unique η_j such that $\mathcal{T}^+$ contains a path from η_j to ∂A. Add all the edges of η to $\mathcal{T}$ except for (η_j, η_{j+1}).

The number of choices for $\mathcal{T}^+$ is $4^{\#(A_\eta^+)+1}\,[F(A_\eta^+)\; H_{\partial A}(\eta, \partial A)]^{-1}$, and the number of choices for $\mathcal{T}^-$ is $4^{\#(A_\eta^-)}\, F(A_\eta^-)^{-1}$. Since A_η^+ and A_η^- are separate connected components and hence no loop in one intersects the other, we have

$$F_\eta(A) = \frac{F(A)}{F(A_\eta^+)F(A_\eta^+)},$$

which gives our claim. An alternative approach would be to use Wilson's algorithm to find the probability that the path from η^{j+1} to ∂A includes the path η with (η_j, η_{j+1}) removed. See Kassel and Kenyon [2017] for another approach to these ideas.

9.3 $0 < \mathbf{c} \le 1$: Conformal loop ensembles. Another measure on loops can be obtained from the random walk loop soup with different intensities. The construction of the Gaussian field used the soup with intensity $1/2$ and we will generalize this to intensities $\mathbf{c}/2$ for $0 < \mathbf{c} \le 1$. Given a realization of the loop soup, there are connected clusters of points. If $\mathbf{c} \le 1/2$, these clusters will be finite. Indeed, if the clusters were not finite for $\mathbf{c} = 1/2$, then the construction of the Gaussian field as described in Section 7 would output a field with all values of the same sign. For each cluster, we can consider the outer boundary (the boundary of the unbounded component of the complement) as a SAP in the dual lattice. For each simply connected A let us write μ_A for this measure on SAPs (on the dual lattice) in A.

In order for η to be the outer boundary of a cluster two things must be true:

- The loop η is not hit by the loop soup.
- All the points "inside" η that are adjacent to η must be in the same connected component.

We call such an η an outermost loop if it also satisfies:

- There is no η' in the annular region between η and ∂D that satisfies the first two conditions and disconnects η from ∂D.

While this measure does not have the exact form (12), we will do some heuristics to see that it is similar. First, if $A \subset A'$, we note that

$$\frac{\mu_{A'}(\eta)}{\mu_A(\eta)} = \frac{m_{A'}(\eta)}{m_A(\eta)} = \left[\frac{F_\eta(A)}{F_\eta(A')}\right]^{\mathbf{c}/2}.$$

where the right-hand side is the probability that the loop soup in A' contains a loop that intersects both η and $A' \setminus A$. Of course an outermost loop in A may no longer be an outermost loop in A'.

The continuous analogue of this construction (as well as a different construction that we will not describe here) was carried out by Sheffield and Werner [2012] focusing on the outermost loops. They used the following property to characterize the measure on outermost loops. Suppose $A \subset A'$ are simply connected and we observe the outermost loops that intersect $A' \setminus A$. Let V be A with the points surrounded by these loops removed. Then the conditional distribution on the remainder of the outermost loops is that of the outermost loops of (the connected components) of V. The exact lattice construction we mention may be unsolved, but there is a closely related construction van de Brug, Camia, and Lis [2016] that focuses only on large (macroscopic and some mesoscopic) loops in the random walk clusters and then shows that the macroscopic clusters are the same as those from Brownian clusters. It is in this regime that the coupling Lawler and Trujillo Ferreras [2007] between the random walk and Brownian loop soups works and hence they can reduce the problem to the Sheffield–Werner construction. One would expect that the exact nature of microscopic loops should not play a big factor in the scaling limit but this is still open.

Another way to get a measure on loops is to observe a field and to consider the loops that separate values of different signs. One case where this has been done is the Gaussian

field. It is useful to consider an equivalent definition of the free field, this time with non-zero boundary conditions, as having the density with respect to normalized Lebesgue measure $\prod_{x\in A}(dz_x/\sqrt{2\pi})$ of

$$\sqrt{\det(I-Q)}\,\exp\left\{-\frac{1}{2}\,\mathcal{E}(\bar{z})\right\},\qquad \mathcal{E}(\bar{z})=\frac{1}{4}\sum_{e}[z_x-z_y]^2,$$

where in this case the sum is over all edges $e=\{x,y\}$ with at least one vertex in A; and $z_x=0$ for $x\in\partial A$ (other boundary conditions can be given).

Suppose a SAP η in the dual lattice is give, and let V_η^+, V_η^- denote the adjacent vertices to η that are outside and inside η respectively. We will consider the event that $z_x<c$ for $x\in V_\eta^+$ and $z_y>c$ for $y\in V_\eta^-$. We first consider the exponential term for edges that cross η. This gives a distribution on $z_x,\in V_\eta:=V_\eta^+\cup V_\eta^-$ up to an additive constant that we then fix so that average of z_x (as seen from far away) in V_η^+ is 0. We let λ be the average value in V_η^- as seen, say, from a point on the inside. (The boundary value will have local microscopic fluctuations but look constant from a macroscopic distance away.)

Given $\{z_x : x\in V\}$, we choose the rest of the Gaussian free field on the remaining points A^+, A^- to be independent fields with zero boundary condition plus a mean given by the harmonic extension of the boundary values. Away from η, this harmonic extension looks like 0 on A^+ and λ on A^-. The energy contribution given by the harmonic extension is local near η and should give a term linear in the length of η. So, roughly speaking, the probability of getting the curve η should look like

$$\frac{e^{-\beta|\eta|}\sqrt{F(A^+)}\sqrt{F(A^-)}}{\sqrt{F(A)}}=e^{-\beta|\eta|}\,F_\eta(A)^{-1/2},$$

for some β. There is a lot of hand waving here, but we can see how a form like (12) arises.

To make arguments like this rigorous in the continuum, one can reverse the operation Schramm and Sheffield [2013] and Wang and Wu [2017] One starts with a measure on loops, one finds a critical value of λ, and then given the loop one constructs independent Gaussian fields on the outside (with boundary value 0) and the inside (with boundary value λ). Then one shows that this construction combines to give a Gaussian field in the large domain. In some since the curve is a level curve for the final Gaussian field (and we can view it as a "function" of that field).

The idea of starting with a Gaussian field and defining curves and loops as a function of the field was proposed in Dubédat [2009] and has been developed by many under the name "imaginary geometry" to get results about SLE and loops, see, e.g., Miller and Sheffield [2016a] A similar result for loops in the Ising model can be found in Benoist and Hongler [2016].

10 SLE_κ loops

There is a direct way to define SLE_κ loops that work for all $\kappa<8$ that is analogous to the definition for the Brownian loop measure. This defines a measure on loops in the

entire plane that is invariant under dilations and rotations, but leaves open the question how to modify the measure for a bounded domain.

Using the Loewner equation with a driving function of a killed process in a quasi-invariant distribution, one can define a σ-finite measure on loops rooted at a particular point. We write $\gamma_t = \gamma[0,t]$ and D_t for the unbounded component of $\mathbb{C} \setminus \gamma_t$. For the moment, we parametrize the curves by capacity in the upper half plane: if $F : \mathbb{C} \setminus \mathbb{D} \to D_t$ is a conformal transformation fixing ∞, then as $z \to \infty$, $|F'(z)| \sim e^t\, |z|$.

- The set of loops with total capacity greater than t is $c\, e^{-t(d-2)}$ for some fixed constant c.
- Conditioned on the total capacity of the loop being greater than t, the conditional distribution of $\gamma(s), s \geq t$ given γ_t is that of chordal SLE_κ from $\gamma(t)$ to $\gamma(0)$ in D_t.

If $f_r(z) = rz$ denotes dilation by r then the measure μ_0 satisfies $f_r \circ \mu_0 = r^{2-d}\, \mu_0$. We can also consider this as a measure on curves with the natural parametrization. Let $T_\gamma = \mathrm{Cont}_d(\gamma)$. Then (by choosing c appropriately) we get a measure on naturally parametrized loops with (recall that a loop of capacity t typically has content of order t^d)

- The set of loops with $T_\gamma \geq T$ has measure $T^{1-\frac{d}{2}}$.

As in the case of Brownian loops, we will try to integrate the rooted loop measure over the starting points to give a measure on unrooted loops. As before, this leads to overcounting so we compensate by considering the measure ν_z given by

$$\frac{d\nu_z}{d\mu_z} = \frac{1}{T_\gamma}, \qquad \nu_z = \int_{\mathbb{C}} \nu_z\, dA(z).$$

Again, we think of this as a measure on unrooted loops. Even for the measure on rooted loops, we get the scaling relation $f_r \circ \nu = \nu$.

Laurie Field and I were studying this and had gotten this far; indeed, one of the motivations for understanding natural parametrization was to try a construction like this in order to give a measure on loops of the type suggested by Kontsevich and Suhov [2007]. However, there was a technical question that we were unable to answer that was necessary to continue this program. There is a property of Brownian loops that it almost "obvious" and is used in the proof of the conformal invariance: if $\gamma(t), 0 \leq t \leq 1$ has the distribution of a Brownian bridge and $0 < s < 1$, then the distribution of $\tilde{\gamma}(t) := \gamma(t + s) - \gamma(s)$ is also that of a Brownian bridge. (Here addition is modulo 1.) The analogue for the SLE_κ loop measure is that the measure conditioned on fixed Minkowski content has the same property. This has recently been proved by Zhan [2017b].

The conformal invariance here is only for the dilations. There is still the hard question about how to restrict the measure to bounded domains. This does not arise for the Brownian loop measure because it satisfies the restriction property. There is not a unique possibility, and the exact version should depend on the particular problem being analyzed.

11 Scaling limit of loop-erased walk

Suppose z, w are distinct boundary points on D. We will consider two processes:

- Chordal SLE_2 γ from z' to w' in D.
- Take the discrete approximation D_N and corresponding boundary points $z, w \in D_N$, and let η be a (scaled) LERW from z to w

These processes are close and, in this section we discuss recent results showing that the "naturally parametrized" curves are close. At the moment this is the only process for which this strong convergence is known.

To establish the result, we start by proving a result about the LERW that can be considered a "local limit theorem". We compare the probability that the LERW goes through the origin with the probability that a chordal SLE_2 path goes through the square $\mathcal{S}_0$ of side length N^{-2}. Let $r_D = r_D(0)$ and $S_D = S_D(0; z, w)$ be the parameters as in Section 6. Using stochastic calculus techniques one can show that there exist c_0, u (independent of D, z, w) such that

$$\mathbf{P}\{\gamma \cap \mathcal{S} \neq \varnothing\} = c_0\, N^{-3/4}\, r_D^{-3/4}\, S_D^3\, [1 + O(N^{-u})].$$

(This was established for a disk rather than a square in Lawler and Rezaei [2015], but the argument can be adapted for a square. The constant c_0, which is different for squares and disks, is not known explicitly.)

We will describe work in Beneš, Lawler, and Viklund [2016] and Lawler [2014] that established the analogous result for LERW: there exists an absolute c_1 such that for all domains

$$\mathbf{P}\{0 \in \eta\} = c_1\, N^{-3/4}\, r_D^{-3/4}\, S_D^3\, [1 + O(N^{-u})].$$

Note that this not only gives the correct scaling exponent (which had been established by Kenyon [2000]) but also the dependence of the constant factor in the asymptotics to the domain, establishing that it is a conformally covariant quantity. The proof combines a key ingredient of Kenyon's proof with the machinery of loop measures, this time with measures that can take negative values.

Let A be a finite, simply connected subset of $\mathbb{Z}^2$ containing the origin, and let D_A be the corresponding "union of squares" domain. We will not scale D_A, so if $f : \mathbb{D} \to A$ is a conformal transformation with $f(0) = 0$, then $r_A := |f_A'(0)| \asymp \operatorname{dist}(0, \partial A)$. if $z_1, z_2 \in \partial A$, we can also find the angle S_A from this map. We will compare three numbers: $H_A(z, w)$, the measure of usual random walks from z to w in A; $\hat{H}_A(z, w; \vec{01})$ the measure of such walks whose loop erasure uses the directed edge $\vec{01}$; and $\hat{H}_A(z, w; \vec{10})$, the measure of such walks that use the edge $\vec{10}$. The probability that the undirected edge $\{0, 1\}$ is used is then

$$\frac{\hat{H}_A(z, w; \vec{01}) + \hat{H}_A(z, w; \vec{10})}{H_A(z, w)}.$$

Using a determinantal formula first given by Fomin [2001], one can give an exact expression for the difference,

$$\hat{H}_A(z, w; \vec{01}) - \hat{H}_A(z, w; \vec{10}) = \frac{1}{4}\, F_{01}(A)\, [H_{A'}(z, 0)\, H_{A'}(w, 1) - H_{A'}(z, 1)\, H_{A'}(w, 0)]$$

where $A' = A \setminus \{0,1\}$. Unfortunately, this is a formula for a difference rather than a sum on the left-hand side. Kenyon's trick is to change some of the weights to negative; more precisely, we can draw a vertical half line ("zipper") on the edges of the dual graph $\{\frac{1}{2} + iy : -y_0 < y < 0\}$ where $\frac{1}{2} - iy_0$ is the first point on ∂D_A reached. Then for each edge of A that crosses the zipper we give weight $-1/4$ rather than $1/4$. This new assignment of edge weights gives a new measure on paths, and hence loops, that we will denote as q. Fomin's identity is a combinatorial bijection that works with any weights on the bonds; in particular,

$$\hat{H}^q_A(z,w;\vec{01}) - \hat{H}^q_A(z,w;\vec{10}) = \frac{1}{4}\, F^q_{01}(A)\, \left[H^q_{A'}(z,0)\, H^q_{A'}(w,1) - H^q_{A'}(z,1)\, H^q_{A'}(w,0)\right]$$

where we use the superscript q to mean quantities computed with that measure.

We use expressions as before

$$\hat{H}_A(z,w;\vec{01}) = \sum_\eta 4^{-|\eta|}\, F_\eta(A), \qquad \hat{H}^q_A(z,w;\vec{01}) = \sum_\eta (-1)^{J(\eta)}\, 4^{-|\eta|}\, F^q_\eta(A),$$

where the sum is over all SAWs η from z to w using the directed edge $\vec{01}$. Here $J(\eta)$ is the number of times that η crosses the zipper. We now use simple connectivity of the domain and some simple topology to observe two facts:

- If z, w are ordered correctly, every SAW from z, w that uses the directed edge $\vec{01}$ crosses the zipper an even number of times while SAWs that use $\vec{10}$ cross an odd number of times.
- Any loop that crosses the zipper an odd number of times must intersect every η using $\vec{01}$ or $\vec{10}$.

This gives

$$\hat{H}^q_A(z,w;\vec{01}) - \hat{H}^q_A(z,w;\vec{10}) = \exp\{-2m(O_A)\}\, \left[\hat{H}_A(z,w;\vec{01}) + \hat{H}_A(z,w;\vec{10})\right],$$

where O_A denotes the set of loops that intersect the zipper an odd number of times. This gives an exact expression for the quantity we want in terms of random walk quantities (including some for the signed measure q):

$$\frac{1}{4}\, F^q_{01}(A)\, e^{2m(O_A)} \left[\frac{H^q_{A'}(z,0)}{H_A(0,z)}\, \frac{H^q_{A'}(w,1)}{H_A(0,w)} - \frac{H^q_{A'}(z,1)}{H_A(0,z)}\, \frac{H^q_{A'}(w,0)}{H_A(0,w)}\right] \left[\frac{H_A(z,w)}{H_A(0,z)\, H_A(0,w)}\right]^{-1}$$

There is a lot of machinery to handle random walk convergence to Brownian motion and in two dimensions one can often get good estimates uniform over all boundary conditions. There is work involved for sure, but we show that

$$\frac{H^q_{A'}(z,0)}{H_A(0,z)}\, \frac{H^q_{A'}(w,1)}{H_A(0,w)} - \frac{H^q_{A'}(z,1)}{H_A(0,z)}\, \frac{H^q_{A'}(w,0)}{H_A(0,w)} = c_1\, r_A^{-1}\, [S_A + O(r_A^{-u})],$$

$$\left[\frac{H_A(z,w)}{H_A(0,z)\, H_A(0,w)}\right]^{-1} = c_2\, [S_A^2 + O(r_A^{-u})],$$

and it is not hard to show that $F^q_{01}(A) = c_3 + O(r_A^{-u})$. The final estimate boils down to

$$m[O(A)] = \frac{1}{8}\log r_A + c_4 + o(r_A^{-u}).$$

This requires comparison to the Brownian loop measure. Suppose A_n is the discrete ball of radius e^n. Then $m[O(A_{n+1})] - m[O(A_n)]$ denotes the measure of loops in A_{n+1} that are not contained in A_n and intersect the zipper an odd number of times. For n large, this boils down to estimating the measure of loops of odd winding number about the origin, and by the strong coupling of random walk and Brownian loop measures, this is about the same as the Brownian motion loop measure of loops in the disk of e^{n+1} that are not in disk of e^n and have odd winding number about the origin. By conformal invariance, this is independent of n and a computation using Brownian bubbles as in (6) gives the value $1/8$. Being more careful about the approximation, we get

$$m[O(A_{n+1})] - m[O(A_n)] = \frac{1}{8} + O(e^{-un}).$$

More general domains than disks are handled similarly, again using the coupling and the conformal invariance of the Brownian loop measure.

Given the sharp estimate we can establish the strong scaling limit for LERW. Let us consider our domain D with two boundary points and let us view the scaled LERW at a macroscopic scale. It was shown in Lawler, Schramm, and Werner [2004] that if we ignore parametrization, the path of the LERW looks like a chordal SLE_2. In Lawler and Viklund [2016] it is shown how to combine these ideas with the sharp estimate for LERW above to show that the scaled natural parametrization of the LERW also converges to (an absolute constant times) the Minkowski content of the SLE path. While the proof is technical, the basic idea is as follows. Suppose we have seen part of the curve. Then the expected total length of a curve given the initial condition is the length of that segment plus the expected length of the remaining curve, see (10). A similar (and more elementary) formula holds for the number of steps of the LERW. The expected length of the remaining curve given the curve is given by the integral of the Green's function (discrete or continuous). Using the estimate in Beneš, Lawler, and Viklund [2016] (and the fact that the estimate does not require smoothness on the boundaries), the two expected lengths are the same. Roughly speaking, the difference of the lengths in the coupling is a martingale whose quadratic variation is very small and hence must be small.

While the structure of the proof in Lawler and Viklund [2016] is potentially applicable to other models, it requires the very sharp estimates for the discrete model. At the moment, there is no other model for which the Green's function can be estimated so precisely. A similar, but at the moment not sufficiently precise, result about the Ising model was shown in Chelkak, Hongler, and Izyurov [2015]; the technique of negative weights above is related to the spinors in that paper. The other model for which there is a relatively strong local theorem is the percolation exploration process, see Garban, Pete, and Schramm [2013].

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HIGH DIMENSIONAL EXPANDERS

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Abstract

Expander graphs have been, during the last five decades, the subject of a most fruitful interaction between pure mathematics and computer science, with influence and applications going both ways. In the last decade, a theory of "high dimensional expanders" has begun to emerge. The goal of the current paper is to describe some paths of this new area of study.

Introduction

Expander graphs are graphs which are, at the same time, sparse and highly connected. These two seemingly contradicting properties are what makes this theory non trivial and useful. The existence of such graphs is not a completely trivial issue, but by now there are many methods to show this: random methods, Kazhdan property (T) from representation theory of semisimple Lie groups, Ramanujan conjecture (as proved by Deligne and Drinfeld) from the theory of automorphic forms, the elementary Zig-Zag method and "interlacing polynomials".

The definition of expander graphs can be expressed in several different equivalent ways (combinatorial, spectral gap etc. - see Lubotzky [1994], Kamber [2016b]). When one comes to develop a high dimensional theory; i.e. a theory of finite simplicial complexes of dimension $d \geq 2$, which resembles that of expander graphs in dimension $d = 1$, the generalizations of the different properties are (usually) not equivalent. One is led to notions like: coboundary expanders, cosystolic expanders, topological expanders, geometric expanders, spectral expanders etc. each of which has its importance and applications.

In section 1, we recall very briefly several of the equivalent definitions of expander graphs (ignoring completely the wealth of their applications). These will serve as pointers to the various high dimensional generalizations.

In section 2, we will start with the spectral definition. For this one needs "discrete Hodge theory" as developed by Eckmann [1945]. In this sense the classical work of Garland [1973], proving Serre's conjecture on the vanishing of the real cohomology groups of arithmetic lattices of p-adic Lie groups, can be considered as the earliest work on high dimensional expanders. His "local to global" method which treats the finite quotients of the Bruhat–Tits building has been rediscovered in recent years, with many applications, some of them will be described in section 2.

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In section 3, we turn our attention to Gromov's topological and geometric expanders (a.k.a. the topological and geometric overlapping properties). These quite intuitive directions were shown to be related to two much more abstract definitions of coboundary and cosystolic (high dimensional) expanders. The last ones are defined using the language of $\mathbb{F}_2$-cohomology. Here also a "local to global" method enables to produce topological expanders from finite quotients of Bruhat–Tits buildings of p-adic Lie groups.

Section 4 will deal with random simplicial complexes, while in section 5 we will briefly mention several applications and connections with computer science.

1 A few words about expander graphs

Let $X = (V, E)$ be a finite connected graph with sets of vertices V and edges E. The Cheeger constant of X, denoted $h(X)$, is:

$$h(X) = \inf_{A,B\subseteq V} \frac{|E(A,B)|}{min(|A|,|B|)}$$

where the infimum runs over all the possibilities of disjoint partitions $V = A \cup B$ and $E(A, B)$ is the set of edges connecting vertices in A to vertices in B.

The graph X is **ε-expander** if $h(X) \geq \varepsilon$.

Let $L^2(V)$ be the space of real functions on V with the inner product $\langle f, g\rangle = \sum_{v\in V} \deg(v) f(v) g(v)$ and $L^2_0(V)$ the subspace of those which are orthogonal to the constant functions. Similarly, $L^2(E)$ is the space of functions on the edges with the standard inner product.

We fix an arbitrary orientation on the edges, and for $e \in E$ we denote its end points by e^- and e^+. Let $d : L^2(V) \to L^2(E)$ be the map $(df)(e) = f(e^+) - f(e^-)$ for $f \in L^2(V)$ and $\Delta = d^*d : L^2(V) \to L^2(V)$ when d^* is the adjoint of d. The operator Δ is called the Laplacian of the graph. One can show (cf. Lubotzky [1994, Chap. 4]), that it is independent of the chosen orientation. One can check that

$$\Delta = I - M$$

when M is the Markov operator on $L^2(V)$, i.e.,

$$(Mf)(x) = \frac{1}{\deg(x)} \sum_{\{y|(x,y)\in E\}} f(y).$$

The smallest eigenvalue of Δ is 0 and it comes with multiplicity one if (and only if) X is connected, which we will always assume. The eigenfunctions with respect to 0 are the constant functions and as Δ is self adjoint, $L^2_0(V)$ is invariant under Δ and the spectral gap

$$\lambda_1(X) = \inf\left\{\frac{\langle \Delta f, f\rangle}{\langle f, f\rangle} \middle| f \in L^2_0(V)\right\}$$

is the smallest eigenvalue of Δ acting on $L^2_0(V)$.

The following result is a discrete analogue of the classical Cheeger inequality (and its converse by Buser). This discrete version was proved by Tanner, Alon and Milman (the reader is referred again to Lubotzky [ibid., Chap. 4] for a detailed history).

Theorem 1.1. *If X is a finite connected k-regular graph, then:*

$$\frac{h^2(X)}{2k^2} \leq \lambda_1(X) \leq \frac{2h(X)}{k}$$

We are usually interested in infinite families of k-regular graphs ("sparse"). Such a family forms a family of expanders (i.e., $h(X) \geq \varepsilon$ for the same $\varepsilon > 0$, for every X) if and only if $\lambda_1(X) \geq \varepsilon' > 0$ for the same ε' for every X. I.e., Theorem 1.1 says that expanders can be defined, equivalently, either by a combinatorial definition or using the spectral gap definition. Expressing this using the adjacency operator A rather than the Laplacian Δ: being expanders means that the second largest eigenvalue $\lambda(X)$ of A is bounded away from k, which is the largest one.

Strictly speaking the notion of expanders requires spectral gap only in one side of the spectrum of A, but in many applications (e.g. if one wants to estimate the rate of convergence of the random walk on X to the uniform distribution) one needs bounds on both sides. Recall that $-k$ is also an eigenvalue of A iff X is bi-partite. We can now define: A k-regular connected graph is *Ramanujan* if all eigenvalues λ of A are either $\lambda = \pm k$ or $|\lambda| \leq 2\sqrt{k-1}$. By the well-known Alon–Boppana theorem, the bound $2\sqrt{k-1}$ is the best one can hope for for an infinite family of k-regular graphs. Let us recall that for the k-regular infinite tree $T = T_k$, the classical result of Kesten asserts that the spectrum of the adjacency operator on $L^2(T_k)$ is exactly the interval $[-2\sqrt{k-1}, 2\sqrt{k-1}]$. In a way Kesten's result lies beyond the Alon–Boppana theorem and there are many generalizations of this philosophy (cf. Grigorchuk and Żuk [1999]).

Ramanujan graphs were presented by Lubotzky, Phillips, and Sarnak [1988], Margulis [1988], Morgenstern and recently by Marcus, Spielman, and Srivastava [2015].

There are several other ways to define expanders. Let us mention here one which has been observed only quite recently and has a natural extension to high dimensional simplicial complexes.

Let X be a finite connected k-regular graph, with adjacency matrix A, denote $k = q+1$ and

$$\mu(X) = \max\{|\lambda| \,\big|\, \lambda \text{ e.v. of } A, \quad \lambda \neq \pm k\}.$$

So X is Ramanujan iff $\mu(X) \leq 2\sqrt{q}$. If X is bipartite, write $V = V_0 \cup V_1$ where V_0 and V_1 are the two sides, and if not $V = V_0 = V_1$. Let

$$L^2_{00}(X) = \{f \in L^2(V) \,\Big|\, \sum_{v \in V_i} f(v) = 0, \quad \text{for } i = 0, 1\}.$$

So, $\mu(X)$ is the largest (in absolute value) eigenvalue of A when acting on $L^2_{00}(X)$. For $\lambda \in [2\sqrt{q}, q+1]$ write $\lambda = q^{1/p} + q^{(p-1)/p}$ for a unique $p \in [2, \infty]$, so $\lambda = 2\sqrt{q}$ when $p = 2$.

Now, let $\pi : T = T_k \to X$ be a covering map. For a fixed $t_0 \in T$, let $S_r = \{t \in T \,\big|\, distance(t, t_0) = r\}$ and for $f \in L^2_{00}(X)$ and $t \in T$, let $\tilde{f}(t) = \frac{1}{|S_r|} \sum_{s \in S_r} f(\pi(s))$ if $r = dist(t, t_0)$, i.e. $\tilde{f}$ is the averaging of the lift of f around t_0.

Theorem 1.2 (Kamber [2016b]). *$\mu(X) \leq \lambda$ if and only if $\tilde{f} \in L^{p+\varepsilon}(T)$ for all $f \in L^2_{00}(X), t_0 \in T$, and $\varepsilon > 0$. As a corollary X is Ramanujan iff*

$$\tilde{f} \in L^{2+\varepsilon}(T),\ \forall t_0, \forall f, \forall \varepsilon.$$

2 High dimensional expanders: spectral gap

As described in section 1, the notion of expander graphs can be expressed via a spectral gap property of the Laplacian. This aspect has a natural high dimension version, but to present it one needs the language of real cohomology. Let us start by recalling the basic notations.

2.1 Simplicial complexes and cohomology. A finite simplicial complex X is a finite collection of subsets, closed under inclusion, of a finite set $X^{(0)}$, called the set of vertices of X. The sets in X are called *simplices* or *faces* and we denote by $X^{(i)}$ the set of simplices of X of dimension i (i-cells), which are the sets in X of size $i+1$. So $X^{(-1)}$ is comprised of the empty set, $X^{(0)}$ - of the vertices, $X^{(1)}$ - the edges, $X^{(2)}$ - the triangles, etc. Let $d = \dim X = \max\{i \,|\, X^{(i)} \neq \varnothing\}$ and assume X is a pure simplicial complex of dimension d, i.e., for every $F \in X$, there exists $G \in X^{(d)}$ with $F \subseteq G$. Throughout this discussion we will assume that $X^{(0)} = \{v_1, \ldots, v_n\}$ is the set of vertices and we fix an order $v_1 < v_2 < \ldots < v_n$ among the vertices. Now, if $F \in X^{(i)}$ we write $F = \{v_{j_0}, \ldots, v_{j_i}\}$ with $v_{j_0} < v_{j_1} < \ldots < v_{j_i}$. If $G \in X^{(i-1)}$, we denote the *oriented incidence number* $[F : G]$ by $(-1)^{\ell}$ if $F \backslash G = \{v_{j_\ell}\}$ and 0 if $G \not\subseteq F$. In particular, for every vertex $v \in X^{(0)}$ and for the unique face $\varnothing \in X^{(-1)}$, $[v : \varnothing] = 1$.

If $\mathbb{F}$ is a field then $C^i(X, \mathbb{F})$ is the $\mathbb{F}$-vector space of the functions from $X^{(i)}$ to $\mathbb{F}$. This is a vector space of dimension $\left|X^{(i)}\right|$ over $\mathbb{F}$ where the characteristic functions $\left\{e_F \,\middle|\, F \in X^{(i)}\right\}$ serve as a basis.

The coboundary map $\delta_i : C^i(X, \mathbb{F}) \to C^{i+1}(X, \mathbb{F})$ is given by:

$$(\delta_i f)(F) = \sum_{G \in X^{(i)}} [F : G]\, f(G).$$

So, if $f = e_G$ for some $G \in X^{(i)}$, $\delta_i e_G$ is a sum of all the simplices of dimension $i+1$ containing G with signs ± 1 according to the relative orientations.

It is well known and easy to prove that $\delta_i \circ \delta_{i-1} = 0$. Thus $B^i(X, \mathbb{F}) = \operatorname{im} \delta_{i-1}$ - "the space of i-coboundaries" is contained in $Z^i(X, \mathbb{F}) = \ker \delta_i$ - the i-cocycles and the quotient $H^i(X, \mathbb{F}) = Z^i(X, \mathbb{F}) / B^i(X, \mathbb{F})$ is the i-th cohomology group of X over $\mathbb{F}$.

In a dual way one can look at $C_i(X, \mathbb{F})$ - the $\mathbb{F}$-vector space spanned by the simplices of dimension i. Let $\partial_i : C_i(X, \mathbb{F}) \to C_{i-1}(X, \mathbb{F})$ be the boundary map defined on the basis element F by: $\partial F = \sum_{G \in X^{(i-1)}} [F : G] \cdot G$, i.e. if $F = \{v_{j_0}, \ldots, v_{j_i}\}$ then $\partial_i F = \sum_{t=0}^{i} (-1)^t \{v_{j_0}, \ldots, \widehat{v_{j_t}}, \ldots, v_{j_i}\}$. Again $\partial_i \circ \partial_{i+1} = 0$ and so the boundaries $B_i(X, \mathbb{F}) = \operatorname{im} \partial_{i+1}$ are inside the cycles $Z_i(X, \mathbb{F}) = \ker \partial_i$ and $H_i(X, \mathbb{F}) = Z_i(X, \mathbb{F}) / B_i(X, \mathbb{F})$ gives the i-th homology group of X over $\mathbb{F}$. As $\mathbb{F}$ is a field, it is not difficult in this case to show that $H_i(X, \mathbb{F}) \simeq H^i(X, \mathbb{F})$.

In the next section, we will need the case $F = \mathbb{F}_2$ - the field of two elements, but for the rest of Section 2 we work with $F = \mathbb{R}$. In this case $C^i(X, \mathbb{R})$ has the natural structure of a Hilbert space, where for $f, g \in C^i(X, \mathbb{R})$, $\langle f, g\rangle = \sum_{F \in X^{(i)}} \deg(F) f(F) g(F)$, when $\deg(F) = \#\{G \in X^{(d)} \big| G \supseteq F\}$. Now, $C_i(X, \mathbb{R})$ is the dual of $C^i(X, \mathbb{R})$ in a natural way and we can identify them and treat the operators $\Delta_i^{up} = \delta_i^* \delta_i$, $\Delta_i^{down} = \delta_{i-1}\delta_{i-1}^*$ and $\Delta_i = \Delta_i^{up} + \Delta_i^{down}$ as operators from C^i to C^i, all are self-adjoint with non-negative eigenvalues. One may check that

$$(\delta_i^* f)(G) = \frac{1}{\deg(G)} \sum_{F \in X^{(i+1)}} [F : G] \deg(F) f(F)$$

for $f \in C^{i+1}(X, \mathbb{R})$ and $G \in X^{(i)}$, so in the regular case δ_i^* is equal to ∂_{i+1} up to a constant multiple. Define $\mathcal{Z}_i = \ker \delta_{i-1}^*$ and $\mathcal{B}_i = \operatorname{im} \delta_i^*$ (so in the regular case $\mathcal{Z}_i = Z_i, \mathcal{B}_i = B_i$). The following proposition, going back to Eckmann [1945], is elementary:

Proposition 2.1 (Hodge decomposition)**.** *$C^i = B^i \oplus \mathcal{H}^i \oplus \mathcal{B}_i$ when $\mathcal{H}^i = Ker(\Delta_i)$ is called the space of Harmonic cycles. In fact $\mathcal{H}^i \simeq H^i(X, \mathbb{R})$. Note that Δ_i^{up} vanishes on $\mathcal{Z}^i = B^i \oplus \mathcal{H}^i$.*

Definition 2.2. The *i-dimensional spectral gap* of X is

$$\lambda^{(i)}(X) = \min \left\{ \lambda \middle| \lambda \text{ e.v. of } \Delta_i^{up}\Big|_{(B^i)^\perp} \right\}$$

One may check that $(B^i)^\perp = \mathcal{Z}_i$, and as $\Delta_i^{up} = \delta_i^* \circ \delta_i$, we have

$$\lambda^{(i)}(X) = \inf_{f \in (B^i)^\perp} \left\{ \frac{|\langle \Delta_i^{up} f, f\rangle|}{\langle f, f\rangle} \right\} = \left(\inf_{f \in \mathcal{Z}_i} \left\{ \frac{\|\delta f\|}{\|f\|} \right\} \right)^2$$

Also, Δ_i^{up} vanishes also on $\mathcal{H}^i$, so $\lambda^{(i)}(X) > 0$ implies $H^i(X, \mathbb{R}) = \{0\}$, and the converse is also true.

For a k-regular graph $(B^0)^\perp = Z_0 = L_0^2(X)$ and so $\lambda_1(X)$ that was defined in section 1 for a graph X, is $\lambda^{(0)}(X)$ in the notations here. We define:

Definition 2.3. A pure d-dimensional simplicial complex will be called *ε-spectral expander* if for every $i = 0, \ldots, d-1$, $\lambda^{(i)} \geq \varepsilon$.

Recall that the Alon–Boppana theorem asymptotically bounds the spectral gap of k-regular graphs by that of their universal cover, the k-regular tree. In higher dimension the situation is more involved:

Theorem 2.4 (Parzanchevski and Rosenthal [2017])**.** *For an infinite complex X, let $\lambda^{(i)}(X)$ be the bottom of the spectrum of $\Delta_i^{up}(X)$ on $(B^i)^\perp$. Let $\{X_n\}$ be a family of quotients of X, such that the injectivity radius of X_n approaches infinity. If zero is not an isolated point in the spectrum of $\Delta_i^{up}(X)$, on $(B^i)^\perp$, then*

$$\liminf_{n \to \infty} \{\lambda^{(i)}(X_n)\} \leq \lambda^{(i)}(X).$$

Note that zero cannot be an isolated point in the spectrum of the Laplacian of an infinite graph, since the constant function is not in L^2. However, for complexes of higher dimensional this can happen, and in this case the Alon–Boppana principle can be violated (see Parzanchevski and Rosenthal [2017, Thm. 3.10] for an example).

2.2 Garland method. The seminal paper of Howard Garland [1973] (see also Borel [1975]), can be considered as the first paper on high dimensional expanders. It gave examples of spectral expanders, by a method which bounds the eigenvalues of the simplicial complex by the eigenvalues of its links. Garland's method has been revisited in recent years with various simplifications and extensions. Let us give here one of them, but we need more definitions: If F is a face of X of dimension i, the *link of F in X* denoted $\ell k_X(F)$, is

$$\ell k_X(F) := \{G \in X \,\big|\, F \cup G \in X, F \cap G = \varnothing\}.$$

One can easily check that if X is a pure simplicial complex of dimension d, $\dim(\ell k_X(F)) = d - i - 1$.

Garland's method can be conveniently summarized by the following theorem. Note that if $dim(X) = d$ and $dim(F) = d - 2$, then $\ell k_X(F)$ is a graph.

Theorem 2.5 (Gundert and Wagner [2016]). *If $dim(X) = d$ and for every face F of dimension $d - 2$, $\lambda^{(0)}(\ell k_X(F)) \geq \varepsilon$, then*

$$\lambda^{(d-1)}(X) \geq 1 + d\,\varepsilon - d.$$

So, Garland's method enables to give a fairly good bound on $\lambda^{(d-1)}(X)$ if all links of $d - 2$ faces are very good expanders. One can use the result to bound also $\lambda^{(j)}(X)$ for $j \leq d - 1$, by replacing X with its $j + 1$ skeleton, i.e., the collection of all the faces of X of dimension at most $j + 1$. In fact, even more: if the links of the $(d - 2)$-faces are excellent expander graphs and the 1-skeleton is connected, then the complex is spectral expander (cf. Oppenheim [2018]). In the next subsection, we will explain Garland's motivation and results. But in recent years his method have been picked up in various different directions. Most of them have to do with vanishing of some cohomology groups.

One of the nicest applications of Garland's method is the work of Żuk [2003], Pansu [1998] and Ballmann and Świątkowski [1997]. The starting point of these works is the well-known result that a discrete group Γ has Kazhdan property (T) iff $H^1(\Gamma, V) = \{0\}$ for every unitary representation of Γ on any Hilbert space. These authors used Garland's work to deduce such a vanishing result for H^1 if Γ acts cocompactly on an infinite contractible simplicial complex of dimension 2 all of whose vertex links are very good expanders. The most amusing is Zuk's method which enables (sometimes) to deduce property (T) from a presentation of Γ by generators and relations. For example it shows property (T) for some random groups (see also Kotowski and Kotowski [2013]). This is very different than the way Kazhdan produced the first groups with property (T) and it shows that property (T) is not such a rare property.

A work of a similar flavor but in a different direction is the work of De Chiffre, Glebsky, Lubotzky, and Thom [2017]. Recall first (vaguely) the basic definition of "group stability": Consider the degree n unitary group $U(n)$ with an invariant metric d_n. We say that a group Γ presented by a finite set of generators S with finitely many relations R, is $(U(n), d_n)$-stable if every *almost representation* ρ of Γ into $U(n)$ is close to a representation $\tilde{\rho}$. By "almost" we mean that $\rho(r)$ is very close to the identity for every $r \in R$ and "close" means that $\rho(s)$ and $\tilde{\rho}(s)$ are close w.r.t. d_n, for every $s \in S$. One can study these questions w.r.t. different distance functions, e.g., the one induced by the Hilbert–Schmidt norm, the operator norm or the L^2-norm, a.k.a. the Frobenius norm.

Let us stick to the L^2-norm. In De Chiffre, Glebsky, Lubotzky, and Thom [ibid.] it is shown that if $H^2(\Gamma, V) = \{0\}$ for every unitary representation of Γ, then Γ is $(U(n), d_{L^2})$-stable. Then the Garland method is used (along the line of the results mentioned above for H^1) to produce many examples of L^2-stable groups by considering actions on 3-dimensional infinite simplicial complexes, whose edge-links are excellent expanders. This implies that many high rank cocompact lattices in simple p-adic Lie groups are $(U(n), d_{L^2})$-stable. The most striking application is proving that there exists a group which is not L^2-approximated (the reader is referred to De Chiffre, Glebsky, Lubotzky, and Thom [ibid.] for the definitions and exact results and to Thom [2018] for background and applications).

In Gundert and Wagner [2016], they used the Garland method to estimate the eigenvalues of random simplicial complexes - see also Section 4. For some stronger versions of Garland's method - see Oppenheim [2017, 2018] and the references therein.

2.3 Bruhat–Tits buildings and their finite quotients. Let K be a non-Archimedean local field, i.e., K is a finite extension of $\mathbb{Q}_p$, the field of p-adic numbers, or K is $\mathbb{F}_q((t))$-the field of Laurent power series over a finite field $\mathbb{F}_q$. Let $\mathcal{O}$ be the ring of integers of K, $\mathfrak{M}$ the (unique) maximal ideal of $\mathcal{O}$, and $\mathbb{F}_q = \mathcal{O}/\mathfrak{M}$ the finite quotient where $q = p^\ell$ for some prime p and $\ell \in \mathbb{N}$. Let $\underline{G}$ be a K-simple simply connected group of K-rank r, e.g., $\underline{G} = \mathrm{SL}_n$ in which case $r = n - 1$, and let $G = \underline{G}(K)$.

Bruhat and Tits developed a theory which associates with G an infinite (if $r \geq 1$) contractible simplicial complex $\mathcal{B} = \mathcal{B}(G)$ of dimension r. Here is a quick description of it: G has $r + 1$ conjugacy classes of maximal compact subgroups (cf. Platonov and Rapinchuk [1994, Theorem 3.13, p. 150]) and a unique class of maximal open pro-p subgroups, called Iwahori subgroups. The vertices of $\mathcal{B}$ are the maximal compact subgroups (so they come with $r + 1$ "colors" according to their conjugacy class) and a set of $i + 1$ such vertices form a cell if their intersection contains an Iwahori subgroup. This is an r-dimensional simplicial complex whose maximal faces can be identified with G/I when I is a fixed Iwahori subgroup (for more see Bruhat and Tits [1972], Platonov and Rapinchuk [1994] and Lubotzky [2014] for a quick explicit description of $\mathcal{B}(\mathrm{SL}_n(\mathbb{Q}_p))$. The case of $\mathcal{B}(\mathrm{SL}_2(K))$, which is a $(q + 1)$-regular tree, is studied in detail in Serre [1980]).

Let Γ be a cocompact lattice in G, i.e., a discrete subgroup with $\Gamma \setminus G$ compact. Assume, for simplicity, that Γ is torsion free, a condition which can always be achieved

by passing to a finite index subgroup. Such Γ is always an arithmetic lattice if $r \geq 2$ by Margulis arithmeticity Theorem (Margulis [1991]) and, at least if $\mathrm{char}(K) = 0$, there are always such lattices by Borel and Harder [1978]. When we fix K and G and run over all such lattices in G, for example, over the infinitely many congruence subgroups of Γ, we obtained a family of *bounded degree* simplicial complexes, i.e. every vertex is included in a bounded number of faces. These simplicial complexes, give the major examples of "high dimensional expanders" discussed in this paper.

Garland's method described in the previous subsection was developed by him in order to prove a conjecture of Serre asserting that if $r \geq 2$, $H^i(\Gamma, \mathbb{R}) = \{0\}$ for every Γ as above and every $1 \leq i \leq r-1$. Indeed, the vertex links of the building $\mathcal{B}$ are the associated spherical building over the finite field $\mathbb{F}_q$ (for example, for $\underline{G} = \mathrm{SL}_n$, this is the flag complex of the proper subspaces of $\mathbb{F}_q^n$). For such buildings, for every cell $F \in X^{(i)}, 0 \leq i \leq d-2$, one has $\lambda^{(0)}(\ell k(F)) \to 1$ when $q \to \infty$ (e.g. for $\underline{G} = \mathrm{SL}_3$, we get the $(q+1)$-regular "points to lines graph" of the projective plane $\mathbb{P}(\mathbb{F}_q^3)$, for which one can check that $\lambda_1 = 1 - \frac{1}{\sqrt{q}}$. See Garland [1973], Ballmann and Świątkowski [1997], and Evra and Kaufman [2016]). One therefore can deduce from Theorem 2.5 that if $q \geq q(\underline{G})$, then $\lambda^{(i)}(X) > \varepsilon'$ for every $i = 1, \ldots, r-1$ and every finite quotient X of $\mathcal{B} = \mathcal{B}(G)$. In particular, all these quotients are spectral expanders as defined in Definition 2.3.

This also implies Serre's conjecture if q is sufficiently large (see Definition 2.2). Serre's conjecture has been proved in full since then (cf. Casselman [1974] and Borel and Wallach [1980, Chap. XI]) by representation theoretic methods, but Garland's method has its own life in various other contexts.

In Section 1, Theorem 1.2, we saw that expander graphs can also be defined as "L_p-expanders" for a suitable $2 \leq p \in \mathbb{R}$. This definition can be extended to high dimensional simplicial complexes and is especially suitable in the context of this subsection.

Let $\mathcal{B}$ be one of the Bruhat–Tits buildings described above and $\pi : \mathcal{B} \to X$ the covering map. Let $f \in L_0^2(X^{(r)})$, i.e. a function orthogonal to the constants on the r-cells of X (one can consider also i-cells for $0 \leq i \leq r$, but we stick to these for simplicity of the exposition, the reader is referred to Kamber [2016a] for a more general setting). Now, using the notion of W-distance on $\mathcal{B}^{(r)}$, when W is the affine Weyl group of G, one can define for a fixed $t_0 \in \mathcal{B}^{(r)}$, a function $\tilde{f}$ on $\mathcal{B}^{(r)}$ - the r-faces of $\mathcal{B}$, as follows: For $t \in \mathcal{B}^{(r)}$, let $\tilde{f}(t) = \frac{1}{|S_t|} \sum_{s \in S_t} f(\pi(s))$ when $S_\ell = \{s \in \mathcal{B}^{(r)} \,\big|\, W\text{-distance}\,(s, t_0) = W\text{-distance}\,(t, t_0)\}$.

Definition 2.6. We say that X is L_p-expander if for every t_0 and f as above $\tilde{f} \in L^{p+\varepsilon}(\mathcal{B}^{(r)})$ for every $\varepsilon > 0$.

Applying Oh's result Oh [2002] which gives the exact rate of decay of the matrix coefficients of the unitary representations of G, the so-called "quantitative property (T)", Kamber deduced that X as above are always L_p-expanders when $p = p(G)$ according to the following table.

W	$\tilde{A}_n$	$\tilde{B}_n$	$\tilde{C}_n$	$\tilde{D}_n$, n even	$\tilde{D}_n$, n odd	$\tilde{E}_6$	$\tilde{E}_7$	$\tilde{E}_8$	$\tilde{F}_4$	$\tilde{G}_2$
p	$2n$	$2n$	$2n$	$2(n-1)$	$2n$	16	18	29	11	6

Let us stress that this is not just an abstract result. From this fact, we can deduce non-trivial inequalities on the eigenvalues of various "Hecke operators" acting on the faces of X. The reader is referred to Kamber [2016a] for more in this direction.

2.4 Ramanujan complexes. Ramanujan graphs stand out among expander graphs as the optimal expanders from a spectral point of view (cf. Valette [1997]). These are the finite connected k-regular graphs X for which every eigenvalue λ of the adjacency matrix $A = A_X$ satisfies either $|\lambda| = k$ or $|\lambda| \leq 2\sqrt{k-1}$. The first constructions of such graphs were presented as an application of the works of Deligne (in characteristic zero) and Drinfeld (in positive characteristic) proving the so called Ramanujan conjecture for GL_2 (see Lubotzky [1994] for a detailed survey). Recently, a new (non-constructive) method has been presented in Marcus, Spielman, and Srivastava [2015].

It is therefore not surprising that following the work of L. Lafforgue [2002] (for which he got the Fields Medal) extending Drinfeld's work from GL_2 to GL_d, general d, several mathematicians have started to develop a high dimensional theory of Ramanujan simplicial complexes, cf. (Cartwright, Solé, and Żuk [2003], Li [2004], Lubotzky, Samuels, and Vishne [2005b], Lubotzky, Samuels, and Vishne [2005a], Sarveniazi [2007]). One may argue what is "the right" definition of Ramanujan complexes (see the above references and Kang, Li, and Wang [2010], Kang [2016], First [2016], Kamber [2016a], Lubetzky, Lubotzky, and Parzanchevski [2017]). This topic deserves a survey of its own. Here we just briefly point out some directions of research which came out in the work of several mathematicians.

In the context of $X = \Gamma \setminus \mathcal{B}$ where $\mathcal{B}$ a Bruhat–Tits building associated with $G = \underline{G}(K)$ as in section 2.3, and Γ a cocompact lattice acting on it, the most sensible definition seems to be the following:

Definition 2.7. In the notation above, $\Gamma \setminus \mathcal{B}$ is called a Ramanujan complex if every infinite dimensional irreducible I-spherical G-subrepresentation of $L^2(\Gamma \setminus G)$ is tempered.

Recall that I is the Iwahori subgroup defined above, a representation is I-spherical if it contains a non-zero I-fixed vector and it is tempered if it is weakly-contained in $L^2(G)$.

This definition can be expressed also in other ways; it is L_2-expander in the notations of Kamber [2016a] and Definition 2.6 above. It can also be expressed in a combinatorial-spectral way. For the group SL_2, in which case $\mathcal{B}$ is a $(q+1)$-regular tree and $X = \Gamma \setminus \mathcal{B}$ is a $(q+1)$-regular graph, this definition is equivalent to the graph being Ramanujan graph. Ramanujan complexes are also optimal among high-dimensional expanders (see Li [2004], Lubotzky, Samuels, and Vishne [2005b] and Parzanchevski and Rosenthal [2017]). For most applications so far (such as the geometric and topological expanders to be presented in section 3) one does not need the full power of the Ramanujan property and quantitative Property (T) (à la Oh [2002], see §2.3) suffices. On the other hand the study of the cut-off phenomenon of Ramanujan complexes in Lubetzky, Lubotzky, and Parzanchevski [2017] did use the full power of the Ramanujan property. The same can be said about the application of Ramanujan graphs and Ramanujan complexes to the study of "golden gates" for quantum computation (see Parzanchevski and Sarnak

[2018] and Evra, Parzanchevski, and Sarnak [n.d.]), where the Ramanujan bounds give a distribution of elements in $SU(2)$ with "optimal entropy".

The Ramanujan graphs of Lubotzky, Phillips, and Sarnak [1988] (a.k.a. the LPS-graphs) have also been used to solve other combinatorial problems. For example they give the best (from a quantitative point of view) known examples of "high girth, high chromatic number" graphs. After finding the appropriate high dimensional notions of "girth" and "chromatic number", these results can indeed be generalized to the Ramanujan complexes constructed in Lubotzky, Samuels, and Vishne [2005a], (see Lubotzky and Meshulam [2007], Golubev and Parzanchevski [2014], Evra, Golubev, and Lubotzky [2015]).

Ramanujan graphs can be characterized as those graphs whose associated zeta functions satisfy "the Riemann Hypothesis (RH)" - see Lubotzky [1994], for an exact formulation and references. An interesting direction of research is to try to associate to high dimensional complexes suitable "zeta functions" with the hope that also in this context the "Ramanujaness" of the complex can be expressed via the RH. For this direction or research - see Storm [2006], Kang and Li [2014], Deitmar and Kang [2014], Kang, Li, and Wang [2010], Kang [2016], Kamber [2016a] and Lubetzky, Lubotzky, and Parzanchevski [2017].

3 Geometric and Topological expanders

In this chapter we will describe a phenomenon which is truly high dimensional; the geometric and topological overlapping properties which lead to geometric and topological expanders. The latter call for coboundary and cosystolic expanders.

3.1 Geometric and Topological overlapping. Our story begins with a result of Boros and Füredi [1984], at the time two undergraduates in Hungary, who proved the following result, as a response to a question of Erdős: If P is a set of n points in $\mathbb{R}^2$, then there exists a point $z \in \mathbb{R}^2$ which is covered by $\left(\frac{2}{9} - o(1)\right)\binom{n}{3}$ of the $\binom{n}{3}$ affine triangles determined by these points. Shortly afterward Bárány [1982] proved the d-dimensional version: For every $d \in \mathbb{N}, \exists\ 0 < C_d \in \mathbb{R}$, such that if $P \subseteq \mathbb{R}^d$ with $|P| = n$, then there exists $z \in \mathbb{R}^d$ which is covered by at least $C_d\binom{n}{d+1}$ of the $\binom{n}{d+1}$ affine simplices determined by these points.

While $2/9$ is optimal for $d = 2$, it is not known what are the optimal C_d's, neither what is their rate of convergence to 0, when d goes to infinity.

Bárany's result can be rephrased as: Let $\Delta_n^{(d)}$ be the complete d-dimensional simplicial complex on n vertices (i.e. the collections of all subsets of $[n]$ of size at most $d+1$) and $f : \Delta_n^{(d)} \to \mathbb{R}^d$ an affine map. Then there exists $z \in \mathbb{R}^d$ which is covered by at least $C_d\binom{n}{d+1}$ of the images of the d-dimensional faces.

In Gromov [2010], Gromov proved the following amazing result: Bárany's theorem above is true for every continuous map $f : \Delta_n^{(d)} \to \mathbb{R}^d$. In fact, he proved it with constants $C_d \in \mathbb{R}$ which were better than what was known before for affine maps. The reader is encouraged to draw the 2-dimensional case to realize how surprising and even

counter-intuitive this theorem is! Gromov also changed the point of view on these types of results; rather than thinking of them as properties of $\mathbb{R}^d$, think of them as properties of the simplicial complex X. Let us now define:

Definition 3.1. A d-dimensional pure simplicial complex X is said to be *ε-geometric* (resp. *ε-topological*) *expander* if for every affine (resp. continuous) map $f : X \to \mathbb{R}^d$, there exists $z \in \mathbb{R}^d$ such that ε-proportion of the images of the d-cells in $X^{(d)}$, covers the point z.

So Bárany (resp., Gromov) Theorem is the claim that $\Delta_n^{(d)}$, the complete simplicial complex of dimension d on n vertices, is C_d-geometric (resp., C_d-topological) expander.

Let us look for a moment at the case of dimension one to see why we call this property "expander": If $X = (V, E)$ is an expander graph and $f : X \to \mathbb{R}$ any continuous map, choose a point $z \in \mathbb{R}$ such that the two disjoint sets

$$A = \{v \in V \,\big|\, f(v) < z\} \text{ and } B = \{v \in V \,\big|\, f(v) > z\}$$

are of size approximately $\frac{|V|}{2}$. By the expansion property, there are many edges in E which connect A and B. The image of each such an edge under f must pass through z by the mean value theorem. Hence X is also a topological expander.

We should mention that a topological expander graph X does not have to be an expander graph. Moreover, it does not even have to be connected. For example, assume X is a union of a large expander graph and another small (say of size $o(|X|)$) connected component. Then X is a topological expander even though it is not an expander graph.

Anyway, Gromov and Bárany Theorems refer to the complete simplicial complexes: note how difficult is the case $d \geq 2$ and how trivial it is to prove that the complete graph is an expander. Gromov also proved that some other interesting simplicial complexes are d-dimensional topological expanders, e.g., the flag complexes of $d+2$ dimensional vector spaces over finite fields or more generally spherical buildings of simple algebraic groups over finite fields (cf. Lubotzky, Meshulam, and Mozes [2016]). All these examples are *not* of bounded degree. Recall (see also Definition 4.1 below) that we say that a family of d-dimensional simplicial complexes are of *bounded degree* (resp. *bounded upper degree*) if for every vertex v (resp., every face F of dimension $d-1$) the number of faces containing it is bounded. The non trivial aspect of expander graphs in dimension one is the construction of such graphs of bounded degree.

Gromov [2010] put forward the basic questions: Let $d \geq 2$, are there bounded degree d-dimensional geometric/topological expanders?

The existence of geometric expanders of bounded degree was shown by Fox, Gromov, V. Lafforgue, Naor, and Pach [2012] in several ways - most notably are two: the random method which we will come back to in section 4 and the second is by showing that for a fixed d, if q is a large enough (depending on d) and fixed, the Ramanujan complexes described in section 2.4 are geometric expanders of bounded degree. A more general version was given by Evra [2017] .

Theorem 3.2. *Given $2 \leq d \in \mathbb{N}$, there exists $q_0 = q_0(d)$ and $\varepsilon = \varepsilon(d)$ such that for every $q > q_0$, if K is a non-Archimedean local field of residue degree q and $\mathcal{G}$ a*

simple K-algebraic group of K-rank d, then the finite quotients of $\mathcal{B} = \mathcal{B}(\mathcal{G}(K))$ - the Bruhat–Tits building associated with $G = \mathcal{G}(K)$ - are all ε-geometric expanders.

Theorem 3.2 is deduced in Evra [2017] in a similar way as the proof in Fox, Gromov, V. Lafforgue, Naor, and Pach [2012] using a "mixing lemma" and a classical convexity result of Pach [1998]. The mixing lemma is deduced there from Oh's "quantitative property (T) (Oh [2002]). The language of L_p-expanders described in section 2 gives a systematic way to express this. (Compare also to Parzanchevski, Rosenthal, and Tessler [2016]). The fact that we have an $\varepsilon = \varepsilon(d)$ which is independent of q, provided $q > q_0$, (which is more than one needs in order to answer Gromov's geometric question) is due to the fact that for a fixed $d \in \mathbb{N}$, one has the same p in the table in section 2.3. which works for all groups of rank d.

The question of bounded degree topological expanders is much more difficult and will be discussed in the next subsections.

3.2 Coboundary expanders. As of now there is only one known method (with several small variants) to prove that a simplicial complex X is a topological expander. This is via "coboundary expander" which requires the language of cohomology as introduced in section 2.1, but this time with $\mathbb{F}_2$-coefficients.

Let X be a finite d-dimensional pure simplicial complex, define on it a *weight function* w as follows: for $F \in X^{(i)}$ let

$$w(F) = \frac{1}{\binom{d+1}{i+1}|X^{(d)}|}\left|\{G \in X^{(d)} | G \supseteq F\}\right|.$$

One could work with a number of different weight functions, but this one is quite pleasant, for example, it is a probability measure on $X^{(i)}$; one easily checks that $\sum_{F \in X^{(i)}} w(F) = 1$. Now for $f \in C^i(X, \mathbb{F}_2)$, denote $\|f\| = \sum_{\{F \in X^{(i)} | f(F) \neq 0\}} w(F)$. We can now define the important notion of "coboundary expanders" - a notion which was independently defined by Linial and Meshulam [2006] and Gromov [2010] (in both cases without calling it coboundary expanders).

Definition 3.3. Let X be as above:

(a) For $0 \leq i \leq d-1$, define the ith coboundary expansion $\hbar_i(X)$ of X as:

$$\hbar_i(X) = \min_{f \in C^i \setminus B^i} \frac{\|\delta_i f\|}{\|[f]\|}$$

where $[f] = f + B^i$ is the coset of f w.r.t. the i-coboundaries and $\|[f]\| = \min_{g \in [f]} \|g\|$. (Note that $\|[f]\|$ is the "normalized distance" of f from B^i). Let $\hbar(X) = \min\{\hbar_i(X) | i = 0, \ldots, d-1\}$.

(b) The complex X is said to be ε-*coboundary expander* if $\hbar(X) \geq \varepsilon$.

A few remarks are in order here:

(i) The reader can easily check that if X is a k-regular graph, then $\hbar(X) = \frac{2}{k} \cdot h(X)$ where $h(X)$ is the Cheeger constant of the graph as defined in section 1. So, indeed the above definition extends the notion of expander graphs.

(ii) The definition of ε_i, and especially the fact that the minimum runs over $f \in C^i \setminus B^i$ looks unnatural at first sight, but if we recall that $\|[f]\|$ is exactly the "norm" of the element in $f + B^i$ which is closest to B^i, we see that this corresponds to going over $(B^i)^\perp$ when we consider real coefficients. Moreover as pointed out in section 2, over $\mathbb{R}$,

$$\frac{\langle \Delta_i^{up} f, f \rangle}{\langle f, f \rangle} = \frac{\|\delta_i f\|}{\|f\|}$$

and so the definition of $\hbar_i$ here is "the characteristic 2 analogue" of the spectral gap defined in Definition 2.2. For the connection between the spectral gap and the coboundary expansion - see Steenbergen, Klivans, and Mukherjee [2014], Parzanchevski, Rosenthal, and Tessler [2016] and Gundert and Szedlák [2015].

(iii) Also here it is easy to see that $\hbar_i(X) > 0$ iff $H^i(X, \mathbb{F}_2) = \{0\}$.

A basic result proved independently in Linial and Meshulam [2006], Meshulam and Wallach [2009] and Gromov [2010] is:

Theorem 3.4. *for the complete d-dimension complex $\Delta_n^{(d)}$, $\hbar_i(\Delta_n^{(d)}) \geq 1 - o_d(1)$ for every $i = 0, \ldots, d-1$.*

Here is Gromov fundamental result on the connection between coboundary expanders and topological expanders:

Theorem 3.5. *Coboundary expanders are topological expanders, namely, for every $d \in \mathbb{N}$ and $0 < \varepsilon \in \mathbb{R}$, there exists $\varepsilon^1 = \varepsilon^1(d, \varepsilon) > 0$ such that if X is a d-dimensional complex which is an ε-coboundary expander then it is an ε^1-topological expander.*

Now, combining Theorem 3.5 with Theorem 3.4, one deduces that $\Delta_n^{(d)}$ are topological expanders as mentioned in section 3.1.

But these are of unbounded degree. Naturally, as the finite quotients of the high rank Bruhat–Tits building are spectral and geometric expanders, one tends to believe that they are also topological expanders. This is still an open problem. Let us say right away that in general these quotients (and even the Ramanujan complexes) are not coboundary expanders. As was explained in Kaufman, Kazhdan, and Lubotzky [2016] for many of the lattices Γ in simple p-adic Lie groups, $H^1(\Gamma \setminus \mathcal{B}, \mathbb{F}_2) \neq \{0\}$ since it is equal to $H^1(\Gamma, \mathbb{F}_2) = \Gamma/[\Gamma, \Gamma]\Gamma^2$ (since $\mathcal{B}$ is contractible) and the latter is often non-zero. Thus $\hbar_i(\Gamma \setminus \mathcal{B}) = 0$ and $\Gamma \setminus \mathcal{B}$ is not a coboundary expanders.

Still, one can overcome this difficulty. For this we need another definition:

Definition 3.6. A d-dimensional complex X is called *ε-cosystolic expander*, if for every $i = 0, \ldots, d-1$, one has $\nu_i(X) \geq \varepsilon$ and $\mu_i(X) \geq \varepsilon$ when:

$$\nu_i(X) = \min_{f \in C^i \setminus Z^i} \frac{\|\delta_i(f)\|}{\|\lceil f \rceil\|}$$

where $\lceil f \rceil = f + Z^i$ and

$$\|\lceil f \rceil\| = \min\{\|g\| \,\big|\, g \in \lceil f \rceil\}$$

and

$$\mu_i = \min_{f \in Z^i \setminus B^i} \|f\|.$$

For later use, let us denote $\mu(X) = \min \mu_i(X)$ and $\nu(X) = \min \nu_i(X)$. So, X is ε-cosystolic expansion if $\mu(X) \geq \varepsilon$ and $\nu(X) \geq \varepsilon$. So, X is "*ε-cocycle expander*"; it may not be coboundary expander if $H^i \neq \{0\}$ (for some $i = 0, \ldots, d-1$) but at least every representative of a non-trivial cohomology class is "large".

An extension of Gromov's Theorem 3.5 is given in Dotterrer, Kaufman, and Wagner [2016]:

Theorem 3.7. *Cosystolic expanders are topological expanders.*

It is natural to conjecture that the Ramanujan complexes and more generally the quotients of the high rank Bruhat–Tits buildings, while not coboundary expanders, in general, are still cosystolic expanders. But also this is open. What is known is a somewhat weaker result which still suffices to answer, in the affirmative, Gromov's question on the existence of bounded degree topological expanders. The following theorem was proved by Kaufman, Kazhdan, and Lubotzky [2016] for $d \leq 3$ and by Evra and Kaufman [2016] for general d.

Theorem 3.8. *Fix $2 \leq d \in \mathbb{N}$, then there exists $\varepsilon = \varepsilon(d) > 0$ and $q_0 = q_0(d)$ such that if K is a local non-Archimedean field of fixed residue degree $q > q_0$ and $G = \underline{G}(K)$ with $\underline{G}$ simple K-group of K-rank d, then the $(d-1)$-skeletons Y of the finite (d-dimensional) quotients X of the Bruhat–Tits building $\mathcal{B} = \mathcal{B}(G)$ form a family of bounded degree $(d-1)$-dimensional ε-cosystolic expanders.*

As this Theorem holds for every d, it solves Gromov's problem, but in a somewhat unexpected way. We do believe that X in the theorem are also cosystolic expanders and not just Y.

Evra and Kaufman [ibid.], give a quite general combinatorial criterion to deduce a result like Theorem 3.8. They prove that if X is a d-dimensional complex of bounded degree all of whose proper links (i.e. $\ell k_X(F)$ for every face $F \neq \varnothing$) are coboundary expanders, and all the underlying graphs of all the links (including $\ell k_X(\varnothing) = X$) are "very good" expander graphs, then the $(d-1)$-skeleton of X is a cosystolic expander. The reader is referred to Evra and Kaufman [ibid.] for the exact quantitative formulation. It is in spirit an "$\mathbb{F}_2$-version" of Garland's local to global method described in section 2.2. It will be interesting to strengthen this result to the same level as Garland's, i.e., to assume only that the proper links are coboundary expanders and connected and X is connected. It will be even more interesting if one could deduce (even with the current hypothesis) that X itself is a cosystolic expander. This will show that the d-dimensional Ramanujan complexes are topological expanders and not merely their $(d-1)$ skeletons as we now know.

The issue discussed in this section is only the tip of the iceberg. There are many more interesting problems (see Gromov [2010], Gromov and Guth [2012]) e.g. every d-dimensional complex can be embedded in $(2d+1)$-dimensional Euclidean space, but only some can be embedded in $2d$. Prove that high dimensional expanders (in some or any of the definitions) can not.

4 Random simplicial complexes

As mentioned briefly above, the easiest way to prove existence of bounded degree expander graphs is by random methods. One may hope that this can be extended to the higher dimensional case of d-dimensional simplicial complexes. But, here the problem is much more difficult. In fact, as of now, there is no known "random model" for d-dimensional simplicial complexes *of bounded degree* (in the strong sense - see below) which gives high dimensional topological expanders. This is surprising as the existence of such topological expanders is known by now by (Kaufman, Kazhdan, and Lubotzky [2016], Evra and Kaufman [2016]) as was explained in section 3. One may start to wonder if such a model exists at all, or maybe topological bounded degree expanders of high dimension are very rare objects. Perhaps there is a kind of rigidity phenomenon analogue to what is well known by now in Lie theory and locally symmetric spaces: While there are many different Riemann surfaces (parameterized by Teichmüller spaces), the higher dimensional case is completely different and rigidity results say that there are "very few" and mainly the ones coming from arithmetic lattices.

Let us now leave aside such a speculation and give a brief background and a short account of the known results:

Erdős and Rényi initiated the study of random graphs in their seminal paper Erdős and Rényi [1960]. Their model is the following: Let $n \in \mathbb{N}$ and $p \in [0,1]$, the random model $X(n,p)$ is the graph X with vertex set $[n] = \{1,\dots,n\}$ and where for every $1 \le i \ne j \le n$, the edge $\{i,j\}$ is in X with probability p, independently of all other edges. They then study the properties of such graphs when $n \to \infty$ (and p can be changed with n). For example, their first famous result is that $p_0 = \frac{\log n}{n}$ is a threshold for the connectedness of $X \in X(n,p)$. Namely, for every $\varepsilon > 0$, if $p \le (1-\varepsilon)\frac{\log n}{n}$ then almost surely (a.s.) such an X is not connected, i.e. Prob. $(X \in X(n,p) : X$ connected$) \underset{n\to\infty}{\longrightarrow} 0$. On the other hand, if $p \ge (1+\varepsilon)\frac{\log n}{n}$ then X is a.s. connected.

Why is $p = \frac{\log n}{n}$ the threshold? Recall the "coupon collector problem" which asserts that if elements of $[m] = \{1,\dots,m\}$ are chosen independently at random with repetition, it will take $t = m \log m$ steps to choose them all. In our process $p\binom{n}{2}$ edges are chosen, and hence $2p\binom{n}{2}$ vertices. Now, if $p < \frac{\log n}{n}$ then less than $2\frac{\log n}{n}\frac{n^2}{2} = n \log n$ vertices are chosen. So w.h.p there is an isolated vertex! The amazing point in the Erdős–Rényi result is the fact that once we cross the threshold, not only are there no isolated vertices, but the graph is connected, and, in fact, even an expander.

This was the starting point of a very elaborate (and very important) theory of random graphs studying more and more delicate properties of such $X \in X(n,p)$.

In Linial and Meshulam [2006] Linial and Meshulam initiated such a theory for 2-dimensional simplicial complexes. A theory which shortly afterward was extended in

Meshulam and Wallach [2009] to the general d-dimensional case. The model studied $X^d(n,p)$ (nowadays called the Linial–Meshulam model for random d-dimensional simplicial complexes) is the following: $X \in X^d(n,p)$ is a d-dimensional complex with $[n]$ as the set of vertices, X contains the full $(d-1)$-skeleton, i.e., every subset of $[n]$ of size at most d is in X and a subset of size $d+1$ is in X with probability p, independently of the other d-cells. So $X^1(n,p)$ is exactly the Erdős–Rényi model. Now, for $d \geq 2$, such an X is always connected. But, note that $X \in X^1(n,p)$ is connected if and only if $H^0(X,\mathbb{F}_2) = \{0\}$, so Linial, Meshulam and Wallach study for $d \geq 2$ and $X \in X^d(n,p)$, when $H^{d-1}(X,\mathbb{F}_2) = \{0\}$ and proved the following far reaching generalization of the Erdős–Rényi theorem.

Theorem 4.1 (Linial and Meshulam [2006] for $d = 2$, Meshulam and Wallach [2009] for all d)**.** *The threshold for the homological connectivity, i.e. the vanishing of $H^{d-1}(X,\mathbb{F}_2)$ for $X \in X^d(n,p)$ is $p_0 = \frac{d \log n}{n}$.*

The heuristic here for $\frac{d \log n}{n}$ is similar to the one above: The process picks $p\binom{n}{d+1}$ d-cells and hence $(d+1)p\binom{n}{d+1}$ $(d-1)$-cells. So, if $p < \frac{d \log n}{n}$ less than $(d+1)\frac{d \log n}{n}\binom{d}{d+1} \approx \binom{n}{d}\log(\binom{n}{d})$ $(d-1)$-cells are chosen and so there is a $(d-1)$-cell τ with no d-cell containing it. Hence the coboundary of e_τ - the characteristic function of τ - is zero, i.e. $e_\tau \in Z^{d-1}(X,\mathbb{F}_2)$. On the other hand e_τ is not a coboundary (note that in the complete d-dimensional complex $\delta(e_\tau) \neq 0$, so it is not even a cocycle) and hence $H^{d-1}(X,\mathbb{F}_2) \neq \{0\}$. Again the interesting aspect of the Linial–Meshulam–Wallach result is that once the threshold is passed, not only does $H^{d-1}(X,\mathbb{F}_2)$ vanish, but X is even a coboundary expander.

A nice theory of random complexes has started to emerge (see Kahle [2014] and the references therein). As our main interest here is in expanders, we refer mainly to Linial and Meshulam [2006], Meshulam and Wallach [2009], and Dotterrer and Kahle [2012], noting that the results there imply (just like in the case of graphs) that above the threshold the complexes are not only homologically connected but also coboundary expanders and therefore topological expanders. The papers Gundert and Wagner [2016] and Knowles and Rosenthal [2017] bring in a very detailed study of the spectrum of the higher dimensional Laplacians Δ_i action on $C^i(X,\mathbb{R})$ for random X.

But our main interest is in bounded degree complexes. Recall that Bollobás [1980] and others (see Wormald [1999] for a comprehensive survey) have developed a theory of random k-regular graphs (for a fixed k) which also got a lot of attention. This model, for $k \geq 3$, gives almost surely expander graphs of bounded degree.

One would like to have such a model for d-dimensional complexes. But first, what do we mean by bounded degree? There are two natural meanings in the literature, which coincide for $d = 1$.

Definition 4.1. A pure d-dimensional simplicial complex X is of degree at most k if every vertex of it is contained in at most k cells of dimension d (and so in at most $2^d \cdot k$ cells of any dimension). It is of *upper-degree* at most k, if every face of dimension $d-1$ is contained in at most k cells of dimension d.

A natural model of bounded degree simplicial complexes $Y^d(n,k)$ is given in Fox, Gromov, V. Lafforgue, Naor, and Pach [2012]: Assume, for simplicity, that $(d+1)|n$ and

take a random partition of $[n]$ into $\frac{n}{d+1}$ subsets each of size $d+1$. Choose independently k such partitions and let Y be the simplicial complex obtained by taking its cells to be all these $k\ \frac{n}{d+1}$ subsets as well as all their subsets. The case $d = 1$ boils down to the standard model of Bollobás.

Theorem 4.2 (Fox, Gromov, V. Lafforgue, Naor, and Pach [ibid.]). *For every fixed $d \in \mathbb{N}$, $\exists k_0 = k_0(d)$, such that for every $k \geq k_0$, a complex $Y \in Y^d(n,k)$ is almost surely d-dimensional geometric expander.*

This theorem is very promising at first sight, but unfortunately, $Y \in Y^d(n,k)$ is typically neither coboundary expander nor topological expander. To visualize this think about the $d = 2$ case: When k is fixed and n very large, for a typical $Y \in Y^2(n,k)$, every edge of Y is contained in at most one triangle. So, homotopically Y looks more like a graph and one can map it into $\mathbb{R}^2$ with only small size overlapping points.

So, altogether, this is a nice model which certainly deserves further study (e.g. what is the threshold for $k_0 = k_0(d)$ in Theorem 4.3?) but it will not give us the stronger versions of expansion (topological, cosystolic, coboundary etc.). As hinted at the beginning of this section, it is still a major open problem to find a random model (if such at all exists) of d-dimensional bounded degree simplicial complexes which will give, say, topological expanders.

The situation with bounded *upper* degree is better: In Lubotzky and Meshulam [2015] Lubotzky and Meshulam gave a model for 2-dimensional complexes of bounded upper degree (using the theory of Latin squares) and it was shown to produce coboundary expanders (and so also topological expanders). This was generalized to all d by Lubotzky, Luria, and Rosenthal [2015], with a slight twist of the construction, replacing the Latin squares by Steiner systems and using the recent breakthrough of Keevash [2014] on existence of designs. Let us briefly describe the general model $W^d(n,k)$.

Let $r \leq q \leq n$ be natural numbers and $\lambda \in \mathbb{N}$. An (n,q,r,λ)-*design* is a collection S of q-element subsets of $[n]$ such that each r-element subset of $[n]$ is contained in exactly λ elements of S. Given $n, d \in \mathbb{N}$, an (n,d)-Steiner system is an $(n,d+1,d,1)$-design, namely, a collection S of subsets of size $d+1$ of $[n]$, such that each set of size d is contained in exactly one element of S. Using the terminology of simplicial complexes, an (n,d)-Steiner system can be considered as a d-dimensional simplicial complex of upper degree one. Recently, in a groundbreaking paper Keevash [ibid.], Peter Keevash gave a randomized construction of Steiner systems for any fixed d and large enough n satisfying certain necessary divisibility conditions (which hold for infinitely many $n \in \mathbb{N}$). From now on, we will assume that given a fixed $d \in \mathbb{N}$, the value of n satisfies the divisibility condition from Keevash's theorem.

Keevash's construction of Steiner systems is based on randomized algorithm which has two stages. We will explicitly describe the first stage and use the second stage as a black box.

Given a set of d-cells $A \subseteq \binom{[n]}{d+1}$, we call a d-cell τ *legal with respect to* A if there is no common $(d-1)$-cell in τ and in any cell in A. Non-legal cells are also called forbidden cells.

In the first stage of Keevash's construction, also known as the greedy stage, one selects a sequence of d-cells according to the following procedure. In the first step, a

d-cell is chosen uniformly at random from $\binom{[n]}{d+1}$. Next, at each step a legal d-cell (with respect to the set of d-cells chosen so far) is chosen uniformly at random and is added to the collection of previously chosen d-cells. If no such d-cell exists the algorithm aborts. The procedure stops when the number of $(d-1)$-cells which do not belong to the boundary of the chosen d-cells is at most $n^{d-\delta_0}$ for some fixed $\delta_0 > 0$ which only depends on d. In particular, if the algorithm does not abort the number of steps is at least $(\binom{n}{d} - n^{d-\delta_0})/(d+1) \geq n^d/(2(d+1)!)$.

In the second stage, Keevash gives a randomized algorithm that adds additional d-cells in order to cover the remaining $(d-1)$-cells that are not contained in any of the d-cells chosen in the greedy stage. We do not need to go into the details of this algorithm. The important thing for us is that with high probability the algorithm produces an (n,d)-Steiner system.

Fix $k \in \mathbb{N}$ and let $S_1, \ldots, S_k$ be k independent copies of (n,d)-Steiner systems chosen according to the above construction, and let W be the d-dimensional simplicial complex whose d-cells are $\bigcup_{i=1}^{k} S_i$, so W contains the complete $(d-1)$-skeleton and it is of upper degree at most k.

We can now state the main result of Lubotzky, Luria, and Rosenthal [2015]:

Theorem 4.3. *Fix $d \in \mathbb{N}$, there exists $k_0 = k_0(d)$ and $\varepsilon = \varepsilon(d)$, such that for every $k \geq k_0$, a random complex $W \in W^d(n,k)$ is almost surely an ε-coboundary expander, and hence also a topological expander.*

It will be of great interest to study various other properties of this model. For example, find the threshold for $k_0(d)$ (the estimates obtained from Lubotzky, Luria, and Rosenthal [ibid.] are huge and it will be very interesting to give more realistic upper bound, note that for $d = 1, k_0(d) = 3$). Another interesting problem is to study $\pi_1(W)$-the fundamental group of W; when is it hyperbolic? has property (T)? trivial? The model W behaves w.r.t. the model X as Bollobás' model w.r.t. Erdős–Rényi, and this suggests many further directions of research on these bounded upper degree complexes.

5 High dimensional expanders and computer science

In recent years high dimensional expanders have captured the interest of computer scientists and various connections and applications have popped up. Most of these works are in their infancy. We will give here only a few short pointers on these developments, with the hope and expectation that the future will bring much more.

Probabilistically Checkable Proofs: The PCP theorem, proven in the early 90's (cf. Arora and Safra [1998] and Arora, Lund, Motwani, Sudan, and Szegedy [1998]), is a cornerstone of modern computational complexity theory stating that proofs can be written in a robust locally-testable format. PCPs are related to many areas within theoretical computer science ranging from hardness of approximation to delegation and efficient cloud computing.

The basic PCP theorem can be proven using an expander-graph-based construction Dinur [2007]. For stronger PCPs, e.g. with unique constraints, or shorter proof length,

or with lower soundness error, stronger forms of expansion seem to be needed, in particular high dimensional expansion might play a pivotal role. Dinur and Kaufman [2017] explore replacing the standard direct product construction (also known as parallel repetition Raz [1998]) by a much more efficient bounded-degree high dimensional expanders as constructed in Lubotzky, Samuels, and Vishne [2005b,a]. Direct products are ubiquitous in complexity, especially as a useful hardness amplification construction, and bounded-degree high dimensional expanders may potentially be useful in many of those settings.

Locally testable codes: LTCs are an information-theoretic analog of PCPs. These error correcting codes have the additional property that it is possible to locally test whether or not a received word is close to being a codeword. Unlike many problems in coding theory, this is a property that random codes *do not* have. This makes it even more challenging to settle the problem whether LTCs can have both linear rate and distance. The current best construction comes from a PCP and its rate is inverse poly-logarithmic Ben-Sasson and Sudan [2008] and Dinur [2007]. High dimensional expanders naturally yield locally testable codes, whose parameters are unfortunately sub-optimal.

Property testing: The central paradigm in property testing is the interplay between local views of an object and its global properties. The object can be a codeword, an NP-proof, or simply a graph. This theory generalizes both PCPs and LTCs and has significant practical applications. It was an unexpected discovery that high dimensional expanders (and especially the cohomological/coboundary expanders mentioned above) fit very naturally into this theory Kaufman and Lubotzky [2014]. Specifically, theorems about high dimensional expanders readily translate to results on property testing.

Quantum computation and quantum error correcting codes: Sipser and Spielman [1996] showed how extremely good expander graphs yield excellent LDPC error-correcting codes. However, the existence of LDPC quantum error-correcting codes (even inexplicitly) remains a major open problem. Recent work by Guth and Lubotzky [2014] is a step in this direction, which is related to our topic: Every simplicial complex gives a "homological error correcting code" (see Bombin and Martin-Delgado [2007], Zémor [2009]) but in general they are of poor quality. High dimensional coboundary expanders are related to local testability of codes (see Aharonov and Eldar [2015]).

Another basic problem in quantum computation seeks a finite *universal set of quantum gates* that can efficiently generate an arbitrary unitary matrix in $U(n)$ to desired accuracy. This is solved by Kitaev and Solovay's classical algorithm, but non-optimally. The generators of Lubotzky–Phillips–Sarnak's Ramanujan graphs Lubotzky, Phillips, and Sarnak [1987] fare better, but come with no efficient generative algorithm. Following the breakthrough of Ross and Selinger [2016], the case $n = 2$ is essentially solved in a recent work by Parzanchevski and Sarnak [2018] who came up with optimal (a.k.a. *golden*) gates and an explicit generative algorithm based on Ramanujan graphs. In ongoing work they use higher dimensional Ramanujan complexes to find such "golden gates" for higher n Evra, Parzanchevski, and Sarnak [n.d.].

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DYNAMICAL SYSTEMS, FRACTAL GEOMETRY AND DIOPHANTINE APPROXIMATIONS

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Abstract

We describe in this survey several results relating Fractal Geometry, Dynamical Systems and Diophantine Approximations, including a description of recent results related to geometrical properties of the classical Markov and Lagrange spectra and generalizations in Dynamical Systems and Differential Geometry.

1 Introduction

The theory of Dynamical Systems is concerned with the asymptotic behaviour of systems which evolve over time, and gives models for many phenomena arising from natural sciences, as Meteorology and Celestial Mechanics. The study of a number of these models had a fundamental impact in the development of the mathematical theory of Dynamical Systems. An important initial stage of the theory of Dynamical Systems was the study by Poincaré of the restricted three-body problem in Celestial Mechanics in late nineteenth century, during which he started to consider the qualitative theory of differential equations and proved results which are also basic to Ergodic Theory, as the famous Poincaré's recurrence lemma. He also discovered during this work the *homoclinic* behaviour of certain orbits, which became very important in the study of the dynamics of a system. The existence of transverse homoclinic points implies that the dynamics is quite complicated, as remarked already by Poincaré: "Rien n'est plus propre à nous donner une idée de la complication du problème des trois corps et en général de tous les problèmes de Dynamique...", in his classic *Les Méthodes Nouvelles de la Mécanique Céleste* (Poincaré [1987]), written in late 19th century. This fact became clearer much decades later, as we will discuss below.

Poincaré's original work on the subject was awarded a famous prize in honour of the 60th birthday of King Oscar II of Sweden. There is an interesting history related to this prize. In fact, there were two versions of Poincaré's work presented for the prize, whose corresponding work was supposed to be published in the famous journal Acta Mathematica - a mistake was detected by the Swedish mathematician Phragmén in the first version. When Poincaré became aware of that, he rewrote the paper, including the quotation above calling the attention to the great complexity of dynamical problems

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related to homoclinic intersections. We refer to the excellent paper Yoccoz [2006] by Jean-Christophe Yoccoz describing such an event.

We will focus our discussion of Dynamical Systems on the study of flows (autonomous ordinary differential equations) and iterations of diffeomorphisms, with emphasis in the second subject (many results for flows are similar to corresponding results for diffeomorphisms).

The first part of this work is related to the interface between Fractal Geometry and Dynamical Systems. We will give particular attention to results related to dynamical bifurcations, specially *homoclinic bifurcations*, perhaps the most important mechanism that creates complicated dynamical systems from simple ones. We will see how the study of the fractal geometry of hyperbolic sets has a central rôle in the study of dynamical bifurcations. We shall discuss recent results and ongoing works on fractal geometry of hyperbolic sets in arbitrary dimensions.

The second part is devoted to the study of the interface between Fractal Geometry and Diophantine Approximations. The main topic of this section will be the study of geometric properties of the classical Markov and Lagrange spectra - we will see how this study is related to the study of sums of regular Cantor sets, a topic which also appears naturally in the study of homoclinic bifurcations (which seems, at a first glance, to be a very distant subject from Diophantine Approximations).

The third part is related to the study of natural generalizations of the classical Markov and Lagrange spectra in Dynamical Systems and in Differential Geometry - for instance, we will discuss properties of generalized Markov and Lagrange spectra associated to geodesic flows in manifolds of negative curvature and to other hyperbolic dynamical systems. This is a subject with much recent activity, and several ongoing relevant works.

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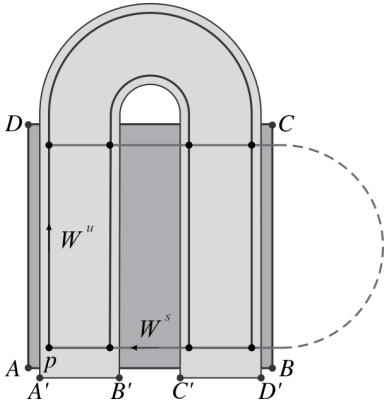
2 Fractal Geometry and Dynamical Systems

2.1 Hyperbolic sets and Homoclinic Bifurcations. The notion of *hyperbolic systems* was introduced by Smale in the sixties, after a global example provided by Anosov, namely the diffeomorphism $f(x, y) = (2x + y, x + y) \pmod 1$ of the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and a famous example, given by Smale himself, of a *horseshoe*, that is a robust example of a dynamical system on the plane with a transverse homoclinic point as above, which implies a rich dynamics - in particular the existence of infinitely many periodic orbits. The figure below depicts a horseshoe.



(the dynamics sends the square $(ABCD)$ onto the domain bounded by $(A'B'C'D')$).

Let $\Lambda \subset M$ be a compact subset of a manifold M. We say that Λ is a hyperbolic set for a diffeomorphism $\varphi\colon M \to M$ if $\varphi(\Lambda) = \Lambda$ and there is a decomposition $T_\Lambda M = E^s \oplus E^u$ of the tangent bundle of M over Λ such that $D\varphi\,|_{E^s}$ is uniformly contracting and $D\varphi\,|_{E^u}$ is uniformly expanding. We say that φ is hyperbolic if the limit set of its dynamics is a hyperbolic set.

It is important to notice that when a diffeomorphism has a transverse homoclinic point then its dynamics contains an non-trivial invariant hyperbolic set which is (equivalent to) a horseshoe - this explains the complicated situation discovered by Poincaré as mentioned above.

The importance of the notion of hyperbolicity is also related to the *stability conjecture* by Palis and Smale, according to which structurally stable dynamical systems are essentially the hyperbolic ones (i.e. the systems whose limit set is hyperbolic). After important contributions by Anosov, Smale, Palis, de Melo, Robbin and Robinson, this conjecture was proved in the C^1 topology by Mañé [1988] for diffeomorphisms and, later, by Hayashi [1997] for flows. The stability conjecture (namely the statement that structural stability implies hyperbolicity) is still open in the C^k topology for $k \geq 2$.

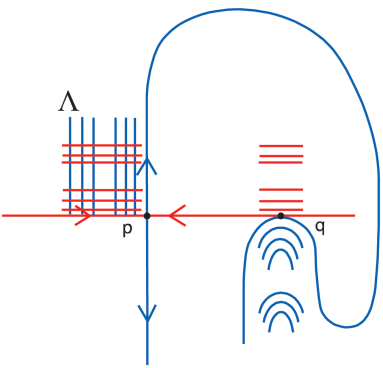
As mentioned in the introduction, homoclinic bifurcations are perhaps the most important mechanism that creates complicated dynamical systems from simple ones. This phenomenon takes place when an element of a family of dynamics (diffeomorphisms or flows) presents a hyperbolic periodic point whose stable and unstable manifolds have a non-transverse intersection. When we connect, through a family, a dynamics with no *homoclinic points* (namely, intersections of stable and unstable manifolds of a hyperbolic periodic point) to another one with a *transverse* homoclinic point (a homoclinic point where the intersection between the stable and unstable manifolds is transverse) by a family of dynamics, we often go through a homoclinic bifurcation.

Homoclinic bifurcations become important when going beyond the hyperbolic theory. In the late sixties, Sheldon Newhouse combined homoclinic bifurcations with the complexity already available in the hyperbolic theory and some new concepts in Fractal Geometry to obtain dynamical systems far more complicated than the hyperbolic ones. Ultimately this led to his famous result on the coexistence of infinitely many periodic attractors (which we will discuss later). Later on, Mora and Viana [1993] proved that any

surface diffeomorphism presenting a homoclinic tangency can be approximated by a diffeomorphism exhibiting a Hénon-like strange attractor (and that such diffeomorphisms appear in any typical family going through a homoclinic bifurcation).

Palis conjectured that any diffeomorphism of a surface can be approximated arbitrarily well in the C^k topology by a hyperbolic diffeomorphism or by a diffeomorphism displaying a homoclinic tangency. This was proved by Pujals and Sambarino [2000] in the C^1 topology. Palis also proposed a general version of this conjecture: any diffeomorphism (in arbitrary ambient dimension) can be approximated arbitrarily well in the C^k topology by a hyperbolic diffeomorphism, by a diffeomorphism displaying a homoclinic tangency or by a diffeomorphism displaying a heteroclinic cycle (a cycle given by intersections of stable and unstable manifolds of periodic points of different indexes). A major advance was done by Crovisier and Pujals [2015], who proved that any diffeomorphism can be approximated arbitrarily well in the C^1 topology by a diffeomorphism displaying a homoclinic tangency, by a diffeomorphism displaying a heteroclinic cycle or by an *essentially hyperbolic* diffeomorphism: a diffeomorphism which displays a finite number of hyperbolic attractors whose union of basins is open and dense.

The first natural problem related to homoclinic bifurcations is the study of *homoclinic explosions* on surfaces: We consider one-parameter families (φ_μ), $\mu \in (-1, 1)$ of diffeomorphisms of a surface for which φ_μ is uniformly hyperbolic for $\mu < 0$, and φ_0 presents a quadratic homoclinic tangency associated to a hyperbolic periodic point (which may belong to a *horseshoe* - a compact, locally maximal, hyperbolic invariant set of saddle type). It unfolds for $\mu > 0$ creating locally two transverse intersections between the stable and unstable manifolds of (the continuation of) the periodic point. A main question is: what happens for (most) positive values of μ? The following figure depicts such a situation for $\mu = 0$.



Fractal sets appear naturally in Dynamical Systems and fractal dimensions when we try to measure fractals. They are essential to describe most of the main results in this presentation.

Given a metric space X, it is often true that the minimum number $N(r)$ of balls of radius r needed to cover X is roughly proportional to $1/r^d$, for some positive constant d, when r becomes small. In this case, d will be the *box dimension* of X. More precisely,

we define the (upper) box dimension of X as

$$d(X) = \limsup_{r \to 0} \frac{\log N(r)}{-\log r}.$$

The notion of *Hausdorff dimension* of a set is more subtle, but more useful. The main difference with the notion of box dimension is that, while the box dimension is related to coverings of X by small balls of equal radius, the Hausdorff dimension deals with arbitrary coverings of X by balls of small (but not necessarily equal) radius.

Given a countable covering $\mathcal{U}$ of X by balls, $\mathcal{U} = (B(x_i, r_i))_{i \in \mathbb{N}}$, we define its norm $||\mathcal{U}||$ as $||\mathcal{U}|| = \max\{r_i,\, i \in \mathbb{N}\}$ (where r_i is the radius of the ball $B(x_i, r_i)$). Given $s \in \mathbb{R}_+$, we define $\mathcal{H}_s(\mathcal{U}) = \sum_{i \in \mathbb{N}} r_i^s$.

The Hausdorff s-measure of X is

$$\mathcal{H}_s(X) = \lim_{\epsilon \to 0} \inf_{\substack{\mathcal{U} \text{ covers } X \\ ||\mathcal{U}|| < \epsilon}} \mathcal{H}_s(\mathcal{U}).$$

One can show that there is an unique real number, the *Hausdorff dimension* of X, which we denote by $HD(X)$, such that $s < HD(X) \Rightarrow \mathcal{H}_s(X) = +\infty$ and $s > HD(X) \Rightarrow \mathcal{H}_s(X) = 0$ (so $HD(X)$ can be defined shortly as

$$HD(X) = \inf\{s > 0;\ \inf_{X \subset \cup B(x_n, r_n)} \sum r_n^s = 0\}).$$

For "well-behaved" sets X - in particular for regular Cantor sets in ambient dimension 1 and horseshoes in ambient dimension 2, the box and Hausdorff dimensions of X coincide.

Regular Cantor sets on the line play a fundamental role in dynamical systems and notably also in some problems in number theory. They are defined by expanding maps and have some kind of self-similarity property: small parts of them are diffeomorphic to big parts with uniformly bounded distortion (we will give a precise definition in a while). In both settings, dynamics and number theory, a key question is whether the arithmetic difference (see definition below) of two such sets has non-empty interior.

A horseshoe Λ in a surface is locally diffeomorphic to the Cartesian product of two regular Cantor sets: the so-called *stable* and *unstable* Cantor sets K^s and K^u of Λ, given by intersections of Λ with local stable and unstable manifolds of some points of the horseshoe. The Hausdorff dimension of Λ, which is equal to the sum of the Hausdorff dimensions of K^s and K^u, plays a fundamental role in several results on homoclinic bifurcations associated to Λ.

From the dynamics side, in the eighties, Palis Jr. and Takens [1985], Palis Jr. and Takens [1987] proved the following theorem about homoclinic bifurcations associated to a hyperbolic set:

Theorem 2.1. *Let (φ_μ), $\mu \in (-1, 1)$ be a family of diffeomorphisms of a surface presenting a homoclinic explosion at $\mu = 0$ associated to a periodic point belonging to a horseshoe Λ. Assume that $HD(\Lambda) < 1$. Then*

$$\lim_{\delta \to 0} \frac{m(H \cap [0, \delta])}{\delta} = 1,$$

where $H := \{\mu > 0 \mid \varphi_\mu$ is uniformly hyperbolic$\}$.

2.2 Regular Cantor sets - a conjecture by Palis. A central fact used in the proof of the above theorem by Palis and Takens is that if K_1 and K_2 are regular Cantor sets on the real line such that the sum of their Hausdorff dimensions is smaller than one, then $K_1 - K_2 = \{x - y \mid x \in K_1, y \in K_2\} = \{t \in \mathbb{R} | K_1 \cap (K_2 + t) \neq \varnothing\}$ (the *arithmetic difference* between K_1 and K_2) is a set of zero Lebesgue measure (indeed of Hausdorff dimension smaller than 1). On the occasion, looking for some kind of characterization property for this phenomenon, Palis conjectured (see Palis Jr. [1987], Palis Jr. [1991]) that for generic pairs of regular Cantor sets (K_1, K_2) of the real line either $K_1 - K_2$ has zero measure or else it contains an interval (the last case should correspond in homoclinic bifurcations to open sets of tangencies). A slightly stronger statement is that, if K_1 and K_2 are generic regular Cantor sets and the sum of their Hausdorff dimensions is bigger than 1, then $K_1 - K_2$ contains intervals.

Another motivation for the conjecture was Newhouse's work in the seventies, when he introduced the concept of thickness of a regular Cantor set, another fractal invariant associated to Cantor sets on the real line. It was used in Newhouse [1970] to exhibit open sets of diffeomorphisms with persistent homoclinic tangencies, therefore with no hyperbolicity. It is possible (Newhouse [1974]) to prove that, under a dissipation hypothesis, in such an open set there is a residual set of diffeomorphisms which present infinitely many coexisting sinks. In Newhouse [1979], it is proved that under generic hypotheses every family of surface diffeomorphisms that unfold a homoclinic tangency goes through such an open set. It is to be noted that in the case described above with $HD(\Lambda) < 1$ (as studied in Palis Jr. and Takens [1987]) these sets have zero density. See Palis Jr. and Takens [1993] for a detailed presentation of these results. An important related question by Palis is whether the sets of parameter values corresponding to infinitely many coexisting sinks have typically zero Lebesgue measure.

An earlier and totally independent development had taken place in number theory. In 1947, M. Hall [1947] proved that any real number can be written as the sum of two numbers whose continued fractions coefficients (of positive index) are bounded by 4. More precisely, if $C(4)$ is the regular Cantor set (see general definition below) formed of such numbers in $[0, 1]$, then one has $C(4) + C(4) = [\sqrt{2} - 1, 4(\sqrt{2} - 1)]$. We will discuss generalizations and consequences of this result in the next section.

A Cantor set $K \subset \mathbb{R}$ is a *C^k-regular Cantor set*, $k \geq 1$, if:

i) there are disjoint compact intervals $I_1, I_2, \ldots, I_r$ such that $K \subset I_1 \cup \cdots \cup I_r$ and the boundary of each I_j is contained in K;

ii) there is a C^k expanding map ψ defined in a neighbourhood of $I_1 \cup I_2 \cup \cdots \cup I_r$ such that, for each j, $\psi(I_j)$ is the convex hull of a finite union of some of these intervals I_s. Moreover, we suppose that ψ satisfies:

   ii.1) for each j, $1 \leq j \leq r$ and n sufficiently big, $\psi^n(K \cap I_j) = K$;

   ii.2) $K = \bigcap_{n \in \mathbb{N}} \psi^{-n}(I_1 \cup I_2 \cup \cdots \cup I_r)$.

Remark 2.2. If k is not an integer, say $k = m + \alpha$, with $m \geq 1$ integer and $0 < \alpha < 1$ we assume that ψ is C^m and $\psi^{(m)}$ is α-Hölder.

We say that $\{I_1, I_2, \dots, I_r\}$ is a *Markov partition* for K and that K is *defined* by ψ.

Remark 2.3. In general, we say that a set $X \subset \mathbb{R}$ is a Cantor set if X is compact, without isolated points and with empty interior. Cantor sets in $\mathbb{R}$ are homeomorphic to the classical ternary Cantor set $K_{1/3}$ of the elements of $[0, 1]$ which can be written in base 3 using only digits 0 and 2. The set $K_{1/3}$ is itself a regular Cantor set, defined by the map $\psi : [0, 1/3] \cup [2/3, 1] \to \mathbb{R}$ given by $\psi[x] = 3x - \lfloor 3x \rfloor$.

An *interval of the construction of the regular Cantor set* K is a connected component of $\psi^{-n}(I_j)$ for some $n \in \mathbb{N}$, $j \le r$.

Given $s \in [1, k]$ and another regular Cantor set $\tilde{K}$, we say that $\tilde{K}$ is close to K in the C^s topology if $\tilde{K}$ has a Markov partition $\{\tilde{I}_1, \tilde{I}_2, \dots, \tilde{I}_r\}$ such that the interval $\tilde{I}_j$ has endpoints close to the endpoints of I_j, for $1 \le j \le r$ and $\tilde{K}$ is defined by a C^s map $\tilde{\psi}$ which is close to ψ in the C^s topology.

The C^{1+}-topology is such that a sequence ψ_n converges to ψ if there is some $\alpha > 0$ such that ψ_n is $C^{1+\alpha}$ for every $n \ge 1$ and ψ_n converges to ψ in the $C^{1+\alpha}$-topology.

The concept of stable intersection of two regular Cantor sets was introduced in Moreira [1996]: two Cantor sets K_1 and K_2 have stable intersection if there is a neighbourhood V of (K_1, K_2) in the set of pairs of C^{1+}-regular Cantor sets such that $(\widetilde{K}_1, \widetilde{K}_2) \in V \Rightarrow \widetilde{K}_1 \cap \widetilde{K}_2 \neq \varnothing$.

In the same paper conditions based on renormalizations were introduced to ensure stable intersections, and applications of stable intersections to homoclinic bifurcations were obtained: roughly speaking, if some translations of the stable and unstable regular Cantor sets associated to the horseshoe at the initial bifurcation parameter $\mu = 0$ have stable intersection then the set $\{\mu > 0 \mid \varphi_\mu$ presents persistent homoclinic tangencies$\}$ has positive Lebesgue density at $\mu = 0$. It was also shown that this last phenomenon can coexist with positive density of hyperbolicity in a persistent way.

Besides, the following question was posed in Moreira [ibid.]: Does there exist a dense (and automatically open) subset $\mathcal{U}$ of

$$\Omega^\infty = \{(K_1, K_2); K_1, K_2 \text{ are } C^\infty\text{−regular Cantor sets and } HD(K_1) + HD(K_2) > 1\}$$

such that $(K_1, K_2) \in \mathcal{U} \Rightarrow \exists\, t \in \mathbb{R}$ such that $(K_1, K_2 + t)$ has stable intersection? A positive answer to this question implies a strong version of Palis' conjecture. Indeed, $K_1 - K_2 = \{t \in \mathbb{R} \mid K_1 \cap (K_2 + t) \neq \varnothing\}$, so, if $(K_1, K_2 + t)$ has stable intersection then t belongs persistently to the interior of $K_1 - K_2$.

Moreira and Yoccoz [2001] gave an affirmative answer to this question, proving the following

Theorem 2.4. *There is an open and dense set* $\mathcal{U} \subset \Omega^\infty$ *such that if* $(K_1, K_2) \in \mathcal{U}$, *then* $I_s(K_1, K_2)$ *is dense in* $K_1 - K_2$ *and* $HD((K_1 - K_2) \backslash I_s(K_1, K_2)) < 1$, *where*

$$I_s(K_1, K_2) := \{t \in \mathbb{R} \mid K_1 \textit{ and } (K_2 + t) \textit{ have stable intersection}\}.$$

Jean-Christophe Yoccoz

The same result works if we replace stable intersection by d-stable intersection, which is defined by asking that any pair $(\widetilde{K}_1, \widetilde{K}_2)$ in some neighbourhood of (K_1, K_2) satisfies $HD(\widetilde{K}_1 \cap \widetilde{K}_2) \geq d$: most pairs of Cantor sets $(K_1, K_2) \in \Omega^\infty$ have d-stable intersection for any $d < HD(K_1) + HD(K_2) - 1$.

The open set $\mathcal{U}$ mentioned in the above theorem is very large in Ω^∞ in the sense that generic n-parameter families in Ω^∞ are actually contained in $\mathcal{U}$.

The proof of this theorem depends on a sufficient condition for the existence of stable intersections of two Cantor sets, related to a notion of renormalization, based on the fact that small parts of regular Cantor sets are diffeomorphic to the whole set: the existence of a *recurrent compact set* of relative positions of limit geometries of them. Roughly speaking, it is a compact set of relative positions of regular Cantor sets such that, for any relative position in such a set, there is a pair of (small) intervals of the construction of the Cantor sets such that the renormalizations of the Cantor sets associated to these intervals belong to the interior of the same compact set of relative positions.

The main result is reduced to prove the existence of recurrent compact sets of relative positions for most pairs of regular Cantor sets whose sum of Hausdorff dimensions is larger than one. A central argument in the proof of this fact is a probabilistic argument à la Erdős: we construct a family of perturbations with a large number of parameters

and show the existence of such a compact recurrent set with large probability in the parameter space (without exhibiting a specific perturbation which works).

Another important ingredient in the proof is the Scale Recurrence Lemma, which, under mild conditions on the Cantor sets (namely that at least one of them is not essentially affine), there is a recurrent compact set for renormalizations at the level of relative scales of limit geometries of the Cantor sets. This lemma is the fundamental tool in the paper Moreira [2016], in which it is proved that, under the same hypothesis above, if K and K' are C^2-regular Cantor sets, then $HD(K+K') = \min\{1, HD(K)+HD(K')\}$.

An important result in fractal geometry which is used in Moreira and Yoccoz [2001] is the famous Marstrand's theorem (Marstrand [1954]), according to which, given a Borel set $X \subset \mathbb{R}^2$ with $HD(X) > 1$ then, for almost every $\lambda \in \mathbb{R}$, $\pi_\lambda(X)$ has positive Lebesgue measure, where $\pi_\lambda : \mathbb{R}^2 \to \mathbb{R}$ is given by $\pi_\lambda(x,y) = x - \lambda y$. In particular, if K_1 and K_2 are regular Cantor sets with $HD(K_1) + HD(K_2) > 1$ then, for almost every $\lambda \in \mathbb{R}$, $K_1 - \lambda K_2$ has positive Lebesgue measure. Moreira and Yuri Lima gave combinatorial alternative proofs of Marstrand's theorem, first in the case of Cartesian products regular Cantor sets (Y. Lima and Moreira [2011a]) and then in the general case (Y. Lima and Moreira [2011b]).

In Moreira and Yoccoz [2010], Moreira and Yoccoz proved the following fact concerning generic homoclinic bifurcations associated to two dimensional saddle-type hyperbolic sets (horseshoes) with Hausdorff dimension bigger than one: typically there are translations of the stable and unstable Cantor sets having stable intersection, and so it yields open sets of stable tangencies in the parameter line with positive density at the initial bifurcation value. Moreover, the union of such a set with the hyperbolicity set in the parameter line generically has full density at the initial bifurcation value. This extends the results of Palis Jr. and Yoccoz [1994].

The situation is quite different in the C^1-topology, in which stable intersections do not exist:

Theorem 2.5 (Moreira [2011]). *Given any pair (K, K') of regular Cantor sets, we can find, arbitrarily close to it in the C^1 topology, pairs $(\tilde{K}, \tilde{K}')$ of regular Cantor sets with $\tilde{K} \cap \tilde{K}' = \varnothing$.*

Moreover, for generic pairs (K, K') of C^1-regular Cantor sets, the arithmetic difference $K - K'$ has empty interior (and so is a Cantor set).

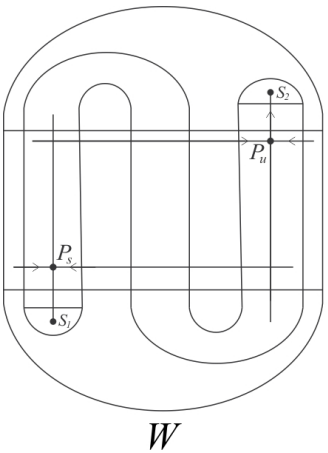
The main technical difference between the C^1 case and the C^2 (or even $C^{1+\alpha}$) cases is the lack of bounded distortion of the iterates of ψ in the C^1 case, and this fact is fundamental for the proof of the previous result.

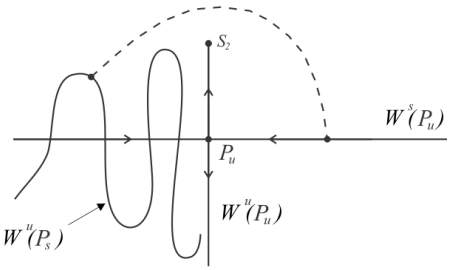
The previous result may be used to show that there are no C^1 robust tangencies between leaves of the stable and unstable foliations of respectively two given hyperbolic horseshoes Λ_1, Λ_2 of a diffeomorphism of a surface. This is also very different from the situation in the C^∞ topology - for instance, in Moreira and Yoccoz [2010] it is proved that, in the unfolding of a homoclinic or heteroclinic tangency associated to two horseshoes, when the sum of the correspondent stable and unstable Hausdorff dimensions is larger than one, there are generically stable tangencies associated to these two horseshoes. This result is done in the following

Theorem 2.6 (Moreira [2011]). *Given a C^1 diffeomorphism ψ of a surface M having two (non necessarily disjoint) horseshoes Λ_1, Λ_2, we can find, arbitrarily close to it in the C^1 topology, a diffeomorphism $\tilde{\psi}$ of the surface for which the horseshoes Λ_1, Λ_2 have hyperbolic continuations $\tilde{\Lambda}_1, \tilde{\Lambda}_2$, and there are no tangencies between leaves of the stable and unstable foliations of $\tilde{\Lambda}_1$ and $\tilde{\Lambda}_2$, respectively. Moreover, there is a generic set $\mathcal{R}$ of C^1 diffeomorphisms of M such that, for every $\check{\psi} \in \mathcal{R}$, there are no tangencies between leaves of the stable and unstable foliations of Λ_1, Λ_2, for any horseshoes Λ_1, Λ_2 of $\check{\psi}$.*

Since stable intersections of Cantor sets are the main known obstructions to density of hyperbolicity for diffeomorphisms of surfaces, the previous result gives some hope of proving density of hyperbolicity in the C^1 topology for diffeomorphisms of surfaces, a well-known question by Smale. In particular in the work Matheus, Moreira, and Pujals [2013] on a family of two-dimensional maps (the so-called Benedicks–Carleson *toy model* for Hénon dynamics) in which we prove that in this family there are diffeomorphisms which present stable homoclinic tangencies (Newhouse's phenomenon) in the C^2-topology but their elements can be arbitrarily well approximated in the C^1-topology by hyperbolic maps.

2.3 Palis–Yoccoz theorem on non-uniformly hyperbolic horseshoes. In a major work by Palis Jr. and Yoccoz [2009]; (see also Palis Jr. and Yoccoz [2001b] and Palis Jr. and Yoccoz [2001a]), they propose to advance considerably the current knowledge on the topic of bifurcations of heteroclinic cycles for smooth (C^∞) parametrized families $\{g_t, t \in \mathbb{R}\}$ of surface diffeomorphisms. They assume that g_t is hyperbolic for $t < 0$ and $|t|$ small and that a quadratic tangency q is formed at $t = 0$ between the stable and unstable lines of two periodic points, not belonging to the same orbit, of a (uniformly hyperbolic) horseshoe Λ and that such lines cross each other with positive relative speed as the parameter evolves, starting at $t = 0$ near the point q. They also assume that, in some neighbourhood W of the union of Λ with the orbit of tangency $O(q)$, the maximal invariant set for $g_0 = g_{t=0}$ is $\Lambda \cup O(q)$, where $O(q)$ denotes the orbit of q for g_0.





A main novelty is that they allowed the Hausdorff dimension $HD(\Lambda)$ to be bigger than one: they assume $HD(\Lambda)$ to be bigger than one, but not much bigger (more precisely, they assume that, if d_s and d_u are the Hausdorff dimensions of, respectively, the stable and unstable Cantor sets of g_0 then $(d_s + d_u)^2 + \max(d_s, d_u)^2 < d_s + d_u + \max(d_s, d_u)$). Then, for most small values of t, g_t is a "non-uniformly hyperbolic" horseshoe in W, and so g_t has no attractors nor repellors in W. Most small values of t, and thus most g_t, here means that t is taken in a set of parameter values with Lebesgue density one at $t = 0$.

The construction of non-uniformly hyperbolic horseshoes for most parameters is a highly non-trivial counterpart of Yoccoz' proof Yoccoz [2015] (based on the so-called *Yoccoz puzzles*) of the celebrated Jakobson's theorem in the context of heteroclinic explosions.

Of course, Palis and Yoccoz do not consider their result as the end of the line. Indeed, they expected the same results to be true for all cases $0 < HD(\Lambda) < 2$. However, to achieve that, it seems that their methods need to be considerably sharpened: it would be necessary to study deeper the dynamical recurrence of points near tangencies of higher order (cubic, quartic...) between stable and unstable curves. They also expected their results to be true in higher dimensions.

Finally, they hoped that the ideas introduced in that work might be useful in broader contexts. In the horizon lies a famous question concerning the standard family of area preserving maps (which is the family $(f_k)_{k\in\mathbb{R}}$ of diffeomorphisms of the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ given by $f_k(x, y) = (-y + 2x + k\sin(2\pi x), x) \pmod 1$): can we find sets of positive Lebesgue measure in the parameter space such that the corresponding maps display non-zero Lyapunov exponents in sets of positive Lebesgue probability in phase space?

In a recent work, Matheus, Moreira and Palis applied the results of Palis Jr. and Yoccoz [2009] to prove the following

Theorem 2.7. *There exists $k_0 > 0$ such that, for all $|k| > k_0$ and $\varepsilon > 0$, the subset of parameters $r \in \mathbb{R}$ such that $|r - k| < \varepsilon$ and f_r exhibits a non-uniformly hyperbolic horseshoe (in the sense of Palis Jr. and Yoccoz [ibid.]) has positive Lebesgue measure.*

2.4 Hyperbolic sets in higher dimensions. The fractal geometric properties of horseshoes of surface diffeomorphisms are quite regular and well-understood. As we mentioned before, a horseshoe Λ in ambient dimension 2 is locally diffeomorphic to a cartesian product $K^s \times K^u$, where K^s and K^u are regular Cantor sets on the real line. McCluskey and Manning proved that $HD(\Lambda)$ is continuous in the C^1–topology (later Palis and Viana gave a more geometrical proof of this fact).

In ambient dimension larger than two, much less is known about the geometry of horseshoes. In general, if Λ is such a horseshoe, the Hausdorff dimension $HD(\Lambda)$ is not always continuous, as shown by Bonatti, Díaz and Viana. A natural question is whether $HD(\Lambda)$ is "typically" continuous. Another natural question is what can be said about the fractal geometry of stable and unstable Cantor sets of horseshoes in arbitrary dimensions. These Cantor sets are no more conformal sets with bounded geometry, which was the case of stable and unstable Cantor sets of horseshoes in surfaces. In what

follows, we will discuss some recent results on the geometry of horseshoes in arbitrary dimensions.

In the first work, in collaboration with J. Palis and M. Viana, we generalize an important lemma from Moreira, Palis Jr., and Viana [2001]: given a horseshoe Λ whose stable spaces have dimension ℓ we define a family of fractal dimensions (the so-called *upper stable dimensions*) $\overline{d}_s^{(j)}(\Lambda), 1 \leq j \leq \ell$ which satisfy $\overline{d}_s^{(1)}(\Lambda) \geq \overline{d}_s^{(2)}(\Lambda) \geq \cdots \geq \overline{d}_s^{(\ell)}(\Lambda) \geq HD(\Lambda \cap W^s(x)), \forall x \in \Lambda$. We define $\hat{d}_s(\Lambda) = \bar{d}_s^{(\ell)}(\Lambda)$. When $r < \overline{d}_s^{(r)}(\Lambda) \leq r+1$ we have $\hat{d}_s(\Lambda) = \overline{d}_s^{(r)}(\Lambda)$ These definitions are inspired in the *affinity dimensions*, introduced by Falconer. We have analogous definitions for upper unstable dimensions. We prove the following results about these dimensions: given $1 \leq r \leq \ell$ and $\varepsilon > 0$ there is a $\varepsilon-$small C^∞ perturbation of the original diffeomorphism for which the hyperbolic continuation of Λ has a subhorseshoe $\tilde{\Lambda}$ which has strong-stable foliations of codimensions j for $1 \leq j \leq r$ and which satisfies $\overline{d}_s^{(r)}(\tilde{\Lambda}) > \overline{d}_s^{(r)}(\Lambda) - \varepsilon$.

In the second work in progress, in collaboration with W. Silva (which extends a previous joint work in codimension 1 - see Moreira and Silva [2012]), we prove that if a horseshoe Λ has strong stable foliations of codimensions j for $1 \leq j \leq r$ and satisfies $\hat{d}_s(\Lambda) > r$ (which is equivalent to $\overline{d}_s^{(r)}(\Lambda) > r$) then it has a small C^∞ perturbation which contains a blender of codimension r: in particular C^1 images of stable Cantor sets of it (of the type $\Lambda \cap W^s(x)$) in $\mathbb{R}^r$ will typically have persistently non-empty interior. We also expect to prove that the Hausdorff dimension of these stable Cantor sets typically coincide with $\hat{d}_s(\Lambda)$, and this dimension depends continuously on Λ on these assumptions, which would imply typical continuity of Hausdorff dimensions of stable and unstable Cantor sets of horseshoes.

Let us recall two main ingredients in the proof by Moreira and Yoccoz of Palis' conjecture:

- A *recurrent compact set criterion* for stable intersections (which implies that arithmetic differences persistently contain intervals).
- An application of Erdős probabilistic method: a family of C^∞ small perturbations of a regular Cantor set (the second Cantor set is fixed) with a large number of parameters such that for most parameters there is a recurrent compact set for the corresponding pair of Cantor sets.

A variation of these ingredients is present in the proof of these results in collaboration of W. Silva: we develop a recurrent compact set criterion which implies that a given horseshoe (with a strong-stable foliation of codimension r) is a r-codimensional blender, and we prove that, if a horseshoe has a strong-stable foliation of codimension r and satisfies $\hat{d}_s(\Lambda) > r$ (which is equivalent to $\overline{d}_s^{(r)}(\Lambda) > r$), then there is a small C^∞ perturbation of it which has a recurrent compact set. In order to do this, we use the probabilistic method: we construct a family of perturbations with a large number of parameters, and show that for most parameters there is such a recurrent compact set.

An important geometrical tool in the proof of these results is the following generalization of Marstrand's theorem, which was proved in works by López, Moreira, Romaña and Silva:

Theorem 2.8. *Let X be a compact metric space, $(\Lambda, \mathcal{P})$ a probability space and $\pi : \Lambda \times X \to \mathbb{R}^r$ a measurable function. Informally, one can think of $\pi_\lambda(\cdot) = \pi(\lambda, \cdot)$ as a family of projections parameterized by λ. We assume that for some positives real numbers α and C the following transversality property is satisfied:*

$$\mathcal{P}[\lambda \in \Lambda : d(\pi_\lambda(x_1), \pi_\lambda(x_2)) \leq \delta d(x_1, x_2)^\alpha] \leq C\delta^r \tag{1}$$

for all $\delta > 0$ and all $x_1, x_2 \in X$. Assume that $\dim X > \alpha r$. *Then $Leb(\pi_\lambda(X)) > 0$ for a.e. $\lambda \in \Lambda$ and $\int_\Lambda Leb(\pi_\lambda(X))^{-1} d\mathcal{P} < +\infty$.*

3 Fractal Geometry and Diophantine Approximations

3.1 The classical Markov and Lagrange spectra. The results discussed in the previous section on regular Cantor sets have, somewhat surprisingly, deep consequences in number theory, which we discuss below. We begin by introducing the classical Lagrange spectrum.

Let α be an irrational number. According to Dirichlet's theorem, the inequality $|\alpha - \frac{p}{q}| < \frac{1}{q^2}$ has infinitely many rational solutions $\frac{p}{q}$. Hurwitz improved this result by proving that $|\alpha - \frac{p}{q}| < \frac{1}{\sqrt{5}q^2}$ also has infinitely many rational solutions $\frac{p}{q}$ for any irrational α, and that $\sqrt{5}$ is the largest constant that works for any irrational α. However, for particular values of α we can improve this constant.

More precisely, we define $k(\alpha) := \sup\{k > 0 \mid |\alpha - \frac{p}{q}| < \frac{1}{kq^2}$ has infinitely many rational solutions $\frac{p}{q}\} = \limsup_{p,q\to+\infty} (q|q\alpha - p|)^{-1}$. We have $k(\alpha) \geq \sqrt{5}$, $\forall \alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $k\left(\frac{1+\sqrt{5}}{2}\right) = \sqrt{5}$. We will consider the set $L = \{k(\alpha) \mid \alpha \in \mathbb{R} \setminus \mathbb{Q}, k(\alpha) < +\infty\}$.

This set is called the Lagrange spectrum. Hurwitz's theorem determines the smallest element of L, which is $\sqrt{5}$. This set L encodes many diophantine properties of real numbers. It is a classical subject the study of the geometric structure of L. Markov proved in 1879 (Markoff [1880]) that

$$L \cap (-\infty, 3) = \{k_1 = \sqrt{5} < k_2 = 2\sqrt{2} < k_3 = \frac{\sqrt{221}}{5} < \dots\}$$

where k_n is a sequence (of irrational numbers whose squares are rational) converging to 3.

The elements of the Lagrange spectrum which are smaller than 3 are exactly the numbers of the form $\sqrt{9 - \frac{4}{z^2}}$ where z is a positive integer for which there are other positive integers x, y such that $1 \leq x \leq y \leq z$ and (x, y, z) is a solution of the *Markov equation*

$$x^2 + y^2 + z^2 = 3xyz.$$

Since this is a quadratic equation in x (resp. in y), whose sum of roots is $3yz$ (resp. $3xz$), given a solution (x, y, z) with $x \leq y \leq z$, we also have the two other solutions $(y, z, 3yz - x)$ and $(x, z, 3xz - y)$ - it is possible to prove that all solutions of the Markov equation appear in the following tree:

$$\begin{tikzcd}
 & (1,1,1) \arrow[d] & \\
 & (1,1,2) \arrow[d] & \\
 & (1,2,5) \arrow[dl] \arrow[dr] & \\
(1,5,13) \arrow[d] \arrow[dr] & & (2,5,29) \arrow[d] \arrow[dr] \\
(1,13,34) & (5,13,194) & (2,29,169) & (5,29,433)
\end{tikzcd}$$

An important open problem related to Markov's equation is the *Unicity Problem*, formulated by Frobenius about 100 years ago: for any positive integers x_1, x_2, y_1, y_2, z with $x_1 \leq y_1 \leq z$ and $x_2 \leq y_2 \leq z$ such that (x_1, y_1, z) and (x_2, y_2, z) are solutions of Markov's equation we always have $(x_1, y_1) = (x_2, y_2)$?

If the Unicity Problem has an affirmative answer then, for every real $t < 3$, its preimage $k^{-1}(t)$ by the function k above consists of a single $GL_2(\mathbb{Z})$-equivalence class (this equivalence relation is such that

$$\alpha \sim \frac{a\alpha + b}{c\alpha + d}, \forall a, b, c, d \in \mathbb{Z}, |ad - bc| = 1.)$$

Despite the "beginning" of the set L is discrete, this is not true for the whole set L. As we mentioned in the introduction, M. Hall proved in 1947 (Hall [1947]) that if $C(4)$ is the regular Cantor set formed by the numbers in $[0, 1]$ whose coefficients in the continued fractions expansion are bounded by 4, then one has $C(4) + C(4) = [\sqrt{2} - 1, 4(\sqrt{2}-1)]$. This implies that L contains a whole half line (for instance $[6, +\infty)$), and G. Freĭman determined in 1975 (Freĭman [1975]) the biggest half line that is contained in L, which is $[c_F, +\infty)$, with

$$c_F = \frac{2221564096 + 283748\sqrt{462}}{491993569} \cong 4,52782956616\ldots.$$

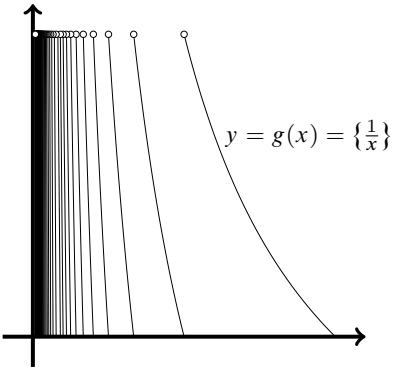
These last two results are based on the study of sums of regular Cantor sets, whose relationship with the Lagrange spectrum will be explained below.

Sets of real numbers whose continued fraction representation has bounded coefficients with some combinatorial constraints, as $C(4)$, are often regular Cantor sets, which we call Gauss–Cantor sets (since they are defined by restrictions of the Gauss map $g(x) = \{1/x\}$ from $(0, 1)$ to $[0, 1)$ to some convenient union of intervals).

We represent below the graphics of the Gauss map $g(x) = \{\frac{1}{x}\}$.

If the continued fraction of α is

$$\alpha = [a_0; a_1, a_2, \ldots] \stackrel{\text{def}}{=} a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}}.$$



then we have the following formula for $k(\alpha)$:

$$k(\alpha) = \limsup_{n\to\infty}(\alpha_n + \beta_n),$$

where $\alpha_n = [a_n; a_{n+1}, a_{n+2}, \dots]$ and $\beta_n = [0; a_{n-1}, a_{n-2}, \dots, a_1]$.

The previous formula follows from the equality

$$|\alpha - \frac{p_n}{q_n}| = \frac{1}{(\alpha_{n+1} + \beta_{n+1})q_n^2}, \quad \forall n \in \mathbb{N},$$

where

$$p_n/q_n = [a_0; a_1, a_2, \dots, a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \frac{1}{a_n}}}, n \in \mathbb{N}$$

are the convergents of the continued fraction of α.

There are many results which relate the dynamics of the Gauss map with the behaviour of continued fractions. For instance, the Khintchine–Lévy theorem, which follows from techniques of Ergodic Theory, states that, for (Lebesgue) almost every $\alpha \in \mathbb{R}$,

$$\lim_{n\to\infty} \sqrt[n]{q_n} = e^{\pi^2/12\log 2}.$$

Remark: The following elementary general facts on Diophantine approximations of real numbers show that the best rational approximations of a given real number are given by convergents of its continued fraction representation:

• For every $n \in \mathbb{N}$,

$$\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{2q_n^2} \text{ or } \left|\alpha - \frac{p_{n+1}}{q_{n+1}}\right| < \frac{1}{2q_{n+1}^2}$$

(moreover, for every positive integer n, there is $k \in \{n-1, n, n+1\}$ with $|\alpha - \frac{p_k}{q_k}| < \frac{1}{\sqrt{5}q_n^2}$).

• If $\left|\alpha - \frac{p}{q}\right| < \frac{1}{2q^2}$ then $\frac{p}{q}$ is a convergent of the continued fraction of α.

This formula for $k(\alpha)$ implies that we have the following alternative definition of the Lagrange spectrum L:

Let $\Sigma = (\mathbb{N}^*)^{\mathbb{Z}}$ be the set of all bi-infinite sequences of positive integers. If $\underline{\theta} = (a_n)_{n\in\mathbb{Z}} \in \Sigma$, let $\alpha_n = [a_n; a_{n+1}, a_{n+2}, \dots]$ and $\beta_n = [0; a_{n-1}, a_{n-2}, \dots], \forall n \in \mathbb{Z}$. We define $f(\underline{\theta}) = \alpha_0 + \beta_0 = [a_0; a_1, a_2, \dots] + [0; a_{-1}, a_{-2}, \dots]$. We have

$$L = \{\limsup_{n\to\infty} f(\sigma^n \underline{\theta}), \underline{\theta} \in \Sigma\},$$

where $\sigma : \Sigma \to \Sigma$ is the shift defined by $\sigma((a_n)_{n\in\mathbb{Z}}) = (a_{n+1})_{n\in\mathbb{Z}}$.

Let us define the Markov spectrum M by

$$M = \{\sup_{n\in\mathbb{Z}} f(\sigma^n \underline{\theta}), \underline{\theta} \in \Sigma\}.$$

It also has an arithmetical interpretation, namely

$$M = \{(\inf_{(x,y)\in\mathbb{Z}^2\setminus(0,0)} |f(x,y)|)^{-1}, \quad f(x,y) = ax^2 + bxy + cy^2, \quad b^2 - 4ac = 1\}.$$

It is well-known (see Cusick and Flahive [1989]) that M and L are closed sets of the real line and $L \subset M$.

We have the following result about the Markov and Lagrange spectra:

Theorem 3.1. *(Moreira [2018]; see also Matheus [2017]) Given $t \in \mathbb{R}$ we have*

$$HD(L \cap (-\infty, t)) = HD(M \cap (-\infty, t)) =: d(t)$$

and $d(t)$ is a continuous surjective function from $\mathbb{R}$ to $[0,1]$. Moreover:

i) $d(t) = \min\{1, 2D(t)\}$, where $D(t) := HD(k^{-1}(-\infty, t)) = HD(k^{-1}(-\infty, t])$ is a continuous function from $\mathbb{R}$ to $[0,1)$.

ii) $\max\{t \in \mathbb{R} \mid d(t) = 0\} = 3$

iii) $d(\sqrt{12}) = 1$.

A fundamental tool in the proof of this result is the theorem below.

We say that a C^2-regular Cantor set on the real line is *essentially affine* if there is a C^2 change of coordinates for which the dynamics that defines the corresponding Cantor set has zero second derivative on all points of that Cantor set. Typical C^2-regular Cantor sets are not essentially affine.

The *scale recurrence lemma*, which is the main technical lemma of Moreira and Yoccoz [2001], can be used in order to prove the following

Theorem 3.2. *(Moreira [2016]) If K and K' are regular Cantor sets of class C^2 and K is non essentially affine, then $HD(K + K') = \min\{HD(K) + HD(K'), 1\}$.*

Remark 3.3. There is a presentation of a version of this result (with a slightly different hypothesis) in Moreira [1999]. That version is also proved by Hochman and Shmerkin [2012].

The results of Markov, Hall and Freĭman mentioned above imply that the Lagrange and Markov spectra coincide below 3 and above c_F. Nevertheless, Freĭman [1968] showed in 1968 that $M \setminus L \neq \varnothing$ by exhibiting a number $\sigma \simeq 3.1181\cdots \in M \setminus L$. Moreira and Matheus described the structure of the intersection of $M \setminus L$ with an interval near this point, and proved that it has positive Hausdorff dimension (see Matheus and Moreira [2019]).

In 1973, Freĭman [1973] showed that

$$\alpha_\infty := \lambda_0(A_\infty) := [2; \overline{1_2, 2_3, 1, 2}] + [0; 1, 2_3, 1_2, 2, 1, \overline{2}] \in M \setminus L$$

In a similar vein, Theorem 4 in Chapter 3 of Cusick and Flahive [1989] book asserts that

$$\alpha_n := \lambda_0(A_n) := [2; \overline{1_2, 2_3, 1, 2}] + [0; 1, 2_3, 1_2, 2, 1, 2_n, \overline{1, 2, 1_2, 2_3}] \in M \setminus L$$

for all $n \geq 4$. In particular, α_∞ is not isolated in $M \setminus L$.

In collaboration with C. Matheus, we proved the following results about $M \setminus L$:

Let X be the Cantor set

$$X := \{[0; \gamma] : \gamma \in \{1, 2\}^{\mathbb{N}} \text{ not containing the subwords in } P\} \tag{2}$$

where

$$P := \{21212, 2121_3, 1_3212, 12121_2, 1_22121, 2_3121_22_21, 12_21_2212_3, 12_3121_22_2, 2_21_2212_31\}$$

Also, let

$$b_\infty := [2; \overline{1_2, 2_3, 1, 2}] + [0; \overline{1, 2_3, 1_2, 2}] = \frac{\sqrt{18229}}{41} = 3.2930442439\ldots$$

and

$$\begin{aligned} B_\infty &:= [2; 1, \overline{1, 2_3, 1, 2, 1_2, 2, 1_2, 2}] + [0; 1, 2_3, 1_2, 2, 1, 2_3, 1_2, 2, 1, 2_2, \overline{1, 2_3, 1, 2, 1_2, 2, 1_2, 2}] \\ &= 3.2930444814\ldots \end{aligned}$$

Theorem 3.4. *(Matheus and Moreira [2017b])* $\{b_\infty, B_\infty\} \subset L$, $L \cap (b_\infty, B_\infty) = \varnothing$ *and* $HD((M \setminus L) \cap (b_\infty, B_\infty)) = HD(X)$.

We implemented the algorithm of Jenkinson and Pollicott [2018] and we obtained the *heuristic* approximation $HD(X) = 0.4816\cdots$. We also we exhibited a Cantor set $K(\{1, 2_2\}) \subset X$ whose Hausdorff dimension can be easily (and rigorously) estimated as $0.353 < HD(K(\{1, 2_2\})) < 0.35792$ via some classical arguments explained in Palis Jr. and Takens [1987] book.

By exploiting the arguments establishing the above Theorem, we are able to exhibit new numbers in $M \setminus L$, including a constant $c \in M \setminus L$ with $c > \alpha_4$:

Proposition 3.5. *The largest element of* $(M \setminus L) \cap (b_\infty, B_\infty)$ *is*

$$c = \frac{77 + \sqrt{18229}}{82} + \frac{17633692 - \sqrt{151905}}{24923467} = 3.29304447990138\ldots.$$

We also found elements of $M \setminus L$ larger than $\sqrt{12}$, disproving a conjecture by Cusick, and giving a new lower estimate for the Hausdorff dimension of $M \setminus L$ (see Matheus and Moreira [2018]):

Proposition 3.6. *The Hausdorff dimension of* $(M \setminus L) \cap (3.7, 3.71)$ *is larger than* 0.53128. *This intersection includes the element*

$$[3; 2, 2, 2, 1, 2, 3, 3, 2, 2, 2, 1, 2, 2, 1, 2, 1, 2, 1, \overline{1, 2}] + [0; 3, \overline{2, 1, 2, 2, 2, 3, 3}] = \tag{3}$$
$$= \frac{7940451225305 - \sqrt{3}}{2326589591051} + \frac{-483 + \sqrt{330629}}{310} = 3.70969985975\ldots.$$

To our best knowledge, this is the largest known element of $M \setminus L$.

We also proved (Matheus and Moreira [2017a]) that $M \setminus L$ *doesn't* have full Hausdorff dimension:

Theorem 3.7. $HD(M \setminus L) < 0.986927$.

One can get better *heuristic* bounds for $HD(M \setminus L)$ thanks to the several methods in the literature to numerically approximate the Hausdorff dimension of Cantor sets of numbers with prescribed restrictions of their continued fraction expansions. By implementing the "thermodynamical method" introduced in Jenkinson and Pollicott [2001], we obtained the heuristic bound $HD(M \setminus L) < 0.888$.

Our proof of Theorem 3.7 relies on the control of several portions of $M \setminus L$ in terms of the sum-set of a Cantor set associated to continued fraction expansions prescribed by a "symmetric block" and a Cantor set of irrational numbers whose continued fraction expansions live in the "gaps" of a "symmetric block". As it turns out, such a control is possible thanks to our key technical Lemma saying that a sufficiently large Markov value given by the sum of two continued fraction expansions systematically meeting a "symmetric block" must belong to the Lagrange spectrum.

It follows that $M \setminus L$ has empty interior, and so, since M and L are closed subsets of $\mathbb{R}$, $\overline{\text{int}(M)} = \overline{\text{int}(L)} \subset L \subset M$. In particular, we have the following

Corollary 3.8. $int(M) = int(L)$.

As a consequence, we recover the fact, proved in Freĭman [1975], that the biggest half-line contained in M coincides with the biggest half-line $[c_F, \infty)$ contained in L.

3.2 Other results on the fractal geometry of Diophantine approximations. In collaboration with Y. Bugeaud (see Bugeaud and Moreira [2011]), we proved some results on sets of exact approximation order by rational numbers:

For a function $\Psi : (0, +\infty) \to (0, +\infty)$, let

$$\mathcal{K}(\Psi) := \left\{\xi \in \mathbb{R} : \left|\xi - \frac{p}{q}\right| < \Psi(q) \quad \text{for infinitely many rational numbers } \frac{p}{q}\right\}$$

denote the set of Ψ-approximable real numbers and let

$$\text{Exact}(\Psi) := \mathcal{K}(\Psi) \setminus \bigcup_{m \geq 2} \mathcal{K}\big((1 - 1/m)\Psi\big)$$

be the set of real numbers approximable to order Ψ and to no better order.

The *lower order at infinity* $\lambda(g)$ of a function $g : (0, +\infty) \to (0, +\infty)$ is defined by

$$\lambda(g) = \liminf_{x \to +\infty} \frac{\log g(x)}{\log x}.$$

We say that a function $\Psi : (0, +\infty) \to (0, +\infty)$ satisfies assumption $(*)$ if $\Psi(x) = o(x^{-2})$ *and there exist real numbers* c, $\tilde{c}$ *and* n_0 *with* $1 \leq \tilde{c} < 4$ *such that, if the positive integers* m, n *satisfy* $m > n \geq n_0$, *then* $\Psi(m)m^c \leq \tilde{c}\Psi(n)n^c$. We emphasize that the real number c occurring in $(*)$ may be negative.

Theorem 3.9. *Let* $\Psi : (0, +\infty) \to (0, +\infty)$ *be a function satisfying assumption* $(*)$. *Then the set* $\mathrm{Exact}(\Psi)$ *is uncountable.*

Theorem 3.10. *Let* $\Psi : (0, +\infty) \to (0, +\infty)$ *be a function satisfying assumption* $(*)$. *If* λ *denotes the lower order at infinity of the function* $1/\Psi$, *then*

$$\dim \mathrm{Exact}(\Psi) = \dim \mathcal{K}(\Psi) = \frac{2}{\lambda}.$$

In another work collaboration with Y. Bugeaud (see Bugeaud and Moreira [2016]), we proved that there are no typical real numbers from the point of view of Diophantine approximations, in a sense that we describe in what follows. Let Ψ be an application from the set of positive integers into the set of nonnegative real numbers. Khintchine established that, if the function $q \mapsto q^2\Psi(q)$ is non-increasing and the series $\sum_{q \geq 1} q\Psi(q)$ diverges, then the set $\mathcal{K}(\Psi)$ has full Lebesgue measure (Beresnevich, Dickinson and Velani proved later the same result assuming that Ψ is just non-increasing). We show that, for almost every real number α, there is a function Ψ which satisfies good "regularity" conditions (on the speed of decreasing of Ψ) - for instance $q \mapsto q^2\Psi(q)$ is non-increasing, such that the series $\sum_{q \geq 1} q\Psi(q)$ diverges but the inequality $|\alpha - p/q| < \Psi(q)$ has no rational solution p/q.

Khintchine also showed that if the series $\sum_{q \geq 1} q\Psi(q)$ converges, then the set $\mathcal{K}(\Psi)$ has zero Lebesgue measure. We show that, for almost every real number α, there is a function Ψ which satisfies good "regularity" conditions (for instance $q \mapsto q^2\Psi(q)$ is non-increasing), such that the series $\sum_{q \geq 1} q\Psi(q)$ converges but the inequality $|\alpha - p/q| < \Psi(q)$ has infinitely many rational solutions p/q.

We also compute Hausdorff dimensions of sets of exceptions to our results (in terms of the regularity conditions on Ψ):

Theorem 3.11. *Let* λ *be a real number in the interval* $[1, 2]$. *We define* X_λ *the set of irrational numbers* ξ *such that, for every function* Ψ *with* $q \mapsto q^\lambda\Psi(q)$ *non-increasing and satisfying*

$$\sum_{q=1}^{\infty} q\Psi(q) = +\infty,$$

the inequality

$$\left|\xi - \frac{p}{q}\right| < \Psi(q)$$

has infinitely many rational solutions. We proved that X_1 is empty and that, for $\lambda \in (1,2]$, the Hausdorff dimension of X_λ is $\lambda/2$ (and so, when λ varies, assume all values in the interval $(1/2,1]$).

Theorem 3.12. *Let b be a positive real number. We define Y_b as the set of the irrational numbers ξ such that, for every function Ψ with$q \mapsto q(\log\log q)^{b\log\log q}\Psi(q)$ non-increasing and satisfying*

$$\sum_{q=1}^{\infty} q\Psi(q) = +\infty,$$

the inequality

$$\left|\xi - \frac{p}{q}\right| < \Psi(q)$$

has infinitely many rational solutions. Then the Hausdorff dimension of Y_b is $\frac{1}{1+e^{1/b}}$ (and so, when b varies, assume all values in the interval $(0,1/2)$).

One of the tools we used in the proof of this last theorem is the results from Łuczak [1997]: given $b, c > 1$ define

$$\tilde{\Xi}(b,c) = \{\xi = [0; a_1, a_2, ...] \in [0,1] : a_n \geq c^{b^n} \text{ for every } n \geq 1\}$$

and

$$\Xi(b,c) = \{\xi = [0; a_1, a_2, ...] \in [0,1] : a_n \geq c^{b^n} \text{ for infinitely many } n \geq 1\}$$

Then $HD(\tilde{\Xi}(b,c)) = HD(\Xi(b,c)) = \frac{1}{1+b}$. In one direction, a more precise result is proved in Feng, Wu, Liang, and Tseng [1997], according to which the Hausdorff dimension of the set

$$K := \{\xi = [0; a_1, a_2, ...] \in [0,1] : \exp(\tilde{b}^n) \leq a_n \leq 3\exp(\tilde{b}^n), \text{ for every } n \geq 1\}$$

is equal to $\frac{1}{1+\tilde{b}}$. We notice that an older result from Good [1941] states that the Hausdorff dimension of the set $\{\xi = [0; a_1, a_2, ...] \in [0,1] : \lim a_n = \infty\}$ is $1/2$.

4 Back to Dynamical Systems (and Differential Geometry)

As we have seen, the sets M and L can be defined in terms of symbolic dynamics. Inspired by these characterizations, we may associate to a dynamical system together with a real function generalizations of the Markov and Lagrange spectra as follows:

Definition 4.1. Given a map $\psi : X \to X$ and a function $f : X \to \mathbb{R}$, we define the associated dynamical Markov and Lagrange spectra as
$M(f,\psi) = \{\sup_{n\in\mathbb{N}} f(\psi^n(x)), x \in X\}$ and
$L(f,\psi) = \{\limsup_{n\to\infty} f(\psi^n(x)), x \in X\}$, respectively.

Given a flow $(\varphi^t)_{t\in\mathbb{R}}$ in a manifold X, we define the associated dynamical Markov and Lagrange spectra as $M(f,(\varphi^t)) = \{\sup_{t\in\mathbb{R}} f(\varphi^t(x)), x \in X\}$ and
$L(f,(\varphi^t)) = \{\limsup_{t\to\infty} f(\varphi^t(x)), x \in X\}$, respectively.

In a work in collaboration with A. Cerqueira and C. Matheus (see Cerqueira, Matheus, and Moreira [2018]), we prove the following result, which generalizes a corresponding fact in the context of the classical Markov and Lagrange spectra:

Lemma 4.2. *Let (φ, f) be a generic pair, where $\varphi\colon M^2 \to M^2$ is a diffeomorphism with $\Lambda \subset M^2$ a hyperbolic set for φ and $f : M \to \mathbb{R}$ is C^2. Let π_s, π_u be the projections of the horseshoe Λ to the stable and unstable regular Cantor sets K^s, K^u associated to it (along the unstable and stable foliations of Λ). Given $t \in \mathbb{R}$, we define*

$$\Lambda_t = \bigcap_{m\in\mathbb{Z}} \varphi^m(\{p \in \Lambda | f(p) \leq t\}),$$

$$K_t^s = \pi_s(\Lambda_t), K_t^u = \pi_u(\Lambda_t).$$

Then the functions $d_s(t) = HD(K_t^s)$ and $d_u(t) = HD(K_t^u)$ are continuous and coincide with the corresponding box dimensions.

The following result is a consequence of the scale recurrence lemma of Moreira and Yoccoz [2001] (see Moreira [2016]):

Lemma 4.3. *Let (φ, f) be a generic pair, where $\varphi\colon M^2 \to M^2$ is a diffeomorphism with $\Lambda \subset M^2$ a hyperbolic set for φ and $f : M \to \mathbb{R}$ is C^2. Then*

$$HD(f(\Lambda)) = \min(HD(\Lambda), 1).$$

Moreover, if $HD(\Lambda) > 1$ then $f(\Lambda)$ has persistently non-empty interior.

Using the previous lemmas we prove a generalization of the results on dimensions of the dynamical spectra:

Theorem 4.4. *(Cerqueira, Matheus, and Moreira [2018]) Let (φ, f) be a generic pair, where $\varphi\colon M^2 \to M^2$ is a* **conservative** *diffeomorphism with $\Lambda \subset M^2$ a hyperbolic set for φ and $f : M \to \mathbb{R}$ is C^2. Then*

$$HD(L(f, \Lambda) \cap (-\infty, t)) = HD(M(f, \Lambda) \cap (-\infty, t)) =: d(t)$$

is a continuous real function whose image is $[0, \min(HD(\Lambda), 1)]$.

(here we use the notation $L(f, \Lambda) := L(f, \varphi|_\Lambda)$).

In Hubert, Marchese, and Ulcigrai [2015], P. Hubert, L. Marchese and C. Ulcigrai introduced in a similar way to the above generalizations the Lagrange spectra of closed-invariant loci for the action of $SL(2, \mathbb{R})$ on the moduli space of translation surfaces, in the context of Teichmüller dynamics, and proved that several of these spectra contain a Hall's ray.

Moreira and Romaña proved the following result on Markov and Lagrange spectra for horseshoes:

Theorem 4.5 (Romaña Ibarra and Moreira [2017a]). *Let Λ be a horseshoe associated to a C^2-diffeomorphism φ such that $HD(\Lambda) > 1$. Then there is, arbitrarily close to φ a diffeomorphism φ_0 and a C^2-neighborhood W of φ_0 such that, if Λ_ψ denotes the*

continuation of Λ *associated to* $\psi \in W$*, there is an open and dense set* $H_\psi \subset C^1(M, \mathbb{R})$ *such that for all* $f \in H_\psi$*, we have*

$$\text{int } L(f, \Lambda_\psi) \neq \varnothing \text{ and int } M(f, \Lambda_\psi) \neq \varnothing,$$

where int A *denotes the interior of* A*.*

Recently, D. Lima and Moreira [2018] proved that, for typical pairs (f, Λ) as in the above theorem for which φ is conservative,

$$\sup\{t \in \mathbb{R} | HD(M(f, \Lambda) \cap (-\infty, t)) < 1\} = \inf\{t \in \mathbb{R} | \text{int}(L(f, \Lambda) \cap (-\infty, t)) \neq \varnothing\},$$

and Moreira [2017] proved that, for typical pairs (f, Λ) as above, the minima of the corresponding Lagrange and Markov dynamical spectra coincide and are given by the image of a periodic point of the dynamics by the real function, solving a question by Yoccoz.

The classical Markov and Lagrange spectra can also be characterized as sets of maximum heights and asymptotic maximum heights, respectively, of geodesics in the modular surface $N = \mathbb{H}^2/PSL(2, \mathbb{Z})$. Moreira and Romaña extend in Romaña Ibarra and Moreira [2015] the fact that these spectra have non-empty interior to the context of negative, non necessarily constant curvature as follows:

Theorem 4.6. *(Romaña Ibarra and Moreira [ibid.]) Let* M *provided with a metric* g_0 *be a complete noncompact surface* M *with finite Gaussian volume and Gaussian curvature bounded between two negative constants,* i.e.*, if* K_M *denotes the Gaussian curvature, then there are constants* $a, b > 0$ *such that*

$$-a^2 \leq K_M \leq -b^2 < 0.$$

Denote by SM *its unitary tangent bundle and by* ϕ *its geodesic flow.*

Then there is a metric g *close to* g_0 *and a dense and* C^2*-open subset* $\mathcal{H} \subset C^2(SM, \mathbb{R})$ *such that*

$$\text{int } M(f, \phi_g) \neq \varnothing \text{ and int } L(f, \phi_g) \neq \varnothing$$

for any $f \in \mathcal{H}$*, where* ϕ_g *is the vector field defining the geodesic flow of the metric* g*.*

Moreover, if X *is a vector field sufficiently close to* ϕ_g *then*

$$\text{int } M(f, X) \neq \varnothing \text{ and int } L(f, X) \neq \varnothing$$

for any $f \in \mathcal{H}$*.*

Recently, in collaboration with Pacífico and Romaña (see Moreira, Pacifico, and Romaña Ibarra [n.d.]), we proved an analogous result for Lorenz flows. Romaña also proved (Romaña Ibarra and Moreira [2017b]) a corresponding result for Anosov flows in compact 3-dimensional manifolds.

Combining the techniques of this result with those of the above results in collaboration with A. Cerqueira and C. Matheus, Cerqueira, Moreira and Romaña proved the following:

Theorem 4.7. *(Cerqueira, Moreira, and Romaña Ibarra [2017]) Let (φ, f) be a generic pair, where $\varphi\colon M^2 \to M^2$ is a* **conservative** *diffeomorphism with $\Lambda \subset M^2$ a hyperbolic set for φ and $f : M \to \mathbb{R}$ is C^2. Then*

$$HD(L(f,\Lambda) \cap (-\infty, t)) = HD(M(f,\Lambda) \cap (-\infty, t)) =: d(t)$$

is a continuous real function whose image is $[0, \min(HD(\Lambda), 1)]$.

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METRIC DIMENSION REDUCTION: A SNAPSHOT OF THE RIBE PROGRAM

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Abstract

The purpose of this article is to survey some of the context, achievements, challenges and mysteries of the field of *metric dimension reduction*, including new perspectives on major older results as well as recent advances.

1 Introduction

From the point of view of theoretical computer science, mathematicians "stumbled upon" metric dimension reduction in the early 1980s, as exemplified by the following quote Vempala [2004].

> *Two decades ago, analysts stumbled upon a surprising fact [...], the Johnson–Lindenstrauss Lemma, as a crucial tool in their project of extending functions in continuous ways. This result [...] says that, if you project n points in some high dimensional space down to a random $O(\log n)$-dimensional plane, the chances are overwhelming that all distances will be preserved within a small relative error. So, if distance is all you care about, there is no reason to stay in high dimensions!*
>
> C. Papadimitriou, 2004 (forward to *The random projection method* by S. Vempala).

The above use of the term "stumbled upon" is justified, because it would be fair to say that at the inception of this research direction mathematicians did not anticipate the remarkable swath of its later impact on algorithms. However, rather than being discovered accidentally, the investigations that will be surveyed here can be motivated by classical issues in metric geometry. From the internal perspective of pure mathematics, it would be more befitting to state that the aforementioned early work stumbled upon the unexpected depth, difficulty and richness of basic questions on the relation between "rough quantitative geometry" and dimension. Despite major efforts by many mathematicians over the past four decades, such questions remain by and large stubbornly open.

We will explain below key ideas of major developments in metric dimension reduction, and also describe the larger mathematical landscape that partially motivates these investigations, most notably the *bi-Lipschitz embedding problem into* $\mathbb{R}^n$ and the *Ribe program*. By choosing to focus on aspects of this area within pure mathematics, we will

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put aside the large (and growing) literature that investigates algorithmic ramifications of metric dimension reduction. Such applications warrant a separate treatment that is far beyond the scope of the present exposition; some aspects of that material are covered in the monographs Matoušek [2002], Vempala [2004], and Har-Peled [2011] and the surveys Indyk [2001] and Linial [2002], as well as the articles of Andoni–Indyk–Razenshteyn and Arora in the present volume.

Remark 1. The broader term *dimension reduction* is used ubiquitously in statistics and machine learning, with striking applications whose full rigorous understanding sometimes awaits the scrutiny of mathematicians (see e.g. Burges [2010]). A common (purposefully vague) description of this term is the desire to decrease the degrees of freedom of a high-dimensional data set while approximately preserving some of its pertinent features; formulated in such great generality, the area includes topics such as neural networks (see e.g. Hinton and Salakhutdinov [2006]). The commonly used term *curse of dimensionality* refers to the perceived impossibility of this goal in many settings, and that the performance (running time, storage space) of certain algorithmic tasks must deteriorate exponentially as the underlying dimension grows. But, sometimes it does seem that certain high-dimensional data sets can be realized faithfully using a small number of latent variables as auxiliary "coordinates." Here we restrict ourselves exclusively to *metric* dimension reduction, i.e., to notions of faithfulness of low-dimensional representations that require the (perhaps quite rough) preservation of pairwise distances, including ways to prove the impossibility thereof.

Roadmap. The rest of the Introduction is an extensive and detailed account of the area of metric dimension reduction, including statements of most of the main known results, background and context, and many important open questions. The Introduction is thus an expository account of the field, so those readers who do not wish to delve into some proofs, could read it separately from the rest of the text. The remaining sections contain further details and complete justifications of those statements that have not appeared in the literature.

1.1 Bi-Lipschitz embeddings. Fix $\alpha \geqslant 1$. A metric space $(\mathfrak{M}, d_{\mathfrak{M}})$ is said to embed with distortion α into a metric space $(\mathfrak{N}, d_{\mathfrak{N}})$ if there is a mapping (an embedding) $f : \mathfrak{M} \to \mathfrak{N}$ and (a scaling factor) $\tau > 0$ such that

$$\forall\, x, y \in \mathfrak{M}, \qquad \tau d_{\mathfrak{M}}(x, y) \leqslant d_{\mathfrak{N}}\big(f(x), f(y)\big) \leqslant \alpha\tau d_{\mathfrak{M}}(x, y). \tag{1}$$

The infimum over $\alpha \in [1, \infty]$ for which $(\mathfrak{M}, d_{\mathfrak{M}})$ embeds with distortion α into $(\mathfrak{N}, d_{\mathfrak{N}})$ is denoted $\mathsf{c}_{(\mathfrak{N}, d_{\mathfrak{N}})}(\mathfrak{M}, d_{\mathfrak{M}})$, or $\mathsf{c}_{\mathfrak{N}}(\mathfrak{M})$ if the underlying metrics are clear from the context. If $\mathsf{c}_{\mathfrak{N}}(\mathfrak{M}) < \infty$, then $(\mathfrak{M}, d_{\mathfrak{M}})$ is said to admit a bi-Lipschitz embedding into $(\mathfrak{N}, d_{\mathfrak{N}})$. Given $p \in [1, \infty)$, if $\mathfrak{N}$ is an $L_p(\mu)$ space into which $\mathfrak{M}$ admits a bi-Lipschitz embedding, then we use the notation $\mathsf{c}_{L_p(\mu)}(\mathfrak{M}) = \mathsf{c}_p(\mathfrak{M})$. The numerical invariant $\mathsf{c}_2(\mathfrak{M})$, which measures the extent to which $\mathfrak{M}$ is close to being a (subset of a) Euclidean geometry, is called the *Euclidean distortion* of $\mathfrak{M}$.

A century of intensive research into bi-Lipschitz embeddings led to a rich theory with many deep achievements, but the following problem, which is one of the first questions that arise naturally in this direction, remains a major longstanding mystery; see e.g. Semmes [1999], Lang and Plaut [2001], Heinonen [2003], and Naor and Neiman [2012]. Analogous issues in the context of topological dimension, differentiable manifolds and Riemannian manifolds were famously settled by Menger [1928] and Nöbeling [1931], Whitney [1936] and Nash [1954], respectively.

Problem 2 (the bi-Lipschitz embedding problem into $\mathbb{R}^k$). Obtain an intrinsic characterization of those metric spaces $(\mathfrak{M}, d_{\mathfrak{M}})$ that admit a bi-Lipschitz embedding into $\mathbb{R}^k$ for some $k \in \mathbb{N}$.

Problem 2 is one of the qualitative underpinnings of the issues that will be surveyed here. We say that it is "qualitative" because it ignores the magnitude of the distortion altogether, and therefore one does not need to specify which norm on $\mathbb{R}^k$ is considered in Problem 2, since all the norms on $\mathbb{R}^k$ are (bi-Lipschitz) equivalent. Problem 2 is also (purposefully) somewhat vague, because the notion of "intrinsic characterization" is not well-defined. We will return to this matter in Section 3 below, where candidates for such a characterization are discussed. At this juncture, it suffices to illustrate what Problem 2 aims to achieve through the following useful example. If one does not impose any restriction on the target dimension and allows for a bi-Lipschitz embedding into an infinite dimensional Hilbert space, then the following intrinsic characterization is available. A metric space $(\mathfrak{M}, d_{\mathfrak{M}})$ admits a bi-Lipschitz embedding into a Hilbert space if and only if there exists $C = C_{\mathfrak{M}} \in [0, 1)$ such that for every $n \in \mathbb{N}$ and every positive semidefinite symmetric matrix $\mathsf{A} = (a_{ij}) \in \mathsf{M}_n(\mathbb{R})$ all of whose rows sum to zero (i.e., $\sum_{j=1}^n a_{ij} = 0$ for every $i \in \{1, \ldots, n\}$), the following quadratic distance inequality holds true.

$$\text{(2)} \quad \forall\, x_1, \ldots, x_n \in \mathfrak{M}, \qquad \sum_{i=1}^n \sum_{j=1}^n a_{ij} d_{\mathfrak{M}}(x_i, x_j)^2 \leqslant C \sum_{i=1}^n \sum_{j=1}^n |a_{ij}| d_{\mathfrak{M}}(x_i, x_j)^2.$$

In fact, one can refine this statement quantitatively as follows. A metric space embeds with distortion $\alpha \in [1, \infty)$ into a Hilbert space if and only if in the setting of (2) we have

(3)

$$\forall\, x_1, \ldots, x_n \in \mathfrak{M}, \qquad \sum_{i=1}^n \sum_{j=1}^n a_{ij} d_{\mathfrak{M}}(x_i, x_j)^2 \leqslant \frac{\alpha^2 - 1}{\alpha^2 + 1} \sum_{i=1}^n \sum_{j=1}^n |a_{ij}| d_{\mathfrak{M}}(x_i, x_j)^2.$$

The case $\alpha = 1$ of (3), i.e., the case of isometric embeddings, is a famous classical theorem of Schoenberg [1938], and the general case of (3) is due to Linial, London, and Rabinovich [1995, Corollary 3.5]. The above characterization is clearly intrinsic, as it is a family of finitary distance inequalities among points of $\mathfrak{M}$ that do not make reference to any other auxiliary/external object. With such a characterization at hand, one could examine the internal structure of a given metric space so as to determine if it can be represented faithfully as a subset of a Hilbert space. Indeed, Linial, London, and Rabinovich [ibid.] uses (3) to obtain an algorithm that takes as input an n-point metric

space $(\mathfrak{M}, d_{\mathfrak{M}})$ and outputs in polynomial time an arbitrarily good approximation to its Euclidean distortion $\mathsf{c}_2(\mathfrak{M})$.

A meaningful answer to Problem 2 could in principle lead to a method for determining if a member of a family $\mathfrak{F}$ of metric spaces admits an embedding with specified distortion into a member of a family $\mathfrak{G}$ of low dimensional normed spaces. Formulated in such great generality, this type of question encompasses all of the investigations into metric dimension reduction that will be discussed in what follows, except that we will also examine analogous issues for embeddings with guarantees that are substantially weaker (though natural and useful) than the "vanilla" bi-Lipschitz requirement (1).

Remark 3. Analogues of the above questions are very natural also when the target low-dimensional geometries are not necessarily normed spaces. Formulating meaningful goals in such a setting is not as straightforward as it is for normed spaces, e.g. requiring that the target space is a manifold of low topological dimension is not very useful, so one must impose geometric restrictions on the target manifold. As another example (to which we will briefly return later), one could ask about embeddings into spaces of probability measures that are equipped with a Wasserstein (transportation cost) metric, with control on the size of the underlying metric space. At present, issues of this type are largely an uncharted terrain whose exploration is likely to be interesting and useful.

1.2 Local theory and the Ribe program. Besides being motivated by the bi-Lipschitz embedding problem into $\mathbb{R}^k$, much of the inspiration for the studies that will be presented below comes from a major endeavour in metric geometry called the *Ribe program*. This is a large and active research area that has been (partially) surveyed in Kalton [2008], Naor [2012b], Ball [2013], M. I. Ostrovskii [2013], and Godefroy [2017]. It would be highly unrealistic to attempt to cover it comprehensively here, but we will next present a self-contained general introduction to the Ribe program that is intended for non-experts, including aspects that are relevant to the ensuing discussion on metric dimension reduction.

Martin Ribe was a mathematician who in the 1970s obtained a few beautiful results in functional analysis, prior to leaving mathematics. Among his achievements is a very influential rigidity theorem Ribe [1976] which shows that the local linear theory of Banach spaces could in principle be described using only distances between points, and hence it could potentially apply to general metric spaces.

Before formulating the above statement precisely, it is instructive to consider a key example. The *infimal cotype* Maurey [2003] q_X of a Banach space $(X, \|\cdot\|)$ is the

infimum over those $q \in [2,\infty]$ for which[1]

$$\text{(4)} \quad \forall\, n \in \mathbb{N},\ \forall\, x_1,\ldots,x_n \in X, \qquad \sum_{i=1}^{n} \|x_i\|^2 \lesssim_{X,q} \frac{n^{1-\frac{2}{q}}}{2^n} \sum_{\varepsilon\in\{-1,1\}^n} \left\| \sum_{i=1}^{n} \varepsilon_i x_i \right\|^2.$$

In the special case $x_1 = \ldots = x_n = x \in X \smallsetminus \{0\}$, the left hand side of (4) is equal to $n\|x\|^2$ and by expanding the squares one computes that the right hand side of (4) is equal to $n^{2(1-1/q)}\|x\|^2$. Hence (4) necessitates that $q \geqslant 2$, which explains why we imposed this restriction on q at the outset. Note also that (4) holds true in any Banach space when $q = \infty$. This is a quick consequence of the convexity of the mapping $x \mapsto \|x\|^2$, since for every $\varepsilon \in \{-1,1\}^n$ and $i \in \{1,\ldots,n\}$ we have

(5)

$$\begin{aligned}\|x_i\|^2 &= \left\| \frac{(\varepsilon_1 x_1 + \ldots + \varepsilon_n x_n) + (-\varepsilon_1 x_1 - \ldots - \varepsilon_{i-1}x_{i-1} + \varepsilon_i x_i - \varepsilon_{i+1}x_{i+1} \ldots - \varepsilon_n x_n)}{2} \right\|^2 \\ &\leqslant \frac{\|\varepsilon_1 x_1 + \ldots + \varepsilon_n x_n\|^2 + \|-\varepsilon_1 x_1 - \ldots - \varepsilon_{i-1}x_{i-1} + \varepsilon_i x_i - \varepsilon_{i+1}x_{i+1} \ldots - \varepsilon_n x_n\|^2}{2}.\end{aligned}$$

By averaging (5) over $\varepsilon \in \{-1,1\}^n$ and $i \in \{1,\ldots,n\}$ we see that (4) holds if $q = \infty$. So, one could view (4) for $q < \infty$ as a requirement that the norm $\|\cdot\| : X \to [0,\infty)$ has a property that is asymptotically stronger than mere convexity. When $X = \ell_\infty$, this requirement does not hold for any $q < \infty$, since if $\{x_i\}_{i=1}^n$ are the first n elements of the coordinate basis, then the left hand side of (4) equals n while its right hand side equals $n^{1-1/q}$.

Maurey and Pisier [1973] proved that the above obstruction to having $q_X < \infty$ is actually the *only possible* such obstruction. Thus, by ruling out the presence of copies of $\{\ell_\infty^n\}_{n=1}^\infty$ in X one immediately deduces the "upgraded" (asymptotically stronger as $n \to \infty$) randomized convexity inequality (4) for some $q < \infty$.

Theorem 4. *The following conditions are equivalent for every Banach space* $(X, \|\cdot\|)$.

- *There is no* $\alpha \in [1,\infty)$ *such that* ℓ_∞^n *is* α*-isomorphic to a subspace of* X *for every* $n \in \mathbb{N}$.
- $q_X < \infty$.

The (standard) terminology that is used in Theorem 4 is that given $\alpha \in [1,\infty)$, a Banach space $(Y, \|\cdot\|_Y)$ is said to be α-isomorphic to a subspace of a Banach space $(Z, \|\cdot\|_Z)$ if there is a linear operator $T : Y \to Z$ satisfying $\|y\|_Y \leqslant \|Ty\|_Z \leqslant \alpha\|y\|_Y$ for every $y \in Y$; this is the same as saying that Y embeds into Z with distortion α via an embedding that is a linear operator.

[1] In addition to the standard O" notation, we will use throughout this article the following standard and convenient asymptotic notation. Given two quantities $Q, Q' > 0$, the notations $Q \lesssim Q'$ and $Q' \gtrsim Q$ mean that $Q \leqslant CQ'$ for some universal constant $C > 0$. The notation $Q \asymp Q'$ stands for $(Q \lesssim Q') \wedge (Q' \lesssim Q)$. If we need to allow for dependence on parameters, we indicate this by subscripts. For example, in the presence of auxiliary objects (e.g. numbers or spaces) $\phi, \mathfrak{Z}$, the notation $Q \lesssim_{\phi,\mathfrak{Z}} Q'$ means that $Q \leqslant C(\phi,\mathfrak{Z})Q'$, where $C(\phi,\mathfrak{Z}) > 0$ is allowed to depend only on $\phi, \mathfrak{Z}$; similarly for the notations $Q \gtrsim_{\phi,\mathfrak{Z}} Q'$ and $Q \asymp_{\phi,\mathfrak{Z}} Q'$.

Suppose that X and Y are Banach spaces that are uniformly homeomorphic, i.e., there is a bijection $f : X \to Y$ such that both f and f^{-1} are uniformly continuous. By the aforementioned rigidity theorem of Ribe (which will be formulated below in full generality), this implies in particular that $q_X = q_Y$. So, despite the fact that the requirement (4) involves linear operations (summation and sign changes) that do not make sense in general metric spaces, it is in fact preserved by purely metric (quantitatively continuous, though potentially very complicated) deformations. Therefore, in principle (4) could be characterized while only making reference to distances between points in X. More generally, Ribe's rigidity theorem makes an analogous assertion for *any* isomorphic local linear property of a Banach space; we will define formally those properties in a moment, but, informally, they are requirements in the spirit of (4) that depend only on the finite dimensional subspaces of the given Banach space and are stable under linear isomorphisms that could potentially incur a large error.

The purely metric reformulation of (4) about which we speculated above is only suggested but not guaranteed by Ribe's theorem. From Ribe's statement we will only infer an indication that there might be a "hidden dictionary" for translating certain linear properties into metric properties, but we will not be certain that any specific "entry" of this dictionary (e.g. the entry for, say, "$q_X = \pi$") does in fact exist, and even if it does exist, we will not have an indication what it says. A hallmark of the Ribe program is that at its core it is a search for a family of analogies and definitions, rather than being a collection of specific conjectures. Once such analogies are made and the corresponding questions are formulated, their value is of course determined by the usefulness/depth of the phenomena that they uncover and the theorems that could be proved about them. Thus far, not all of the steps of this endeavour turned out to have a positive answer, but the vast majority did. This had major impact on the study of metric spaces that a priori have nothing to do with Banach spaces, such as graphs, manifolds, groups, and metrics that arise in algorithms (e.g. as continuous relaxations).

The first written formulation of the plan to uncover a hidden dictionary between normed spaces and metric spaces is the following quote of Bourgain [1986], a decade after Ribe's theorem appeared.

> *It follows in particular from Ribe's result [...] that the notions from local theory of normed spaces are determined by the metric structure of the space and thus have a purely metrical formulation. The next step consists in studying these metrical concepts in general metric spaces in an attempt to develop an analogue of the linear theory. A detailed exposition of this program will appear in J. Lindenstrauss's forthcoming survey [...] in our "dictionary" linear operators are translated in Lipschitz maps, the operator norm by the Lipschitz constant of the map [...] The translations of "Banach–Mazur distance" and "finite-representability" in linear theory are immediate. At the roots of the local theory of normed spaces are properties such as type, cotype, superreflexivity [...] The analogue of type in the geometry of metric spaces is [...] A simple metrical invariant replacing the notion of cotype was not yet discovered.*
>
> J. Bourgain, 1986.

Unfortunately, the survey of Lindenstrauss that is mentioned above never appeared. Nonetheless, Lindenstrauss had massive impact on this area as a leader who helped

set the course of the Ribe program, as well as due to the important theorems that he proved in this direction. In fact, the article Johnson and Lindenstrauss [1984] of Johnson and Lindenstrauss, where the aforementioned metric dimension reduction lemma was proved, appeared a few years before Bourgain [1986] and contained inspirational (even prophetic) ideas that had major subsequent impact on the Ribe program (including on Bourgain's works in this area). In the above quote, we removed the text describing "the analogue of type in the geometry of metric spaces" so as to not digress; it refers to the influential work of Bourgain, V. Milman, and Wolfson [1986] (see also the earlier work of Enflo [1978] and the subsequent work of Pisier [1986]). "Superreflexivity" was the main focus of Bourgain [1986], where the corresponding step of the Ribe program was completed (we will later discuss and use a refinement of this solution). An answer to the above mentioned question on cotype, which we will soon describe, was subsequently found by Mendel and the author Mendel and Naor [2008]. We will next explain the terminology "finite-representability" in the above quote, so as to facilitate the ensuing discussion.

1.2.1 Finite representability. The first decades of work on the geometry of Banach spaces focused almost entirely on an inherently infinite dimensional theory. This was governed by Banach's partial ordering of Banach spaces Banach [1993, Chapter 7], which declares that a Banach space X has "linear dimension" at most that of a Banach space Y if there exists $\alpha \geqslant 1$ such that X is α-isomorphic to a subspace of Y. In a remarkable feat of foresight, the following quote of Grothendieck [1953b] heralded the *local theory of Banach spaces*, by shifting attention to the geometry of the finite dimensional subspaces of a Banach space as a way to understand its global structure.

> *assouplissons la notion de "dimension linéaire" de Banach, en disant que l'espace normé E a un* type linéaire *inférieur à celui d'un espace normé F, si on peut trouver un $M > 0$ fixe tel que tout sous-espace de dimension finie E_1 de E soit isomorphe "à M près" à un sous-espace F_1 de F (i.e. il existe une application linéaire biunivoque de E_1 sur F_1 , de norme $\leqslant 1$, dont lápplication inverse a une norme $\leqslant 1 + M$).*
>
> A. Grothendieck, 1953.

Grothendieck's work in the 1950s exhibited astounding (technical and conceptual) ingenuity and insight that go well-beyond merely defining a key concept, as he did above. In particular, in Grothendieck [ibid.] he conjectured an important phenomenon[2] that was later proved by Dvoretzky [1961] (see the discussion in Schechtman [2017]), and his contributions in Grothendieck [1953a] were transformative (e.g. Lindenstrauss and Pełczyński [1968], Diestel, J. H. Fourie, and Swart [2008], Khot and Naor [2012], and

[2]This phenomenon was situated within the Ribe program by Bourgain, Figiel, and V. Milman [1986], and as such it eventually had ramifications to a well-studied (algorithmic) form of metric dimension reduction through its use to "compress" a finite metric space into a data structure called an *approximate distance oracle* Thorup and Zwick [2005]. To date, the only known way to construct such a data structure with constant query time (and even, by now, conjecturally sharp approximation factor Chechik [2014]) is via the nonlinear Dvoretzky theorem of Mendel and Naor [2007a], and thus through the Ribe program. For lack of space, we will not discuss this direction here; see the survey Naor [2012b].

Pisier [2012]). The above definition set the stage for decades of (still ongoing) work on the local theory of Banach spaces which had major impact on a wide range of mathematical areas.

The above "softening" of Banach's "linear dimension" is called today *finite representability*, following the terminology of James [1972] (and his important contributions on this topic). Given $\alpha \in [1,\infty)$, a Banach space X is said to be α-finitely representable in a Banach space Y if for any $\beta > \alpha$, any finite dimensional subspace of X is β-isomorphic to a subspace of Y (in the notation of the above quote, $\beta = 1 + M$); X is (crudely) finitely representable in Y if there is some $\alpha \in [1,\infty)$ such that X is α-finitely representable in Y. This means that the finite dimensional subspaces of X are not very different from subspaces of Y; if each of X and Y is finitely representable in the other, then this should be viewed as saying that X and Y have the same finite dimensional subspaces (up to a global allowable error that does not depend on the finite dimensional subspace in question). As an important example of the "taming power" of this definition, the *principle of local reflexivity* of Lindenstrauss and Rosenthal [1969] asserts that even though sometimes $X^{**} \neq X$, it is always true that X^{**} is 1-finitely representable in X. Thus, while in infinite dimensions X^{**} can be much larger than X, passing to the bidual cannot produce substantially new finite dimensional structures. The aforementioned Dvoretzky theorem Dvoretzky [1961] asserts that ℓ_2 is 1-finitely representable in any infinite dimensional Banach space. As another example of a landmark theorem on finite representability, Maurey and Pisier [1976] strengthened Theorem 4 by showing that ℓ_{q_X} is 1-finitely representable in any infinite dimensional Banach space X.

Isomorphic local linear properties of Banach spaces are defined to be those properties that are preserved under finite representability. As an example, one should keep in mind finitary inequalities such as the cotype condition (4). The formal statement of Ribe's rigidity theorem Ribe [1976] is

Theorem 5. *Uniformly homeomorphic Banach spaces X and Y are finitely representable in each other.*

The "immediate translation" of finite representability in the above quoted text from Bourgain [1986] is to define that for $\alpha \in [1,\infty)$ a metric space $\mathfrak{M}$ is α-finitely representable in a metric space $\mathfrak{N}$ if $\mathsf{c}_{\mathfrak{N}}(\mathcal{C}) \leqslant \alpha$ for *every* finite subset $\mathcal{C} \subseteq \mathfrak{M}$. By doing so one does not induce any terminological conflict, because one can show that a Banach space X is (linearly) α-finitely representable in a Banach space Y if and only if X is α-finitely representable in Y when X and Y are viewed as metric spaces. This statement follows from "soft" reasoning that is explained in Giladi, Naor, and Schechtman [2012] (relying on a w^*-differentiation argument of Heinrich and Mankiewicz [1982] as well as properties of ultrapowers of Banach spaces and the aforementioned principle of local reflexivity), though it also follows from Ribe's original proof of Theorem 5 in Ribe [1976], and a different (quantitative) approach to this statement was obtained in Bourgain [1987].

1.2.2 Universality and dichotomies. Say that a metric space $\mathfrak{M}$ is (finitarily) universal if there is $\alpha \geqslant 1$ such that $\mathsf{c}_{\mathfrak{M}}(\mathcal{F}) \leqslant \alpha$ for *every* finite metric space $\mathcal{F}$. By Matoušek [1992b] and Mendel and Naor [2008], this requirement holds for some $\alpha \geqslant 1$

if and only if it holds for $\alpha = 1$ (almost-isometric embeddings), so the notion of universality turns out to be insensitive to the underlying distortion bound. Since for every $n \in \mathbb{N}$, any n-point metric space $(\mathcal{F} = \{x_1, \ldots, x_n\}, d_{\mathcal{F}})$ is isometric to a subset of ℓ_∞ via the embedding $(x \in \mathcal{F}) \mapsto (d_{\mathcal{F}}(x, x_i))_{i=1}^n$ (Fréchet embedding Fréchet [1910]), a different way to state the notion of universality is to say that $\mathfrak{M}$ is universal if ℓ_∞ is finitely representable in $\mathfrak{M}$.

Determining whether a given metric space is universal is a subtle matter. By Theorem 4, for a Banach space X this is the same as asking to determine whether $q_X = \infty$. Such questions include major difficult issues in functional analysis that have been studied for a long time; as notable examples, see the works Pełczyński [1977] and Bourgain [1984] on the (non)universality of the dual of the disc algebra, and the characterization Bourgain and V. Milman [1985] of Sidon subsets of the dual of a compact Abelian group G in terms of the universality of their span is the space of continuous functions on G. Here are three famous concrete situations in which it is unknown if a certain specific space is universal.

Question 6 (Pisier's dichotomy problem). For each $n \in \mathbb{N}$ let X_n be an arbitrary linear subspace of ℓ_∞^n satisfying

$$\limsup_{n\to\infty} \frac{\dim(X_n)}{\log n} = \infty. \tag{6}$$

Pisier [1981] conjectured that (6) forces the ℓ_2 (Pythagorean) direct sum $(X_1 \oplus X_2 \oplus \ldots)_2$ to be universal. By duality, a positive answer to this question is equivalent to the following appealing statement on the geometry of polytopes. For $n \in \mathbb{N}$, suppose that $K \subseteq \mathbb{R}^n$ is an origin-symmetric polytope with $e^{o(n)}$ faces. Then, for each $\delta > 0$ there is $k = k(n, \delta) \in \{1, \ldots, n\}$ with $\lim_{n\to\infty} k(n, \delta) = \infty$, a subspace $F = F(n, \delta)$ of $\mathbb{R}^n$ with $\dim(F) = k$ and a parallelepiped $Q \subseteq F$ (thus, Q is an image of $[-1, 1]^k$ under an invertible linear transformation) such that $Q \subseteq K \cap F \subseteq (1 + \delta)Q$. Hence, a positive answer to Pisier's dichotomy conjecture implies that every centrally symmetric polytope with $e^{o(n)}$ faces has a central section of dimension k going to ∞ (as a function of the specific $o(n)$ dependence in the underlying assumption), which is $(1+\delta)$-close to a polytope (a parallelepiped) with only $O(k)$ faces. The use of "dichotomy" in the name of this conjecture is due to the fact that this conclusion does not hold with $o(n)$ replaced by $O(n)$, as seen by considering polytopes that approximate the Euclidean ball. More generally, by the "isomorphic version" of the Dvoretzky theorem due to V. D. Milman and Schechtman [1995], for every sequence of normed spaces $\{Y_n\}_{n=1}^\infty$ with $\dim(Y_n) = n$ (not only $Y_n = \ell_\infty^n$, which is the case of interest above), and every $k(n) \in \{1, \ldots, n\}$ with $k(n) = O(\log n)$, there is a subspace $X_n \subseteq Y_n$ with $\dim(X_n) = k(n)$ such that the space $(X_1 \oplus X_2 \oplus \ldots)_2$ is isomorphic to a Hilbert space, and hence in particular it is not universal. The best-known bound in Pisier's dichotomy conjecture appears in the forthcoming work of Schechtman and Tomczak-Jaegermann [2018], where it is shown that the desired conclusion does indeed hold true if (6) is replaced by the stronger assumption $\limsup_{n\to\infty} \dim(X_n)/((\log n)^2(\log\log n)^2) = \infty$; this is achieved Schechtman and Tomczak-Jaegermann [ibid.] by building on ideas of Bourgain [1985b], who obtained the same conclusion if $\limsup_{n\to\infty} \dim(X_n)/(\log n)^4 > 0$. Thus, due to Schechtman

and Tomczak-Jaegermann [2018] the above statement about almost-parallelepiped central sections of centrally symmetric polytopes does hold true if the initial polytope is assumed to have $\exp(o(\sqrt{n}/\log n))$ faces.

Prior to stating the next question on universality (which, apart from its intrinsic interest, plays a role in the ensuing discussion on metric dimension reduction), we need to very briefly recall some basic notation and terminology from optimal transport (see e.g. Ambrosio, Gigli, and Savaré [2008] and Villani [2009]). Suppose that $(\mathfrak{M}, d_{\mathfrak{M}})$ is a separable complete metric space and fix $p \in [1,\infty)$. Denote by $\mathsf{P}_1(\mathfrak{M})$ the set of all Borel probability measures μ on $\mathfrak{M}$ of finite p'th moment, i.e., those Borel probability measure μ on $\mathfrak{M}$ for which $\int_{\mathfrak{M}} d_{\mathfrak{M}}(x,y)^p \,\mathrm{d}\mu(y) < \infty$ for all $x \in \mathfrak{M}$. A probability measure $\pi \in \mathsf{P}_p(\mathfrak{M}\times\mathfrak{M})$ is a called a coupling of $\mu,\nu \in \mathsf{P}_p(\mathfrak{M})$ if $\mu(A) = \pi(A\times\mathfrak{M})$ and $\nu(A) = \pi(\mathfrak{M}\times A)$ for every Borel measurable subset $A \subseteq \mathfrak{M}$. The Wasserstein-p distance between $\mu,\nu \in \mathsf{P}_p(\mathfrak{M})$, denoted $\mathsf{W}_p(\mu,\nu)$, is defined to be the infimum of $(\iint_{\mathfrak{M}\times\mathfrak{M}} d_{\mathfrak{M}}(x,y)^p \,\mathrm{d}\pi(x,y))^{1/p}$ over all couplings π of μ,ν. Below, $\mathsf{P}_p(\mathfrak{M})$ is always assumed to be endowed with the metric W_p. The following question is from Bourgain [1986].

Question 7 (Bourgain's universality problem). Is $\mathsf{P}_1(\mathbb{R}^2)$ universal? This formulation may seem different from the way it is asked in Bourgain [ibid.], but, as explained Andoni, Naor, and Neiman [2015, Section 1.5], it is equivalent to it. More generally, is $\mathsf{P}_1(\mathbb{R}^k)$ universal for some integer $k \geqslant 2$ (it is simple to see that $\mathsf{P}_1(\mathbb{R})$ is not universal)? In Bourgain [1986] it was proved that $\mathsf{P}_1(\ell_1)$ is universal (see also the exposition in M. I. Ostrovskii [2013]). So, it is important here that the underlying space is finite dimensional, though to the best of our knowledge it is also unknown whether $\mathsf{P}_1(\ell_2)$ is universal, or, for that matter, if $\mathsf{P}_1(\ell_p)$ is universal for any $p \in (1,\infty)$. See Andoni, Naor, and Neiman [2015] for a (sharp) universality property of $\mathsf{P}_p(\mathbb{R}^3)$ if $p \in (1,2]$.

For the following open question about universality (which will also play a role in the subsequent discussion on metric dimension reduction), recall the notion Grothendieck [1952] and Grothendieck [1955] of *projective tensor product* of Banach spaces. Given two Banach spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$, their projective tensor product $X\widehat{\otimes}Y$ is the completion of their algebraic tensor product $X\otimes Y$ under the norm whose unit ball is the convex hull of the simple tensors of vectors of norm at most 1, i.e., the convex hull of the set $\{x\otimes y \in X\otimes Y : \|x\|_X, \|y\|_Y \leqslant 1\}$. For example, $\ell_1\widehat{\otimes}X$ can be naturally identified with $\ell_1(X)$, and $\ell_2\widehat{\otimes}\ell_2$ can be naturally identified with Schatten–von Neumann trace class S_1 (recall that for $p \in [1,\infty]$, the Schatten–von Neumann trace class S_p is the Banach space von Neumann [1937] of those compact linear operators $T : \ell_2 \to \ell_2$ for which $\|T\|_{\mathsf{S}_p} = (\sum_{j=1}^{\infty} \mathsf{s}_j(T)^p)^{1/p} < \infty$, where $\{\mathsf{s}_j(T)\}_{j=1}$ are the singular values of T); see the monograph Ryan [2002] for much more on tensor products of Banach spaces.

It is a longstanding endeavour in Banach space theory to understand which properties of Banach spaces are preserved under projective tensor products; see Diestel, J. Fourie, and Swart [2003] and Briët, Naor, and Regev [2012] and the references therein for more on this research direction. Deep work of Pisier [1983] shows that there exist two Banach spaces X, Y that are not universal (even of cotype 2) such that $X\widehat{\otimes}Y$ is universal. The following question was posed by Pisier [1992b].

Question 8 (universality of projective tensor products). Suppose that $p \in (1,2)$. Is $\ell_p \widehat{\otimes} \ell_2$ universal? We restricted the range of p here because it is simple to check that $\ell_1 \widehat{\otimes} \ell_2 \cong \ell_1(\ell_2)$ is not universal, Tomczak-Jaegermann [1974] proved that $\ell_2 \otimes \ell_2 \cong \mathsf{S}_1$ is not universal, and Pisier [1992a] proved that $\ell_p \widehat{\otimes} \ell_q$ is not universal when $p, q \in [2, \infty)$. It was also asked in Pisier [1992b] if $\ell_2 \widehat{\otimes} \ell_2 \widehat{\otimes} \ell_2$ is universal. The best currently available result in this direction (which will be used below) is that, using the local theory of Banach spaces and recent work on locally decodable codes, it was shown in Briët, Naor, and Regev [2012] that if $a, b, c \in (1, \infty)$ satisfy $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leqslant 1$, then $\ell_a \widehat{\otimes} \ell_b \widehat{\otimes} \ell_c$ is universal.

The following theorem is a union of several results of Mendel and Naor [2008].

Theorem 9. *The following conditions are equivalent for a metric space $(\mathfrak{M}, d)$.*

- *$\mathfrak{M}$ is not universal.*
- *There exists $q = q(\mathfrak{M}) \in (0, \infty)$ with the following property. For every $n \in \mathbb{N}$ there is $m = m(n, \mathfrak{M}, q) \in \mathbb{N}$ such that any collection of points $\{x_w\}_{w \in \mathbb{Z}_{2m}^n}$ in $\mathfrak{M}$ satisfies the inequality*

$$\sum_{i=1}^{n} \sum_{w \in \mathbb{Z}_{2m}^n} \frac{d(x_{w+me_i}, x_w)^2}{m^2} \lesssim_{X,q} \frac{n^{1-\frac{2}{q}}}{3^n} \sum_{\varepsilon \in \{-1,0,1\}^n} \sum_{w \in \mathbb{Z}_{2m}^n} d(x_{w+\varepsilon}, x)^2. \tag{7}$$

  Here $e_1, \ldots, e_n$ are the standard basis of $\mathbb{R}^n$ and addition (in the indices) is modulo $2m$.

- *There is $\theta(\mathfrak{M}) \in (0, \infty)$ with the following property. For arbitrarily large $n \in \mathbb{N}$ there exists an n-point metric space $(\mathcal{B}_n, d_n)$ such that*

$$\mathsf{c}_{\mathfrak{M}}(\mathcal{B}_n) \gtrsim_{\mathfrak{M}} (\log n)^{\theta(\mathfrak{M})}. \tag{8}$$

Moreover, if we assume that $\mathfrak{M}$ is a Banach space rather than an arbitrary metric space, then for any $q \in (0, \infty]$ the validity of (7) (as stated, i.e., for each $n \in \mathbb{N}$ there is $m \in \mathbb{N}$ for which (7) holds for every configuration $\{x_w\}_{w \in \mathbb{Z}_{2m}^n}$ of points in $\mathfrak{M}$) is equivalent to (4), and hence, in particular, the infimum over those $q \in (0, \infty]$ for which (7) holds true is equal to the infimal cotype $q_{\mathfrak{M}}$ of $\mathfrak{M}$.

The final sentence of Theorem 9 is an example of a successful step in the Ribe program, because it reformulates the local linear invariant (4) in purely metric terms, namely as the quadratic geometric inequality (7) that imposes a restriction on the behavior of pairwise distances within any configuration of $(2m)^n$ points $\{x_w\}_{w \in \mathbb{Z}_{2m}^n}$ (indexed by the discrete torus $\mathbb{Z}_{2m}^n$) in the given Banach space.

With this at hand, one can consider (7) to be a property of a metric space, while initially (4) made sense only for a normed space. As in the case of (4), if $q = \infty$, then (7) holds in any metric space $(\mathfrak{M}, d_{\mathfrak{M}})$ (for any $m \in \mathbb{Z}$, with the implicit constant in (7) being universal); by its general nature, such a statement must of course be nothing more than a formal consequence of the triangle inequality, as carried out in Mendel and Naor [ibid.]. So, the validity of (7) for $q < \infty$ could be viewed as an asymptotic

(randomized) enhancement of the triangle inequality in $(\mathfrak{M}, d_{\mathfrak{M}})$; by considering the the canonical realization of $\mathbb{Z}_{2m}^n$ in $\mathbb{C}^n$, namely the points

$$\{(\exp(\pi i\, w_1/m), \ldots, \exp(\pi i\, w_n/m))\}_{w \in \mathbb{Z}_{2m}^n}$$

equipped with the metric inherited from $\ell_\infty^n(\mathbb{C})$, one checks that not every metric space satisfies this requirement. The equivalence of the first two bullet points in Theorem 9 shows that once one knows that a metric space is not universal, one deduces the validity of such an enhancement of the triangle inequality. This is an analogue of Theorem 4 of Maurey and Pisier for general metric spaces.

The equivalence of the first and third bullet points in Theorem 9 yields the following dichotomy. If one finds a finite metric space $\mathcal{F}$ such that $\mathsf{c}_{\mathcal{F}}(\mathfrak{M}) > 1$, then there are arbitrary large finite metric spaces whose minimal distortion in $\mathfrak{M}$ is at least a fixed positive power (depending on $\mathfrak{M}$) of the logarithm their cardinality. Hence, for example, if every n-point metric space embeds into $\mathfrak{M}$ with distortion $O(\log\log n)$, then actually for any $\delta > 0$, every finite metric space embeds into $\mathfrak{M}$ with distortion $1+\delta$. See Mendel [2009], Mendel and Naor [2011], Mendel and Naor [2013], and Andoni, Naor, and Neiman [2015] for more on metric dichotomies of this nature, as well as a quite delicate counterexample Mendel and Naor [2013] for a natural variant for trees (originally asked by C. Fefferman). It remains a mystery Mendel and Naor [2008] if the power of the logarithm $\theta(\mathfrak{M})$ in Theorem 9 could be bounded from below by a universal positive constant, as formulated in the following open question.

Question 10 (metric cotype dichotomy problem). Is there a universal constant $\theta > 0$ such that in Theorem 9 one could take $\theta(\mathfrak{M}) > \theta$. All examples that have been computed thus far leave the possibility that even $\theta(\mathfrak{M}) \geqslant 1$, which would be sharp (for $\mathfrak{M} = \ell_2$) by Bourgain's embedding theorem Bourgain [1985a]. Note, however, that in Andoni, Naor, and Neiman [2015] it is asked whether for the Wasserstein space $\mathsf{P}_p(\mathbb{R}^3)$ we have $\liminf_{p\to 1} \theta(\mathsf{P}_p(\mathbb{R}^3)) = 0$. If this were true, then it would resolve the metric cotype dichotomy problem negatively. It would be interesting to understand the bi-Lipschitz structure of these spaces of measures on $\mathbb{R}^3$ regardless of this context, due to their independent importance.

Theorem 9 is a good illustration of a "vanilla" accomplishment of the Ribe program, since it obtains a metric reformulation of a key isomorphic linear property of metric spaces, and also proves statements about general metric spaces which are inspired by the analogies with the linear theory that the Ribe program is aiming for. However, even in this particular setting of metric cotype, Theorem 9 is only a part of the full picture, as it has additional purely metric ramifications. Most of these rely on a delicate issue that has been suppressed in the above statement of Theorem 9, namely that of understanding the asymptotic behavior of $m = m(n, \mathfrak{M}, q)$ in (7). This matter is not yet fully resolved even when $\mathfrak{M}$ is a Banach space Mendel and Naor [2008] and Giladi, Mendel, and Naor [2011], and generally such questions seem to be quite challenging (see Mendel and Naor [2007b], Giladi and Naor [2010], and Naor [2016] for related issues). Thus far, whenever this question was answered for specific (classes of) metric spaces, it led to interesting geometric applications; e.g. its resolution for certain Banach spaces in Mendel and Naor [2008] was used in Naor [2012a] to answer a longstanding question Väisälä [1999]

about quasisymmetric embeddings, and its resolution for Alexandrov spaces of (global) nonpositive curvature Aleksandrov [1951] (see e.g. Burago, Gromov, and Perelman [1992] and Sturm [2003] for the relevant background) in the forthcoming work Eskenazis, Mendel, and Naor [2017b] is used there to answer a longstanding question about the coarse geometry of such Alexandrov spaces.

1.3 Metric dimension reduction. By its nature, many aspects of the local theory of Banach spaces involve describing phenomena that rely on dimension-dependent estimates. In the context of the Ribe program, the goal is to formulate/conjecture analogous phenomena for metric spaces, which is traditionally governed by asking Banach space-inspired questions about a finite metric space $(\mathfrak{M}, d_{\mathfrak{M}})$ in which $\log|\mathfrak{M}|$ serves as a replacement for the dimension. This analogy arises naturally also in the context of the bi-Lipschitz embedding problem into $\mathbb{R}^k$ (Problem 2); see Remark 39 below. Early successful instances of this analogy can be found in the work of Marcus and Pisier [1984], as well as the aforementioned work of Johnson and Lindenstrauss [1984]. However, it should be stated at the outset that over the years it became clear that while making this analogy is the right way to get "on track" toward the discovery of fundamental metric phenomena, from the perspective of the Ribe program the reality is much more nuanced and, at times, even unexpected and surprising.

Johnson and Lindenstrauss [ibid., Problem 3] asked whether every finite metric space $\mathfrak{M}$ embeds with distortion $O(1)$ into some normed space $X_{\mathfrak{M}}$ (which is allowed to depend on $\mathfrak{M}$) of dimension $\dim(X_{\mathfrak{M}}) \lesssim \log|\mathfrak{M}|$. In addition to arising from the above background, this question is motivated by Johnson and Lindenstrauss [ibid., Problem 4], which asks if the Euclidean distortion of every finite metric space $\mathfrak{M}$ satisfies $\mathsf{c}_2(\mathfrak{M}) \lesssim \sqrt{\log|\mathfrak{M}|}$. If so, this would have served as a very satisfactory metric analogue of John's theorem John [1948], which asserts that any finite dimensional normed space X is $\sqrt{\dim(X)}$-isomorphic to a subspace of ℓ_2. Of course, John's theorem shows that a positive answer to the former question Johnson and Lindenstrauss [1984, Problem 3] formally implies a positive answer to the latter question Johnson and Lindenstrauss [ibid., Problem 4].

The aforementioned Johnson–Lindenstrauss lemma Johnson and Lindenstrauss [ibid.] (JL lemma, in short) shows that, at least for finite subsets of a Hilbert space, the answer to the above stated Johnson and Lindenstrauss [ibid., Problem 3] is positive.

Theorem 11 (JL lemma)**.** *For each $n \in \mathbb{N}$ and $\alpha \in (1,\infty)$, there is $k \in \{1,\ldots,n\}$ with $k \lesssim_\alpha \log n$ such that any n-point subset of ℓ_2 embeds into ℓ_2^k with distortion α.*

We postpone discussion of this fundamental geometric fact to Section 2 below, where it is examined in detail and its proof is presented. Beyond Hilbert spaces, there is only one other example (and variants thereof) of a Banach space for which it is currently known that Johnson and Lindenstrauss [ibid., Problem 3] has a positive answer for any of its finite subsets, as shown in the following theorem from Johnson and Naor [2010].

Theorem 12. *There is a Banach space $\mathcal{T}^{(2)}$ which is not isomorphic to a Hilbert space yet it has the following property. For every finite subset $\mathcal{C} \subseteq \mathcal{T}^{(2)}$ there is*

$k \in \{1,\ldots,n\}$ with $k \lesssim \log|\mathcal{C}|$ and a k-dimensional linear subspace F of $\mathcal{T}^{(2)}$ such that $\mathcal{C}$ embeds into F with $O(1)$ distortion.

The space $\mathcal{T}^{(2)}$ of Theorem 12 is not very quick to describe, so we refer to Johnson and Naor [2010] for the details. It suffices to say here that this space is the 2-convexification of the classical Tsirelson space Cirel'son [1974] and Figiel and Johnson [1974], and that the proof that it satisfies the stated dimension reduction result is obtained in Johnson and Naor [2010] via a concatenation of several (substantial) structural results in the literature; see Section 4 in Johnson and Naor [ibid.] for a discussion of variants of this construction, as well as related open questions. The space $\mathcal{T}^{(2)}$ of Theorem 12 is not isomorphic to a Hilbert space, but barely so: it is explained in Johnson and Naor [ibid.] that for every $n \in \mathbb{N}$ there exists an n-dimensional subspace F_n of $\mathcal{T}^{(2)}$ with $\mathsf{c}_2(F_n) \geqslant e^{c\mathsf{Ack}^{-1}(n)}$, where $c > 0$ is a universal constant and $\mathsf{Ack}^{-1}(\cdot)$ is the inverse of the Ackermann function from computability theory (see e.g. Alon, Kaplan, Nivasch, Sharir, and Smorodinsky [2008, Appendix B]). So, indeed $\lim_{n\to\infty} \mathsf{c}_2(F_n) = \infty$, but at a tremendously slow rate.

Remarkably, despite major scrutiny for over 3 decades, it remains unknown if Johnson and Lindenstrauss [1984, Problem 3] has a positive answer for subsets of *any* non-universal classical Banach space. In particular, the following question is open.

Question 13. Suppose that $p \in [1,\infty) \smallsetminus \{2\}$. Are there $\alpha = \alpha(p), \beta = \beta(p) \in [1,\infty)$ such that for any $n \in \mathbb{N}$, every n-point subset of ℓ_p embeds with distortion α into some k-dimensional normed space with $k \leqslant \beta \log n$?

It is even open if in Question 13 one could obtain a bound of $k = o(n)$ for any fixed $p \in [1,\infty) \smallsetminus \{2\}$. Using John's theorem as above, a positive answer to Question 13 would imply that $\mathsf{c}_2(\mathcal{C}) \lesssim_p \sqrt{\log|\mathcal{C}|}$ for any finite subset $\mathcal{C}$ of ℓ_p. At present, such an embedding statement is not known for *any* $p \in [1,\infty) \smallsetminus \{2\}$, though for $p \in [1,2]$ it is known Arora, Lee, and Naor [2008] that any n-point subset of ℓ_p embeds into ℓ_2 with distortion $(\log n)^{1/2+o(1)}$; it would be interesting to obtain any $o(\log n)$ bound here for any fixed $p \in (2,\infty)$, which would be a "nontrivial" asymptotic behavior in light of the following general theorem Bourgain [1985a].

Theorem 14 (Bourgain's embedding theorem)**.** $\mathsf{c}_2(\mathfrak{M}) \lesssim \log|\mathfrak{M}|$ *for every finite metric space* $\mathfrak{M}$.

The above questions from Johnson and Lindenstrauss [1984] were the motivation for the influential work Bourgain [1985a], where Theorem 14 was proved. Using a probabilistic construction and the JL lemma, it was shown in Bourgain [ibid.] that Theorem 14 is almost sharp in the sense that there are arbitrarily large n-point metric spaces $\mathfrak{M}_n$ for which $\mathsf{c}_2(\mathfrak{M}_n) \gtrsim (\log n)/\log\log n$. By John's theorem, for every $\alpha \geqslant 1$, if X is a finite dimensional normed space and $\mathsf{c}_X(\mathfrak{M}_n) \leqslant \alpha$, then $\mathsf{c}_2(\mathfrak{M}_n) \leqslant \alpha\sqrt{\dim(X)}$. Therefore the above lower bound on $\mathsf{c}_2(\mathfrak{M}_n)$ implies that $\dim(X) \gtrsim (\log n)^2/(\alpha^2(\log\log n)^2)$.

The achievement of Bourgain [ibid.] is thus twofold. Firstly, it discovered Theorem 14 (via the introduction of an influential randomized embedding method), which is the "correct" metric version of John's theorem in the Ribe program. The reality turned out to be more nuanced in the sense that the answer is not quite as good as the $O(\sqrt{\log n})$ that was predicted in Johnson and Lindenstrauss [1984], but the $O(\log n)$ of Theorem 14

is still a strong and useful phenomenon that was discovered through the analogy that the Ribe program provided. Secondly, we saw above that Johnson and Lindenstrauss [ibid., Problem 3] was *disproved* in Bourgain [1985a], though the "bad news" that follows from Bourgain [ibid.] is only mildly worse than the $O(\log n)$ dimension bound that Johnson and Lindenstrauss [1984, Problem 3] predicted, namely a dimension lower bound that grows quite slowly, not faster than $(\log n)^{O(1)}$. Curiously, the very availability of strong dimension reduction in ℓ_2 through the JL lemma is what was harnessed in Bourgain [1985a] to deduce that any "host normed space" that contains $\mathfrak{M}_n$ with $O(1)$ distortion must have dimension at least of order $(\log n/\log\log n)^2 \gg \log n$. Naturally, in light of these developments, the question of understanding what is the correct asymptotic behavior of the smallest $k(n) \in \mathbb{N}$ such that any n-point metric space embeds with distortion $O(1)$ into a $k(n)$-dimensional normed space was raised in Bourgain [ibid.].

In order to proceed, it would be convenient to introduce some notation and terminology.

Definition 15 (metric dimension reduction modulus)**.** *Fix $n \in \mathbb{N}$ and $\alpha \in [1,\infty)$. Suppose that $(X, \|\cdot\|_X)$ is a normed space. Denote by $\mathsf{k}_n^\alpha(X)$ the minimum $k \in \mathbb{N}$ such that for* any *$\mathcal{C} \subseteq X$ with $|\mathcal{C}| = n$ there exists a k-dimensional linear subspace $F_{\mathcal{C}}$ of X into which $\mathcal{C}$ embeds with distortion α.*

The quantity $\mathsf{k}_n^\alpha(\ell_\infty)$ was introduced by Bourgain [ibid.] under the notation $\psi_\alpha(n) = \mathsf{k}_n^\alpha(\ell_\infty)$; see also Arias-de-Reyna and Rodríguez-Piazza [1992] and Matoušek [1996] where this different notation persists, though for the sake of uniformity of the ensuing discussion we prefer not to use it here because we will treat $X \neq \ell_\infty$ extensively. Bourgain [1985a] focused for concreteness on the arbitrary value $\alpha = 2$, and asked for the asymptotic behavior $\mathsf{k}_n^2(\ell_\infty)$ as $n \to \infty$.

An n-point subset of $X = \ell_\infty$ is nothing more than a general n-point metric space, via the aforementioned isometric Fréchet embedding. In the same vein, a k-dimensional linear subspace of ℓ_∞ is nothing more than a general k-dimensional normed space $(F, \|\cdot\|_F)$ via the linear isometric embedding $(x \in F) \mapsto (x_i^*(x))_{i=1}^\infty$, where $\{x_i^*\}_{i=1}^\infty$ is an arbitrary sequence of linear functionals on F that are dense in the unit sphere of the dual space F^*. Thus, the quantity $\mathsf{k}_n^\alpha(\ell_\infty)$ is the smallest $k \in \mathbb{N}$ such that every n-point metric space $(\mathfrak{M}, d_{\mathfrak{M}})$ can be realized with distortion at most α as a subset of $(\mathbb{R}^k, \|\cdot\|_{\mathfrak{M}})$ for some norm $\|\cdot\|_{\mathfrak{M}} : \mathbb{R}^k \to \mathbb{R}^k$ on $\mathbb{R}^k$ (which, importantly, is allowed to be adapted to the initial metric space $\mathfrak{M}$), i.e., $\mathsf{c}_{(\mathbb{R}^k, \|\cdot\|_{\mathfrak{M}})}(\mathfrak{M}, d_{\mathfrak{M}}) \leqslant \alpha$. This is precisely the quantity that Johnson and Lindenstrauss [1984, Problem 3] asks about, and above we have seen that Bourgain [1985a] gives the lower bound

$$\mathsf{k}_n^\alpha(\ell_\infty) \gtrsim \left(\frac{\log n}{\alpha \log\log n}\right)^2. \tag{9}$$

Theorem 16 below is a summary of the main nontrivial[3] bounds on the modulus $\mathsf{k}_n^\alpha(X)$ that are currently known for specific Banach spaces X. Since such a "combined statement" contains a large amount information and covers a lot of the literature on this

[3]Trivially $\mathsf{k}_n^\alpha(X) \leqslant n-1$, by considering in Definition 15 the subspace $F_{\mathcal{C}} = \mathbf{span}(\mathcal{C} - x_0)$ for any fixed $x_0 \in \mathcal{C}$.

topic, we suggest reading it in tandem with the subsequent discussion, which includes further clarifications and explanation of the history of the respective results. A "take home" message from the statements below is that despite major efforts by many researchers, apart from information on metric dimension reduction for ℓ_2, the space $\mathcal{T}^{(2)}$ of Theorem 12, ℓ_∞, ℓ_1 and S_1, nothing is known for other spaces (even for ℓ_1 and S_1 more remains to be done, notably with respect to bounding $\mathsf{k}_n^\alpha(\ell_1), \mathsf{k}_n^\alpha(\mathsf{S}_1)$ from above).

Theorem 16 (summary of the currently known upper and lower bounds on metric dimension reduction)**.** *There exist universal constants $c, C > 0$ such that the following assertions hold true for every integer $n \geqslant 20$.*

1. *In the Hilbertian setting, we have the sharp bounds*
(10)
$$\forall\, \alpha \geqslant 1 + \frac{1}{\sqrt[3]{n}}, \qquad \mathsf{k}_n^\alpha(\ell_2) \asymp \frac{\log n}{\log(1+(\alpha-1)^2)} \asymp \max\left\{\frac{\log n}{(\alpha-1)^2}, \frac{\log n}{\log \alpha}\right\}.$$

2. *For the space $\mathcal{T}^{(2)}$ of Theorem 12, there exists $\alpha_0 \in [1,\infty)$ for which $\mathsf{k}_n^{\alpha_0}(\mathcal{T}^{(2)}) \asymp \log n$.*

3. *For ℓ_∞, namely in the setting of Johnson and Lindenstrauss [1984, Problem 3], we have*
$$\forall\, \alpha \in [1,2), \qquad \mathsf{k}_n^\alpha(\ell_\infty) \asymp n, \tag{11}$$
and
$$\forall\, \alpha \geqslant 2, \qquad n^{\frac{c}{\alpha}} + \frac{\log n}{\log\left(\frac{\log n}{\alpha} + \frac{\alpha \log\log n}{\log n}\right)} \lesssim \mathsf{k}_n^\alpha(\ell_\infty) \lesssim \frac{n^{\frac{C}{\alpha}} \log n}{\log\left(1 + \frac{\alpha}{\log n}\right)}. \tag{12}$$

4. *For ℓ_1, we have*
$$\forall\, \alpha \geqslant 1, \qquad n^{\frac{c}{\alpha^2}} + \frac{\log n}{\log(\alpha+1)} \lesssim \mathsf{k}_n^\alpha(\ell_1) \lesssim \frac{n}{\alpha}. \tag{13}$$
Moreover, if $\alpha \geqslant 2C\sqrt{\log n}\log\log n$, then we have the better upper bound
$$\mathsf{k}_n^\alpha(\ell_1) \lesssim \frac{\log n}{\log\left(\frac{\alpha}{C\sqrt{\log n}\log\log n}\right)}. \tag{14}$$

5. *For the Schatten–von Neumann trace class S_1, we have*
$$\forall\, \alpha \geqslant 1, \qquad \mathsf{k}_n^\alpha(\mathsf{S}_1) \gtrsim n^{\frac{c}{\alpha^2}} + \frac{\log n}{\log(\alpha+1)}. \tag{15}$$

The bound $\mathsf{k}_n^\alpha(\ell_2) \lesssim (\log n)/\log(1+(\alpha-1)^2)$ in (10) restates Theorem 11 (the JL lemma) with the implicit dependence on α now stated explicitly; it actually holds for every $\alpha > 1$, as follows from the original proof in Johnson and Lindenstrauss [ibid.]

and explained in Section 2 below. The restriction $\alpha \geqslant 1 + 1/\sqrt[3]{n}$ in (10) pertains only to the corresponding lower bound on $\mathsf{k}_n^\alpha(\ell_2)$, which exhibits different behaviors in the low-distortion and high-distortion regimes.

Despite scrutiny of many researchers over the past 3 decades, only very recently the dependence on α in the JL lemma when α is arbitrarily close to 1 but independent of n was proved to be sharp by Larsen and Nelson [2017] (see Alon [2003, 2009] and Larsen and Nelson [2016] for earlier results in this direction, as well as the subsequent work Alon and Klartag [2017]). This is so even when α is allowed to tend to 1 with n, and even in a somewhat larger range than the requirement $\alpha \geqslant 1 + 1/\sqrt[3]{n}$ in (10) (see Larsen and Nelson [2017] for the details), though there remains a small range of values of α (n-dependent, very close to 1) for which it isn't currently known what is the behavior of $\mathsf{k}_n^\alpha(\ell_2)$. The present article is focused on embeddings that permit large errors, and in particular in ways to prove impossibility results even if large errors are allowed. For this reason, we will not describe here the ideas of the proof in Larsen and Nelson [ibid.] that pertains to the almost-isometric regime.

For, say, $\alpha \geqslant 2$, it is much simpler to see that the $\mathsf{k}_n^\alpha(\ell_2) \gtrsim (\log n)/\log\alpha$, in even greater generality that also explains the appearance of the term $(\log n)/\log(\alpha+1)$ in (13) and (15). One could naturally generalize Definition 15 so as to introduce the following notation for *relative metric dimension reduction moduli*. Let $\mathfrak{F}$ be a family of metric spaces and $\mathfrak{Y}$ be a family of normed spaces. For $n \in \mathbb{N}$ and $\alpha \in [1,\infty)$, denote by $\mathsf{k}_n^\alpha(\mathfrak{F},\mathfrak{Y})$ the minimum $k \in \mathbb{N}$ such that for every $\mathfrak{M} \in \mathfrak{F}$ with $|\mathfrak{M}| = n$ there exists $Y \in \mathfrak{Y}$ with $\dim(Y) = k$ such that $\mathsf{c}_Y(\mathfrak{M}) \leqslant \alpha$. When $\mathfrak{F}$ is the collection of all the finite subsets of a fixed Banach space X, and $\mathfrak{Y}$ is the collection of all the finite-dimensional linear subspaces of a fixed Banach space Y, we use the simpler notation $\mathsf{k}_n^\alpha(\mathfrak{F},\mathfrak{Y}) = \mathsf{k}_n^\alpha(X,Y)$. Thus, the modulus $\mathsf{k}_n^\alpha(X)$ of Definition 15 coincides with $\mathsf{k}_n^\alpha(X,X)$. Also, under this notation Question (13) asks if for $p \in [1,\infty)\smallsetminus\{2\}$ we have $\mathsf{k}_n^\alpha(\ell_p,\ell_\infty) \lesssim_p \log n$ for some $1 \leqslant \alpha \lesssim_p 1$. The study of the modulus $\mathsf{k}_n^\alpha(\mathfrak{F},\mathfrak{Y})$ is essentially a completely unexplored area, partially because even our understanding of the "vanilla" dimension reduction modulus $\mathsf{k}_n^\alpha(X)$ is currently very limited. By a short volumetric argument that is presented in Section 2 below, every infinite dimensional Banach space X satisfies

$$\forall (n,\alpha) \in \mathbb{N}\times[1,\infty), \qquad \mathsf{k}_n^\alpha(X,\ell_\infty) \geqslant \frac{\log n}{\log(\alpha+1)}. \tag{16}$$

Hence also $\mathsf{k}_n^\alpha(X) \geqslant (\log n)/\log(\alpha+1)$, since (16) rules out embeddings into any normed space of dimension less than $(\log n)/\log(\alpha+1)$, rather than only into such spaces that are also subspaces of X.

Using an elegant Fourier-analytic argument, Arias-de-Reyna and Rodríguez–Piazza proved in Arias-de-Reyna and Rodríguez-Piazza [1992] that for every $\alpha \in [1,2)$ we have $\mathsf{k}_n^\alpha(\ell_\infty) \gtrsim (2-\alpha)n$. This was slightly improved by Matoušek [1996] to $\mathsf{k}_n^\alpha(\ell_\infty) \gtrsim n$, i.e., he showed that the constant multiple of n actually remains bounded below by a positive constant as $\alpha \to 2^-$ (curiously, the asymptotic behavior of $\mathsf{k}_n^2(\ell_\infty)$ remains unknown). These results establish (11). So, for sufficiently small distortions one cannot hope to embed every n-point metric space into some normed space of dimension

$o(n)$. For larger distortions (our main interest), it was conjectured in Arias-de-Reyna and Rodríguez-Piazza [1992] that $\mathsf{k}_n^\alpha(\ell_\infty) \lesssim (\log n)^{O(1)}$ if $\alpha > 2$.

The bounds in (12) refute this conjecture of Arias-de-Reyna and Rodríguez-Piazza [ibid.], since they include the lower bound $\mathsf{k}_n^\alpha(\ell_\infty) \gtrsim n^{c/\alpha}$, which is a landmark achievement of Matoušek [1996] (obtained a decade after Bourgain asked about the asymptotics here and over a decade after Johnson and Lindenstrauss posed the question whether $\mathsf{k}_n^\alpha(\ell_\infty) \lesssim_\alpha \log n$). It is, of course, an exponential improvement over Bourgain's bound (9). Actually, in the intervening period Linial, London, and Rabinovich [1995] removed the iterated logarithm in the lower bound of Bourgain [1985a] by showing that Theorem 14 (Bourgain's embedding) is sharp up to the value of the implicit universal constant. By the same reasoning as above (using John's theorem), this also removed the iterated logarithm from the denominator in (9), i.e., Linial, London, and Rabinovich [1995] established that $\mathsf{k}_n^\alpha(\ell_\infty) \gtrsim (\log n)^2/\alpha^2$. This was the best-known bound prior to Matoušek [1996].

Beyond proving a fundamental geometric theorem, which, as seen in (12), is optimal up to the constant in the exponent, this work of Matoušek is important because it injected a refreshing approach from real algebraic geometry into this area, which was previously governed by considerations from analysis, geometry, probability and combinatorics. Section 4 covers this outstanding contribution in detail, and obtains the following stronger statement that wasn't previously noticed in the literature but follows from an adaptation of Matoušek's ideas.

Theorem 17 (impossibility of coarse dimension reduction)**.** *There is a universal constant $c \in (0,\infty)$ with the following property. Suppose that $\omega, \Omega : [0,\infty) \to [0,\infty)$ are increasing functions that satisfy $\omega(s) \leqslant \Omega(s)$ for all $s \in [0,\infty)$, as well as $\lim_{s\to\infty} \omega(s) = \infty$. Define*

$$\beta(\omega,\Omega) \stackrel{\mathrm{def}}{=} \sup_{s\in(0,\infty)} \frac{s}{\omega^{-1}\big(2\Omega(s)\big)} \in (0,1). \tag{17}$$

For arbitrarily large $n \in \mathbb{N}$ there is a metric space

$$(\mathfrak{M}, d_{\mathfrak{M}}) = \big(\mathfrak{M}(n,\omega,\Omega), d_{\mathfrak{M}(n,\omega,\Omega)}\big)$$

with $|\mathfrak{M}| = 3n$ such that for any normed space $(X, \|\cdot\|_X)$, if there exists $f : \mathfrak{M} \to X$ which satisfies

$$\forall\, x,y \in \mathfrak{M}, \qquad \omega\big(d_{\mathfrak{M}}(x,y)\big) \leqslant \|f(x)-f(y)\|_X \leqslant \Omega\big(d_{\mathfrak{M}}(x,y)\big), \tag{18}$$

then necessarily

$$\dim(X) \gtrsim n^{c\beta(\omega,\Omega)}. \tag{19}$$

A mapping that satisfies (18) is called a *coarse embedding* (with moduli ω, Ω), as introduced in Gromov's seminal work Gromov [1993] and studied extensively ever since, with a variety of interesting applications (see the monographs Roe [2003], Nowak and Yu [2012], and M. I. Ostrovskii [2013] and the references therein). The bi-Lipschitz requirement (1) corresponds to $\omega(s) = \tau s$ and $\Omega(s) = \alpha\tau s$ in (18), in which case (19)

becomes Matoušek's aforementioned lower bound on $\mathsf{k}_n^\alpha(\ell_\infty)$. Theorem 17 asserts that there exist arbitrarily large finite metric spaces that cannot be embedded even with a very weak (coarse) guarantee into any low-dimensional normed space, with the dimension of the host space being forced to be at least a power of their cardinality, which is exponentially larger than the logarithmic behavior that one would predict from the natural ball-covering requirement that is induced by low-dimensionality (see the discussion of the doubling condition in Section 3, as well as the proof of (16) in Section 2).

Remark 18. Consider the following special case of Theorem 17. Fix $\theta \in (0,1]$ and let $(\mathfrak{M}, d_{\mathfrak{M}})$ be a metric space. It is straightforward to check that $d_{\mathfrak{M}}^\theta : \mathfrak{M} \times \mathfrak{M} \to [0,\infty)$ is also a metric on $\mathfrak{M}$. The metric space $(\mathfrak{M}, d_{\mathfrak{M}}^\theta)$ is commonly called the θ-*snowflake* of $\mathfrak{M}$ (in reference to the von Koch snowflake curve; see e.g. David and Semmes [1997]) and it is denoted $\mathfrak{M}^\theta$. Given $\alpha \geqslant 1$, the statement that $\mathfrak{M}^\theta$ embeds with distortion α into a normed space $(X, \|\cdot\|_X)$ is the same as the requirement (18) with $\omega(s) = s^\theta$ and $\Omega(s) = \alpha s^\theta$. Hence, by Theorem 17 there exist arbitrarily large n-point metric spaces $\mathfrak{M}_n = \mathfrak{M}_n(\alpha, \theta)$ such that if $\mathfrak{M}_n^\theta$ embeds with distortion α into some k-dimensional normed space, then $k \geqslant n^{c/(2\alpha)^{1/\theta}}$. Conversely, Remark 21 below shows that for every $n \in \mathbb{N}$ and $\alpha > 1$, the θ-snowflake of any n-point metric space embeds with distortion α into a normed space X with $\dim(X) \lesssim_{\alpha,\theta} n^{C/\alpha^{1/\theta}}$. So, the bound (19) of Theorem 17 is quite sharp even for embeddings that are not bi-Lipschitz, though we did not investigate the extent of its sharpness for more general moduli $\omega, \Omega : [0,\infty) \to [0,\infty)$.

At this juncture, it is natural to complement the (coarse) strengthening in Theorem 17 of Matoušek's bound $\mathsf{k}_n^\alpha(\ell_\infty) \geqslant n^{c/\alpha}$ by stating the following different type of strengthening, which we recently obtained in Naor [2017].

Theorem 19 (impossibility of average dimension reduction)**.** *There is a universal constant $c \in (0,\infty)$ with the following property. For arbitrarily large $n \in \mathbb{N}$ there is an n-point metric space $(\mathfrak{M}, d_{\mathfrak{M}})$ such that for any normed space $(X, \|\cdot\|_X)$ and any $\alpha \in [1,\infty)$, if there exists $f : \mathfrak{M} \to X$ which satisfies $\|f(x) - f(y)\|_X \leqslant \alpha d_{\mathfrak{M}}(x,y)$ for all $x, y \in X$, yet $\frac{1}{n^2}\sum_{x,y\in X}\|f(x)-f(y)\|_X \geqslant \frac{1}{n^2}\sum_{x,y\in X} d_{\mathfrak{M}}(x,y)$, then necessarily* $\dim(X) \geqslant n^{c/\alpha}$.

An n-point metric space $\mathfrak{M}$ as in Theorem 19 is intrinsically high dimensional even on average, in the sense that if one wishes to assign in an α-Lipschitz manner to each point in $\mathfrak{M}$ a vector in some normed space X such that the average distance in the image is the same as the average distance in $\mathfrak{M}$, then this forces the ambient dimension to satisfy $\dim(X) \geqslant n^{c/\alpha}$. Prior to Theorem 19, the best-known bound here was $\dim(X) \gtrsim (\log n)^2/\alpha^2$, namely the aforementioned lower bound $\mathsf{k}_n^\alpha(\ell_\infty) \gtrsim (\log n)^2/\alpha^2$ of Linial, London, and Rabinovich [1995] actually treated the above "average distortion" requirement rather than only the (pairwise) bi-Lipschitz requirement.

Remark 20. The significance of Theorem 19 will be discussed further in Section 5 below; see also Andoni, Nguyen, Nikolov, Razenshteyn, and Waingarten [2017] and Andoni, Naor, Nikolov, Razenshteyn, and Waingarten [2018a]. In Section 5 we will present a new proof of Theorem 19 that is different from (though inspired by) its proof in Naor [2017]. It suffices to say here that the proof of Theorem 19 is conceptually

different from Matoušek's approach Matoušek [1996]. Namely, in contrast to the algebraic/topological argument of Matoušek [ibid.], the proof of Theorem 19 relies on the theory of nonlinear spectral gaps, which is also an outgrowth of the Ribe program; doing justice to this theory and its ramifications is beyond the scope of the present article (see Mendel and Naor [2014] and the references therein), but the basics are recalled in Section 5. Importantly, the proof of Theorem 19 obtains a criterion for determining if a given metric space $\mathfrak{M}$ satisfies its conclusion, namely $\mathfrak{M}$ can be taken to be the shortest-path metric of any bounded degree graph with a spectral gap. This information is harnessed in the forthcoming work Andoni, Naor, Nikolov, Razenshteyn, and Waingarten [2018a] to imply that finite-dimensional normed spaces have a structural property (a new type of hierarchical partitioning scheme) which has implications to the design of efficient data structures for approximate nearest neighbor search, demonstrating that the omnipresent "curse of dimensionality" is to some extent absent from this fundamental algorithmic task.

In the intervening period between Bourgain's work Bourgain [1985a] and Matousek's solution Matoušek [1996], the question of determining the asymptotic behavior of $\mathsf{k}_n^\alpha(\ell_\infty)$ was pursued by Johnson, Lindenstrauss and Schechtman, who proved in Johnson, Lindenstrauss, and Schechtman [1987] that $\mathsf{k}_n^\alpha(\ell_\infty) \lesssim_\alpha n^{C/\alpha}$ for some universal constant $C > 0$. They demonstrated this by constructing for every n-point metric space $\mathfrak{M}$ a normed space $X_{\mathfrak{M}}$, which they (probabilistically) tailored to the given metric space $\mathfrak{M}$, with $\dim(X_{\mathfrak{M}}) \lesssim_\alpha n^{C/\alpha}$ and such that $\mathfrak{M}$ embeds into $X_{\mathfrak{M}}$ with distortion α. Subsequently, Matoušek showed Matoušek [1992a] via a different argument that one could actually work here with $X_{\mathfrak{M}} = \ell_\infty^k$ for $k \in \mathbb{N}$ satisfying $k \lesssim_\alpha n^{C/\alpha}$, i.e., in order to obtain this type of upper bound on the asymptotic behavior of $\mathsf{k}_n^\alpha(\ell_\infty)$ one does not need to adapt the target normed space to the metric space $\mathfrak{M}$ that is being embedded. The implicit dependence on α here, as well as the constant C in the exponent, were further improved in Matoušek [1996]. For $\alpha = O((\log n)/\log\log n)$, the upper bound on $\mathsf{k}_n^\alpha(\ell_\infty)$ that appears in (12) is that of Matoušek [ibid.], and for the remaining values of α it is due to a more recent improvement over Matoušek [ibid.] by Abraham, Bartal, and Neiman [2011] (specifically, the upper bound in (12) is a combination of Theorem 5 and Theorem 6 of Abraham, Bartal, and Neiman [ibid.]).

Remark 21. An advantage of the fact Matoušek [1992a] that one could take $X_{\mathfrak{M}} = \ell_\infty^k$ rather than the more general normed space of Johnson, Lindenstrauss, and Schechtman [1987] is that it quickly implies the optimality of the lower bound from Remark 18 on dimension reduction of snowflakes. Fix $n \in \mathbb{N}$, $\alpha > 1$ and $\theta \in (0,1]$. Denote $\delta = \min\{\sqrt{\alpha} - 1, 1\}$, so that $\alpha \asymp \alpha/(1+\delta) > 1$. By Matoušek [1992a], given an n-point metric space $(\mathfrak{M}, d_{\mathfrak{M}})$ there is an integer $k \lesssim n^{c/\alpha^{1/\theta}}$ and $f = (f_1, \ldots, f_k) : \mathfrak{M} \to \mathbb{R}^k$ such that

$$\forall\, x, y \in \mathfrak{M}, \qquad d_{\mathfrak{M}}(x,y) \leqslant \|f(x) - f(y)\|_{\ell_\infty^k} \leqslant \left(\frac{\alpha}{1+\delta}\right)^{\frac{1}{\theta}} d_{\mathfrak{M}}(x,y).$$

Hence (here it becomes useful that we are dealing with the ℓ_∞^k norm, as it commutes with powering),

$$\forall\, x, y \in \mathfrak{M}, \qquad d_{\mathfrak{M}}(x,y)^\theta \leqslant \max_{i\in\{1,\ldots,k\}} |f_i(x) - f_i(y)|^\theta \leqslant \frac{\alpha}{1+\delta} d_{\mathfrak{M}}(x,y)^\theta.$$

By works of Kahane [1981] and Talagrand [1992], there is $m = m(\delta,\theta)$ and a mapping (a quasi-helix) $h : \mathbb{R} \to \mathbb{R}^m$ such that $|s-t|^\theta \leqslant \|h(s) - h(t)\|_{\ell_\infty^m} \leqslant (1+\delta)|s-t|^\theta$ for all $s, t \in \mathbb{R}$. The mapping

$$(x \in \mathfrak{M}) \mapsto \bigoplus_{i=1}^k h \circ f_i(x) \in \bigoplus_{i=1}^k \ell_\infty^m$$

is a distortion-α embedding of the θ-snowflake $(\mathfrak{M}, d_{\mathfrak{M}}^\theta)$ into a normed space of dimension $mk \lesssim_{\alpha,\theta} n^{c/\alpha^{1/\theta}}$. The implicit dependence on α, θ that Kahane [1981] and Talagrand [1992] imply here is quite good, but likely not sharp as $\alpha \to 1^+$ when $\theta \neq \frac{1}{2}$.

Since the expressions in (12) are somewhat involved, it is beneficial to restate them on a case-by-case basis as follows. For sufficiently large α, we have a bound[4] that is sharp up to universal constant factors.

$$\alpha \geqslant (\log n)\log\log n \implies \mathsf{k}_n^\alpha(\ell_\infty) \asymp \frac{\log n}{\log\left(\frac{\alpha}{\log n}\right)}. \tag{20}$$

For a range of smaller values of α, including those α that do not tend to ∞ with n, we have

$$1 \leqslant \alpha \leqslant \frac{\log n}{\log\log n} \implies n^{\frac{c}{\alpha}} \lesssim \mathsf{k}_n^\alpha(\ell_\infty) \lesssim n^{\frac{C}{\alpha}}. \tag{21}$$

(21) satisfactorily shows that the asymptotic behavior of $\mathsf{k}_n^\alpha(\ell_\infty)$ is of power-type, but it is not as sharp as (20). We suspect that determining the correct exponent of n in the power-type dependence of $\mathsf{k}_n^\alpha(\ell_\infty)$ would be challenging (there is indication Matoušek [1996, 2002], partially assuming a positive answer to a difficult conjecture of Erdős [1964] and Bollobás [1978], that this exponent has infinitely many jump discontinuities as a function of α). In an intermediate range $(\log n)/\log\log n \lesssim \alpha \lesssim (\log n)\log\log n$ the bounds (12) are less satisfactory. The case $\alpha \asymp \log n$, corresponding to the distortion in Bourgain's embedding theorem, is especially intriguing, with (12) becoming

$$\frac{\log n}{\log\log\log n} \lesssim \mathsf{k}_n^{\Theta(\log n)}(\ell_\infty) \lesssim \log n. \tag{22}$$

[4]One can alternatively justify the upper bound in (20) (for sufficiently large n) by first using Theorem 14 (Bourgain's embedding theorem) to embed an n-point metric space $\mathfrak{M}$ into ℓ_2 with distortion $A\log n$ for some universal constant $A \geqslant 1$, and then using Theorem 11 (the JL lemma) with the dependence on the distortion as stated in (10) to reduce the dimension of the image of $\mathfrak{M}$ under Bourgain's embedding to $O((\log n)/\log(\alpha/(A\log n)))$ while incurring a further distortion of $\alpha/(A\log n)$, thus making the overall distortion be at most α. The right hand side of (20) is therefore in fact an upper bound on $\mathsf{k}_n^\alpha(\ell_\infty, \ell_2)$; see also Corollary 50.

The first inequality in (22) has not been stated in the literature, and we justify it in Section 5 below. A more natural lower bound here would be a constant multiple of $(\log n)/\log\log n$, as this corresponds to the volumetric restriction (16), and moreover by the upper bound (10) in the JL lemma we know that any n-point subset of a Hilbert space does in fact embed with distortion $\log n$ into ℓ_2^k with $k \lesssim (\log n)/\log\log n$. The triple logarithm in (22) is therefore quite intriguing/surprising, thus leading to the following open question.

Question 22. Given an integer $n \geqslant 2$, what is the asymptotic behavior of the smallest $k = k_n \in \mathbb{N}$ for which any n-point metric space $\mathfrak{M}$ embeds with distortion $O(\log n)$ into some k-dimensional normed space $X_{\mathfrak{M}}$.

There is a dearth of available upper bounds on $\mathsf{k}_n^\alpha(\cdot)$, i.e., positive results establishing that metric dimension reduction is possible. This is especially striking in the case of $\mathsf{k}_n^\alpha(\ell_1)$, due to the importance of ℓ_1 from the perspective of pure mathematics and algorithms. The upper bound on $\mathsf{k}_n^\alpha(\ell_1)$ in the large distortion regime (14) follows from combining the Euclidean embedding of Arora, Lee, and Naor [2008] with the JL lemma. The only general dimension reduction result in ℓ_1 that lowers the dimension below the trivial bound $\mathsf{k}_n^\alpha(\ell_1) \leqslant n-1$ is the forthcoming work Austin and Naor [2017], where the estimate $\mathsf{k}_n^\alpha(\ell_1) \lesssim n/\alpha$ in (13) is obtained; even this modest statement requires effort (among other things, it relies on the sparsification method of Batson, Spielman, and Srivastava [2012]).

The bound $\mathsf{k}_n^\alpha(\ell_1) \geqslant n^{c/\alpha^2}$ in (13) is a remarkable theorem of Brinkman and M. Charikar [2005] which answered a question that was at the time open for many years. To avoid any possible confusion, it is important to note that Brinkman and M. Charikar [ibid.] actually exhibits an n-point subset $\mathscr{C}_{\mathsf{BC}}$ of ℓ_1 for which it is shown in Brinkman and M. Charikar [ibid.] that if $\mathscr{C}_{\mathsf{BC}}$ embeds with distortion α into ℓ_1^k, then necessarily $k \geqslant n^{c/\alpha^2}$. On the face of it, this seems weaker than (13), because the lower bound on $\mathsf{k}_n^\alpha(\ell_1)$ in (13) requires showing that if $\mathscr{C}_{\mathsf{BC}}$ embeds into an *arbitrary* finite-dimensional linear subspace F of ℓ_1, then necessarily $\dim(F) \geqslant n^{c/\alpha^2}$. However, Talagrand [1990] proved that in this setting for every $\beta > 1$ the subspace F embeds with distortion β into ℓ_1^k, where $k \lesssim_\beta \dim(F)\log\dim(F)$. From this, an application of the above stated result of Brinkman and M. Charikar [2005] gives that $\dim(F)\log\dim(F) \gtrsim n^{c/\alpha^2}$, and so the lower bound in (13) follows from the formulation in Brinkman and M. Charikar [ibid.]. Satisfactory analogues of the above theorem of Talagrand are known Schechtman [1987], Bourgain, Lindenstrauss, and V. Milman [1989], and Talagrand [1995] (see also the survey Johnson and Schechtman [2001] for more on this subtle issue) when ℓ_1 is replaced by ℓ_p for some $p \in (1,\infty)$, but such reductions to "canonical" linear subspaces are not available elsewhere, so the above reasoning is a rare "luxury" and in general one must treat arbitrary low-dimensional linear subspaces of the Banach space in question.

The above difficulty was overcome for S_1 in Naor, Pisier, and Schechtman [2018], where (15) was proven. The similarity of the lower bounds in (13) and (15) is not coincidental. One can view the Brinkman–Charikar example $\mathscr{C}_{\mathsf{BC}} \subseteq \ell_1$ also as a collection of diagonal matrices in S_1, and Naor, Pisier, and Schechtman [ibid.] treats this very same subset by strengthening the assertion of Brinkman and M. Charikar [2005] that

$\mathcal{C}_{\mathsf{BC}}$ does not well-embed into low-dimensional subspaces of S_1 which consist entirely of diagonal matrices, to the same assertion for low-dimensional subspaces of S_1 which are now allowed to consist of any matrices whatsoever. Using our notation for relative dimension reduction moduli, this gives the stronger assertion $\mathsf{k}_n^{\alpha}(\ell_1, \mathsf{S}_1) \geqslant n^{c/\alpha^2}$.

A geometric challenge of the above discussion is that, even after one isolates a candidate n-point subset $\mathcal{C}$ of ℓ_1 that is suspected not to be realizable with $O(1)$ distortion in low-dimensions (finding such a suspected intrinsically high-dimensional set is of course a major challenge in itself), one needs to devise a way to somehow argue that if one could find a configuration of n points in a low-dimensional subspace F of ℓ_1 (or S_1) whose pairwise distances are within a fixed, but potentially very large, factor $\alpha \geqslant 1$ of the corresponding pairwise distances within $\mathcal{C}$ itself, then this would force the ambient dimension $\dim(F)$ to be very large. In Brinkman and M. Charikar [ibid.] this was achieved via a clever proof that relies on linear programming; see also Andoni, M. S. Charikar, Neiman, and Nguyen [2011] for a variant of this linear programming approach in the almost isometric regime $\alpha \to 1^+$. In Regev [2013] a different proof of the Brinkman–Charikar theorem was found, based on information-theoretic reasoning. Another entirely different geometric method to prove that theorem was devised in Lee and Naor [2004]; see also Lee, Mendel, and Naor [2005] and Johnson and Schechtman [2009] for more applications of the approach of Lee and Naor [2004].

Very recently, a further geometric approach was obtained in Naor, Pisier, and Schechtman [2018], where it was used to derive a stronger statement that, as shown in Naor, Pisier, and Schechtman [ibid.], cannot follow from the method of Lee and Naor [2004] (the statement is that the n-point subset $\mathcal{C}_{\mathsf{BC}} \subseteq \ell_1$ is not even an α-Lipschitz quotient of any subset of a low dimensional subspace of S_1; see Naor, Pisier, and Schechtman [2018] for the relevant definition an a complete discussion). The approach of Naor, Pisier, and Schechtman [ibid.] relies on an invariant that arose in the Ribe program and is called *Markov convexity*. Fix $q > 0$. Let $\{\chi_t\}_{t\in\mathbb{Z}}$ be a Markov chain on a state space Ω. Given an integer $k \geqslant 0$, denote by $\{\widetilde{\chi}_t(k)\}_{t\in\mathbb{Z}}$ the process that equals χ_t for time $t \leqslant k$, and evolves independently of χ_t (with respect to the same transition probabilities) for time $t > k$. Following Lee, Naor, and Peres [2009], the Markov q-convexity constant of a metric space $(\mathfrak{M}, d_{\mathfrak{M}})$, denoted $\Pi_q(\mathfrak{M})$, is the infimum over those $\Pi \in [0,\infty]$ such that for every Markov chain $\{\chi_t\}_{t\in\mathbb{Z}}$ on a state space Ω and every $f : \Omega \to \mathfrak{M}$ we have

$$\left(\sum_{k=1}^{\infty}\sum_{t\in\mathbb{Z}}\frac{1}{2^{qk}}\mathbb{E}\left[d_{\mathfrak{M}}\big(f(\widetilde{\chi}_t(t-2^k)), f(\chi_t)\big)^q\right]\right)^{\frac{1}{q}} \leqslant \Pi\left(\sum_{t\in\mathbb{Z}}\mathbb{E}\left[d_{\mathfrak{M}}\big(f(\chi_t), f(\chi_{t-1})\big)^q\right]\right)^{\frac{1}{q}}$$

By Lee, Naor, and Peres [2009] and Mendel and Naor [2013], a Banach space X satisfies $\Pi_q(X) < \infty$ if and only if it has an equivalent norm $|||\cdot||| : X \to [0,\infty)$ whose modulus of uniform convexity has power type q, i.e., $|||x+y||| \leqslant 2 - \Omega_X(|||x-y|||^q)$ for every $x, y \in X$ with $|||x||| = |||y||| = 1$. This completes the step in the Ribe program which corresponds to the local linear property "X admits an equivalent norm whose modulus of uniform convexity has power type q," and it is a refinement of the aforementioned characterization of superreflexivity in Bourgain [1986] (which by deep results of Enflo [1972] and Pisier [1975] corresponds to the cruder local linear property "there is a finite $q \geqslant 2$ for which X admits an equivalent norm whose modulus of uniform convexity has power type q"). By Mendel and Naor [2013] and Eskenazis, Mendel,

and Naor [2017a], the Brinkman–Charikar subset $\mathcal{C}_{\mathsf{BC}} \subseteq \ell_1$ (as well as a variant of it due to Laakso [2000] which has Lee, Mendel, and Naor [2005] the same non-embeddability property into low-dimensional subspaces of ℓ_1) satisfies $\Pi_q(\mathcal{C}_{\mathsf{BC}}) \gtrsim (\log n)^{1/q}$ for every $q \geqslant 2$ (recall that in our notation $|\mathcal{C}_{\mathsf{BC}}| = n$). At the same time, it is proved in Naor, Pisier, and Schechtman [2018] that $\Pi_2(F) \lesssim \sqrt{\log \dim(F)}$ for every finite dimensional subset of S_1. It remains to contrast these asymptotic behaviors (for $q = 2$) to deduce that if $\mathsf{c}_F(\mathcal{C}_{\mathsf{BC}}) \leqslant \alpha$, then necessarily $\dim(F) \geqslant n^{c/\alpha^2}$.

Prior to the forthcoming work Naor and Young [2017a], the set $\mathcal{C}_{\mathsf{BC}}$ (and variants thereof of a similar nature) was the only known example that demonstrates that there is no ℓ_1 analogue of the JL-lemma. The following theorem is from Naor and Young [ibid.].

Theorem 23. *There is a universal constant $c \in (0,\infty)$ with the following property. For arbitrarily large $n \in \mathbb{N}$ there exists an n-point $O(1)$-doubling subset $\mathcal{H}_n$ of ℓ_1 satisfying $\mathsf{c}_4(\mathcal{H}_n) \lesssim 1$, such that for every $\alpha \in [1,\infty)$ and every finite-dimensional linear subspace F of S_1, if $\mathcal{H}_n$ embeds into F with distortion α, then necessarily*

$$\dim(F) \geqslant \exp\Bigl(\frac{c}{\alpha^2}\sqrt{\log n}\Bigr). \tag{23}$$

See Section 3 for the (standard) terminology "doubling" that is used in Theorem 23. While (23) is weaker than the lower bound of Brinkman–Charikar in terms of the dependence on n, it nevertheless rules out metric dimension reduction in ℓ_1 (or S_1) in which the target dimension is, say, a power of $\log n$. The example $\mathcal{H}_n$ of Theorem 23 embeds with distortion $O(1)$ into ℓ_4, and hence in particular $\sup_{n\in\mathbb{N}} \Pi_4(\mathcal{H}_n) \lesssim \Pi_4(\ell_4) < \infty$, by Lee, Naor, and Peres [2009]. This makes $\mathcal{H}_n$ be qualitatively different from all the previously known examples which exhibit the impossibility of metric dimension reduction in ℓ_1, and as such its existence has further ramifications that answer long-standing questions; see Naor and Young [2017a] for a detailed discussion. The proof of Theorem 23 is markedly different from (and more involved than) previous proofs of impossibility of dimension reduction in ℓ_1, as it relies on new geometric input (a subtle property of the 3-dimensional Heisenberg group which fails for the 5-dimensional Heisenberg group) that is obtained in Naor and Young [ibid.], in combination with results from Austin, Naor, and Tessera [2013], Lafforgue and Naor [2014b,a], and Hytönen and Naor [2016]; full details appear in Naor and Young [2017a].

1.4 Spaces admitting bi-Lipschitz and average metric dimension reduction. Say that an infinite dimensional Banach space $(X, \|\cdot\|_X)$ *admits metric dimension reduction* if there is $\alpha = \alpha_X \in [1,\infty)$ such that

$$\lim_{n\to\infty} \frac{\log \mathsf{k}_n^{\alpha}(X)}{\log n} = 0.$$

In other words, the requirement here is that for some $\alpha = \alpha_X \in [1,\infty)$ and every $n \in \mathbb{N}$, any n-point subset $\mathcal{C} \subseteq X$ embeds with (bi-Lipschitz) distortion α into some linear subspace F of X with $\dim(F) = n^{o_X(1)}$.

Analogously, we say that $(X, \|\cdot\|_X)$ *admits average metric dimension reduction* if there is $\alpha = \alpha_X \in [1,\infty)$ such that for any $n \in \mathbb{N}$ there is $k_n = n^{o_X(1)}$, i.e.,

$\lim_{n\to\infty}(\log k_n)/\log n = 0$, such that for any n-point subset $\mathcal{C}$ of X there is a linear subspace F of X with $\dim(F) = k_n$ and a mapping $f : \mathcal{C} \to F$ which is α-Lipschitz, i.e., $\|f(x) - f(y)\|_X \leqslant \alpha\|x - y\|_X$ for all all $x, y \in \mathcal{C}$, yet

$$\frac{1}{n^2}\sum_{x\in\mathcal{C}}\sum_{y\in\mathcal{C}}\|f(x) - f(y)\|_X \geqslant \frac{1}{n^2}\sum_{x\in\mathcal{C}}\sum_{y\in\mathcal{C}}\|x - y\|_X. \tag{24}$$

Our choice here of the behavior $n^{o_X(1)}$ for the target dimension is partially motivated by the available results, based on which this type of asymptotic behavior appears to be a benchmark. We stress, however, that since the repertoire of spaces that are known to admit metric dimension reduction is currently very limited, finding any new setting in which one could prove that reducing dimension to $o_X(n)$ is possible would be a highly sought after achievement. In the same vein, finding new spaces for which one could prove a metric dimension reduction lower bound that tends to ∞ faster than $\log n$ (impossibility of a JL-style guarantee) would be very interesting.

Remark 24. In the above definition of spaces admitting average metric dimension reduction we imposed the requirement (24) following the terminology that was introduced by Rabinovich [2008], and due to the algorithmic usefulness of this notion of embedding. However, one could also consider natural variants such as $(\frac{1}{n^2}\sum_{x\in\mathcal{C}}\sum_{y\in\mathcal{C}}\|f(x) - f(y)\|_X^p)^{1/p} \geqslant (\frac{1}{n^2}\sum_{x\in\mathcal{C}}\sum_{y\in\mathcal{C}}\|x - y\|_X^q)^{1/q}$ for any $p, q \in (0, \infty]$, and much of the ensuing discussion holds mutatis mutandis in this setting as well.

The only Banach spaces that are currently known to admit metric dimension reduction are Hilbert spaces and the space $\mathcal{T}^{(2)}$ of Theorem 12 (and variants thereof). These examples allow for the possibility that if $(X, \|\cdot\|_X)$ admits metric dimension reduction, i.e., $\mathsf{k}_n^{O_X(1)}(X) = n^{o_X(1)}$, then actually $\mathsf{k}_n^{O_X(1)}(X) = O_X(\log n)$. Such a dichotomy would of course be remarkable, but there is insufficient evidence to conjecture that this is so.

The available repertoire of spaces that admit average metric dimension reduction is larger, since if $p \in [2, \infty)$, then ℓ_p and even S_p satisfy the assumption of the following theorem, by Mazur [1929] and Ricard [2015], respectively.

Theorem 25. *Let $(X, \|\cdot\|_X)$ be an infinite dimensional Banach space with unit ball $B_X = \{x \in X : \|x\|_X \leqslant 1\}$. Suppose that there is a Hilbert space $(H, \|\cdot\|_H)$ and a one-to-one mapping $f : B_X \to H$ such that f is Lipschitz and $f^{-1} : f(B_X) \to X$ is uniformly continuous. Then X admits average metric dimension reduction. In fact, this holds for embeddings into a subspace of logarithmic dimension, i.e., there is $\alpha = \alpha_X \in [1, \infty)$ such that for any $n \in \mathbb{N}$ and any n-point subset $\mathcal{C}$ of X there is a linear subspace F of X with $\dim(F) \lesssim \log n$ and a mapping $f : \mathcal{C} \to F$ which satisfies both (24) and $\|f(x) - f(y)\|_X \leqslant \alpha\|x - y\|_X$ for all $x, y \in \mathcal{C}$.*

Proof. This statement is implicit in Naor [2014]. By combining Naor [ibid., Proposition 7.5] and Naor [ibid., Lemma 7.6] there is a $O_X(1)$-Lipschitz mapping $f : \mathcal{C} \to \ell_2$ which satisfies $\frac{1}{n^2}\sum_{x\in\mathcal{C}}\sum_{y\in\mathcal{C}}\|f(x) - f(y)\|_2 \geqslant \frac{1}{n^2}\sum_{x\in\mathcal{C}}\sum_{y\in\mathcal{C}}\|x - y\|_X$. By the JL lemma we may assume that f actually takes values in ℓ_2^k for some $k \lesssim \log n$. Since X is infinite dimensional, Dvoretzky's theorem Dvoretzky [1961] ensures that ℓ_2^k is 2-isomorphic to a k-dimensional subspace F of X. $\square$

Remark 26. By Mazur [1929], for $p \in [2, \infty)$ the assumption of Theorem 25 holds for $X = \ell_p$. An inspection of the proofs in Naor [2014] reveals that the dependence of the Lipschitz constant $\alpha = \alpha_p$ on p that Theorem 25 provides for $X = \ell_p$ grows to ∞ exponentially with p. As argued in Naor [ibid., Section 5.1] (using metric cotype), this exponential behavior is unavoidable using the above proof. However, in this special case a more sophisticated argument of Naor [ibid.] yields $\alpha_p \lesssim p^{5/2}$; see equation (7.40) in Naor [ibid.]. Motivated by Naor [ibid., Corollary 1.6], we conjecture that this could be improved to $\alpha_p \lesssim p$, and there is some indication (see Naor [ibid., Lemma 1.11]) that this would be sharp.

Prior to Naor [2017], it was not known if there exists a Banach space which fails to admit average metric dimension reduction. Now we know (Theorem 19) that ℓ_∞ fails to admit average metric dimension reduction, and therefore also any universal Banach space fails to admit average metric dimension reduction. A fortiori, the same is true also for (non-average) metric dimension reduction, but this statement follows from the older work Matoušek [1996]. Failure of average metric dimension reduction is not known for *any* non-universal (finite cotype) Banach space, and it would be very interesting to provide such an example. By Brinkman and M. Charikar [2005] and Naor, Pisier, and Schechtman [2018] we know that ℓ_1 and S_1 fail to admit metric dimension reduction, but this is not known for average distortion, thus leading to the following question.

Question 27. Does ℓ_1 admit average metric dimension reduction? Does ℓ_p have this property for any $p \in [1, 2)$?

All of the available examples of n-point subsets of ℓ_1 for which the ℓ_1 analogue of the JL lemma fails (namely if $k = O(\log n)$, then they do not embed with $O(1)$ distortion into ℓ_1^k) actually embed into the real line $\mathbb{R}$ with $O(1)$ average distortion; this follows from Rabinovich [2008]. Specifically, the examples in Brinkman and M. Charikar [2005] and Lee, Mendel, and Naor [2005] are the shortest-path metric on planar graphs, and the example in Theorem 23 is $O(1)$-doubling, and both of these classes of metric spaces are covered by Rabinovich [2008]; see also Naor [2014, Section 7] for generalizations. Thus, the various known proofs which demonstrate that the available examples cannot be embedded into a low dimensional subspace of ℓ_1 argue that any such low-dimensional embedding must highly distort some distance, but this is not so for a typical distance. A negative answer to Question 27 would therefore require a substantially new type of construction which exhibits a much more "diffuse" intrinsic high-dimensionality despite it being a subset of ℓ_1. In the reverse direction, a positive answer to Question 27, beyond its intrinsic geometric/structural interest, could have algorithmic applications.

1.4.1 Lack of stability under projective tensor products. Prior to the recent work Naor, Pisier, and Schechtman [2018], it was unknown whether the property of admitting metric dimension reduction is preserved under projective tensor products.

Corollary 28. *There exist Banach spaces X, Y that admit metric dimension reduction yet $X \widehat{\otimes} Y$ does not.*

Since S_1 is isometric to $\ell_2 \widehat{\otimes} \ell_2$ and Naor, Pisier, and Schechtman [ibid.] establishes that S_1 fails to admit metric dimension reduction, together with the JL lemma this implies Corollary 28 (we can thus even have $X = Y$ and $\mathsf{k}_n^{\alpha}(n) \lesssim_{\alpha} \log n$ for all $\alpha > 1$).

Since we do not know whether S_1 admits average metric dimension reduction (the above comments pertaining to Question 27 are valid also for S_1), the analogue of Corollary 28 for average metric dimension reduction was previously unknown. Here we note the following statement, whose proof is a somewhat curious argument.

Theorem 29. *There exist Banach spaces X, Y that admit average metric dimension reduction yet $X \widehat{\otimes} Y$ does not. Moreover, for every $p \in (2, \infty)$ we can take here $X = \ell_p$.*

Proof. By Briët, Naor, and Regev [2012] (which relies on major input from the theory of locally decodable codes Efremenko [2009] and an important inequality of Pisier [1980]), the 3-fold product $\ell_3 \widehat{\otimes} \ell_3 \widehat{\otimes} \ell_3$ is universal. So, by the recent work Naor [2017] (Theorem 19), $\ell_3 \widehat{\otimes} \ell_3 \widehat{\otimes} \ell_3$ does not admit average metric dimension reduction. At the same time, by Theorem 25 we know that ℓ_3 admits average metric dimension reduction. So, if $\ell_3 \widehat{\otimes} \ell_3$ fails to admit average metric dimension reduction, then we can take $X = Y = \ell_3$ in Theorem 29. Otherwise, $\ell_3 \widehat{\otimes} \ell_3$ does admit average metric dimension reduction, in which case we can take $X = \ell_3$ and $Y = \ell_3 \widehat{\otimes} \ell_3$. Thus, in either of the above two cases, the first assertion of Theorem 29 holds true. The second assertion of Theorem 29 follows by repeating this argument using the fact Briët, Naor, and Regev [2012] that $\ell_p \widehat{\otimes} \ell_p \widehat{\otimes} \ell_q$ is universal if $2/p + 1/q \leqslant 1$, or equivalently $q \geqslant p/(p-2)$. If we choose, say, $q = \max\{2, p/(p-2)\}$, then by Theorem 25 we know that both ℓ_p and ℓ_q admit average metric dimension reduction, while $\ell_p \widehat{\otimes} \ell_p \widehat{\otimes} \ell_q$ does not. So, the second assertion of Theorem 29 holds for either $Y = \ell_p$ or $Y = \ell_p \widehat{\otimes} \ell_q$. $\square$

The proof of Theorem 29 establishes that at least one of the pairs $(X = \ell_3, Y = \ell_3)$ or $(X = \ell_3, Y = \ell_3 \widehat{\otimes} \ell_3)$ satisfies its conclusion, but it gives no indication which of these two options occurs. This naturally leads to

Question 30. Does $\ell_3 \widehat{\otimes} \ell_3$ admit average metric dimension reduction?

A positive answer to Question 30 would yield a new space that admits average metric dimension reduction. In order to claim that $\ell_3 \widehat{\otimes} \ell_3$ is indeed new in this context, one must show that it does not satisfy the assumption of Theorem 25. This is so because S_1 (hence also ℓ_1) is finitely representable in $\ell_3 \widehat{\otimes} \ell_3$; see e.g. Diestel, J. Fourie, and Swart [2003, page 61]. The fact that no Banach space in which ℓ_1 is finitely representably satisfies the assumption of Theorem 25 follows by combining Naor [2014, Lemma 1.12], Naor [ibid., Proposition 7.5], and Naor [ibid., Lemma 7.6]. This also shows that a positive answer to Question 30 would imply that any n-point subset of ℓ_1 (or S_1) embeds with $O(1)$ average distortion into some normed space (a subspace of $\ell_3 \widehat{\otimes} \ell_3$) of dimension $n^{o(1)}$, which is a statement in the spirit of Question 27. If the answer to Question 30 were negative, then $\ell_3 \widehat{\otimes} \ell_3$ would be the first example of a non-universal space which fails to admit average metric dimension reduction, because Pisier proved Pisier [1992a,b] that $\ell_3 \widehat{\otimes} \ell_3$ is not universal.

Another question that arises naturally from Theorem 29 is whether its conclusion holds true also for $p = 2$.

Question 31. Is there a Banach space Y that admits average metric dimension reduction yet $\ell_2 \widehat{\otimes} Y$ does not?

1.4.2 Wasserstein spaces. Let $(\mathfrak{M}, d_{\mathfrak{M}})$ be a metric space and $p \in [1, \infty)$. The Wasserstein space $\mathsf{P}_p(\mathfrak{M})$ is not a Banach space, but there is a natural version of the metric dimension reduction question in this context.

Question 32. Fix $\alpha > 1$, $n \in \mathbb{N}$ and $\mu_1, \ldots, \mu_n \in \mathsf{P}_p(\mathfrak{M})$. What is the asymptotic behavior of the smallest $k \in \mathbb{N}$ for which there is $\mathcal{S} \subseteq \mathfrak{M}$ with $|\mathcal{S}| \leqslant k$ such that $(\{\mu_1, \ldots, \mu_n\}, \mathsf{W}_p)$ embeds with distortion α into $\mathsf{P}_p(\mathcal{S})$?

Spaces of measures with the Wasserstein metric W_p are of major importance in pure and applied mathematics, as well as in computer science (mainly for $p = 1$, where they are used in graphics and vision, but also for other values of p; see e.g. the discussion in Andoni, Naor, and Neiman [2016]). However, their bi-Lipschitz structure is poorly understood, especially so in the above context of metric dimension reduction. If k were small in Question 32, then this would give a way to "compress" collections of measures using measures with small support while approximately preserving Wasserstein distances. In the context of, say, image retrieval (mainly $\mathfrak{M} = \{1, \ldots, n\}^2 \subseteq \mathbb{R}^2$ and $p = 1$), this could be viewed as obtaining representations of images using a small number of "pixels."

M. S. Charikar [2002] and Indyk and Thaper [2003] proved that if $\mathfrak{M}$ is a finite metric space, then $\mathsf{P}_1(\mathfrak{M})$ embeds into ℓ_1 with distortion $O(\log |\mathfrak{M}|)$. Hence, if the answer to Question 32 were $k = n^{o(1)}$ for some $\alpha = O(1)$, then it would follow that any n-point subset of $\mathsf{P}_1(\mathfrak{M})$ embeds into ℓ_1 with distortion $o(\log n)$, i.e., better distortion than the general bound that is provided by Bourgain's embedding theorem (actually the ℓ_1-variant of that theorem, which is also known to be sharp in general Linial, London, and Rabinovich [1995]). This shows that one cannot hope to answer Question 32 with $k = n^{o(1)}$ and $\alpha = O(1)$ without imposing geometric restrictions on the underlying metric space $\mathfrak{M}$, since if $(\mathfrak{M} = \{x_1, \ldots, x_n\}, d_{\mathfrak{M}})$ is a metric space for which $\mathsf{c}_1(\mathfrak{M}) \asymp \log n$, then we can take $\mu_1, \ldots, \mu_n$ to be the point masses $\delta_{x_1}, \ldots, \delta_{x_n}$, so that $(\{\mu_1, \ldots, \mu_n\}, \mathsf{W}_1)$ is isometric to $(\mathfrak{M}, d_{\mathfrak{M}})$. The pertinent issue is therefore to study Question 32 when the $\mathfrak{M}$ is "nice." For example, sufficiently good bounds here for $\mathfrak{M} = \mathbb{R}^2$ would be relevant to Question 7, but at this juncture such a potential approach to Question 7 is quite speculative.

The above "problematic" example relied inherently on the fact that the underlying metric space $\mathfrak{M}$ is itself far from being embeddable in ℓ_1, but the difficulty persists even when $\mathfrak{M} = \ell_1$. Indeed, we recalled in Question 7 that Bourgain proved Bourgain [1986] that $\mathsf{P}_1(\ell_1)$ is universal, and hence the spaces of either Theorem 17 or Theorem 19 embed into $\mathsf{P}_1(\ell_1)$ with $O(1)$ distortion. So, for arbitrarily large $n \in \mathbb{N}$ we can find probability measures $\mu_1, \ldots, \mu_n$ on ℓ_1 (actually on a sufficiently high dimensional Hamming cube $\{0,1\}^N$) such that $(\{\mu_1, \ldots, \mu_n\}, \mathsf{W}_1)$ does not admit a good embedding into any normed space of dimension $n^{o(1)}$. This rules out an answer of $k = n^{o(1)}$ to Question 32 (even for average distortion) for $(\mu_1, \ldots, \mu_n, \mathsf{W}_1)$, because in the setting of Question 32, while $\mathsf{P}_1(\mathcal{S})$ is not a normed space, it embeds isometrically into a normed space of dimension $|\mathcal{S}| - 1$ (the dual of the mean-zero Lipschitz functions on

$(\mathcal{S}, d_{\mathfrak{M}})$; see e.g. Naor and Schechtman [2007] and Villani [2009] for an explanation of this standard fact). In the case of average distortion, one could see this using a different approach of Khot and the author Khot and Naor [2006] which constructs a collection of $n = e^{O(d)}$ probability measures on the Hamming cube $\{0,1\}^d$ that satisfy the conclusion of Theorem 19, as explained in Naor [2017, Remark 5]. This shows that even though these n probability measures reside on a Hamming cube of dimension $O(\log n)$, one cannot realize their Wasserstein-1 geometry with $O(1)$ distortion (even on average) in any normed space of dimension $n^{o(1)}$, let alone in $\mathsf{P}_1(\mathcal{S})$ with $|\mathcal{S}| = n^{o(1)}$.

It is therefore natural to investigate Question 32 when $\mathfrak{M}$ is low-dimensional. When $p = 1$, this remains an (important) uncharted terrain. When $p > 1$ and $\mathfrak{M} = \mathbb{R}^3$, partial information on Question 32 follows from Andoni, Naor, and Neiman [2015]. To see this, focus for concreteness on the case $p = 2$. Fix $\alpha \geqslant 1$ and $n \in \mathbb{N}$. Suppose that $(\mathfrak{N}, d_{\mathfrak{N}})$ is an n-point metric space for which the conclusion of Theorem 17 holds true with $\omega(t) = \sqrt{t}$ and $\Omega(t) = 2\alpha\sqrt{t}$. By Andoni, Naor, and Neiman [ibid.], the metric space $(\mathfrak{N}, \sqrt{d_{\mathfrak{N}}})$ embeds with distortion 2 into $\mathsf{P}_2(\mathbb{R}^3)$, where $\mathbb{R}^3$ is equipped with the standard Euclidean metric. Hence, if the image under this embedding of $\mathfrak{N}$ in $\mathsf{P}_2(\mathbb{R}^3)$ embedded into some k-dimensional normed space with distortion α, then by Theorem 17 necessarily $k \geqslant n^{c/\alpha^2}$ for some universal constant c. This does not address Question 32 as stated, because to the best of our knowledge it is not known whether $\mathsf{P}_2(\mathcal{S})$ embeds with $O(1)$ distortion into some "low-dimensional" normed space for every "small" $\mathcal{S} \subseteq \mathbb{R}^3$ (the relation between "small" and "low-dimensional" remains to be studied). In the case of average distortion, repeat this argument with $\mathfrak{N}$ now being the metric space of Theorem 19. By Remark 48 below, if the image in $\mathsf{P}_2(\mathbb{R}^3)$ of $(\mathfrak{N}, \sqrt{d_{\mathfrak{N}}})$ embedded with average distortion α into some k-dimensional normed space, then necessarily $k \geqslant \exp(\frac{c}{\alpha}\sqrt{\log n})$.

2 Finite subsets of Hilbert space

The article Johnson and Lindenstrauss [1984] is devoted to proving a theorem on the extension of Lipschitz functions from finite subsets of metric spaces.[5] Over the ensuing decades, the classic Johnson and Lindenstrauss [ibid.] attained widespread prominence outside the rich literature on the *Lipschitz extension problem*, due to two components of Johnson and Lindenstrauss [ibid.] that had major conceptual significance and influence, but are technically simpler than the proof of its main theorem.

The first of these components is the JL lemma, which we already stated in the Introduction. Despite its wide acclaim and applicability, this result is commonly called a "lemma" rather than a "theorem" because within the context of Johnson and Lindenstrauss [ibid.] it was just that, i.e., a relatively simple step toward the proof of the main theorem of Johnson and Lindenstrauss [ibid.].

The second of these components is a section of Johnson and Lindenstrauss [ibid.] that is devoted to formulating open problems in the context of the Ribe program; we

[5] Stating this theorem here would be an unnecessary digression, but we highly recommend examining the accessible geometric result of Johnson and Lindenstrauss [1984]; see Naor and Rabani [2017] for a review of the current state of the art on Lipschitz extension from finite subsets.

already described a couple of the questions that were raised there, but it contains more questions that proved to be remarkably insightful and had major impact on subsequent research (see e.g. Ball [1992] and Naor, Peres, Schramm, and Sheffield [2006]). Despite its importance, the impact of Johnson and Lindenstrauss [1984] on the Ribe program will not be pursued further in the present article, but we will next proceed to study the JL lemma in detail (including some new observations).

Recalling Theorem 11, the JL lemma Johnson and Lindenstrauss [ibid.] asserts that for every integer $n \geqslant 2$ and (distortion/error tolerance) $\alpha \in (1,\infty)$, if $x_1,\ldots,x_n$ are distinct vectors in a Hilbert space $(H, \|\cdot\|_H)$, then there exists (a target dimension) $k \in \{1,\ldots,n\}$ and a new n-tuple of k-dimensional vectors $y_1,\ldots,y_n \in \mathbb{R}^k$ such that

$$k \lesssim_\alpha \log n, \tag{25}$$

and the assignment $x_i \mapsto y_i$, viewed as a mapping into ℓ_2^k, has distortion at most α, i.e.,

$$\forall\, i,j \in \{1,\ldots,n\}, \qquad \|x_i - x_j\|_H \leqslant \|y_i - y_j\|_{\ell_2^k} \leqslant \alpha \|x_i - x_j\|_H. \tag{26}$$

It is instructive to take note of the "compression" that this statement entails. By tracking the numerical value of the target dimension k that the proof in Section 2.1 below yields (see Remark 38), one concludes that given an arbitrary collection of, say, a billion vectors of length a billion (i.e., $1\,000\,000\,000$ elements of $\mathbb{R}^{1\,000\,000\,000}$), one can find a billion vectors of length 329 (i.e., elements of $\mathbb{R}^{329}$), all of whose pairwise distances are within a factor 2 of the corresponding pairwise distances among the initial configuration of billion-dimensional vectors. Furthermore, if one wishes to maintain the pairwise distances of those billion vectors within a somewhat larger constant factor, say, a factor of 10 or 450, then one could do so in dimension 37 or 9, respectively.

The logarithmic dependence on n in (25) is optimal, up to the value of the implicit (α-dependent) constant factor. This is so even when one considers the special case when $x_1,\ldots,x_n \in H$ are the vertices of the standard $(n-1)$-simplex, i.e., $\|x_i - x_j\|_H = 1$ for all distinct $i,j \in \{1,\ldots,n\}$, and even when one allows the Euclidean norm in (26) to be replaced by any norm $\|\cdot\| : \mathbb{R}^k \to [0,\infty)$, namely if instead of (26) we have $1 \leqslant \|y_i - y_j\| \leqslant \alpha$ for all distinct $i,j \in \{1,\ldots,n\}$. Indeed, denote the unit ball of $\|\cdot\|$ by $B = \{z \in \mathbb{R}^k : \|z\| \leqslant 1\}$ and let $\mathbf{vol}_k(\cdot)$ be the Lebesgue measure on $\mathbb{R}^k$. If $i,j \in \{1,\ldots,n\}$ are distinct, then by the triangle inequality the assumed lower bound $\|y_i - y_j\| \geqslant 1$ implies that the interiors of $y_i + \frac{1}{2}B$ and $y_j + \frac{1}{2}B$ are disjoint. Hence, if we denote $A = \bigcup_{i=1}^n (y_i + \frac{1}{2}B)$, then $\mathbf{vol}_k(A) = \sum_{i=1}^n \mathbf{vol}_k(y_i + \frac{1}{2}B) = \frac{n}{2^k}\mathbf{vol}_k(B)$. At the same time, for every $u,v \in A$ there are $i,j \in \{1,\ldots,n\}$ for which $u \in y_i + \frac{1}{2}B$ and $v \in y_j + \frac{1}{2}B$, so by another application of the triangle inequality we have $\|u-v\| \leqslant \|y_i - y_j\| + 1 \leqslant \alpha + 1$. This implies that $A - A \subseteq (\alpha+1)B$. Hence,

$$(\alpha+1)\sqrt[k]{\mathbf{vol}_k(B)} = \sqrt[k]{\mathbf{vol}_k((\alpha+1)B)} \geqslant \sqrt[k]{\mathbf{vol}_k(A-A)} \geqslant 2\sqrt[k]{\mathbf{vol}_k(A)} = \sqrt[k]{n\mathbf{vol}_k(B)},$$

where the penultimate step uses the Brunn–Minkowski inequality Schneider [2014]. This simplifies to give

$$k \geqslant \frac{\log n}{\log(\alpha+1)}. \tag{27}$$

By Braß [1999] and Dekster [2000], the vertices of $(n-1)$-simplex embed isometrically into any infinite dimensional Banach space, so we have thus justified the bound (16), and hence in particular the first lower bound on $\mathsf{k}_n^\alpha(\ell_2)$ in (10). As we already explained, the second lower bound (for the almost-isometric regime) on $\mathsf{k}_n^\alpha(\ell_2)$ in (10) is due to the very recent work Larsen and Nelson [2017]. The upper bound on $\mathsf{k}_n^\alpha(\ell_2)$ in (10), namely that in (25) we can take

$$k \lesssim \frac{\log n}{\log\left(1+(\alpha-1)^2\right)} \asymp \max\left\{\frac{\log n}{(\alpha-1)^2}, \frac{\log n}{\log \alpha}\right\}, \tag{28}$$

follows from the original proof of the JL lemma in Johnson and Lindenstrauss [1984]. A justification of (28) appears in Section 2.1 below.

Question 33 (dimension reduction for the vertices of the simplex). Fix $\delta \in (0, \frac{1}{2})$. What is the order of magnitude (up to universal constant factors) of the smallest $\mathfrak{S}(\delta) \in (0,\infty)$ such that for every $n \in \mathbb{N}$ there is $k \in \mathbb{N}$ with $k \leqslant \mathfrak{S}(\delta)\log n$ and $y_1,\ldots,y_n \in \mathbb{R}^k$ that satisfy $1 \leqslant \|y_i - y_j\|_2 \leqslant 1+\delta$ for all distinct $i,j \in \{1,\ldots,n\}$? By (28) we have $\mathfrak{S}(\delta) \lesssim 1/\delta^2$. The best-known lower bound here is $\mathfrak{S}(\delta) \gtrsim 1/(\delta^2 \log(1/\delta))$, due to Alon [2003].

Remark 34. The upper bound (28) treats the target dimension in the JL lemma for an *arbitrary* subset of a Hilbert space. The lower bound (27) was derived in the special case of the vertices of the regular simplex, but it is also more general as it is valid for embeddings of these vertices into an *arbitrary* k-dimensional norm. In this (both special, and more general) setting, the bound (27) is quite sharp for large α. Indeed, by Arias-de-Reyna, Ball, and Villa [1998] (see also M. Ostrovskii and Randrianantoanina [2016, Corollary 2.4]), for each $n \in \mathbb{N}$ and $\alpha > \sqrt{2}$, if we write $k = \lceil (\log(4n))/\log(\alpha^2/(2\sqrt{\alpha^2-1}))\rceil$, then for *every* norm $\|\cdot\|$ on $\mathbb{R}^k$ there exist $y_1,\ldots,y_n \in \mathbb{R}^k$ satisfying $1 \leqslant \|y_i - y_j\| \leqslant \alpha$ for distinct $i,j \in \{1,\ldots,n\}$. See Füredi and Loeb [1994, Theorem 4.3] for an earlier result in this direction. See also Arias-de-Reyna, Ball, and Villa [1998] and the references therein (as well as M. Ostrovskii and Randrianantoanina [2016, Problem 2.5]) for partial results towards understanding the analogous issue (which is a longstanding open question) in the small distortion regime $\alpha \in (1,\sqrt{2}]$.

2.1 Optimality of re-scaled random projections. To set the stage for the proof of the JL lemma, note that by translation-invariance we may assume without loss of generality that one of the vectors $\{x_i\}_{i=1}^n$ vanishes, and then by replacing the Hilbert space H with the span of $\{x_i\}_{i=1}^n$, we may further assume that $H = \mathbb{R}^{n-1}$.

Let $\mathsf{Proj}_{\mathbb{R}^k} \in \mathsf{M}_{k\times(n-1)}(\mathbb{R})$ be the k by $n-1$ matrix of the orthogonal projection from $\mathbb{R}^{n-1}$ onto $\mathbb{R}^k$, i.e., $\mathsf{Proj}_{\mathbb{R}^k} z = (z_1,\ldots,z_k) \in \mathbb{R}^k$ is the first k coordinates of $z = (z_1,\ldots,z_{n-1}) \in \mathbb{R}^{n-1}$. One could attempt to simply truncate the vectors vectors $x_1,\ldots,x_n$ so as to obtain k-dimensional vectors, namely to consider the vectors $\{y_i = \mathsf{Proj}_{\mathbb{R}^k} x_i\}_{i=1}^n$ in (26). This naïve (and heavy-handed) way of forcing low-dimensionality can obviously fail miserably, e.g. we could have $\mathsf{Proj}_k x_i = 0$ for all $i \in \{1,\ldots,n\}$. Such a simplistic idea performs poorly because it makes two arbitrary and unnatural choices, namely it does not take advantage of rotation-invariance and

scale-invariance. To remedy this, let $\mathsf{O}_{n-1} \subseteq \mathsf{M}_{n-1}(\mathbb{R})$ denote the group of $n-1$ by $n-1$ orthogonal matrices, and fix (a scaling factor) $\sigma \in (0,\infty)$. Let $\mathsf{O} \in \mathsf{O}_{n-1}$ be a random orthogonal matrix distributed according to the Haar probability measure on O_{n-1}. In Johnson and Lindenstrauss [1984] it was shown that if k is sufficiently large (yet satisfying (25)), then for an appropriate $\sigma > 0$ with positive probability (26) holds for the following random vectors.

$$\{y_i = \sigma \mathsf{Proj}_{\mathbb{R}^k} \mathsf{O} x_i\}_{i=1}^n \subseteq \mathbb{R}^k. \tag{29}$$

We will do more than merely explain why the randomly projected vectors in (29) satisfy the desired conclusion (26) of the JL lemma with positive probability. We shall next demonstrate that such a procedure is the *best possible* (in a certain sense that will be made precise) among all the possible choices of random assignments of $x_1,\ldots,x_n$ to $y_1,\ldots,y_n$ via multiplication by a random matrix in $M_{k\times(n-1)}(\mathbb{R})$, provided that we optimize so as to use the best scaling factor $\sigma \in (0,\infty)$ in (29).

Let μ be any Borel probability measure on $\mathsf{M}_{k\times(n-1)}(\mathbb{R})$, i.e., μ represents an arbitrary (reasonably measurable) distribution over $k \times (n-1)$ random matrices $\mathsf{A} \in \mathsf{M}_{k\times(n-1)}(\mathbb{R})$. For $\alpha \in (1,\infty)$ define

$$\mathfrak{p}_\mu^\alpha \stackrel{\text{def}}{=} \inf_{z\in \mathbf{S}^{n-2}} \mu\Big[\big\{\mathsf{A} \in \mathsf{M}_{k\times(n-1)}(\mathbb{R}) :\ 1 \leqslant \|\mathsf{A}z\|_{\ell_2^k} \leqslant \alpha\big\}\Big], \tag{30}$$

where $\mathbf{S}^{n-2} = \{z \in \mathbb{R}^{n-1} :\ \|z\|_{\ell_2^{n-1}} = 1\}$ denotes the unit Euclidean sphere in $\mathbb{R}^{n-1}$. Then

$$\begin{aligned}
&\mu\Big[\bigcap_{i,j\in\{1,\ldots,n\}} \big\{\mathsf{A} \in \mathsf{M}_{k\times(n-1)}(\mathbb{R}) : \|x_i - x_j\|_{\ell_2^{n-1}} \leqslant \|\mathsf{A}x_i - \mathsf{A}x_j\|_{\ell_2^k} \leqslant \alpha\|x_i - x_j\|_{\ell_2^{n-1}}\big\}\Big] \\
&\quad = 1 - \mu\Big[\bigcup_{i=1}^n \bigcup_{j=i+1}^n \Big(\mathsf{M}_{k\times(n-1)}(\mathbb{R}) \smallsetminus \Big\{\mathsf{A} \in \mathsf{M}_{k\times(n-1)}(\mathbb{R}) : 1 \leqslant \Big\|\mathsf{A}\frac{x_i - x_j}{\|x_i - x_j\|_{\ell_2^{n-1}}}\Big\|_{\ell_2^k} \leqslant \alpha\Big\}\Big)\Big] \\
&\quad \geqslant 1 - \sum_{i=1}^n \sum_{j=i+1}^n \Big(1 - \mu\Big[\Big\{\mathsf{A} \in \mathsf{M}_{k\times(n-1)}(\mathbb{R}) : 1 \leqslant \Big\|\mathsf{A}\frac{x_i - x_j}{\|x_i - x_j\|_{\ell_2^{n-1}}}\Big\|_{\ell_2^k} \leqslant \alpha\Big\}\Big]\Big) \\
&\quad \geqslant 1 - \binom{n}{2}(1 - \mathfrak{p}_\mu^\alpha).
\end{aligned} \tag{31}$$

Hence, the random vectors $\{y_i = \mathsf{A}x_i\}_{i=1}^n$ will satisfy (26) with positive probability if $\mathfrak{p}_\mu^\alpha > 1 - \frac{2}{n(n-1)}$.

In order to succeed to embed the largest possible number of vectors into $\mathbb{R}^k$ via the above randomized procedure while using the estimate (31), it is in our best interest to work with a probability measure μ on $\mathsf{M}_{k\times(n-1)}(\mathbb{R})$ for which $\mathfrak{p}_\mu^\alpha$ is as large as possible. To this end, define

$$\mathfrak{p}_{n,k}^\alpha \stackrel{\text{def}}{=} \sup\Big\{\mathfrak{p}_\mu^\alpha :\ \mu \text{ is a Borel probability measure on } \mathsf{M}_{k\times(n-1)}(\mathbb{R})\Big\}. \tag{32}$$

Then, the conclusion (26) of the JL lemma will be valid provided $k \in \{1,\ldots,n\}$ satisfies

$$\mathfrak{p}_{n,k}^\alpha > 1 - \frac{2}{n(n-1)}. \tag{33}$$

The following proposition asserts that the supremum in the definition (32) of $\mathfrak{p}^{\alpha}_{n,k}$ is attained at a distribution over random matrices that has the aforementioned structure (29).

Proposition 35 (multiples of random orthogonal projections are JL-optimal)**.** *Fix $\alpha \in (1,\infty)$, an integer $n \geqslant 4$ and $k \in \{1,\ldots,n-3\}$. Let $\mu = \mu^{\alpha}_{n,k}$ be the probability distribution on $\mathsf{M}_{k\times(n-1)}(\mathbb{R})$ of the random matrix*

$$\sqrt{\frac{\alpha^{\frac{2n-6}{n-k-3}}-1}{\alpha^{\frac{2k}{n-k-3}}-1}}\cdot \mathsf{Proj}_{\mathbb{R}^k}\mathsf{O}, \tag{34}$$

that is obtained by choosing $\mathsf{O} \in \mathsf{O}_{n-1}$ according to the normalized Haar measure on O_{n-1}. Then $\mathfrak{p}^{\alpha}_{\mu} = \mathfrak{p}^{\alpha}_{n,k}$.

Obviously (34) is *not* a multiple of a uniformly random rank k orthogonal projection $\mathsf{Proj} : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ (chosen according to the normalized Haar measure on the appropriate Grassmannian). To obtain such a distribution, one should multiply the matrix in (34) on the left by O^*. That additional rotation does not influence the Euclidean length of the image, and hence it does not affect the quantity (30). For this reason and for simplicity of notation, we prefer to work with (34) rather than random projections as was done in Johnson and Lindenstrauss [ibid.].

Proof of Proposition 35. Given $\mathsf{A} \in \mathsf{M}_{k\times(n-1)}(\mathbb{R})$, denote its singular values by $\mathsf{s}_1(\mathsf{A}) \geqslant \ldots \geqslant \mathsf{s}_k(\mathsf{A})$, i.e., they are the eigenvalues (with multiplicity) of the symmetric matrix $\sqrt{\mathsf{A}\mathsf{A}^*} \in \mathsf{M}_k(\mathbb{R})$. Then,

$$\mathfrak{S}^{\mathsf{O}_{n-1}}\Big[\big\{\mathsf{O} \in \mathsf{O}_{n-1} : 1 \leqslant \|\mathsf{A}\mathsf{O}z\|_{\ell_2^k} \leqslant \alpha\big\}\Big] = \int_{\mathbf{S}^{k-1}} \psi^{\alpha}_{n,k}\Big(\Big(\sum_{i=1}^{k}\mathsf{s}_i(\mathsf{A})^2\omega_i^2\Big)^{\frac{1}{2}}\Big)\,\mathrm{d}\mathfrak{S}^{\mathbf{S}^{k-1}}(\omega) \tag{35}$$

where $\mathfrak{S}^{\mathsf{O}_{n-1}}$ and $\mathfrak{S}^{\mathbf{S}^{k-1}}$ are the Haar probability measures on the orthogonal group O_{n-1} and the unit Euclidean sphere $\mathbf{S}^{k-1}$, respectively, and the function $\psi^{\alpha}_{n,k} : [0,\infty) \to \mathbb{R}$ is defined by

$$\forall\, \sigma \in [0,\infty), \qquad \psi^{\alpha}_{n,k}(\sigma) \stackrel{\text{def}}{=} \frac{2\pi^{\frac{k}{2}}}{\Gamma\big(\frac{k}{2}\big)}\int_{\max\{1,\frac{\sigma}{\alpha}\}}^{\max\{1,\sigma\}} \frac{(s^2-1)^{\frac{n-k-3}{2}}}{s^{n-2}}\,\mathrm{d}s. \tag{36}$$

To verify the identity (35), consider the singular value decomposition

$$\mathsf{A} = \mathsf{U}\begin{pmatrix} \mathsf{s}_1(\mathsf{A}) & 0 & \ldots & \ldots & 0 \\ 0 & \mathsf{s}_2(\mathsf{A}) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \ldots & \ldots & 0 & \mathsf{s}_k(\mathsf{A}) \end{pmatrix}\mathsf{Proj}_{\mathbb{R}^k}\mathsf{V}, \tag{37}$$

where $\mathsf{U} \in \mathsf{O}_k$ and $\mathsf{V} \in \mathsf{O}_{n-1}$. If $\mathsf{O} \in \mathsf{O}_{n-1}$ is distributed according to $\mathfrak{S}^{\mathsf{O}_{n-1}}$, then by the left-invariance of $\mathfrak{S}^{\mathsf{O}_{n-1}}$ we know that $\mathsf{V}\mathsf{O}$ is distributed according to $\mathfrak{S}^{\mathsf{O}_{n-1}}$.

By rotation-invariance and uniqueness of Haar measure on $\mathbf{S}^{n-2}$ (e.g. V. D. Milman and Schechtman [1986]), it follows that for every $z \in \mathbf{S}^{n-1}$ the random vector $\mathsf{V}\mathsf{O}z$ is distributed according to the normalized Haar measure on $\mathbf{S}^{n-2}$. So, $\mathsf{Proj}_{\mathbb{R}^k}\mathsf{V}\mathsf{O}z$ is distributed on the Euclidean unit ball of $\mathbb{R}^k$, with density

$$\big(u \in \mathbb{R}^k\big) \mapsto \frac{\Gamma\big(\frac{n-1}{2}\big)}{\pi^{\frac{k}{2}}\Gamma\big(\frac{n-1-k}{2}\big)}\Big(1-\|u\|_{\ell_2^k}^2\Big)^{\frac{n-k-3}{2}}\mathbf{1}_{\{\|u\|_{\ell_2^k}\leqslant 1\}}. \tag{38}$$

See Stam [1982] for a proof of this distributional identity (or Barthe, Guédon, Mendelson, and Naor [2005, Corollary 4] for a more general derivation); in codimension 2, namely $k = n-3$, this is a higher-dimensional analogue of Archimedes' theorem that the projection to $\mathbb{R}$ of the uniform surface area measure on the unit Euclidean sphere in $\mathbb{R}^3$ is the Lebesgue measure on $[-1,1]$. Recalling (37), it follows from this discussion that the Euclidean norm of $\mathsf{A}\mathsf{O}z$ has the same distribution as $(\sum_{i=1}^k \mathsf{s}_i(\mathsf{A})^2 u_i^2)^{1/2}$, where $u = (u_1,\ldots,u_k) \in \mathbb{R}^k$ is distributed according to the density (38). The identity (35) now follows by integration in polar coordinates $(\omega, r) \in \mathbf{S}^{k-1}\times[0,\infty)$, followed by the change of variable $s = 1/r$.

Next, $\psi_{n,k}^{\alpha}$ vanishes on $[0,1]$, increases on $[1,\alpha]$, and is smooth on $[\alpha,\infty)$. The integrand in (36) is at most s^{-k-1}, so $\lim_{\sigma\to\infty}\psi_{n,k}^{\alpha}(\sigma) = 0$. By directly differentiating (36) and simplifying the resulting expression, one sees that if $\sigma \in [\alpha,\infty)$, then $(\psi_{n,k}^{\alpha})'(\sigma) = 0$ if and only if $\sigma = \sigma_{\max}(n,k,\alpha)$, where

$$\sigma_{\max}(n,k,\alpha) \stackrel{\text{def}}{=} \sqrt{\frac{\alpha^{\frac{2n-6}{n-k-3}}-1}{\alpha^{\frac{2k}{n-k-3}}-1}}. \tag{39}$$

Therefore, the global maximum of $\psi_{n,k}^{\alpha}$ is attained at $\sigma_{\max}(n,k,\alpha)$, and by (35) we have $\forall\, \mathsf{A} \in \mathsf{M}_{k\times(n-1)}(\mathbb{R})$

$$\mathfrak{H}^{\mathsf{O}_{n-1}}\Big[\big\{\mathsf{O}\in\mathsf{O}_{n-1}:\ 1\leqslant\|\mathsf{A}\mathsf{O}z\|_{\ell_2^k}\leqslant\alpha\big\}\Big] \leqslant \psi_{n,k}^{\alpha}\big(\sigma_{\max}(n,k,\alpha)\big) = \mathfrak{p}_{\mu}^{\alpha}. \tag{40}$$

The final step of (40) is another application (35), this time in the special case $\mathsf{A} = \sigma_{\max}(n,k,\alpha)\mathsf{Proj}_{\mathbb{R}^k}$, while recalling (30) and (39), and that μ is the distribution of the random matrix appearing in (34).

To conclude the proof of Proposition (35), take any Borel probability measure ν on $\mathsf{M}_{k\times(n-1)}(\mathbb{R})$ and integrate (40) while using Fubini's theorem to obtain the estimate

$$\begin{aligned}\mathfrak{p}_{\mu}^{\alpha} &\geqslant \int_{\mathsf{M}_{k\times(n-1)}(\mathbb{R})}\mathfrak{H}^{\mathsf{O}_{n-1}}\Big[\big\{\mathsf{O}\in\mathsf{O}_{n-1}:\ 1\leqslant\|\mathsf{A}\mathsf{O}z_0\|_{\ell_2^k}\leqslant\alpha\big\}\Big]\,\mathrm{d}\nu(\mathsf{A})\\ &= \int_{\mathsf{O}_{n-1}}\nu\Big[\big\{\mathsf{A}\in\mathsf{M}_{k\times(n-1)}(\mathbb{R}):\ 1\leqslant\|\mathsf{A}\mathsf{O}z_0\|_{\ell_2^k}\leqslant\alpha\big\}\Big]\,\mathrm{d}\mathfrak{H}^{\mathsf{O}_{n-1}}(\mathsf{O})\\ &\stackrel{(30)}{\geqslant} \int_{\mathsf{O}_{n-1}}\mathfrak{p}_{\nu}^{\alpha}\,\mathrm{d}\mathfrak{H}^{\mathsf{O}_{n-1}}(\mathsf{O}) = \mathfrak{p}_{\nu}^{\alpha}.\end{aligned}$$

So, the maximum of $\mathfrak{p}_{\nu}^{\alpha}$ over the Borel probability measures ν on $\mathsf{M}_{k\times(n-1)}(\mathbb{R})$ is attained at μ. $\square$

Remark 36. Recalling (33), due to (40) the conclusion (26) of the JL lemma holds if k satisfies

$$\frac{2\pi^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}\right)} \int_{\frac{1}{\alpha}\sigma_{\max}(n,k,\alpha)}^{\sigma_{\max}(n,k,\alpha)} \frac{(s^2-1)^{\frac{n-k-3}{2}}}{s^{n-2}}\, \mathrm{d}s > 1 - \frac{2}{n(n-1)}, \tag{41}$$

where $\sigma_{\max}(n,k,\alpha)$ is given in (39). This is the best-known bound on k in the JL lemma, which, due to Proposition 35, is the best-possible bound that is obtainable through the reasoning (31). In particular, the asymptotic estimate (28) follows from (41) via straightforward elementary calculus.

Remark 37. The JL lemma was reproved many times; see Frankl and Maehara [1988], Gordon [1988], Indyk and Motwani [1999], Arriaga and Vempala [1999], Dasgupta and Gupta [2003], Achlioptas [2003], Klartag and Mendelson [2005], Indyk and Naor [2007], Matoušek [2008], Ailon and Chazelle [2009], Dmitriyuk and Gordon [2009], Krahmer and Ward [2011], Ailon and Liberty [2013], Dmitriyuk and Gordon [2014], Kane and Nelson [2014], Bourgain, Dirksen, and Nelson [2015], and Dirksen [2016], though we make no claim that this is a comprehensive list of references. There were several motivations for these further investigations, ranging from the desire to obtain an overall better understanding of the JL phenomenon, to obtain better bounds, and to obtain distributions on random matrices A as in (31) with certain additional properties that are favorable from the computational perspective, such as ease of simulation, use of fewer random bits, sparsity, and the ability to evaluate the mapping $(z \in \mathbb{R}^{n-1}) \mapsto \mathsf{A}z$ quickly (akin to the fast Fourier transform). This body of work represents ongoing efforts by computer scientists and applied mathematicians to further develop improved "JL transforms," driven by their usefulness as a tool for data-compression. We will not survey these ideas here, partially because we established that using random projections yields the best-possible bound on the target dimension k (moreover, this procedure is natural and simple). We speculate that working with the Haar measure on the orthogonal group O_{n-1} as in (29) could have benefits that address the above computational issues, but leave this as an interesting open-ended direction for further research. A specific conjecture towards this goal appears in Ailon and Chazelle [2009, page 320], and we suspect that the more recent work Bourgain and Gamburd [2012] on the spectral gap of Hecke operators of orthogonal Cayley graphs should be relevant in this context as well (e.g. for derandomization and fast implementation of (29); see Brandão, Harrow, and Horodecki [2016] and Kothari and Meka [2015] for steps in this direction).

Remark 38. In the literature there is often a preference to use random matrices with independent entries in the context of the JL lemma, partially because they are simple to generate, though see the works Stewart [1980], Genz [2000], and Mezzadri [2007] on generating elements of the orthogonal group O_{n-1} that are distributed according to its Haar measure. In particular, the best bound on k in (25) that was previously available in the literature Dasgupta and Gupta [2003] arose from applying (31) when A is replaced by the random matrix $\sigma\mathsf{G}$, where $\sigma = 1/\sqrt{k}$ and the entries of $\mathsf{G} = (\mathsf{g}_{ij}) \in M_{k\times(n-1)}(\mathbb{R})$ are independent standard Gaussian random variables. We can, however, optimize over the scaling factor σ in this setting as well, in analogy to the above optimization over the scaling factor in (29), despite the fact that we know that working with the Gaussian

matrix G is inferior to using a random rotation. A short calculation reveals that the optimal scaling factor is now $\sqrt{(\alpha^2-1)/(2k\log\alpha)}$, i.e., the best possible re-scaled Gaussian matrix for the purpose of reasoning as in (31) is not $\frac{1}{\sqrt{k}}\mathsf{G}$ but rather the random matrix

$$\mathsf{G}_k^\alpha \stackrel{\mathrm{def}}{=} \sqrt{\frac{\alpha^2-1}{2k\log\alpha}}\cdot\mathsf{G}. \tag{42}$$

For this optimal multiple of a Gaussian matrix, one computes that for every $z\in\mathbf{S}^{n-2}$ we have

$$\begin{aligned}1-\mathbb{P}\Big[1\leqslant\|\mathsf{G}_k^\alpha z\|_{\ell_2^k}\leqslant\alpha\Big] &= \frac{2k^{\frac{k}{2}}}{\Gamma\big(\frac{k}{2}\big)}\int_{\log\alpha}^{\infty}\left(\frac{\beta}{e^{2\beta}-1}\right)^{\frac{k}{2}}\exp\left(-\frac{k\beta}{e^{2\beta}-1}\right)\mathrm{d}\beta \\ &< \frac{4k^{\frac{k}{2}-1}}{\Gamma\big(\frac{k}{2}\big)}\left(\frac{\alpha^2-1}{\log\alpha}\alpha^{\frac{2}{\alpha^2-1}}\right)^{-\frac{k}{2}}\frac{(\alpha^2-1)^2\log\alpha}{2\alpha^4\log\alpha+2\alpha^2-\alpha^4-4\alpha^2(\log\alpha)^2-2\log\alpha-1}.\end{aligned} \tag{43}$$

The first step in (43) follows from a straightforward computation using the fact that the squared Euclidean length of $\mathsf{G}_k^\alpha z$ is distributed according to a multiple of the χ^2 distribution with k degrees of freedom (see e.g. Durrett [2010]), i.e., one can write the leftmost term of (43) explicitly as a definite integral, and then check that it indeed equals the middle term of (43), e.g., by verifying the the derivatives with respect to α of both expressions coincide. The final estimate in (43) can be justified via a modicum of straightforward calculus. We deduce from this that the conclusion (26) of the JL lemma is holds with positive probability if for each $i\in\{1,\ldots,n\}$ we take y_i to be the image of x_i under the re-scaled Gaussian matrix in (42), provided that k is sufficiently large so as to ensure that

$$\frac{\Gamma\big(\frac{k}{2}\big)}{k^{\frac{k}{2}-1}}\left(\frac{\alpha^2-1}{\log\alpha}\alpha^{\frac{2}{\alpha^2-1}}\right)^{\frac{k}{2}} \geqslant \frac{2n^2(\alpha^2-1)^2\log\alpha}{2\alpha^4\log\alpha+2\alpha^2-\alpha^4-4\alpha^2(\log\alpha)^2-2\log\alpha-1}. \tag{44}$$

The values that we stated for the target dimension k in the JL lemma with a billion vectors were obtained by using (44), though even better bounds arise from an evaluation of the integral in (43) numerically, which is what we recommend to do for particular settings of the parameters. As $\alpha\to1$, the above bounds improve over those of Dasgupta and Gupta [2003] only in the second-order terms. For larger α these bounds yield substantial improvements that might matter in practice, e.g. for embedding a billion vectors with distortion 2, the target dimension that is required using the best-available estimate in the literature Dasgupta and Gupta [ibid.] is $k=768$, while (44) shows that $k=329$ suffices.

3 Infinite subsets of Hilbert space

The JL lemma provides a quite complete understanding of the metric dimension reduction problem for finite subsets of Hilbert space. For infinite subsets of Hilbert space,

the research splits into two strands. The first is to understand those subsets $\mathcal{C} \subseteq \mathbb{R}^n$ for which certain random matrices in $\mathsf{M}_{k\times n}(\mathbb{R})$ (e.g. random projections, or matrices whose entries are i.i.d. independent sub-Gaussian random variables) yield with positive probability an embedding of $\mathcal{C}$ into $\mathbb{R}^k$ of a certain pre-specified distortion; this was pursued in Gordon [1988], Klartag and Mendelson [2005], Indyk and Naor [2007], Mendelson, Pajor, and Tomczak-Jaegermann [2007], Mendelson and Tomczak-Jaegermann [2008], Bourgain, Dirksen, and Nelson [2015], Dirksen [2016], and Puy, Davies, and Gribonval [2017], yielding a satisfactory answer which relies on multi-scale chaining criteria Talagrand [2014] and Nelson [2016] .

The second (and older) research strand focuses on the mere *existence* of a low-dimensional embedding rather than on the success of the specific embedding approach of (all the known proofs of) the JL lemma. Specifically, given a subset $\mathcal{C}$ of a Hilbert space and $\alpha \in [1,\infty)$, could one understand when does $\mathcal{C}$ admit an embedding with distortion α into ℓ_2^k for some $k \in \mathbb{N}$? If one ignores the dependence on the distortion α, then this qualitative question coincides with Problem 2 (the bi-Lipschitz embedding problem into $\mathbb{R}^k$), since if a metric space $(\mathfrak{M}, d_{\mathfrak{M}})$ satisfies $\inf_{k\in\mathbb{N}} \mathsf{c}_{\mathbb{R}^k}(\mathfrak{M}) < \infty$, then in particular it admits a bi-Lipschitz embedding into a Hilbert space.

We shall next describe an obvious necessary condition for bi-Lipschitz embeddability into $\mathbb{R}^k$ for some $k \in \mathbb{N}$. In what follows, all balls in a metric space $(\mathfrak{M}, d_{\mathfrak{M}})$ will be closed balls, i.e., for $x \in \mathfrak{M}$ and $r \in [0,\infty)$ we write $B_{\mathfrak{M}}(x,r) = \{y \in \mathfrak{M} : d_{\mathfrak{M}}(x,y) \leqslant r\}$. Given $K \in [2,\infty)$, a metric space $(\mathfrak{M}, d_{\mathfrak{M}})$ is said to be *K-doubling* (e.g. Bouligand [1928] and Coifman and Weiss [1971]) if every ball in $\mathfrak{M}$ (centered anywhere in $\mathfrak{M}$ and of any radius) can be covered by at most K balls of half its radius, i.e., for every $x \in \mathfrak{M}$ and $r \in [0,\infty)$ there is $m \in \mathbb{N}$ with $m \leqslant K$ and $y_1,\ldots,y_m \in \mathfrak{M}$ such that $B_{\mathfrak{M}}(x,r) \subseteq B_{\mathfrak{M}}(y_1,\frac{1}{2}r) \cup \ldots \cup B_{\mathfrak{M}}(y_m,\frac{1}{2}r)$. A metric space is doubling if it is K-doubling for some $K \in [2,\infty)$.

Fix $k \in \mathbb{N}$ and $\alpha \geqslant 1$. If a metric space $(\mathfrak{M}, d_{\mathfrak{M}})$ embeds with distortion α into a normed space $(\mathbb{R}^k, \|\cdot\|)$, then $\mathfrak{M}$ is $(4\alpha+1)^k$-doubling. Indeed, fix $x \in \mathfrak{M}$ and $r > 0$. Let $\{z_1,\ldots,z_n\} \subseteq B_{\mathfrak{M}}(x,r)$ be a maximal subset (with respect to inclusion) of $B_{\mathfrak{M}}(x,r)$ satisfying $d_{\mathfrak{M}}(z_i,z_j) > \frac{1}{2}r$ for distinct $i,j \in \{1,\ldots,n\}$. The maximality of $\{z_1,\ldots,z_n\}$ ensures that for any $w \in B_{\mathfrak{M}}(x,r) \smallsetminus \{z_1,\ldots,z_n\}$ we have $\min_{i\in\{1,\ldots,n\}} d_{\mathfrak{M}}(w,z_i) \leqslant \frac{1}{2}r$, i.e., $B_{\mathfrak{M}}(x,r) \subseteq B_{\mathfrak{M}}(z_1,\frac{1}{2}r) \cup \ldots \cup B_{\mathfrak{M}}(z_n,\frac{1}{2}r)$. We are assuming that there is an embedding $f : \mathfrak{M} \to \mathbb{R}^k$ that satisfies $d_{\mathfrak{M}}(u,v) \leqslant \|f(u)-f(v)\| \leqslant \alpha d_{\mathfrak{M}}(u,v)$ for all $u,v \in \mathfrak{M}$. So, for distinct $i,j \in \{1,\ldots,n\}$ we have $\frac{r}{2} < d_{\mathfrak{M}}(z_i,z_j) \leqslant \|f(z_i)-f(z_j)\| \leqslant \alpha d_{\mathfrak{M}}(z_i,z_j) \leqslant \alpha(d_{\mathfrak{M}}(z_i,x) + d_{\mathfrak{M}}(x,z_j)) \leqslant 2\alpha r$. The reasoning that led to (27) with $y_1 = \frac{2}{r}f(z_1),\ldots,y_n = \frac{2}{r}f(z_n)$ and α replaced by 4α gives $k \geqslant (\log n)/\log(4\alpha+1)$, i.e., $n \leqslant (4\alpha+1)^k$.

Remark 39. In Section 1.3 we recalled that in the context of the Ribe program $\log|\mathfrak{M}|$ was the initial (in hindsight somewhat naïve, though still very useful) replacement for the "dimension" of a finite metric space $\mathfrak{M}$. This arises naturally also from the above discussion. Indeed, $\mathfrak{M}$ is trivially $|\mathfrak{M}|$-doubling (simply cover each ball in $\mathfrak{M}$ by singletons), and this is the best bound that one could give on the doubling constant of $\mathfrak{M}$ in terms of $|\mathfrak{M}|$. So, from the perspective of the doubling property, the natural restriction

on $k \in \mathbb{N}$ for which there exists an embedding of $\mathfrak{M}$ into some k-dimensional normed space with $O(1)$ distortion is that $k \gtrsim \log|\mathfrak{M}|$.

Using terminology that was recalled in Remark 18, the definition of the doubling property directly implies that for every $\theta \in (0,1)$ a metric space $\mathfrak{M}$ is doubling if and only if its θ-snowflake $\mathfrak{M}^\theta$ is doubling. With this in mind, Theorem 40 below is a very important classical achievement of Assouad [1983].

Theorem 40. *The following assertions are equivalent for every metric space* $(\mathfrak{M}, d_{\mathfrak{M}})$.

- $\mathfrak{M}$ *is doubling.*
- *For every* $\theta \in (0,1)$ *there is* $k \in \mathbb{N}$ *such that* $\mathfrak{M}^\theta$ *admits a bi-Lipschitz embedding into* $\mathbb{R}^k$.
- *Some snowflake of* $\mathfrak{M}$ *admits a bi-Lipschitz embedding into* $\mathbb{R}^k$ *for some* $k \in \mathbb{N}$.

Theorem 40 is a qualitative statement, but its proof in Assouad [ibid.] shows that for every $K \in [2,\infty)$ and $\theta \in (0,1)$, there are $\alpha(K,\theta) \in [1,\infty)$ and $k(K,\theta) \in \mathbb{N}$ such that if $\mathfrak{M}$ is K-doubling, then $\mathfrak{M}^\theta$ embeds into $\mathbb{R}^{k(K,\theta)}$ with distortion $\alpha(K,\theta)$; the argument of Assouad [ibid.] inherently gives that as $\theta \to 1$, i.e., as the θ-snowflake $\mathfrak{M}^\theta$ approaches the initial metric space $\mathfrak{M}$, we have $\alpha(K,\theta) \to \infty$ and $k(K,\theta) \to \infty$. A meaningful study of the best-possible asymptotic behavior of the distortion $\alpha(K,\theta)$ here would require specifying which norm on $\mathbb{R}^k$ is being considered. Characterizing the quantitative dependence in terms of geometric properties of the target norm on $\mathbb{R}^k$ has not been carried out yet (it isn't even clear what should the pertinent geometric properties be), though see Har-Peled and Mendel [2006] for an almost isometric version when one considers the ℓ_∞ norm on $\mathbb{R}^k$ (with the dimension k tending to ∞ as the distortion approaches 1); see also Gottlieb and Krauthgamer [2015] for a further partial step in this direction. In Naor and Neiman [2012] it was shown that for $\theta \in [\frac{1}{2},1)$ one could take $k(K,\theta) \lesssim k(K)$ to be bounded by a constant that depends only on K; the proof of this fact in Naor and Neiman [ibid.] relies on a probabilistic construction, but in David and Snipes [2013] a clever and instructive deterministic proof of this phenomenon was found (though, yielding asymptotically worse estimates on $\alpha(K,\theta), k(K)$ than those of Naor and Neiman [2012]).

Assouad's theorem is a satisfactory characterization of the doubling property in terms of embeddability into finite-dimensional Euclidean space. However, it is a "near miss" as an answer to Problem 2: the same statement with $\theta = 1$ would have been a wonderful resolution of the bi-Lipschitz embedding problem into $\mathbb{R}^k$, showing that a simple intrinsic ball covering property is equivalent to bi-Lipschitz embeddability into some $\mathbb{R}^k$. It is important to note that while the snowflaking procedure does in some sense "tend to" the initial metric space as $\theta \to 1$, for $\theta < 1$ it deforms the initial metric space substantially (e.g. such a θ-snowflake does not contain any non-constant rectifiable curve). So, while Assouad's theorem with the stated snowflaking is useful (examples of nice applications appear in Bonk and Schramm [2000] and Har-Peled and Mendel [2006]), its failure to address the bi-Lipschitz category is a major drawback.

Alas, more than a decade after the publication of Assouad's theorem, it was shown in Semmes [1996] (relying a on a rigidity theorem of Pansu [1989]) that Assouad's theorem does not hold with $\theta = 1$, namely there exists a doubling metric space that does not admit a bi-Lipschitz embedding into $\mathbb{R}^k$ for any $k \in \mathbb{N}$. From the qualitative perspective, we now know that the case $\theta = 1$ of Assouad's theorem fails badly in the sense that there exists a doubling metric space (the continuous 3-dimensional Heisenberg group, equipped with the Carnot–Carathéodory metric) that does not admit a bi-Lipschitz embedding into any Banach space with the Radon–Nikodým property Lee and Naor [2006] and Cheeger and Kleiner [2006] (in particular, it does not admit a bi-Lipschitz embedding into any reflexive or separable dual Banach space, let alone a finite dimensional Banach space), into any $L_1(\mu)$ space Cheeger and Kleiner [2010], or into any Alexandrov space of curvature bounded above or below Pauls [2001] (a further strengthening appears in the forthcoming work Austin and Naor [2017]). From the quantitative perspective, by now we know that balls in the discrete 5-dimensional Heisenberg group equipped with the word metric (which is doubling) have the asymptotically worst-possible bi-Lipschitz distortion (as a function of their cardinality) in uniformly convex Banach spaces Lafforgue and Naor [2014b] (see also Austin, Naor, and Tessera [2013]) and $L_1(\mu)$ spaces Naor and Young [2017c] and Naor and Young [2017b]; interestingly, the latter assertion is not true for the 3-dimensional Heisenberg group Naor and Young [2017a], while the former assertion does hold true for the 3-dimensional Heisenberg group Lafforgue and Naor [2014b].

All of the known "bad examples" (including, in addition to the Heisenberg group, those that were subsequently found in Laakso [2000], Laakso [2002], Bourdon and Pajot [1999], and Cheeger [1999]) which show that the doubling property is not the sought-after answer to Problem 2 do not even embed into an infinite-dimensional Hilbert space. This leads to the following natural and intriguing question that was stated by Lang and Plaut in Lang and Plaut [2001].

Question 41. Does every doubling subset of a Hilbert admit a bi-Lipschitz embedding into $\mathbb{R}^k$ for some $k \in \mathbb{N}$?

As stated, Question 41 is qualitative, but by a compactness argument (see Naor and Neiman [2012, Section 4]) if its answer were positive, then for every $K \in [2, \infty)$ there would exist $d_K \in \mathbb{N}$ and $\alpha_K \in [1, \infty)$ such that any K-doubling subset of a Hilbert space would embed into $\ell_2^{d_K}$ with distortion α_K. If Question 41 had a positive answer, then it would be very interesting to determine the asymptotic behavior of d_K and α_K as $K \to \infty$. A positive answer to Question 41 would be a solution of Problem 2, though the intrinsic criterion that it would provide would be quite complicated, namely it would say that a metric space $(\mathfrak{M}, d_{\mathfrak{M}})$ admits a bi-Lipschitz embedding into $\mathbb{R}^k$ for some $k \in \mathbb{N}$ if and only if it is doubling and satisfies the family of quadratic distance inequalities (2). More importantly, it seems that any positive answer to Question 41 would devise a procedure that starts with a subset in a very high-dimensional Euclidean space and, if that subset is $O(1)$-doubling, produce a bi-Lipschitz embedding into $\mathbb{R}^{O(1)}$; such a procedure, if possible, would be a quintessential metric dimension reduction result that is bound to be of major importance. It should be noted that, as proved in Indyk and Naor [2007, Remark 4.1], any such general procedure cannot be an embedding into

low-dimensions via a linear mapping as in the JL lemma, i.e., Question 41 calls for a genuinely nonlinear dimension reduction technique.[6]

Despite the above reasons why a positive answer to Question 41 would be very worthwhile, we suspect that Question 41 has a negative answer. A specific doubling subset of a Hilbert space which is a potential counterexample to Question 41 was constructed in Naor and Neiman [2012, Question 3], but to date it remains unknown whether or not this subset admits a bi-Lipschitz embedding into $\mathbb{R}^{O(1)}$. If the answer to Question 41 is indeed negative, then the next challenge would be to formulate a candidate conjectural characterization for resolving the bi-Lipschitz embedding problem into $\mathbb{R}^k$.

The analogue of Question 41 is known to fail in some non-Hilbertian settings. Specifically, it follows from Lafforgue and Naor [2014a], Naor and Young [2017c], and Naor and Young [2017b] that for every $p \in (2, \infty)$ there exists a doubling subset $\mathcal{D}_p$ of $L_p(\mathbb{R})$ that does not admit a bi-Lipschitz embedding into any $L_q(\mu)$ space for any $q \in [1, p)$. So, in particular there is no bi-Lipschitz embedding of $\mathcal{D}_p$ into any finite-dimensional normed space, and a fortiori there is no such embedding into any finite-dimensional subspace of $L_p(\mathbb{R})$. Note that in Lafforgue and Naor [2014a] this statement is made for embeddings of $\mathcal{D}_p$ into $L_q(\mu)$ in the reflexive range $q \in (1, p)$, and the case $q = 1$ is treated in Lafforgue and Naor [ibid.] only when $p \geqslant p_0$ for some universal constant $p_0 > 2$. The fact that $\mathcal{D}_p$ does not admit a bi-Lipschitz embedding into any $L_1(\mu)$ space follows by combining the argument of Lafforgue and Naor [ibid.] with the more recent result[7] of Naor and Young [2017c] and Naor and Young [2017b] when the underlying group in the construction of Lafforgue and Naor [2014a] is the 5-dimensional Heisenberg group; interestingly we now know Naor and Young [2017a] that if one carries out the construction of Lafforgue and Naor [2014a] for the 3-dimensional Heisenberg group, then the reasoning of Lafforgue and Naor [ibid.] would yield the above conclusion only when $p > 4$. A different example of a doubling subset of $L_p(\mathbb{R})$ that fails to embed bi-Lipschitzly into ℓ_p^k for any $k \in \mathbb{N}$ was found in Bartal, Gottlieb, and Neiman [2015]. In $L_1(\mathbb{R})$, there is an even stronger counterexample Lafforgue and Naor [2014a, Remark 1.4]: By Gupta, Newman, Rabinovich, and Sinclair [2004], the spaces considered in Laakso [2000] and Laakso [2002] yields a doubling subset of $L_1(\mathbb{R})$ that by Cheeger and Kleiner [2009] (see also M. I. Ostrovskii [2011]) does not admit a bi-Lipschitz embedding into any Banach space with the Radon–Nikodým property Lee and Naor [2006] and Cheeger and Kleiner [2006], hence it does not admit

[6] On its own, the established necessity of obtaining a genuinely nonlinear embedding method into low dimensions should not discourage attempts to answer Question 41, because some rigorous nonlinear dimension reduction methods have been devised in the literature; see e.g. Assouad [1983], Semmes [1999], M. Charikar and Sahai [2002], Gupta, Krauthgamer, and Lee [2003], Bartal, Linial, Mendel, and Naor [2004], Bartal and Mendel [2004], Krauthgamer, Lee, Mendel, and Naor [2005], Lang and Schlichenmaier [2005], Brinkman, Karagiozova, and Lee [2007], Lee, Naor, and Peres [2009], Chan, Gupta, and Talwar [2010], Abraham, Bartal, and Neiman [2011], Bartal, Recht, and Schulman [2011], Gupta and Talwar [2011], Naor and Neiman [2012], David and Snipes [2013], Lee, de Mesmay, and Moharrami [2013], Newman and Rabinovich [2013], Gottlieb and Krauthgamer [2015], Bartal and Gottlieb [2016], Neiman [2016], M. Ostrovskii and Randrianantoanina [2016], and Andoni, Naor, and Neiman [2017]. However, all of these approaches seem far from addressing Question 41.

[7] When Lafforgue and Naor [2014a] was written, only a weaker bound of Cheeger, Kleiner, and Naor [2011] was known.

a bi-Lipschitz embedding into any reflexive or separable dual Banach space. The potential validity of the above statement for $p \in (1,2)$ remains an intriguing open problem, and the case $p = 2$ is of course Question 41.

4 Matoušek's random metrics, Milnor–Thom, and coarse dimension reduction

Fix two moduli $\omega, \Omega : [0,\infty) \to [0,\infty)$ as in Theorem 17, i.e., they are increasing functions and $\omega \leqslant \Omega$ point-wise. For a metric space $(\mathfrak{M}, d_{\mathfrak{M}})$ define $\dim_{(\omega,\Omega)}(\mathfrak{M}, d_{\mathfrak{M}})$ to be the smallest dimension $k \in \mathbb{N}$ for which there exists a k-dimensional normed space $(X, \|\cdot\|_X) = (X(\mathfrak{M}), \|\cdot\|_{X(\mathfrak{M})})$ and a mapping $f : \mathfrak{M} \to X$ that satisfies (18). If no such $k \in \mathbb{N}$ exists, then write $\dim_{(\omega,\Omega)}(\mathfrak{M}, d_{\mathfrak{M}}) = \infty$. For $\alpha \in [1,\infty)$, this naturally generalizes the notation $\dim_\alpha(\mathfrak{M}, d_{\mathfrak{M}})$ of Linial, London, and Rabinovich [1995] in the bi-Lipschitz setting, which coincides with $\dim_{(t,\alpha t)}(\mathfrak{M}, d_{\mathfrak{M}})$.

Recalling (17), the goal of this section is to show that $\dim_{(\omega,\Omega)}(\mathfrak{M}, d_{\mathfrak{M}}) \geqslant n^{c\beta(\omega,\Omega)}$ for arbitrarily large $n \in \mathbb{N}$, some universal constant $c \in (0,\infty)$ and some metric space $(\mathfrak{M}, d_{\mathfrak{M}})$ with $|\mathfrak{M}| = 3n$, thus proving Theorem 17. We will do so by following Matoušek's beautiful ideas in Matoušek [1996], yielding a probabilistic argument for the existence of such an intrinsically (coarsely) high-dimensional metric space $(\mathfrak{M}, d_{\mathfrak{M}})$.

The collections of subsets of a set S of size $\ell \in \mathbb{N}$ will be denoted below $\binom{S}{\ell} = \{e \subseteq S : |e| = \ell\}$. Fix $n \in \mathbb{N}$ and a bipartite graph $\mathsf{G} = (\mathsf{L}, \mathsf{R}, \mathsf{E})$ with $|\mathsf{L}| = |\mathsf{R}| = n$. Thus, L and R are disjoint n-point sets (the "left side" and "right side" of G) and E is a subset of $\binom{\mathsf{L}\cup\mathsf{R}}{2}$ consisting only of $e \subseteq \mathsf{L}\cup\mathsf{R}$ such that $|\mathsf{L}\cap e| = |\mathsf{R}\cap e| = 1$. Following Matoušek Matoušek [ibid.], any such graph G can used as follows as a "template" for obtaining a family $2^{|\mathsf{E}|}$ graphs, each of which having $3n$ vertices. For each $\lambda \in \mathsf{L}$ introduce two new elements λ^+, λ^-. Denote $\mathsf{L}^+ = \{\lambda^+ : \lambda \in \mathsf{L}\}$ and $\mathsf{L}^- = \{\lambda^- : \lambda \in \mathsf{L}\}$. Assume that the sets $\mathsf{L}^+, \mathsf{L}^-, \mathsf{R}$ are disjoint. For every $\sigma : \mathsf{E} \to \{-,+\}$ define

$$\mathsf{E}_\sigma \stackrel{\text{def}}{=} \Big\{\big\{\lambda^{\sigma(\{\lambda,\rho\})}, \rho\big\} : (\lambda,\rho) \in \mathsf{L}\times\mathsf{R} \wedge \{\lambda,\rho\} \in \mathsf{E}\Big\} \subseteq \binom{\mathsf{L}^+\cup\mathsf{L}^-\cup\mathsf{R}}{2}. \tag{45}$$

We thus obtain a bipartite graph $\mathsf{G}_\sigma = (\mathsf{L}^+\cup\mathsf{L}^-, \mathsf{R}, \mathsf{E}_\sigma)$. By choosing $\sigma : \mathsf{E} \to \{-,+\}$ uniformly at random, we think of G_σ as a random graph; let $\mathbb{P}$ denote the uniform probability measure on the set of all such σ. In other words, consider $\sigma : \mathsf{E} \to \{-,+\}$ to be independent tosses of a fair coin, one for each edge of G. Given an outcome of the coin tosses σ, each edge $e \in \mathsf{E}$ of G induces an element of E_σ as follows. If λ is the endpoint of e in L and ρ is the endpoint of e in R, then E_σ contains exactly one of the unordered pairs $\{\lambda^+,\rho\}, \{\lambda^-,\rho\}$ depending on whether $\sigma(e) = +$ or $\sigma(e) = -$, respectively; see Figure 1 below for a schematic depiction of this construction.

Let $d_{\mathsf{G}_\sigma} : (\mathsf{L}^+\cup\mathsf{L}^-\cup\mathsf{R}) \times (\mathsf{L}^+\cup\mathsf{L}^-\cup\mathsf{R}) \to [0,\infty]$ be the shortest-path metric corresponding to G_σ, with the convention that $d_{\mathsf{G}_\sigma}(x,y) = \infty$ if $x, y \in \mathsf{L}^+\cup\mathsf{L}^-\cup\mathsf{R}$ belong to different connected components of G_σ. We record for convenience of later use the following very simple observation.

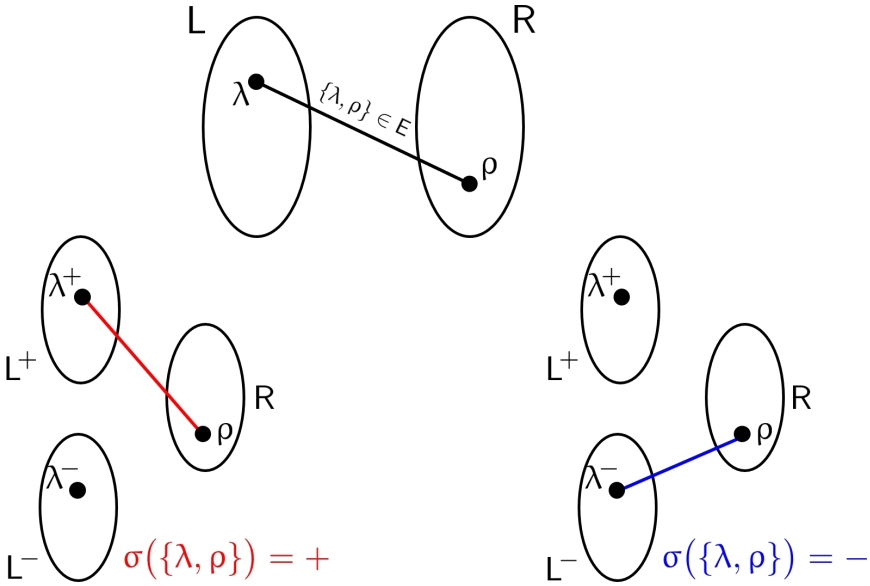

Figure 1 – The random bipartite graph $\mathsf{G}_\sigma = (\mathsf{L}^+ \cup \mathsf{L}^-, \mathsf{R}, \mathsf{E}_\sigma)$ that is associated to the bipartite graph $\mathsf{G} = (\mathsf{L}, \mathsf{R}, \mathsf{E})$ and coin flips $\sigma : \mathsf{E} \to \{-, +\}$. Suppose that $(\lambda, \rho) \in \mathsf{L} \times \mathsf{R}$ and $e = \{\lambda, \rho\} \in \mathsf{E}$. If the outcome of the coin that was flipped for the edge e is $+$, then include in E_σ the red edge between λ^+ and ρ and do not include an edge between λ^- and ρ. If the outcome of the coin that was flipped for the edge e is $-$, then include in E_σ the blue edge between λ^- and ρ and do not include an edge between λ^+ and ρ.

Claim 42. *Fix $\lambda \in \mathsf{L}$ and $\sigma : \mathsf{E} \to \{-, +\}$. Suppose that $k \stackrel{\text{def}}{=} d_{\mathsf{G}_\sigma}(\lambda^+, \lambda^-) < \infty$. Then the original "template graph" G contains a cycle of length at most k.*

Proof. Denote by $\pi : \mathsf{L}^+ \cup \mathsf{L}^- \cup \mathsf{R} \to \mathsf{L} \cup \mathsf{R}$ the canonical "projection," i.e., π is the identity mapping on R and $\pi(\lambda^+) = \pi(\lambda^-) = \lambda$ for every $\lambda \in \mathsf{L}$. The natural induced mapping $\pi : \mathsf{E}_\sigma \to \mathsf{E}$ (given by $\pi(\{x, y\}) = \{\pi(x), \pi(y)\}$ for each $\{x, y\} \in \mathsf{E}_\sigma$) is one-to-one, because by construction E_σ contains one and only one of the unordered pairs $\{\mu^+, \rho\}, \{\mu^-, \rho\}$ for each $(\mu, \rho) \in \mathsf{L} \times \mathsf{R}$ with $\{\mu, \rho\} \in \mathsf{E}$.

Let $\gamma : \{0, \ldots, k\} \to \mathsf{L}^+ \cup \mathsf{L}^- \cup \mathsf{R}$ be a geodesic in G_σ that joins λ^+ and λ^-. Thus $\gamma(0) = \lambda^+$, $\gamma(k) = \lambda^-$ and $\{\{\gamma(i-1), \gamma(i)\}\}_{i=1}^k$ are distinct edges in E_σ (they are distinct because γ is a shortest path joining λ^+ and λ^- in G_σ). By the injectivity of π on E_σ, the unordered pairs $\{\{\pi(\gamma(i-1)), \pi(\gamma(i))\}\}_{i=1}^k$ are distinct edges in E. So, the subgraph H of G that is induced on the vertices $\{\pi(\gamma(i))\}_{i=0}^k$ has at least k edges. But, H has at most k vertices, because $\pi(\gamma(0)) = \pi(\gamma(k)) = \lambda$. Hence H is not a tree, i.e., it contains a cycle of length at most k. $\square$

Even though d_{G_σ} is not necessarily a metric due to its possible infinite values, for every $s, T \in (0, \infty)$ we can re-scale and truncate it so as to obtain a (random) metric

$d_\sigma^{s,T} : (\mathsf{L}^+ \cup \mathsf{L}^- \cup \mathsf{R}) \times (\mathsf{L}^+ \cup \mathsf{L}^- \cup \mathsf{R}) \to [0,\infty]$ by defining

$$\forall\, x, y \in \mathsf{L}^+ \cup \mathsf{L}^- \cup \mathsf{R}, \qquad d_\sigma^{s,T}(x,y) \stackrel{\text{def}}{=} \min\left\{s d_{\mathsf{G}_\sigma}(x,y), T\right\}. \tag{46}$$

The following lemma shows that if in the above construction G has many edges and no short cycles, then with overwhelmingly high probability the random metric in (46) has large coarse metric dimension.

Lemma 43. *There is a universal constant $\eta > 0$ with the following property. For every $\omega, \Omega : [0,\infty) \to [0,\infty)$ as above, every $n \in \mathbb{N}$ and every template graph $\mathsf{G} = (\mathsf{L}, \mathsf{R}, \mathsf{E})$ as above, suppose that $g \in \mathbb{N}$ and $s, T > 0$ satisfy*

$$\frac{\omega^{-1}(2\Omega(s))}{s} < g \leqslant \frac{T}{s}, \tag{47}$$

and that the shortest cycle in G has length at least g. Then for every $\delta \in (0, \frac{1}{3}]$ we have

$$\mathbb{P}\left[\sigma : \mathsf{E} \to \{-,+\} : \dim_{\omega,\Omega}\left(\mathsf{L}^+ \cup \mathsf{L}^- \cup \mathsf{R}, d_\sigma^{s,T}\right) \leqslant \delta\eta \frac{|\mathsf{E}|}{n}\right] < \left(2\delta^\delta\right)^{-|\mathsf{E}|}. \tag{48}$$

In particular, by choosing $\delta = \frac{1}{3}$ in (48) we deduce that

$$\mathbb{P}\left[\sigma : \mathsf{E} \to \{-,+\} : \dim_{\omega,\Omega}\left(\mathsf{L}^+ \cup \mathsf{L}^- \cup \mathsf{R}, d_\sigma^{s,T}\right) > \frac{\eta|\mathsf{E}|}{3n}\right] > 1 - e^{-\frac{1}{5}|\mathsf{E}|}. \tag{49}$$

Prior to proving Lemma 43, we shall now explain how it implies Theorem 17.

Proof of Theorem 17 assuming Lemma 43. Recalling (17), we can fix $s \in (0,\infty)$ such that

$$g \stackrel{\text{def}}{=} \left\lfloor \frac{\omega^{-1}(2\Omega(s))}{s} \right\rfloor + 1 \leqslant \frac{2}{\beta(\omega,\Omega)}. \tag{50}$$

There is a universal constant $\kappa \in (0,\infty)$ such that for arbitrarily large $n \in \mathbb{N}$ there exists a bipartite graph $\mathsf{G} = (\mathsf{L}, \mathsf{R}, \mathsf{E})$ with $|\mathsf{L}| = |\mathsf{R}| = n$, girth at least g (i.e., G does not contain any cycle of length smaller than g) and $|\mathsf{E}| \geqslant n^{1+\kappa/g}$. Determining the largest possible value of κ here is a well-studied and longstanding open problem in graph theory (see e.g. the discussions in Bollobás [2001], Matoušek [2002], and M. I. Ostrovskii [2013]), but for the present purposes any value of κ suffices. For the latter (much more modest) requirement, one can obtain G via a simple probabilistic construction (choosing each of the edges independently at random and deleting an arbitrary edge from each cycle of length at most $g-1$). See Lazebnik, Ustimenko, and Woldar [1995] for the best known lower bound on κ here (arising from an algebraic construction).

We shall use the above graph G as the template graph for the random graphs $\{\mathsf{G}_\sigma\}_{\sigma:\mathsf{E}\to\{-,+\}}$. Our choice of g in (50) ensures that if we write $T = sg$, then (47) holds true and we obtain a distribution over metric spaces $(\mathsf{L}^+ \cup \mathsf{L}^- \cup \mathsf{R}, d_\sigma^{s,T})$ for which the conclusion (49) of Lemma 43 holds true. Hence, by choosing $c = \kappa/2$ and substituting the bound $|\mathsf{E}| \geqslant n^{1+\kappa/g}$ into (49) while using (50) we have

$$\mathbb{P}\left[\sigma : \mathsf{E} \to \{-,+\} : \dim_{\omega,\Omega}\left(\mathsf{L}^+ \cup \mathsf{L}^- \cup \mathsf{R}, d_\sigma^{s,T}\right) \gtrsim n^{c\beta(\omega,\Omega)}\right] \geqslant 1 - \exp\left(-\frac{1}{5} n^{1+c\beta(\omega,\Omega)}\right)$$

Consequently, by the definition of $\dim_{\omega,\Omega}(\cdot)$, with probability exponentially close to 1 the random metric space $(\mathsf{L}^+ \cup \mathsf{L}^- \cup \mathsf{R}, d_\sigma^{s,T})$ satisfies the assertion of Theorem 17. $\square$

The proof of Lemma 43 relies on the following lemma that was obtained implicitly by Matoušek [1996]. Its proof takes as input a clever argument of Alon [1986] which uses the classical bound of Milnor [1964] and Thom [1965] on the number of connected components of a real algebraic variety.

Lemma 44. *Fix $m, n \in \mathbb{N}$ and $\mathsf{E} \subseteq \{1,\ldots,n\}^2$. Suppose that $\mathsf{A}_1 = (a_{ij}^1), \ldots, \mathsf{A}_m = (a_{ij}^m) \in \mathsf{M}_n(\mathbb{R})$ are matrices that satisfy $a_{ij}^k \neq 0$ for all $(i,j) \in \mathsf{E}$ and $k \in \{1,\ldots,m\}$, and that the sign vectors*

$$\big(\mathbf{sign}(a_{ij}^1)\big)_{(i,j)\in\mathsf{E}}, \big(\mathbf{sign}(a_{ij}^2)\big)_{(i,j)\in\mathsf{E}} \ldots, \big(\mathbf{sign}(a_{ij}^m)\big)_{(i,j)\in\mathsf{E}} \subseteq \{-1,1\}^{\mathsf{E}} \tag{51}$$

are distinct. Then there exists $k \in \{1,\ldots,m\}$ such that

$$\mathbf{rank}(\mathsf{A}_k) \gtrsim \frac{\log m}{n \log\left(\frac{|\mathsf{E}|}{\log m}\right)}. \tag{52}$$

Proof. Let $\alpha \in \mathbb{N}$ be an auxiliary parameter that will be specified later so as to optimize the ensuing argument. Write $h = \lceil |\mathsf{E}|/\alpha \rceil$ and fix any partition of E into subsets $\mathsf{J}_1, \ldots, \mathsf{J}_h \neq \varnothing$ (i.e., $\mathsf{J}_1, \ldots, \mathsf{J}_h \subseteq \mathsf{E}$ are pairwise disjoint and $\mathsf{E} = \mathsf{J}_1 \cup \ldots \cup \mathsf{J}_h$) that satisfy $|\mathsf{J}_u| \leqslant \alpha$ for all $u \in \{1,\ldots,h\}$.

Denote

$$\mu \stackrel{\text{def}}{=} \min_{\substack{(i,j)\in\mathsf{E}\\ k\in\{1,\ldots,m\}}} \left|a_{ij}^k\right| \qquad \text{and} \qquad r \stackrel{\text{def}}{=} \max_{k\in\{1,\ldots,m\}} \mathbf{rank}(\mathsf{A}_k). \tag{53}$$

Lemma 44 assumes that $\mu > 0$, and its goal is to show that r is at least a universal constant multiple of the quantity that appears in the right hand side on (52). The definition of r means that for every $k \in \{1,\ldots,m\}$ there exist n-by-r and r-by-n matrices $\mathsf{B}_k \in \mathsf{M}_{n\times r}(\mathbb{R})$ and $\mathsf{C}_k \in \mathsf{M}_{r\times n}(\mathbb{R})$, respectively, such that $\mathsf{A}_k = \mathsf{B}_k\mathsf{C}_k$. Define vectors $\{\zeta_k = (\zeta_1^k, \ldots, \zeta_h^k) \in \mathbb{R}^h\}_{k=1}^m$ by setting

$$\forall\, (k,u) \in \{1,\ldots,m\} \times \{1,\ldots,h\}, \qquad \zeta_u^k \stackrel{\text{def}}{=} \sqrt{\prod_{(i,j)\in\mathsf{J}_u} (a_{ij}^k)^2 - \frac{1}{2}\mu^{2|\mathsf{J}_u|}}. \tag{54}$$

Observe that the definition of μ in (53) ensures that the quantity under the square root in (54) is positive, so $\zeta_u \in (0,\infty)$. Define polynomials $\{\mathfrak{p}_u : \mathsf{M}_{n\times r}(\mathbb{R}) \times \mathsf{M}_{r\times n}(\mathbb{R}) \times \mathbb{R}^h \to \mathbb{R}\}_{u=1}^h$ by setting

$$\mathfrak{p}_u(X,Y,z) \stackrel{\text{def}}{=} \prod_{(i,j)\in\mathsf{J}_u} (XY)_{ij}^2 - z_u^2 - \frac{1}{2}\mu^{2|\mathsf{J}_u|} = \prod_{(i,j)\in\mathsf{J}_u} \left(\sum_{k=1}^n x_{ik}y_{kj}\right)^2 - z_u^2 - \frac{1}{2}\mu^{2|\mathsf{J}_u|}, \tag{55}$$

for all $u \in \{1,\ldots,h\}$, $X = (x_{is}) \in \mathsf{M}_{n\times r}(\mathbb{R})$, $Y = (y_{sj}) \in \mathsf{M}_{r\times n}(\mathbb{R})$ and $z = (z_i) \in \mathbb{R}^h$. The above notation ensures that $\mathfrak{p}_u(\mathsf{B}_k, \mathsf{C}_k, \zeta_k) = 0$ for all $k \in \{1,\ldots,m\}$

and $u \in \{1, \dots, h\}$. In other words, $\{(\mathsf{B}_k, \mathsf{C}_k, \zeta_k)\}_{k=1}^m \subseteq \mathcal{V}$, where $\mathcal{V} \subseteq \mathsf{M}_{n\times r}(\mathbb{R}) \times \mathsf{M}_{r\times n}(\mathbb{R}) \times \mathbb{R}^h$ is the variety

$$\mathcal{V} \stackrel{\text{def}}{=} \bigcap_{u=1}^{h} \left\{(X, Y, z) \in \mathsf{M}_{n\times r}(\mathbb{R}) \times \mathsf{M}_{r\times n}(\mathbb{R}) \times \mathbb{R}^h;\ \mathfrak{p}_u(X, Y, z) = 0\right\}. \tag{56}$$

We claim that $\mathcal{V}$ has at least m connected components. In fact, if $k, \ell \in \{1, \dots, m\}$ are distinct, then $(\mathsf{B}_k, \mathsf{C}_k, \zeta_k)$ and $(\mathsf{B}_\ell, \mathsf{C}_\ell, \zeta_\ell)$ belong to different connected component of $\mathcal{V}$. Indeed, suppose for the sake of obtaining a contradiction that $\mathcal{C} \subseteq \mathcal{V}$ is a connected subset of $\mathcal{V}$ and $(\mathsf{B}_k, \mathsf{C}_k, \zeta_k), (\mathsf{B}_\ell, \mathsf{C}_\ell, \zeta_\ell) \in \mathcal{C}$. Since $k \neq \ell$, by switching the roles of k and ℓ if necessary, the assumption of Lemma 44 ensures that there exists $(i, j) \in \mathsf{E}$ such that $(\mathsf{B}_k\mathsf{C}_k)_{ij} = a_{ij}^k < 0 < a_{ij}^\ell = (\mathsf{B}_\ell\mathsf{C}_\ell)_{ij}$. So, if we denote $\psi : \mathcal{C} \to \mathbb{R}$ by $\psi(X, Y, z) = (XY)_{ij}$, then $\psi(\mathsf{B}_k, \mathsf{C}_k, \zeta_k) < 0 < \psi(\mathsf{B}_\ell, \mathsf{C}_\ell, \zeta_\ell)$. Since $\mathcal{C}$ is connected and ψ is continuous, it follows that $\psi(X, Y, z) = 0$ for some $(X, Y, z) \in \mathcal{C}$. Let $u \in \{1, \dots, h\}$ be the index for which $(i, j) \in \mathsf{J}_u$. By the definition (55) of $\mathfrak{p}_u$, the fact that $\psi(X, Y, z) = (XY)_{ij} = 0$ implies that $\mathfrak{p}_u(X, Y, z) = -z_u^2 - \frac{1}{2}\mu^{|J_u|} \leqslant -\frac{1}{2}\mu^{|J_u|} < 0$, since $\mu > 0$. Hence $(X, Y, z) \notin \mathcal{V}$, in contradiction to our choice of (X, Y, z) as an element of $\mathcal{C} \subseteq \mathcal{V}$.

Recalling (55), for all $u \in \{1, \dots, h\}$ the degree of $\mathfrak{p}_u$ is $4|\mathsf{J}_u| \leqslant 4\alpha$. So, the variety $\mathcal{V}$ in (56) is defined using h polynomials of degree at most 4α in $2nr + h$ variables. By (a special case of) a theorem of Milnor [1964] and Thom [1965], the number of connected components of $\mathcal{V}$ is at most $4\alpha(8\alpha-1)^{2nr+h-1} = 4\alpha(8\alpha-1)^{2nr+\lceil|\mathsf{E}|/\alpha\rceil-1}$. Since we already established that this number of connected components is at least m, it follows that

$$m \leqslant 4\alpha(8\alpha-1)^{2nr+\left\lceil\frac{|\mathsf{E}|}{\alpha}\right\rceil-1} \iff r \geqslant \frac{1}{2n}\left(\frac{\log\left(\frac{m}{4\alpha}\right)}{\log(8\alpha-1)} - \left\lceil\frac{|\mathsf{E}|}{\alpha}\right\rceil + 1\right). \tag{57}$$

The value of $\alpha \in \mathbb{N}$ that maximizes the rightmost quantity in (57) satisfies

$$\alpha \asymp \frac{|\mathsf{E}|}{\log(2m)}\left(\log\left(\frac{|\mathsf{E}|}{\log(2m)}\right)\right)^2.$$

For this α the estimate (57) simplifies to imply the desired bound

$$r \gtrsim \log m/(n\log(|\mathsf{E}|/\log m)).$$

□

Proof of Lemma 43. For notational convenience, write $\mathsf{L} = \{\lambda_1, \dots, \lambda_n\}$ and $\mathsf{R} = \{\rho_1, \dots, \rho_n\}$, and think of E as a subset of $\{1, \dots, n\}^2$ (i.e., $(i, j) \in \mathsf{E}$ if and only if $\{\lambda_i, \rho_j\}$ is an edge of G).

For every $\Delta \in \mathbb{N}$ denote

$$\mathcal{B}_\Delta \stackrel{\text{def}}{=} \left\{\sigma : \mathsf{E} \to \{-, +\} :\ \dim_{\omega,\Omega}\left(\mathsf{L}^+ \cup \mathsf{L}^- \cup \mathsf{R}, d_\sigma^{s,T}\right) \leqslant \Delta\right\}. \tag{58}$$

Then, by the definition of $\dim_{\omega,\Omega}(\cdot)$, if $\sigma \in \mathcal{B}_\Delta$, we can fix a normed space $(X_\sigma, \|\cdot\|_{X_\sigma})$ with $\dim(X_\sigma) = \Delta$ and a mapping $f_\sigma : \mathsf{L}^+ \cup \mathsf{L}^- \cup \mathsf{R} \to X_\sigma$ that satisfies

(59)
$$\forall\, x, y \in \mathsf{L}^+ \cup \mathsf{L}^- \cup \mathsf{R}, \qquad \omega\big(d_\sigma^{s,T}(x,y)\big) \leqslant \|f_\sigma(x) - f_\sigma(y)\|_{X_\sigma} \leqslant \Omega\big(d_\sigma^{s,T}(x,y)\big).$$

Using the Hahn–Banach theorem, for each $i \in \{1,\ldots,n\}$ and $\sigma \in \mathcal{B}_\Delta$ we can fix a linear functional $z^*_{\sigma,i} \in X^*_\sigma$ of unit norm that normalizes the vector $f_\sigma(\lambda_i^+) - f_\sigma(\lambda_i^-) \in X_\sigma$, i.e.,

$$z^*_{\sigma,i}\big(f_\sigma(\lambda_i^+) - f_\sigma(\lambda_i^-)\big) = \big\|f_\sigma(\lambda_i^+) - f_\sigma(\lambda_i^+)\big\|_{X_\sigma} \qquad \text{and} \tag{60}$$

$$\|z^*_{\sigma,i}\|_{X^*_\sigma} = \sup_{w \in X_\sigma \smallsetminus \{0\}} \frac{\big|z^*_{\sigma,i}(w)\big|}{\|w\|_{X_\sigma}} = 1. \tag{61}$$

Using these linear functionals, define an $n \times n$ matrix $\mathsf{A}_\sigma = (a_{ij}^\sigma) \in \mathsf{M}_n(\mathbb{R})$ by setting

$$\forall (i,j) \in \{1,\ldots,n\}^2, \qquad a_{ij}^\sigma \stackrel{\text{def}}{=} z^*_{\sigma,i}\big(f_\sigma(\rho_j)\big) - \frac{1}{2} z^*_{\sigma,i}\big(f_\sigma(\lambda_i^+) + f_\sigma(\lambda_i^-)\big). \tag{62}$$

Observe in passing that the following identity holds true for every $i, j \in \{1,\ldots,n\}$ and $\sigma \in \mathcal{B}_\Delta$.

$$\sigma(i,j) a_{ij}^\sigma = \frac{1}{2} z^*_{\sigma,i}\big(f_\sigma(\lambda_i^+) - f_\sigma(\lambda_i^-)\big) + \sigma(i,j) z^*_{\sigma,i}\Big(f_\sigma(\rho_j) - f_\sigma\Big(\lambda_i^{\sigma(i,j)}\Big)\Big). \tag{63}$$

(Simply verify (63) for the cases $\sigma(i,j) = +$ and $\sigma(i,j) = -$ separately, using the linearity of $z^*_{\sigma,i}$.)

Since we are assuming in Lemma 43 that the shortest cycle in the template graph G has length at least g, it follows from Claim 42 that $d_{\mathsf{G}_\sigma}(\lambda_i^+, \lambda_i^-) \geqslant g$ for all $i \in \{1,\ldots,n\}$ and $\sigma : \mathsf{E} \to \{-,+\}$. So,

$$d_\sigma^{s,T}(\lambda_i^+, \lambda_i^-) \stackrel{(46)}{=} \min\big\{s d_{\mathsf{G}_\sigma}(\lambda_i^+, \lambda_i^-), T\big\} \geqslant \min\{sg, T\} \stackrel{(47)}{=} sg. \tag{64}$$

Recalling (45), we have $\{\lambda_i^{\sigma(i,j)}, \rho_j\} \in \mathsf{E}_\sigma$ for all $(i,j) \in \mathsf{E}$. Hence $d_{\mathsf{G}_\sigma}(\lambda_i^{\sigma(i,j)}, \rho_j) = 1$ and therefore

$$d_\sigma^{s,T}\big(\lambda_i^{\sigma(i,j)}, \rho_j\big) \stackrel{(46)}{\leqslant} s d_{\mathsf{G}_\sigma}\big(\lambda_i^{\sigma(i,j)}, \rho_j\big) = s. \tag{65}$$

Consequently, for every $(i,j) \in \mathsf{E}$ and $\sigma \in \mathcal{B}_\Delta$ we have

$$\begin{aligned}
\sigma(i,j) a_{ij}^\sigma &\stackrel{(63)}{\geqslant} \frac{1}{2} z^*_{\sigma,i}\big(f_\sigma(\lambda_i^+) - f_\sigma(\lambda_i^-)\big) - \|z^*_{\sigma,i}\|_{X^*_\sigma} \cdot \Big\|f_\sigma(\rho_j) - f_\sigma\Big(\lambda_i^{\sigma(i,j)}\Big)\Big\|_{X_\sigma} \\
&\stackrel{(60)}{=} \frac{1}{2}\big\|f_\sigma(\lambda_i^+) - f_\sigma(\lambda_i^-)\big\|_{X_\sigma} - \Big\|f_\sigma(\rho_j) - f_\sigma\Big(\lambda_i^{\sigma(i,j)}\Big)\Big\|_{X_\sigma} \\
&\stackrel{(59)}{\geqslant} \frac{1}{2}\omega\big(d_\sigma^{s,T}(\lambda_i^+, \lambda_i^-)\big) - \Omega\Big(d_\sigma^{s,T}\big(\lambda_i^{\sigma(i,j)}, \rho_j\big)\Big) \\
&\stackrel{(64)\wedge(65)}{\geqslant} \frac{1}{2}\omega(sg) - \Omega(s) \stackrel{(47)}{>} 0.
\end{aligned}$$

Hence, $a_{ij}^{\sigma} \neq 0$ and $\mathbf{sign}(a_{ij}^{\sigma}) = \sigma(i,j)$ for all $(i,j) \in \mathsf{E}$ and $\sigma \in \mathcal{B}_{\Delta}$. This is precisely the setting of Lemma 44 (with $m = |\mathcal{B}_{\Delta}|$), from which we conclude that there exists $\tau \in \mathcal{B}_{\Delta}$ such that

$$\mathbf{rank}(\mathsf{A}_{\tau}) \geqslant \frac{c \log |\mathcal{B}_{\Delta}|}{n \log\left(\frac{|\mathsf{E}|}{\log |\mathcal{B}_{\Delta}|}\right)}, \tag{66}$$

where $c \in (0,\infty)$ is a universal constant. Henceforth, we shall fix a specific $\tau \in \mathcal{B}_{\Delta}$ as in (66).

Since $\tau \in \mathcal{B}_{\Delta}$ we have $\dim(X_{\tau}) = \Delta$, so we can fix a basis $e_{\tau}^{1}, \ldots, e_{\tau}^{\Delta}$ of X_{τ} and for every $j \in \{1,\ldots,n\}$ write $f_{\tau}(\rho_j) = \gamma_{\tau,j}^{1} e_{\tau}^{1} + \ldots + \gamma_{\tau,j}^{\Delta} e_{\tau}^{\Delta}$ for some scalars $\gamma_{\tau,j}^{1}, \ldots, \gamma_{\tau,j}^{\Delta} \in \mathbb{R}$. Hence,

$$(a_{ij}^{\tau})_{i=1}^{n} \overset{(62)}{=} \gamma_{\tau,j}^{1}\left(z_{\tau,i}^{*}(e_{\tau}^{1})\right)_{i=1}^{n} + \ldots + \gamma_{\tau,j}^{\Delta}\left(z_{\tau,i}^{*}(e_{\tau}^{\Delta})\right)_{i=1}^{n} - \frac{1}{2}\left(z_{\sigma,i}^{*}\left(f_{\tau}(\lambda_i^{+}) + f_{\tau}(\lambda_i^{-})\right)\right)_{i=1}^{n}$$

We have thus expressed the columns of the matrix A_{τ} as elements of the span of the $\Delta + 1$ vectors

$$\left(z_{\tau,i}^{*}(e_{\tau}^{1})\right)_{i=1}^{n}, \left(z_{\tau,i}^{*}(e_{\tau}^{2})\right)_{i=1}^{n}, \ldots, \left(z_{\tau,i}^{*}(e_{\tau}^{\Delta})\right)_{i=1}^{n}, \left(z_{\sigma,i}^{*}\left(f_{\tau}(\lambda_i^{+}) + f_{\tau}(\lambda_i^{-})\right)\right)_{i=1}^{n} \in \mathbb{R}^{n}.$$

Consequently, the rank of A_{τ} is at most $\Delta + 1 \leqslant 2\Delta$. By contrasting this with (66), we see that

$$\frac{|\mathsf{E}|}{\log |\mathcal{B}_{\Delta}|} \log\left(\frac{|\mathsf{E}|}{\log |\mathcal{B}_{\Delta}|}\right) \geqslant \frac{c|\mathsf{E}|}{2\Delta n}. \tag{67}$$

We shall now conclude by showing that Lemma 43 holds true with $\eta = c/2$. Indeed, fix $\delta \in (0, \frac{1}{3}]$ and observe that we may assume also that $\delta\eta|\mathsf{E}|/n \geqslant 1$, since otherwise the left hand side of (48) vanishes. Then, by choosing $\Delta = \lfloor \delta\eta|\mathsf{E}|/n \rfloor \in \mathbb{N}$ in the above reasoning it follows from (67) that

$$\frac{|\mathsf{E}|}{\log |\mathcal{B}_{\Delta}|} \log\left(\frac{|\mathsf{E}|}{\log |\mathcal{B}_{\Delta}|}\right) \geqslant \frac{c}{2\delta\eta} = \frac{1}{\delta} \geqslant 3 > e.$$

This implies that $|\mathcal{B}_{\Delta}| \leqslant \delta^{-\delta|\mathsf{E}|}$. Equivalently, $\mathbb{P}[\mathcal{B}_{\Delta}] \leqslant (2\delta)^{-\delta|\mathsf{E}|}$, which is the desired bound (48). □

5 Nonlinear spectral gaps and impossibility of average dimension reduction

Fix $n \in \mathbb{N}$ and an irreducible reversible row-stochastic matrix $\mathsf{A} = (a_{ij}) \in \mathsf{M}_n(\mathbb{R})$. This implies that there is a unique[8] A-stationary probability measure $\pi = (\pi_1, \ldots, \pi_n) \in$

[8] We are assuming irreducibility only for notational convenience, namely so that π will be unique and could therefore be suppressed in the ensuing notation. Our arguments work for any stochastic matrix and any probability measure π on $\{1,\ldots,n\}$ with respect to which A is reversible. We suggest focusing initially on the case when A is symmetric and π is the uniform measure on $\{1,\ldots,n\}$, though the general case is useful for treating graphs that are not regular, e.g. those of Section 4. See Levin, Peres, and Wilmer [2017] for the relevant background.

$[0,1]^n$ on $\{1,\ldots,n\}$, namely $\pi\mathsf{A}=\pi$, and we have the reversibility condition $\pi_i a_{ij} = \pi_j a_{ji}$ for all $i,j \in \{1,\ldots,n\}$. Then A is a self-adjoint contraction on $L_2(\pi)$, and we denote by $1=\lambda_1(\mathsf{A}) \geqslant \lambda_2(\mathsf{A}) \geqslant \ldots \geqslant \lambda_n(\mathsf{A}) \geqslant -1$ the decreasing rearrangement of its eigenvalues.

Given a metric space $(\mathfrak{M}, d_{\mathfrak{M}})$ and $p>0$, define $\gamma(\mathsf{A}, d_{\mathfrak{M}}^p)$ to be the infimum over those $\gamma>0$ such that

(68)
$$\forall\, x_1,\ldots,x_n \in \mathfrak{M}, \qquad \sum_{i=1}^n\sum_{j=1}^n \pi_i\pi_j d_{\mathfrak{M}}(x_i,x_j)^p \leqslant \gamma \sum_{i=1}^n\sum_{j=1}^n \pi_i a_{ij} d_{\mathfrak{M}}(x_i,x_j)^p.$$

This definition is implicit in Gromov [2003], and appeared explicitly in Naor and Silberman [2011]; see Mendel and Naor [2014], Naor [2014], and Mendel and Naor [2015] for a detailed treatment. It suffices to note here that if $(H, \|\cdot\|_H)$ is a Hilbert space and $p=2$, then by expanding the squares one directly sees that $\gamma(\mathsf{A}, \|\cdot\|_H^2) = 1/(1-\lambda_2(\mathsf{A}))$ is the reciprocal of the *spectral gap* of A. In general, we think of $\gamma(\mathsf{A}, d_{\mathfrak{M}}^p)$ as measuring the magnitude of the nonlinear spectral gap of A with respect to the kernel $d_{\mathfrak{M}}^p : \mathfrak{M}\times\mathfrak{M} \to [0,\infty)$.

Using the notation that was recalled in Section 1.1, the definition (68) immediately implies that nonlinear spectral gaps are bi-Lipschitz invariants in the sense that $\gamma(\mathsf{A}, d_{\mathfrak{M}}^p) \leqslant \mathsf{c}_{\mathfrak{N}}(\mathfrak{M})^p \gamma(\mathsf{A}, d_{\mathfrak{N}}^p)$ for every two metric spaces $(\mathfrak{M}, d_{\mathfrak{M}})$ and $(\mathfrak{N}, d_{\mathfrak{N}})$, every matrix A as above and every $p>0$. In particular, if $(H, \|\cdot\|_H)$ is a Hilbert space into which $(\mathfrak{M}, d_{\mathfrak{M}})$ admits a bi-Lipschitz embedding, then we have the following general (trivial) bound.

$$\sqrt{\gamma(\mathsf{A}, d_{\mathfrak{M}}^2)} \leqslant \mathsf{c}_2(\mathfrak{M})\sqrt{\gamma(\mathsf{A}, \|\cdot\|_H^2)}. \tag{69}$$

In the recent work Naor [2017] we proved the following theorem, which improves over (69) when $\mathfrak{M}$ is a Banach space.

Theorem 45. *Suppose that $(X, \|\cdot\|_X)$ is a Banach space and that $(H, \|\cdot\|_H)$ is a Hilbert space. Then for every $M \in (0,\infty)$ and every matrix A as above for which $\lambda_2(\mathsf{A}) \leqslant 1 - M^2/\mathsf{c}_2(X)^2$ we have*

$$\sqrt{\gamma(\mathsf{A}, \|\cdot\|_X^2)} \lesssim \frac{\log(M+1)}{M}\mathsf{c}_2(X)\sqrt{\gamma(\mathsf{A}, \|\cdot\|_H^2)}. \tag{70}$$

In the setting of Theorem 45, since $\gamma(\mathsf{A}, \|\cdot\|_H^2) = 1/(1-\lambda_2(\mathsf{A}))$, the bound (70) can be rewritten as

$$\gamma(\mathsf{A}, \|\cdot\|_X^2) \lesssim \left(\frac{\log(\mathsf{c}_2(X)\sqrt{1-\lambda_2(\mathsf{A})}+1)}{1-\lambda_2(\mathsf{A})}\right)^2,$$

which is how Theorem 45 was stated in Naor [ibid.]. Note that (70) coincides (up to the implicit constant factor) with the trivial bound (69) if $M = O(1)$, but (70) is an asymptotic improvement over (69) as $M\to\infty$.

The proof of Theorem 45 in Naor [ibid.] is a short interpolation argument that takes as input a theorem from Naor [2014]. While we do not know of a different proof of (70),

below we will present a new and self-contained derivation of the following weaker estimate (using the same notation as in Theorem 45) that suffices for deducing Theorem 19.

$$\sqrt{\gamma(\mathsf{A}, \|\cdot\|_X^2)} \lesssim \frac{\log(\mathsf{c}_2(X)+1)}{M}\mathsf{c}_2(X)\sqrt{\gamma(\mathsf{A}, \|\cdot\|_H^2)}. \tag{71}$$

The proof of (71) appears in Section 5.1 below. We will next show how (71) implies Theorem 19 as well as the leftmost inequality in (12), which includes as a special case the triple logarithmic estimate in (22).

Given $n \in \mathbb{N}$ and an n-vertex connected graph $\mathsf{G} = (\{1,\ldots,n\}, \mathsf{E}_{\mathsf{G}})$, let A_{G} be its random walk matrix, i.e., if $\deg_{\mathsf{G}}(i)$ is the degree in G of the vertex $i \in \{1,\ldots,n\}$, then $(\mathsf{A}_{\mathsf{G}})_{ij} = \mathbf{1}_{\{i,j\}\in\mathsf{E}_{\mathsf{G}}}/\deg_{\mathsf{G}}(i)$ for all $i, j \in \{1,\ldots,n\}$. For $i \in \{1,\ldots,n\}$ we also denote $\pi_i^{\mathsf{G}} = \deg_{\mathsf{G}}(i)/(2|\mathsf{E}_{\mathsf{G}}|)$. Thus, $\pi^{\mathsf{G}} \in \mathbb{R}^n$ is the probability measure on $\{1,\ldots,n\}$ with respect to which A_{G} is reversible. We will use the simpler notation $\lambda_i(\mathsf{A}_{\mathsf{G}}) = \lambda_i(\mathsf{G})$ for every $i \in \{1,\ldots,n\}$. For $p \in (0,\infty)$and a metric space $(\mathfrak{M}, d_{\mathfrak{M}})$, we will write $\gamma(\mathsf{G}, d_{\mathfrak{M}}^2) = \gamma(\mathsf{A}_{\mathsf{G}}, d_{\mathfrak{M}}^2)$. The shortest-path metric that is induced by G on $\{1,\ldots,n\}$ will be denoted $d_{\mathsf{G}} : \{1,\ldots,n\}\times\{1,\ldots,n\} \to \mathbb{N}\cup\{0\}$.

Theorem 46. *There is a universal constant $K > 1$ with the following property. Fix $n \in \mathbb{N}$ and $\alpha \geqslant 1$. Let $\mathsf{A} \in \mathsf{M}_n(\mathbb{R})$ and $\pi \in [0,1]^n$ be as above. For every normed space $(X, \|\cdot\|_X)$, if $f : \{1,\ldots,n\} \to X$ satisfies*

$$\left(\sum_{i=1}^n\sum_{j=1}^n \pi_i a_{ij}\|f(i)-f(j)\|_X^2\right)^{\frac{1}{2}} \leqslant \alpha, \tag{72}$$

then necessarily

$$\dim(X) \gtrsim K^{\frac{1-\lambda_2(\mathsf{A})}{\alpha}\sqrt{\sum_{i=1}^n\sum_{j=1}^n \pi_i\pi_j\|f(i)-f(j)\|_X^2}}. \tag{73}$$

In particular, in the special case when $\mathsf{G} = (\{1,\ldots,n\}, \mathsf{E}_{\mathsf{G}})$ is a connected graph we have

$$\left(\frac{1}{|\mathsf{E}_{\mathsf{G}}|}\sum_{\{i,j\}\in\mathsf{E}_{\mathsf{G}}}\|f(i)-f(j)\|_X^2\right)^{\frac{1}{2}} \leqslant \alpha \implies \dim(X) \gtrsim K^{\frac{1-\lambda_2(\mathsf{G})}{\alpha}\sqrt{\sum_{i=1}^n\sum_{j=1}^n \pi_i^{\mathsf{G}}\pi_j^{\mathsf{G}}\|f(i)-f(j)\|_X^2}}$$

In the case of regular graphs with a spectral gap, Theorem 46 has the following corollary.

Corollary 47. *Fix two integers $n, r \geqslant 3$ and let $\mathsf{G} = (\{1,\ldots,n\}, \mathsf{E}_{\mathsf{G}})$ be a connected r-regular graph. If $(X, \|\cdot\|_X)$ is a normed space into which there is a mapping $f : \{1,\ldots,n\} \to X$ that satisfies*

$$\left(\frac{1}{|\mathsf{E}_{\mathsf{G}}|}\sum_{\{i,j\}\in\mathsf{E}_{\mathsf{G}}}\|f(i)-f(j)\|_X^2\right)^{\frac{1}{2}} \leqslant \alpha \qquad \text{and} \tag{74}$$

$$\left(\frac{1}{n^2}\sum_{i=1}^n\sum_{j=1}^n \|f(i)-f(j)\|_X^2\right)^{\frac{1}{2}} \geqslant \frac{1}{n^2}\sum_{i=1}^n\sum_{j=1}^n d_{\mathsf{G}}(i,j), \tag{75}$$

then necessarily

$$\dim(X) \gtrsim n^{\frac{c(1-\lambda_2(\mathsf{G}))}{\alpha\log r}},$$

where $c \in (0,\infty)$ is a universal constant.

Proof. This is nothing more than a special case of Theorem 46 once we note that by a straightforward and standard counting argument (see e.g. Matoušek [1997]) we have $\frac{1}{n}\sum_{i=1}^n\sum_{j=1}^n d_{\mathsf{G}}(i,j) \gtrsim \log_r n$. □

For every integer $r \geqslant 3$ there exist arbitrarily large r-regular graphs G with $\lambda_2(\mathsf{G}) = 1-\Omega(1)$; see Hoory, Linial, and Wigderson [2006] for this and much more on such *spectral expanders*. Corollary 47 shows that the shortest-path metric on any such graph with $r = O(1)$ satisfies the conclusion of Theorem 19, because the α-Lipschitz assumption of Theorem 19 implies the first inequality in (74) and the assumption

$$\frac{1}{n^2}\sum_{i=1}^n\sum_{j=1}^n \|f(i)-f(j)\|_X \geqslant \frac{1}{n^2}\sum_{i=1}^n\sum_{j=1}^n d_{\mathsf{G}}(i,j)$$

of Theorem 19 implies the second inequality in (74) (using Jensen's inequality).

Note that we actually proved above that any expander is "metrically high dimensional" in a stronger sense. Specifically, if $\mathsf{G} = (\{1,\ldots,n\},\mathsf{E}_{\mathsf{G}})$ is a $O(1)$-spectral expander, i.e., it is $O(1)$-regular and $\lambda_2(\mathsf{G}) \leqslant 1-\Omega(1)$, and one finds vectors $x_1,\ldots,x_n$ in a normed space $(X,\|\cdot\|_X)$ for which the averages $\frac{1}{|\mathsf{E}_{\mathsf{G}}|}\sum_{\{i,j\}\in\mathsf{E}_{\mathsf{G}}}\|x_i-x_j\|_X^2$ and $\frac{1}{n^2}\sum_{i,j=1}^n\|x_i-x_j\|_X^2$ are within a $O(1)$ factor of the averages

$$\frac{1}{|\mathsf{E}_{\mathsf{G}}|}\sum_{\{i,j\}\in\mathsf{E}_{\mathsf{G}}} d_{\mathsf{G}}(i,j)^2 = 1$$

and $\frac{1}{n^2}\sum_{i,j=1}^n d_{\mathsf{G}}(i,j)^2$, respectively, then this "finitary average distance information" (up to a fixed but potentially very large multiplicative error) forces the ambient space X to be very high (worst-possible) dimensional, namely $\dim(X) \geqslant n^{\Omega(1)}$.

Remark 48. If one replaces (74) by the requirement that for an increasing modulus $\omega : [0,\infty) \to [0,\infty)$ we have,

$$\left(\frac{1}{|\mathsf{E}_{\mathsf{G}}|}\sum_{\{i,j\}\in\mathsf{E}_{\mathsf{G}}}\|f(i)-f(j)\|_X^2\right)^{\frac{1}{2}} \leqslant 1 \qquad \text{and} \tag{76}$$

$$\left(\frac{1}{n^2}\sum_{i=1}^n\sum_{j=1}^n\|f(i)-f(j)\|_X^2\right)^{\frac{1}{2}} \geqslant \frac{1}{n^2}\sum_{i=1}^n\sum_{j=1}^n \omega\big(d_{\mathsf{G}}(i,j)\big), \tag{77}$$

then the above argument applies mutatis mutandis to yield the conclusion

$$\dim(X) \gtrsim e^{c(1-\lambda_2(\mathsf{G}))\omega(c\log_r n)}. \tag{78}$$

Indeed, the aforementioned counting argument shows that least 50% of the pairs $(i,j) \in \{1,\ldots,n\}^2$ satisfy $d_{\mathsf{G}}(i,j) \gtrsim \log_r n$. Compare (78) to Theorem 17 which provides a stronger bound if the average requirement (76) is replaced by its pairwise counterpart (18). Nevertheless, the bound (78) is quite sharp (at least when $r = O(1)$ and $\lambda_2(\mathsf{G}) = 1-\Omega(1)$), in the sense that there is a normed space $(X,\|\cdot\|_X)$ for which (76) holds and

$$\dim(X) \lesssim e^{C(\log r)\omega\left(\frac{C\log n}{\sqrt{1-\lambda_2(\mathsf{G})}}\right)}\log n, \tag{79}$$

where $C > 0$ is a universal constant. Indeed, by Chung, Faber, and Manteuffel [1994] the diameter of the metric space $(\{1,\ldots,n\}, d_{\mathsf{G}})$ satisfies

$$\mathbf{diam}(\mathsf{G}) \lesssim (\log n)/\sqrt{1-\lambda_2(\mathsf{G})}.$$

By an application of (12) with $\alpha \asymp (\log_r n)/\omega(\mathbf{diam}(\mathsf{G}))$ there exists a normed space $(X, \|\cdot\|_X)$ with $\dim(X) \lesssim n^{O(1/\alpha)}\log n$, thus (79) holds, and a mapping $f : \{1,\ldots,n\} \to X$ that satisfies $d_{\mathsf{G}}(i,j)/\alpha \leqslant \|f(i)-f(j)\|_X \leqslant d_{\mathsf{G}}(i,j)$ for all $i,j \in \{1,\ldots,n\}$. Hence, the first inequality in (76) holds, and

$$\left(\frac{1}{n^2}\sum_{i=1}^n\sum_{j=1}^n \|f(i)-f(j)\|_X^2\right)^{\frac12} \geqslant \frac{1}{\alpha}\left(\frac{1}{n^2}\sum_{i=1}^n\sum_{j=1}^n d_{\mathsf{G}}(i,j)^2\right)^{\frac12} \tag{80}$$

$$\gtrsim \frac{\log_r n}{\alpha} \tag{81}$$

$$\asymp \omega\big(\mathbf{diam}(\mathsf{G})\big) \tag{82}$$

$$\geqslant \frac{1}{n^2}\sum_{i=1}^n\sum_{j=1}^n \omega\big(d_{\mathsf{G}}(i,j)\big). \tag{83}$$

Proof of Theorem 46 assuming (71). Let $C \in (0,\infty)$ be the implicit universal constant in (71). Then

$$\left(\sum_{i=1}^n\sum_{j=1}^n \pi_i\pi_j\|f(i)-f(j)\|_X^2\right)^{\frac12} \overset{(68)}{\leqslant} \sqrt{\gamma(\mathsf{A},\|\cdot\|_X^2)}\left(\sum_{i=1}^n\sum_{j=1}^n \pi_i a_{ij}\|f(i)-f(j)\|_X^2\right)^{\frac12} \tag{84}$$
$$\overset{(72)}{\leqslant} \alpha\sqrt{\gamma(\mathsf{A},\|\cdot\|_X^2)} \overset{(71)}{\leqslant} \frac{C\alpha\log(\mathsf{c}_2(X)+1)}{1-\lambda_2(\mathsf{A})},$$

where the last step of (84) is an application of (71) with $M = \mathsf{c}_2(X)\sqrt{1-\lambda_2(\mathsf{A})}$, while using that for a Hilbert space H we have $\gamma(\mathsf{A},\|\cdot\|_H^2) = 1/(1-\lambda_2(\mathsf{A}))$. It follows that

$$2\sqrt{\dim(X)} \geqslant 2\mathsf{c}_2(X) \geqslant \mathsf{c}_2(X)+1 \overset{(84)}{\geqslant} e^{\frac{1-\lambda_2(\mathsf{A})}{C\alpha}\sqrt{\sum_{i=1}^n\sum_{j=1}^n \pi_i\pi_j\|f(i)-f(j)\|_X^2}}, \tag{85}$$

where the first step of (85) uses John's theorem John [1948]. This establishes (73) with $K = e^{2/C} > 1$. □

For non-contracting embeddings (in particular, for bi-Lipschitz embedding), the proof of the following lemma is an adaptation of the proof of Abraham, Bartal, and Neiman [2011, Theorem 13].

Lemma 49. *Fix two integers $n, r \geqslant 3$ and let $\mathsf{G} = (\{1,\ldots,n\}, \mathsf{E}_{\mathsf{G}})$ be a connected r-regular graph. If $(X, \|\cdot\|_X)$ is a normed space into which there is a mapping $f : \{1,\ldots,n\} \to X$ that satisfies*

$$\min_{\substack{i,j\in\{1,\ldots,n\}\\ i\neq j}} \frac{\|f(i)-f(j)\|_X}{d_{\mathsf{G}}(i,j)} \geqslant 1, \qquad \text{and} \qquad \left(\frac{1}{|\mathsf{E}_{\mathsf{G}}|}\sum_{\{i,j\}\in\mathsf{E}_{\mathsf{G}}} \|f(i)-f(j)\|_X^2\right)^{\frac12} \leqslant \alpha. \tag{86}$$

Then necessarily

$$\frac{\log n}{\log r} n^{\frac{1}{2\dim(X)}} \lesssim \alpha \sqrt{\gamma(\mathsf{G}, \|\cdot\|_X^2)}. \tag{87}$$

Prior to proving Lemma 49, we will derive some of its corollaries. For the terminology of Corollary 50 below, recall that a Banach space Y is said to be B-convex Beck [1962] if ℓ_1 is *not* finitely representable in Y; see the survey Maurey [2003] for more on this important notion, including useful analytic, geometric and probabilistic characterizations.

Corollary 50. *There is a universal constant $C \in (0,\infty)$ for which the following assertion holds true. Let Y be an infinite dimensional B-convex Banach space. For arbitrarily large $n \in \mathbb{N}$, if $\alpha \geqslant C \log n$, then we have*

$$\mathsf{k}_n^{\alpha}(\ell_\infty, Y) \asymp_Y \frac{\log n}{\log\left(\frac{\alpha}{\log n}\right)}. \tag{88}$$

Thus, we have in particular $\mathsf{k}_n^{C\log n}(\ell_\infty, Y) \asymp_Y \log n$.

Proof. The upper bound $\mathsf{k}_n^{\alpha}(\ell_\infty, Y) \lesssim (\log n)/\log(\alpha/\log n)$ actually holds for *any* infinite dimensional Banach space Y. Indeed, by Bourgain's embedding theorem Bourgain [1985a] any n-point metric space $\mathfrak{M}$ admits an embedding f into ℓ_2 with distortion $A\log n$, where $A \in (0,\infty)$ is a universal constant. If $\alpha \geqslant 4A\log n$, then by applying the JL-Lemma Johnson and Lindenstrauss [1984] we know that $f(\mathfrak{M})$ embeds with distortion $\alpha/(2A\log n)$ into ℓ_2^k, where $k \lesssim (\log n)/\log(\alpha/\log n)$. By Dvoretzky's theorem Dvoretzky [1961], we know that ℓ_2^k embeds with distortion 2 into Y, so overall we obtain an embedding of $\mathfrak{M}$ into a k-dimensional subspace of Y with distortion at most $2(A\log n)(\alpha/(2A\log n)) = \alpha$.

Conversely, suppose that $\alpha \geqslant 2\log n$ and that Y is a B-convex Banach space. By a theorem of V. Lafforgue [2009] (see also Mendel and Naor [2014] for a different approach), for arbitrarily large $n \in \mathbb{N}$ there is a $O(1)$-regular graph $\mathsf{G} = (\{1,\ldots,n\}, \mathsf{E}_{\mathsf{G}})$ such that $\gamma(\mathsf{G}, \|\cdot\|_Y^2) \lesssim_Y 1$. If $(\{1,\ldots,n\}, d_{\mathsf{G}})$ embeds with distortion α into a k-dimensional subspace of Y, then by Lemma 49 we have $n^{1/(2k)} \lesssim_Y \alpha/\log n$. Thus $k \gtrsim_Y (\log n)/\log(\alpha/\log n)$, as required. □

Question 51. Is the assumption of B-convexity needed for the conclusion (88) of Corollary 50? Perhaps finite cotype suffices for this purpose? This matter is of course closely related to Question 22.

Corollary 52. *Under the assumptions and notation of Lemma 49, we have*

$$\frac{\log n}{\log r} n^{\frac{1}{2\dim(X)}} \lesssim \frac{\alpha \log(\mathsf{c}_2(X)+1)}{1-\lambda_2(\mathsf{G})}.$$

Proof. This is nothing more than a substitution of (71) with $M = \mathsf{c}_2(X)\sqrt{1-\lambda_2(\mathsf{G})}$ into (87). □

Since by John's theorem John [1948] we have $\mathsf{c}_2(X) \leqslant \sqrt{\dim(X)}$ and for every $n \in \mathbb{N}$ there exists a graph G as in Lemma 49 with $r = O(1)$ and $\lambda_2(\mathsf{G}) = 1 - \Omega(1)$, it follows from Corollary 52 that

$$\alpha \log(\mathsf{k}_n^{\alpha}(\ell_\infty) + 1) \gtrsim n^{\frac{1}{2\mathsf{k}_n^{\alpha}(\ell_\infty)}} \log n.$$

This implies the lower bound on $\mathsf{k}_n^{\alpha}(\ell_\infty)$ in (12). In particular, for $\alpha \asymp \log n$ it gives the first inequality in (22).

Proof of Lemma 49. Denote $\gamma = \gamma(\mathsf{G}, \|\cdot\|_X^2)$. For $i \in \{1, \ldots, n\}$ write

(89)
$$\mathcal{U}_i = f^{-1}\Big(B_X\big(f(i), \alpha\sqrt{2\gamma}\big)\Big) = \Big\{j \in \{1, \ldots, n\} : \|f(i) - f(j)\|_X \leqslant \alpha\sqrt{2\gamma}\Big\}.$$

Let $m \in \{1, \ldots, n\}$ satisfy $|\mathcal{U}_m| = \max_{i \in \{1,\ldots,n\}} |\mathcal{U}_i|$. Then

$$n^2\gamma\alpha^2 \overset{(68)\wedge(86)}{\geqslant} \sum_{i=1}^n \sum_{j=1}^n \|f(i) - f(j)\|_X^2 \geqslant \sum_{i=1}^n \sum_{j \in \{1,\ldots,n\} \smallsetminus \mathcal{U}_i} \|f(i) - f(j)\|_X^2 \overset{(89)}{>} \sum_{i=1}^n \big(\alpha\sqrt{2\gamma}\big)^2 (n - |\mathcal{U}_i|) \geqslant 2n\gamma\alpha^2(n - |\mathcal{U}_m|). \tag{90}$$

This simplifies to $|\mathcal{U}_m| \geqslant \frac{1}{2}n$. Also, since

$$\frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n \|f(i) - f(j)\|_X^2 \geqslant \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n d_{\mathsf{G}}(i, j)^2 \gtrsim (\log_r n)^2$$

the first inequality in (90) implies the a priori lower bound $\alpha\sqrt{\gamma} \gtrsim \log_r n$.

Next, fix $\rho \in (0, \infty)$ and let $\mathcal{N}_{2\rho} \subseteq \mathcal{U}_m$ be a maximal (with respect to inclusion) 2ρ-separated subset of $\mathcal{U}_m$. Then $\mathcal{U}_m \subseteq \cup_{i \in \mathcal{N}_\Delta} B_{\mathsf{G}}(i, 2\rho)$, where $B_{\mathsf{G}}(i, 2\rho)$ denotes the ball centered at i of radius 2ρ in the shortest-path metric d_{G}. Since G is r-regular, for each $i \in \{1, \ldots, n\}$ we have the (crude) bound $|B_{\mathsf{G}}(i, 2\rho)| \leqslant 2r^{2\rho}$. Hence, $\frac{1}{2}n \leqslant |\mathcal{U}_m| \leqslant 2r^{2\rho}|\mathcal{N}_{2\rho}|$. So, if we choose $\rho = \frac{1}{4}\log_r n$, then $|\mathcal{N}_{2\rho}| \gtrsim \sqrt{n}$. Since by (86) distinct $i, j \in \mathcal{N}_{2\rho}$ satisfy $\|f(i) - f(j)\|_X \geqslant d_{\mathsf{G}}(i, j) \geqslant 2\rho$, the X-balls $\{B_X(f(i), \rho) : i \in \mathcal{N}_{2\rho}\}$ have pairwise disjoint interiors. At the same time, since each $i \in \mathcal{N}_{2\rho}$ belongs to $\mathcal{U}_m$, we have $\|f(i) - f(m)\|_X \leqslant \alpha\sqrt{2\gamma}$ (by the definition of $\mathcal{U}_m$), and hence $B_X(f(i), \rho) \subseteq B_X(f(m), \alpha\sqrt{2\gamma} + \rho)$. So, writing $\dim(X) = k$, we have the following volume comparison.

$$\begin{aligned}(\alpha\sqrt{2\gamma} + \rho)^k \mathbf{vol}_k\big(B_X(0, 1)\big) &= \mathbf{vol}_k\big(B_X(f(m), \alpha\sqrt{2\gamma} + \rho)\big) \geqslant \mathbf{vol}_k\Big(\bigcup_{i \in \mathcal{N}_{2\rho}} B_X(f(i), \rho)\Big) \\ &= \sum_{i \in \mathcal{N}_{2\rho}} \mathbf{vol}_k\big(B_X(f(i), \rho)\big) = \rho^k \mathbf{vol}_k\big(B_X(0, 1)\big)|\mathcal{N}_{2\rho}| \gtrsim \rho^k \mathbf{vol}_k\big(B_X(0, 1)\big)\sqrt{n}.\end{aligned}$$

This simplifies to give $n^{\frac{1}{2k}} \lesssim \frac{\alpha\sqrt{2\gamma}}{\rho} + 1 \asymp \frac{\alpha\sqrt{\gamma}}{\log_r n}$, where we used the definition of ρ, and that $\alpha\sqrt{\gamma} \gtrsim \log_r n$. $\square$

5.1 Nonlinear Rayleigh quotient inequalities. Our goal in this section is to present a proof of (71). As we stated earlier, the proof that appears below is different from the proof of Theorem 45 in Naor [2017]. However, the reason that underlies its validity is the same as that of the original argument in Naor [ibid.]. Specifically, we arrived at the ensuing proof because we were driven by an algorithmic need that arose in Andoni, Naor, Nikolov, Razenshteyn, and Waingarten [2018a]. This need required proving a point-wise strengthening of an upper bound on nonlinear spectral gaps, which is called in Andoni, Naor, Nikolov, Razenshteyn, and Waingarten [ibid.] a "nonlinear Rayleigh quotient inequality." We will clarify what we mean by this later; a detailed discussion appears in Andoni, Naor, Nikolov, Razenshteyn, and Waingarten [ibid.].

The need to make the interpolation-based proof in Naor [2017] constructive / algorithmic led us to merge the argument in Naor [ibid.] with the *proof of* a theorem from Naor [2014], rather than quoting and using the latter as a "black box" as we did in Naor [2017]. In doing so, we realized that for the purpose of obtaining only the weaker bound (71) one could more efficiently combine Naor [2014] and Naor [2017] so as to skip the use of complex interpolation and to obtain the estimate (71) as well as its nonlinear Rayleigh quotient counterpart. Thus, despite superficial differences, the argument below amounts to unravelling the proofs in Naor [2014, 2017] and removing steps that are needed elsewhere but not for (71). At present, we do not have a proof of the stronger inequality (70) that differs from its proof in Naor [2017], and the interpolation-based approach of Naor [ibid.] is used for more refined algorithmic results in the forthcoming work Andoni, Naor, Nikolov, Razenshteyn, and Waingarten [2018b].

We will continue using the notation/conventions that were set at the beginning of Section 5. Fix $p \geqslant 1$ and a metric space $(\mathfrak{M}, d_{\mathfrak{M}})$. Let $L_p(\pi; \mathfrak{M})$ be the metric space $(\mathfrak{M}^n, d_{L_p(\pi;\mathfrak{M})})$, where $d_{L_p(\pi;\mathfrak{M})} : \mathfrak{M}^n \times \mathfrak{M}^n \to [0, \infty)$ is $\forall\, x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n) \in \mathfrak{M}^n$

$$d_{L_p(\pi;\mathfrak{M})}(x, y) \stackrel{\text{def}}{=} \left(\sum_{i=1}^n \pi_i d_{\mathfrak{M}}(x_i, y_i)^p \right)^{\frac{1}{p}}.$$

Throughout what follows, it will be notationally convenient to slightly abuse notation by considering $\mathfrak{M}$ as a subset of $L_p(\pi; \mathfrak{M})$ through its identification with the *diagonal* subset of $\mathfrak{M}^n$, which is an isometric copy of $\mathfrak{M}$ in $L_p(\pi; \mathfrak{M})$. Namely, we identify each $x \in \mathfrak{M}$ with the n-tuple $(x, x \ldots, x) \in \mathfrak{M}^n$.

If $x = (x_1, \ldots, x_n) \in L_p(\pi; \mathfrak{M}) \smallsetminus \mathfrak{M}$, then the corresponding *nonlinear Rayleigh quotient* is defined to be

$$\mathcal{R}(x; \mathsf{A}, d_{\mathfrak{M}}^p) \stackrel{\text{def}}{=} \frac{\sum_{i=1}^n \sum_{j=1}^n \pi_i a_{ij} d_{\mathfrak{M}}(x_i, x_j)^p}{\sum_{i=1}^n \sum_{j=1}^n \pi_i \pi_j d_{\mathfrak{M}}(x_i, x_j)^p}. \tag{91}$$

The restriction $x \notin \mathfrak{M}$ was made here only to ensure that the denominator in (91) does not vanish. By definition,

$$\gamma(\mathsf{A}, d_{\mathfrak{M}}^p) = \sup_{x \in L_p(\pi;\mathfrak{M}) \smallsetminus \mathfrak{M}} \frac{1}{\mathcal{R}(x; \mathsf{A}, d_{\mathfrak{M}}^p)}. \tag{92}$$

Note that $L_p(\pi; X)$ is a Banach space for every Banach space $(X, \|\cdot\|_X)$. In this case, the matrix $\mathsf{A} \in \mathsf{M}_n(\mathbb{R})$ induces a linear operator $\mathsf{A} \otimes \mathsf{Id}_X : L_p(\pi; X) \to L_p(\pi; X)$ that is given by $(\mathsf{A} \otimes \mathsf{Id}_X)(x_1, \ldots x_n) = (\sum_{j=1}^n a_{ij} x_j)_{i=1}^n$.

The following lemma records some simple and elementary general properties of nonlinear Rayleigh quotients.

Lemma 53. *Suppose that $(\mathfrak{M}, d_{\mathfrak{M}})$ is a metric space, $n \in \mathbb{N}$, $p \in [1, \infty)$ and $\delta \in [0, 1]$. Let $\pi = (\pi_1, \ldots, \pi_n)$ be a probability measure on $\{1, \ldots, n\}$ and $\mathsf{A}, \mathsf{B} \in \mathsf{M}_n(\mathbb{R})$ be row-stochastic matrices that are reversible with respect to π. For any $x \in L_p^n(\pi; \mathfrak{M}) \smallsetminus \mathfrak{M}$ we have*

1. $\mathcal{R}\big(x; \delta\mathsf{A} + (1-\delta)\mathsf{B}, d_{\mathfrak{M}}^p\big) = \delta\mathcal{R}\big(x; \mathsf{A}, d_{\mathfrak{M}}^p\big) + (1-\delta)\mathcal{R}\big(x; \mathsf{B}, d_{\mathfrak{M}}^p\big)$.

2. $\mathcal{R}\big(x; (1-\delta)\mathsf{Id}_n + \delta\mathsf{A}, d_{\mathfrak{M}}^p\big) = \delta\mathcal{R}\big(x; \mathsf{A}, d_{\mathfrak{M}}^p\big)$, *where* $\mathsf{Id}_n \in \mathsf{M}_n(\mathbb{R})$ *is the identity matrix.*

3. $\mathcal{R}(x; \mathsf{A}, d_{\mathfrak{M}}^p) \leqslant 2^p$.

4. $\mathcal{R}(x; \mathsf{AB}, d_{\mathfrak{M}}^p)^{\frac{1}{p}} \leqslant \mathcal{R}(x; \mathsf{A}, d_{\mathfrak{M}}^p)^{\frac{1}{p}} + \mathcal{R}(x; \mathsf{B}, d_{\mathfrak{M}}^p)^{\frac{1}{p}}$.

5. $\mathcal{R}\big(x; \mathsf{A}^{\mathsf{t}}, d_{\mathfrak{M}}^p\big) \leqslant \mathsf{t}^p \mathcal{R}\big(x; \mathsf{A}, d_{\mathfrak{M}}^p\big)$ *for every* $\mathsf{t} \in \mathbb{N}$.

Proof. The first assertion is an immediate consequence of the definition of nonlinear Rayleigh quotients. The second assertion is a special case of the first assertion, since by definition $\mathcal{R}(x; \mathsf{Id}_n, d_{\mathfrak{M}}^p) = 0$. The third assertion is justified by noting that by the triangle inequality, for every $i, j, k \in \{1, \ldots, n\}$ we have

$$d_{\mathfrak{M}}(x_i, x_j)^p \leqslant \big(d_{\mathfrak{M}}(x_i, x_k) + d_{\mathfrak{M}}(x_k, x_j)\big)^p \leqslant 2^{p-1} d_{\mathfrak{M}}(x_i, x_k)^p + 2^{p-1} d_{\mathfrak{M}}(x_k, x_j)^p \tag{93}$$

where the last step of (93) uses the convexity of the function $(t \in [0, \infty)) \mapsto t^p$. By multiplying (93) by $\pi_i a_{ij}/n^2$, summing over $i, j, k \in \{1, \ldots, n\}$ and using the fact that A is reversible with respect to π, we get

$$\sum_{i=1}^n \sum_{j=1}^n \pi_i a_{ij} d_{\mathfrak{M}}(x_i, x_j)^p \leqslant 2^p \sum_{i=1}^n \sum_{j=1}^n \pi_i \pi_j d_{\mathfrak{M}}(x_i, x_j)^p.$$

Recalling the notation (91), this is precisely the third assertion of Lemma 53.

It remains to justify the fourth assertion of Lemma 53, because its fifth assertion follows from iterating its fourth assertion $\mathsf{t} - 1$ times (with B a power of A). To this end,

writing $\mathsf{A} = (a_{ij})$ and $\mathsf{B} = (b_{ij})$, we have

$$\begin{aligned}
&\Bigg(\sum_{i=1}^n\sum_{j=1}^n \pi_i(\mathsf{AB})_{ij} d_{\mathfrak{M}}(x_i,x_j)^p\Bigg)^{\frac{1}{p}} \leqslant \\
&\leqslant \Bigg(\sum_{i=1}^n\sum_{j=1}^n \pi_i\Bigg(\sum_{k=1}^n a_{ik}b_{kj}\Bigg)\big(d_{\mathfrak{M}}(x_i,x_k)+d_{\mathfrak{M}}(x_k,x_j)\big)^p\Bigg)^{\frac{1}{p}} \\
&\leqslant \Bigg(\sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^n \pi_i a_{ik}b_{kj} d_{\mathfrak{M}}(x_i,x_k)^p\Bigg)^{\frac{1}{p}} + \Bigg(\sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^n \pi_i a_{ik}b_{kj} d_{\mathfrak{M}}(x_k,x_j)^p\Bigg)^{\frac{1}{p}} \\
&= \Bigg(\sum_{i=1}^n\sum_{k=1}^n \pi_i a_{ik} d_{\mathfrak{M}}(x_i,x_k)^p\Bigg)^{\frac{1}{p}} + \Bigg(\sum_{j=1}^n\sum_{k=1}^n \pi_k b_{kj} d_{\mathfrak{M}}(x_k,x_j)^p\Bigg)^{\frac{1}{p}},
\end{aligned} \tag{94}$$

where the first step of (94) uses the triangle inequality in $\mathfrak{M}$, the second step of (94) uses the triangle inequality in $L_p(\mu)$ with μ being the measure on $\{1,\ldots,n\}^3$ given by $\mu(i,j,k) = \pi_i a_{ik} b_{kj}$ for all $i,j,k \in \{1,\ldots,n\}$, and the final step of (94) uses the fact that A and B are both row-stochastic and reversible with respect to π. □

The identity in the following claim is a consequence of a very simple and standard Hilbertian computation that we record here for ease of later references.

Claim 54. *For every Hilbert space $(H, \|\cdot\|_H)$ and every $x \in L_2(\pi; H) \smallsetminus H$ we have $\mathcal{R}(x; \mathsf{A}^2, \|\cdot\|_H^2) \leqslant 1$. Moreover, if $\sum_{i=1}^n \pi_i x_i = 0$, then*

$$\frac{\|(\mathsf{A}\otimes \mathsf{Id}_H)x\|_{L_2(\pi;H)}}{\|x\|_{L_2(\pi;H)}} = \sqrt{1-\mathcal{R}(x;\mathsf{A}^2,\|\cdot\|_H^2)}.$$

Proof. Let $\langle\cdot,\cdot\rangle : H\times H \to \mathbb{R}$ be the scalar product that induces the Hilbertian norm $\|\cdot\|_H$. Then, the scalar product that induces the norm $\|\cdot\|_{L_2(\pi;H)}$ is given by $\langle y,z\rangle_{L_2(\pi;H)} = \sum_{i=1}^n \pi_i\langle y_i,z_i\rangle$. By expanding the squares while using the fact that A is row-stochastic, reversible relative to π, and $\sum_{i=1}^n \pi_i x_i = 0$, we get that

$$\begin{aligned}
&\sum_{i=1}^n\sum_{j=1}^n \pi_i(\mathsf{A}^2)_{ij}\|x_i-x_j\|_H^2 = 2\|x\|_{L_2(\pi;H)}^2 - 2\sum_{i=1}^n \pi_i\Bigg\langle x_i, \sum_{j=1}^n(\mathsf{A}^2)_{ij}x_j\Bigg\rangle \\
&= 2\|x\|_{L_2(\pi;H)}^2 - 2\big\langle x, (\mathsf{A}^2\otimes\mathsf{Id}_H)x\big\rangle_{L_2(\pi;H)} = 2\|x\|_{L_2(\pi;H)}^2 - 2\big\|(\mathsf{A}\otimes\mathsf{Id}_H)x\big\|_{L_2(\pi;H)}^2,
\end{aligned}$$

and

$$\sum_{i=1}^n\sum_{j=1}^n \pi_i\pi_j\|x_i-x_j\|_H^2 = 2\sum_{i=1}^n \pi_i\|x_i\|_H^2 - 2\Bigg\|\sum_{i=1}^n \pi_i x_i\Bigg\|_H^2 = 2\|x\|_{L_2(\pi;H)}^2.$$

Therefore, recalling the definition (91), we have

$$\mathcal{R}(x;\mathsf{A}^2,\|\cdot\|_H^2) = 1 - \frac{\big\|(\mathsf{A}\otimes\mathsf{Id}_H)x\big\|_{L_2(\pi;H)}^2}{\|x\|_{L_2(\pi;H)}^2} \leqslant 1.$$ □

Lemma 55 (point-wise Rayleigh quotient estimate for Hilbert isomorphs)**.** *Let* $(X, \|\cdot\|_X)$ *be a normed space and fix* $\mathsf{d} \in [1,\infty)$*. Suppose that* $\|\cdot\|_H : X \to [0,\infty)$ *is a Hilbertian norm on* X *that satisfies*

$$\forall\, y \in X, \qquad \|y\|_H \leqslant \|y\|_X \leqslant \mathsf{d}\|y\|_H. \tag{95}$$

For every $x \in L_2(\pi; X) \smallsetminus X$ *define a quantity* $\mathsf{t}(x, \mathsf{A}) = \mathsf{t}(x; \mathsf{A}, \|\cdot\|_H, \mathsf{d})$ *to be the minimum* $\mathsf{t} \in \mathbb{N}$ *such that*

$$\mathcal{R}\left(x; \left(\frac{1}{2}\mathsf{Id}_n + \frac{1}{2}\mathsf{A}\right)^{2\mathsf{t}}, \|\cdot\|_H^2\right) \geqslant 1 - \frac{1}{4\mathsf{d}^2}, \tag{96}$$

with the convention that $\mathsf{t}(x; \mathsf{A}) = \infty$ *if no such* t *exists. Then,*

$$\frac{1}{\mathcal{R}(x; \mathsf{A}, \|\cdot\|_X^2)} \lesssim \mathsf{t}(x; \mathsf{A})^2. \tag{97}$$

Proof. We may assume without loss of generality that $\sum_{i=1}^n \pi_i x_i = 0$ and $\mathsf{t}(x; \mathsf{A}) < \infty$. Define a matrix

$$\mathsf{B}_x \stackrel{\text{def}}{=} \left(\frac{1}{2}\mathsf{Id}_n + \frac{1}{2}\mathsf{A}\right)^{\mathsf{t}(x;\mathsf{A})} \in \mathsf{M}_n(\mathbb{R}). \tag{98}$$

Then B_x is also a row-stochastic matrix which is reversible with respect to π, and, by the definition of $\mathsf{t}(x; \mathsf{A})$,

$$\mathcal{R}(x; \mathsf{B}_x^2, \|\cdot\|_H^2) \geqslant 1 - \frac{1}{4\mathsf{d}^2}.$$

By Claim 54, since $\sum_{i=1}^n \pi_i x_i = 0$, this implies that

$$\frac{\|(\mathsf{B}_x \otimes \mathsf{Id}_H)x\|_{L_2(\pi;H)}}{\|x\|_{L_2(\pi;H)}} \leqslant \sqrt{1 - \left(1 - \frac{1}{4\mathsf{d}^2}\right)} = \frac{1}{2\mathsf{d}}. \tag{99}$$

At the same time, due to (95) we have

$$\frac{\|(\mathsf{B}_x \otimes \mathsf{Id}_X)x\|_{L_2(\pi;X)}}{\|x\|_{L_2(\pi;X)}} \leqslant \mathsf{d}\frac{\|(\mathsf{B}_x \otimes \mathsf{Id}_H)x\|_{L_2(\pi;H)}}{\|x\|_{L_2(\pi;H)}}. \tag{100}$$

By combining (99) and (100) we see that $\|(\mathsf{B}_x \otimes \mathsf{Id}_H)x\|_{L_2(\pi;X)} \leqslant \frac{1}{2}\|x\|_{L_2(\pi;X)}$. Consequently,

$$\|x - (\mathsf{B}_x \otimes \mathsf{Id}_X)\, x\|_{L_2(\pi;X)} \geqslant \|x\|_{L_2(\pi;X)} - \|(\mathsf{B}_x \otimes \mathsf{Id}_X)\, x\|_{L_2(\pi;X)} \geqslant \frac{1}{2}\|x\|_{L_2(\pi;X)}. \tag{101}$$

Observe that

$$\left(\sum_{i=1}^n \sum_{j=1}^n \pi_i \pi_j \|x_i - x_j\|_X^2\right)^{\frac{1}{2}} \leqslant \left(\sum_{i=1}^n \sum_{j=1}^n \pi_i \pi_j \left(\|x_i\|_X + \|x_j\|_X\right)^2\right)^{\frac{1}{2}} \leqslant 2\|x\|_{L_2(\pi;X)} \tag{102}$$

where in the first step of (102) we used the triangle inequality in X and the second step of (102) is an application of the triangle inequality in $L_2(\pi\otimes\pi)$. Also, since B_x is row-stochastic,

$$\|x-(\mathsf{B}_x\otimes\mathsf{Id}_X)\,x\|_{L_2(\pi;X)}=\Bigg(\sum_{i=1}^n\pi_i\Bigg\|\sum_{j=1}^n(\mathsf{B}_x)_{ij}(x_i-x_j)\Bigg\|_X^2\Bigg)^{\frac12}\leqslant\Bigg(\sum_{i=1}^n\sum_{j=1}^n\pi_i(\mathsf{B}_x)_{ij}\|x_i-x_j\|_X^2\Bigg)^{\frac12},\tag{103}$$

where in the final step of (103) we used the convexity of the function $\|\cdot\|_X^2:X\to\mathbb{R}$.

Recalling the definition (91), by substituting (102) and (103) into (101) we see that

$$\mathcal{R}\Bigg(x;\Big(\frac12\mathsf{Id}_n+\frac12\mathsf{A}\Big)^{\mathsf{t}(x;\mathsf{A})},\|\cdot\|_X^2\Bigg)\overset{(98)}{=}\mathcal{R}\big(x;\mathsf{B}_x,\|\cdot\|_X^2\big)\geqslant\frac{1}{16}.\tag{104}$$

We now conclude the proof of the desired estimate (97) as follows.

$$\begin{aligned}1\overset{(104)}{\lesssim}\mathcal{R}\Bigg(x;\Big(\frac12\mathsf{Id}_n+\frac12\mathsf{A}\Big)^{\mathsf{t}(x;\mathsf{A})},\|\cdot\|_X^2\Bigg)&\leqslant\mathsf{t}(x;\mathsf{A})^2\mathcal{R}\Bigg(x;\frac12\mathsf{Id}_n+\frac12\mathsf{A},\|\cdot\|_X^2\Bigg)\\&=\frac12\mathsf{t}(x;\mathsf{A})^2\mathcal{R}\big(x;\mathsf{A},\|\cdot\|_X^2\big),\end{aligned}\tag{105}$$

where the second step of (105) uses the fifth assertion of Lemma 53, and the final step uses its second assertion. □

The quantity $\mathsf{t}(x;\mathsf{A})$ of Lemma 55 can be bounded as follows in terms of the spectral gap of A.

Lemma 56. *Continuing with the notation of Lemma 55, the following estimate holds true.*

$$\mathsf{t}(x;\mathsf{A})\leqslant\left\lceil\frac{\log(2\mathsf{d})}{\log\left(\frac{2}{1+\lambda_2(\mathsf{A})}\right)}\right\rceil\lesssim\frac{\log(2\mathsf{d})}{1-\lambda_2(\mathsf{A})}.\tag{106}$$

Proof. Since A is row-stochastic, $\lambda_n(\mathsf{A})\geqslant-1$. Therefore $\frac12\mathsf{Id}_n+\frac12\mathsf{A}$ is a positive semidefinite self-adjoint operator on $L_2(\pi)$ that preserves the hyperplane $L_2^0(\pi)=\{u\in\mathbb{R}^n;\ \sum_{i=1}^n\pi_iu_i=0\}$. The largest eigenvalue of $\frac12\mathsf{Id}_n+\frac12\mathsf{A}$ on $L_2^0(\pi)$ is $\frac12+\frac12\lambda_2(\mathsf{A})$, and therefore $\|(\frac12\mathsf{Id}_n+\frac12\mathsf{A})^{\mathsf{t}}u\|_{L_2(\pi)}\leqslant(\frac12+\frac12\lambda_2(\mathsf{A}))^{\mathsf{t}}\|u\|_{L_2(\pi)}$ for $u\in L_2^0(\pi)$ and $\mathsf{t}\in\mathbb{N}$.

If $x\in L_2(\pi;X)\smallsetminus X$ satisfies $\sum_{i=1}^n\pi_ix_i=0$, then we may apply the above observation to the coordinates of x with respect to some orthonormal basis of H, each of

which is an element of $L_2^0(\pi)$, and deduce that

$$\left(\frac{1}{2}+\frac{1}{2}\lambda_2(\mathsf{A})\right)^{\dagger} \geqslant \frac{\left\|\left(\left(\frac{1}{2}\mathsf{Id}_n+\frac{1}{2}\mathsf{A}\right)^{\dagger}\otimes \mathsf{Id}_H\right)x\right\|_{L_2(\pi;H)}}{\|x\|_{L_2(\pi;H)}} = \sqrt{1-\mathcal{R}\left(x;\left(\frac{1}{2}\mathsf{Id}_n+\frac{1}{2}\mathsf{A}\right)^{2\dagger},\|\cdot\|_H^2\right)}, \tag{107}$$

where in the second step of (107) we applied Lemma 54 with A replaced by $(\frac{1}{2}\mathsf{Id}_n+\frac{1}{2}\mathsf{A})^{\dagger}$. Hence

$$\mathcal{R}\left(x;\left(\frac{1}{2}\mathsf{Id}_n+\frac{1}{2}\mathsf{A}\right)^{2\dagger},\|\cdot\|_H^2\right) \geqslant 1-\left(\frac{1}{2}+\frac{1}{2}\lambda_2(\mathsf{A})\right)^{2\dagger}.$$

Consequently, if $\dagger \geqslant (\log(2\mathsf{d}))/\log(2/(\lambda_2(\mathsf{A})+1))$, then $\mathcal{R}(x;(\frac{1}{2}\mathsf{Id}_n+\frac{1}{2}\mathsf{A})^{2\dagger},\|\cdot\|_H^2) \geqslant 1-\frac{1}{4\mathsf{d}^2}$. By the definition of $\dagger(x;\mathsf{A})$, this implies the first inequality in (106). The second inequality in (106) follows by elementary calculus. □

Proof of (71). By a classical linearization argument Enflo [1970] (see Benyamini and Lindenstrauss [2000, Chapter 7] for a modern treatment), for every $\mathsf{d} > \mathsf{c}_2(X)$ there is a Hilbertian norm $\|\cdot\|_H$ on X that satisfies (95). We therefore see that for every A as above

$$\begin{aligned}\sqrt{\gamma(\mathsf{A},\|\cdot\|_X^2)} &\overset{(92)}{=} \sup_{x\in L_2(\pi;X)\setminus X}\frac{1}{\sqrt{\mathcal{R}(x;\mathsf{A},\|\cdot\|_X^2)}} \overset{(97)}{\leqslant} \sup_{x\in L_2(\pi;X)\setminus X}\dagger(x;\mathsf{A}) \\ &\overset{(106)}{\lesssim} \frac{\log(\mathsf{c}_2(X)+1)}{1-\lambda_2(\mathsf{A})} = \frac{\log(\mathsf{c}_2(X)+1)}{\sqrt{1-\lambda_2(\mathsf{A})}}\sqrt{\gamma(\mathsf{A},\|\cdot\|_H^2)} \\ &\leqslant \frac{\log(\mathsf{c}_2(X)+1)}{M}\mathsf{c}_2(X)\sqrt{\gamma(\mathsf{A},\|\cdot\|_H^2)},\end{aligned} \tag{108}$$

where, for the final step of (108) recall that in the context of (71) we assume that $\lambda_2(\mathsf{A}) \leqslant 1-M^2/\mathsf{c}_2(X)^2$. □

5.1.1 Structural implications of nonlinear Rayleigh quotient inequalities. Fix integers $n,k,r \geqslant 3$ (think of n as much larger than k). Let $(X,\|\cdot\|_X)$ be a k-dimensional normed space. Suppose that $\mathsf{G}=(\{1,\ldots,n\},\mathsf{E}_{\mathsf{G}})$ is a connected r-regular graph. Although we phrased (and used) Corollary 47 as an impossibility result that provides an obstruction (spectral gap) for faithfully realizing (on average) the metric space $(\{1,\ldots,n\},d_{\mathsf{G}})$ in X, a key insight of the recent work Andoni, Naor, Nikolov, Razenshteyn, and Waingarten [2018a] by Andoni, Nikolov, Razenshteyn, Waingarten and the author is that one could "flip" this point of view to deduce from Corollary 47 useful information on those graphs that do happen to admit such a faithful geometric realization in X, namely they satisfy (74). Clearly there are plenty of graphs with this property, including those graphs that arise from discrete approximations of subsets of X (as a "vanilla" example to keep in mind, fix a small parameter $\delta > 0$, consider a δ-net in the

unit ball of X as the vertices, and join two net points by an edge if their distance in X is $O(\delta)$). The conclusion of Corollary 47 for any such graph is that it cannot have a large spectral gap, and by Cheeger's inequality Cheeger [1970], Tanner [1984], Alon and V. D. Milman [1985], and Sinclair and Jerrum [1989] it follows that this graph can be partitioned into two pieces with a small (relative) "discrete boundary." On the other hand, if we are given a mapping $f : \{1, \dots, n\} \to X$ that satisfies the first condition in (74) but not the second condition in (74), then there must be a ball in X of relatively small radius that contains a substantial fraction of the vectors $\{f(i)\}_{i=1}^n$. The partition of $\{1, \dots, n\}$ that corresponds to this dense ball and its complement encodes useful geometric "clustering" information. We have thus observed a dichotomic behavior that allows one to partition geometrically-induced graphs using either a "spectral partition" or a "dense ball partition."

In Andoni, Naor, Nikolov, Razenshteyn, and Waingarten [2018a], the above idea is used iteratively to construct a hierarchical partition of X. Our overview suppresses important technical steps, which include both randomization and a re-weighting procedure of the graphs that arise at later stages of the construction (the start of the construction is indeed the above "net graph"); see Andoni, Naor, Nikolov, Razenshteyn, and Waingarten [ibid.] for the full details. In particular, one needs to use general row-stochastic matrices due to the re-weighting procedure, i.e., one uses the full strength of Theorem 46 rather than only the case of graphs as in Corollary 47.

In summary, one can use the bound (71) on nonlinear spectral gaps to provide a "cutting rule" that governs an iterative partitioning procedure in which each inductive step is either geometric (a ball and its complement) or a less explicit existential step that follows from spectral information which is deduced from a contrapositive assumption of (rough, average) embeddability. This structural information is used in Andoni, Naor, Nikolov, Razenshteyn, and Waingarten [ibid.] to design a new data structure for approximate nearest neighbor search in arbitrary norms (see the article of Andoni, Indyk and Razenshteyn in the present volume for an extensive account of approximate nearest neighbor search). Although this yields important (and arguably unexpected) progress on an algorithmic question of central importance, the non-explicitness and potential high complexity of the spectral partitioning step raises issues of efficiency that are not yet fully resolved. Specifically, the most general data structure that is designed in Andoni, Naor, Nikolov, Razenshteyn, and Waingarten [ibid.] is efficient only in the so-called "cell probe model," but not in the full polynomial-time sense; we refer to Andoni, Naor, Nikolov, Razenshteyn, and Waingarten [ibid.] for an explanation of these complexity-theoretic issues and their significance, because they are beyond the scope of the present article.

While the above issue of efficiency does not occur in our initial investigation within pure mathematics, it is very important from the algorithmic perspective. This is what initially led to the desire to obtain a nonlinear Rayleigh quotient inequality rather than to merely bound the nonlinear spectral gap, though (in hindsight) such inequalities are interesting from the mathematical perspective as well. We did not formally define what we mean by a "nonlinear Rayleigh quotient inequality" because there is some flexibility here, but the basic desire is, given $x_1, \dots, x_n \in X$, to bound their Rayleigh quotient in X by a Rayleigh quotient of points in a *Euclidean space*.

The inequality (97) of Lemma 55 is of the above form, because the parameter $\dagger(x;\mathsf{A})$ defined by (96) involves only examining a certain Rayleigh quotient in a Hilbert space. It should be noted, however, that to date we have not succeeded to use the specific nonlinear Rayleigh quotient inequality of Lemma 55 for algorithmic purposes (though with more work this may be possible), despite the fact that it was found with this motivation in mind.

Other nonlinear Rayleigh quotient inequalities were obtained in Andoni, Naor, Nikolov, Razenshteyn, and Waingarten [2018a,b] and used to address issues of algorithmic efficiency. Very roughly, the drawback of (97) is that the matrix A is changed in the Hilbertian Rayleigh quotient of (96) (the main problem is the potentially high power $2\dagger$). A more directly algorithmically-useful nonlinear Rayleigh quotient inequality would be to change the point $x \in L_2(\pi; X)$ but not change the matrix A. Namely, suppose that we could control $\mathcal{R}(x;\mathsf{A},\|\cdot\|_X^2)$ from below by a function of $\mathcal{R}(\phi_{\mathsf{A}}(x);\mathsf{A},\|\cdot\|_H^2)$, for some mapping $\phi_{\mathsf{A}} : L_2(\pi; X) \to L_2(\pi; H)$. Nonlinear Rayleigh quotient inequalities of this type are proved in Andoni, Naor, Nikolov, Razenshteyn, and Waingarten [2018a,b], though the associated mappings ϕ_{A} turn out to be highly nonlinear and quite complicated.[9]

The upshot of the latter type of nonlinear Rayleigh quotient inequality is that if (due to existence of a faithful embedding into X) we know that $\mathcal{R}(x;\mathsf{A},\|\cdot\|_X^2)$ is small, then it follows that also $\mathcal{R}(\phi_{\mathsf{A}}(x);\mathsf{A},\|\cdot\|_H^2)$ is small. The *proof of* Cheeger's inequality (via examination of level sets of the second eigenvector) would now provide a sparse "spectral partition" of $\{1,\ldots,n\}$ that has the following auxiliary structure: The partition is determined by thresholding one of the coordinates of H (in some fixed orthonormal basis), namely the part to which each $i \in \{1,\ldots,n\}$ belongs depends only on whether the coordinate in question of the transformed vector $\phi_{\mathsf{A}}(x)_i \in H$ is above or below a certain value. If in addition $(\mathsf{A},x) \mapsto \phi_{\mathsf{A}}(x)$ has favorable computational properties (see Andoni, Naor, Nikolov, Razenshteyn, and Waingarten [2018a] for a formulation; roughly, what is important here is that after a "preprocessing step" one can decide quickly to which piece of the partition each $i \in \{1,\ldots,n\}$ belongs), then this would lead to fast "query time."

The above description of the algorithmic role of nonlinear Rayleigh quotient inequalities is impressionistic, but it conveys the core ideas while not delving into (substantial) details. Such inequalities are interesting in their own right, partially because they necessitate making mathematical arguments constructive, thus leading to new proofs, as we did for (71), and also leading to intrinsically meaningful studies, such as obtaining Andoni, Naor, Nikolov, Razenshteyn, and Waingarten [2018b] algorithmic versions of existential statements that arise from the use of the maximum principle in complex interpolation.

[9]Specifically, in Andoni, Naor, Nikolov, Razenshteyn, and Waingarten [2018a] such a mapping ϕ_{A} is constructed for Schatten-von Neumann trace classes using the Brouwer fixed-point theorem and estimates from Ricard [2015]. In Andoni, Naor, Nikolov, Razenshteyn, and Waingarten [2018b], ϕ_{A} is constructed for general normed spaces using, in addition to Brouwer's theorem, convex programming and (algorithmic variants of) complex interpolation. These lead to data structures that are efficient in all respects other than the "preprocessing stage," which at present remains potentially time-consuming due to the complexity of ϕ_{A}.

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ON THE CROSSROADS OF ENUMERATIVE GEOMETRY AND GEOMETRIC REPRESENTATION THEORY

Andrei Okounkov

Abstract

The subjects in the title are interwoven in many different and very deep ways. I recently wrote several expository accounts that reflect a certain range of developments, but even in their totality they cannot be taken as a comprehensive survey. In the format of a 30-page contribution aimed at a general mathematical audience, I have decided to illustrate some of the basic ideas in one very interesting example – that of $Hilb(\mathbb{C}^2, n)$, hoping to spark the curiosity of colleagues in those numerous fields of study where one should expect applications.

1 The Hilbert scheme of points in $\mathbb{C}^2$

1.1 Classical geometry. distinct points in the plane $\mathbb{C}^2$ is uniquely specified by the corresponding ideal

$$I_P = \{f(p_1) = \cdots = f(p_n) = 0\} \subset \mathbb{C}[x_1, x_2]$$

in the coordinate ring $\mathbb{C}^2$. The codimension of this ideal, i.e. the dimension of the quotient

$$\mathbb{C}[x_1, x_2]/I_P = \text{functions on } P \stackrel{\text{def}}{=} \mathcal{O}_P\,,$$

clearly equals n.

If the points $\{p_i\}$ merge, the limit of I_P stores more information than just the location of points. For instance, for two points, it remembers the direction along which they came together. One defines

$$\mathrm{Hilb}(\mathbb{C}^2, n) = \{\text{ideals } I \subset \mathbb{C}[x_1, x_2] \text{ of codimension } n\}\,,$$

and with its natural scheme structure Fantechi, Göttsche, Illusie, Kleiman, Nitsure, and Vistoli [2005] and Kollár [1996] this turns out to be a smooth irreducible algebraic variety — a special feature of the Hilbert schemes of surfaces that fails very badly in higher dimensions.

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The map $I_P \mapsto P$ extends to a natural map

$$\pi_{\mathrm{Hilb}} : \mathrm{Hilb}(\mathbb{C}^2, n) \to (\mathbb{C}^2)^n / S(n) \tag{1}$$

which is proper and birational, in other words, a resolution of singularities of $(\mathbb{C}^2)^n/S(n)$. This makes $\mathrm{Hilb}(\mathbb{C}^2, n)$ an instance of an *equivariant symplectic resolution* — a very special class of algebraic varieties Kaledin [2009] and Beauville [2000] that plays a central role in the current development of both enumerative geometry and geometric representation theory. This general notion axiomatizes two key features of π_{Hilb}:

- the source of π is an algebraic symplectic variety (here, with the symplectic form induced from that of $\mathbb{C}^2$),
- the map is equivariant for an action of a torus T that contracts the target to a point (here, T are the diagonal matrices in $GL(2)$ and the special point is the origin in $(\mathbb{C}^2)^n$).

Both enumerative geometry and geometric representation theory really work with algebraic varieties X and correspondences, that is, cycles (or sheaves, etc.) in $X_1 \times X_2$, considered up to a certain equivalence. These one can compose geometrically and they form a nonlinear analog of matrices of linear algebra and classical representation theory. To get to vector spaces and matrices, one considers functors like the equivariant cohomology $H^\bullet_{\mathsf{T}}(X)$, the equivariant K-theory etc., with the induced action of the correspondences.

Working with equivariant cohomology whenever there is a torus action available is highly recommended, in particular, because:

- equivariant cohomology is in many ways simpler than the ordinary, while also more general. E.g. the spectrum of the ring $H^\bullet_{\mathsf{T}}(\mathrm{Hilb}(\mathbb{C}^2, n))$ is a union of explicit essentially linear subvarieties over all partitions of n.
- the base ring $H^\bullet_{\mathsf{T}}(\mathrm{pt}, \mathbb{Q}) = \mathbb{Q}[\mathrm{Lie}\,\mathsf{T}]$ of equivariant cohomology introduces parameters in the theory, on which everything depends in a very rich and informative way,
- equivariance is a way to trade global geometry for local parameters. For instance, all formulas in the classical (that is, not quantum) geometry of Hilbert schemes of points generalize Ellingsrud, Göttsche, and Lehn [2001] to the general surface $\mathcal{S}$ with the substitution

$$c_1(\mathcal{S}) = t_1 + t_2\,, \quad c_2(\mathcal{S}) = t_1 t_2\,,$$

  where $\mathrm{diag}(t_1, t_2) \in \mathrm{Lie}\,\mathsf{T}$.

The fundamental correspondence between Hilbert schemes of points is the Nakajima correspondence

$$\alpha_{-k}(\gamma) \subset \mathrm{Hilb}(\mathcal{S}, n+k) \times \mathrm{Hilb}(\mathcal{S}, n)$$

formed for $k > 0$ by pairs of ideals (I_1, I_2) such that $I_1 \subset I_2$ and the quotient I_2/I_1 is supported at a single point located along a cycle γ in the surface $\mathcal{S}$. This cycle is

Lagrangian if $\gamma \subset \mathcal{S}$ is an algebraic curve. For $\mathcal{S} = \mathbb{C}^2$, there are very few T-equivariant choices for γ and they are all proportional, e.g.

$$[\{x_2 = 0\}] = t_2[\mathbb{C}^2], \quad [0] = t_1 t_2[\mathbb{C}^2]\,.$$

One defines $\alpha_k(\gamma)$ for $k > 0$ as $(-1)^k$ times the transposed correspondence (see section 3.1.3 in Maulik and Okounkov [2012] on the author's preferred way to deal with signs in the subject).

The fundamental result of Nakajima [1997] is the following commutation relation[1]

$$[\alpha_n(\gamma), \alpha_m(\gamma')] = -(\gamma, \gamma')\, n\, \delta_{n+m}\,, \tag{2}$$

where the intersection pairing (γ, γ') for $\mathcal{S} = \mathbb{C}^2$ is $([0], [\mathbb{C}^2]) = 1$. One recognizes in (2) the commutation relation for the Heisenberg Lie algebra $\widehat{\mathfrak{gl}}(1)$ — a central extension of the commutative Lie algebra of Laurent polynomials with values in $\mathfrak{gl}(1)$. The representation theory of this Lie algebra is very simple, yet very constraining, and one deduces the identification

$$\bigoplus_{n \geq 0} H_{\mathsf{T}}^{\bullet}(\mathrm{Hilb}(\mathcal{S}, n)) \cong \mathsf{S}^{\bullet}\,(\text{span of } \{\alpha_{-k}(\gamma)\}_{k>0}) \tag{3}$$

with the Fock module generated by the vacuum $H_{\mathsf{T}}^{\bullet}(\mathrm{Hilb}(\mathcal{S}, 0)) = H_{\mathsf{T}}^{\bullet}(\mathrm{pt})$.

Fock spaces (equivalently, symmetric functions) are everywhere in mathematics and mathematical physics and many remarkable computations and phenomena are naturally expressed in this language. My firm belief is that geometric construction, in particular the DT theory of 3-folds to be discussed below, are the best known way to think about them.

The identification (3) is a good example to illustrate the general idea that the best way to understand an algebraic variety X and, in particular, its equivariant cohomology $H_{\mathsf{T}}^{\bullet}(X)$, is to construct interesting correspondences acting on it.

For a general symplectic resolution

$$\pi : X \to X_0 \tag{4}$$

the irreducible components of the *Steinberg* variety $X \times_{X_0} X$ give important correspondences. For $X = \mathrm{Hilb}(\mathbb{C}^2, n)$, these will be quadratic in Nakajima correspondences, and hence not as fundamental. In Section 2 we will see one general mechanism into which $\alpha_k(\gamma)$ fit.

Since $\widehat{\mathfrak{gl}}(1)$ acts irreducibly in (3), it is natural to express all other geometrically defined operators in terms of $\alpha_k(\gamma)$. Of special importance in what follows will be the operator of cup product (and also of the quantum product) by Chern classes of the tautological bundle $\mathsf{Taut} = \mathbb{C}[x_1, x_2]/I$ over the Hilbert scheme.

[1] in which the commutator is the supercommutator for odd-dimensional cycles γ, γ'. Similarly, the symmetric algebra in (3) is taken in the $\mathbb{Z}/2$-graded sense.

The operator of multiplication by the divisor $c_1(\mathsf{Taut})$ was computed by Lehn [1999] as follows

$$c_1(\mathsf{Taut})\cup = -\tfrac{1}{2}\sum_{n,m>0}\big(\alpha_{-n}\,\alpha_{-m}\,\alpha_{n+m}+\alpha_{-n-m}\,\alpha_n\,\alpha_m\big)$$
$$+\,(t_1+t_2)\sum_{n>0}\frac{n-1}{2}\,\alpha_{-n}\,\alpha_n \tag{5}$$

with the following convention[2] about the arguments of the α's. If, say, we have 3 alphas, and hence need 3 arguments, we take the Künneth decomposition of

$$[\text{small diagonal}] = [0]\times[0]\times[\mathbb{C}^2]\in H^\bullet_{\mathsf{T}}((\mathbb{C}^2)^3)\,.$$

Similarly, $\alpha_{-n}\,\alpha_n$ is short for $\alpha_{-n}([0])\,\alpha_n([\mathbb{C}^2])$. With this convention, (5) is clearly an operator of cohomological degree 2. There is a systematic way to prove formulas like (5) in the framework of Section 2, see Smirnov [2016a].

A remarkable observation, made independently by several people, is that the operator (5) is identical to the second-quantized trigonometric Calogero–Sutherland operator — an classical object in many-body systems and symmetric functions (its eigenfunctions being the Jack symmetric polynomials), see e.g. Costello and Grojnowski [2003] for a comprehensive discussion.

The quantum CS Hamiltonian[3]

$$H_{CS} = \tfrac{1}{2}\sum_{i=1}^{N}\left(w_i\frac{\partial}{\partial w_i}\right)^2+\theta(\theta-1)\sum_{i<j\le N}\frac{1}{|w_i-w_j|^2}\,,\qquad \theta=-\big(t_1/t_2\big)^{\pm 1}\,, \tag{6}$$

describes a system of N identical particles interacting with $|w|^{-2}$-potential on the unit circle $|w|=1$. After conjugation by an eigenfunction $\prod_{i<j}(w_i-w_j)^\theta$, it preserves symmetric Laurent polynomials in w_i and stabilizes as $N\to\infty$ to a limit in which the left-movers and right-movers (that is, symmetric polynomials in w_i and polynomials in w_i^{-1}) decouple. This limit is (5) with α_{-k} proportional to multiplication by $\sum w_i^k$.

Another interpretation of the same equation (5) is an integrable quantum version of the Benjamin–Ono equation of 1-dimensional hydrodynamics, see in particular Abanov and Wiegmann [2005]. The BO equation describes waves on a 1-dimensional surface of a fluid of infinite depth, and it involves the Hilbert transform — a nonlocal operation. This nonlocal operation is precisely responsible for the term $\sum_{n>0} n\,\alpha_{-n}\,\alpha_n$ which is present in (5) and looks a bit unconventional when expressed in terms of the field $\boldsymbol{\alpha}(\zeta)=\sum\alpha_{-n}\zeta^n$. Note that other terms in (5) are the normally ordered constant terms in $\boldsymbol{\alpha}(\zeta)^3$ and $\boldsymbol{\alpha}(\zeta)^2$, respectively.

These by now classical connections are only a preview of the kind of connections that exists between enumerative problems and quantum integrable systems in the full unfolding of the theory. Crucial insights into this connection were made in the pioneering work of N. A. Nekrasov and Shatashvili [2009, 2010].

[2]Note that it differs by a sign from the convention used in Maulik and Okounkov [2012].

[3]Note that the well-known, but still remarkable strong/weak duality $\theta\mapsto 1/\theta$ in the CS model becomes simply the permutation of the coordinates in $\mathbb{C}^2$ in the geometric interpretation.

1.2 Counting curves in $\mathrm{Hilb}(\mathbb{C}^2, n)$**.** Enumerative geometry of curves in an algebraic variety Y is a very old subject in mathematics, with the counts like the 27 lines on a smooth cubic surface going as far back as the work of Cayley from 1849. While superficially the subject may be likened to counting points of Y over some field , the actual framework that the geometers have to construct to do the counts looks very different from the number-theoretic constructions. In particular, the counts are defined treating curves in Y as an excess intersection problem, with the result that the counts are invariant under deformation even though there may be no way to deform actual curves.

Also, the subject draws a lot of inspiration from mathematical physics, where various curve counts are interpreted as counts (more precisely, indices) of supersymmetric states in certain gauge or string theories. This leaves a very visible imprint on the field, ranging from how one organizes the enumerative data to what is viewed as an important goal/result in the subject. In particular, any given count, unless it is something as beautiful as 27 lines, is viewed as only an intermediate step in the quest to uncover universal structures that govern a certain totality of the counts.

Depending on what is meant by a "curve" in Y, enumerative theories come in several distinct flavors, with sometimes highly nontrivial interrelations between them. My personal favorite among them is the Donaldson–Thomas theory of 3-folds Donaldson and Thomas [1996] and Thomas [2000], see, in particular, Okounkov [2017c] for a recent set of lecture notes.

The DT theory views a curve $C \subset Y$ as something defined by equations in Y, that is, as a subsheaf $\mathcal{I}_C \subset \mathcal{O}_Y$ of regular functions on Y formed by those functions that vanish on C. The DT moduli space for Y is thus

$$\mathrm{Hilb}(Y, \text{curves}) = \bigsqcup_{\beta, n} \mathrm{Hilb}(Y, \beta, n) \tag{7}$$

where $\beta \in H_2(Y, \mathbb{Z})_{\mathrm{eff}}$ is the degree of the curve and $n = \chi(\mathcal{O}_C)$ is the holomorphic Euler characteristic of $\mathcal{O}_C = \mathcal{O}_Y / \mathcal{I}_C$. In particular, if C is a smooth connected curve of genus g then $n = 1 - g$.

While similar in construction and universal properties to the Hilbert schemes of point in surfaces, Hilbert schemes of 3-folds are, in general, highly singular varieties and nearly nothing is known about their dimension or irreducible components. However, viewed as moduli of sheaves of on Y of the form $\{\mathcal{I}_C\}$, they have a good deformation theory and thus a virtual fundamental cycle of (complex) dimension $\beta \cdot c_1(Y)$ Thomas [2000]. The DT curve counts are defined by pairing this virtual cycle against natural cohomology classes, such as those pulled back from $\mathrm{Hilb}(D, \text{points})$ via a map that assigns to C its pattern of tangency to a fixed smooth divisor $D \subset Y$, see Figure 1 and the discussion in Section 2 of Okounkov [2017c]. Note that the divisor $D = \bigsqcup D_i$ may be disconnected, as in Figure 1, in which case curve counts really define a tensor in a tensor product of several Fock spaces, e.g. an operator from one Fock space to another if there are two components. I believe that this is the geometric source of great many, if not all, interesting tensors in Fock spaces.

There is a broader world of DT counts, in which one counts other 1-dimensional sheaves on 3-folds (a very important example being the Pandharipande–Thomas moduli spaces of stable pairs Pandharipande and Thomas [2009]), and even more broadly stable

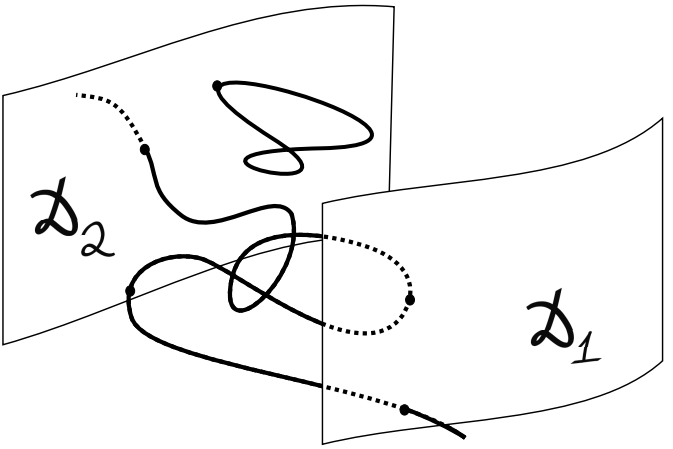

Figure 1: A fundamental object to count in DT theory are algebraic curves, or more precisely, subschemes $C \subset Y$ of given (β, χ) constrained by how they meet a fixed divisor $D \subset Y$.

objects in other categories that look like coherent sheaves on a smooth 3-fold. For all of these, the deformation theory has certain self-duality features that makes K-theoretic and otherwise refined enumerative information a well-behaved and a very interesting object to study.

Since the virtual dimension does not depend on n, it is convenient to organize the DT counts by summing over all n with a weight z^n, where z is a new variable. These are conjectured to be rational functions of z with poles at roots of unity Maulik, N. Nekrasov, Okounkov, and Pandharipande [2006a,b]. This is known in many important cases Maulik, Oblomkov, Okounkov, and Pandharipande [2011], Pandharipande and Pixton [2013], Smirnov [2016b], and Toda [2010] and may be put into a larger conjectural framework as in N. Nekrasov and Okounkov [2016], see Section 3.1.

An algebraic analog of cutting Y into pieces is a degeneration of Y to a transverse union of Y_1 and Y_2 along a smooth divisor D_0 as in Figure 2. A powerful result of Levine and Pandharipande [2009] shows that any smooth projective 3-fold can be linked to a product of projective spaces (or any other basis in algebraic cobordism) by a sequence of such moves. The DT counts satisfy a certain gluing formula for such degeneration Li and Wu [2015] in which the divisor D_0 is added to the divisors from Figure 1. This highlights the importance of understanding the DT counts in certain basic geometries which can serve as building blocks for arbitrary 3-folds.

One set of basic geometries is formed by $\mathcal{S}$-bundles over a curve B

$$\begin{tikzcd} \mathcal{S} \arrow[r, hook] & Y \arrow[d, "g"'] \\ & B\,, \end{tikzcd} \tag{8}$$

where $\mathcal{S}$ is a smooth surface[4]. One take $D = \bigcup g^{-1}(b_i)$ for $\{b_i\} \subset B$ and, by degeneration, this defines a TQFT on B with the space of states $H_{\mathsf{T}}^{\bullet}(\mathrm{Hilb}(\mathcal{S}))$, where $\mathsf{T} \subset \mathrm{Aut}(\mathcal{S})$ is a maximal torus. This TQFT structure is captured by the counts for $B = \mathbb{P}^1 \supset \{b_1, b_2, b_3\}$, which define a new, z- and β-dependent supercommutative multiplication in this Fock space. It is a very interesting question to describe this multiplication explicitly[5].

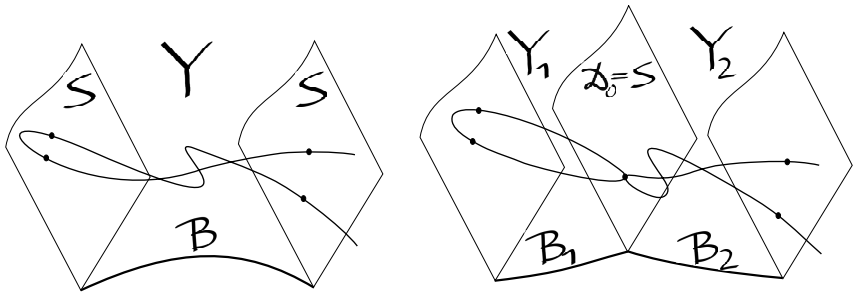

Figure 2: Basic building blocks of DT theory are $\mathcal{S}$ bundles over a curve B as in (8). As B degenerates to a nodal curve $B_1 \cup B_2$, Y degenerates to a transverse union of Y_1 and Y_2 along a smooth divisor $D_0 \cong \mathcal{S}$. DT counts satisfy a gluing formula for such degenerations.

Geometric representation theory provides an answer when $\mathcal{S}$ itself is a symplectic resolution, which concretely means an ADE surface — a minimal resolution of the corresponding surface singularity $\mathcal{S}_0$. In this case, $\mathrm{Hilb}(\mathcal{S}, n)$ is a symplectic resolution of $(\mathcal{S}_0)^n/S(n)$, just like (1), and the multiplication above is nothing but the quantum multiplication in $H_{\mathsf{T}}^{\bullet}(\mathrm{Hilb}(\mathcal{S}))$, see Maulik and Oblomkov [2009] and Okounkov and Pandharipande [2010].

Recall that the quantum product $\star$ is a supercommutative, associative deformation of the classical $\cup$-product in $H_{\mathsf{T}}^{\bullet}(X)$ whose structure constants are the counts of 3-pointed rational curves in X

$$(\alpha \star \beta, \gamma) = \sum_{d \in H_2(X,\mathbb{Z})_{\mathrm{eff}}} z^d \quad \begin{array}{c} \#\text{ of degree } d \text{ rational curves} \\ \text{meeting cycles dual to } \alpha, \beta, \gamma\,. \end{array} \tag{9}$$

This is made mathematically precise using the correspondence

$$\sum_d z^d \, \mathrm{ev}_* \left(\left[\overline{\mathcal{M}}_{0,3}(X, d) \right]_{\mathrm{vir}} \right) \in H_\bullet(X \times X \times X)[[z]] \tag{10}$$

obtained from the virtual fundamental cycle of the moduli space of 3-pointed stable rational maps to X, see Hori, Katz, Klemm, Pandharipande, Thomas, Vafa, Vakil, and Zaslow [2003] for an introduction.

[4]It suffices to take $\mathcal{S} \in \{A_0 = \mathbb{C}^2, A_1, A_2\}$, where A_n is the minimal resolution of the corresponding surface singularity, to generate a basic set of counts.

[5]In particular, there are very interesting results and conjectures for K3 surface fibrations, see Oberdieck [2018]. Note that the dimension counts work out best when $c_1(\mathcal{S}) = 0$.

A closely related structure is the *quantum differential equation*, or Dubrovin connection ∇_X, which is a flat connection on trivial bundle over $H^2(X,\mathbb{C}) \ni \lambda$ with fiber $H_{\mathsf{T}}^{\bullet}(X,\mathbb{C})$. Its flat sections satisfy

$$\frac{d}{d\lambda}\Psi(z) = \lambda \star \Psi(z)\,, \qquad \frac{d}{d\lambda} z^d = (\lambda, d)\, z^d\,, \tag{11}$$

and contain very important enumerative information. For $X = \mathrm{Hilb}(\mathbb{C}^2)$, the quantum multiplication ring is generated by the divisor, so the two structures are really the same.

With the $z \mapsto -z$ substitution, the $\star$-deformation of (5) for $X = \mathrm{Hilb}(\mathbb{C}^2)$ was computed in Okounkov and Pandharipande [2010] as follows

$$c_1(\mathsf{Taut})_\star = c_1(\mathsf{Taut}) \cup +(t_1+t_2) \sum_{d>0} d\, \frac{z^d}{1-z^d}\, \alpha_{-d}\, \alpha_d + \dots\,, \tag{12}$$

with the same convention about the arguments of α_n as in (5) and with dots denoting a scalar operator of no importance for us now. Note the simplicity of the purely quantum terms[6].

The largest and the richest class of equivariant symplectic resolutions known to date is formed by the Nakajima quiver varieties Nakajima [1994], of which $\mathrm{Hilb}(\mathbb{C}^2)$ is an example. Formula (12) illustrates many general features of the quantum cohomology of Nakajima quiver varieties proven in Maulik and Okounkov [2012], such as:

- the purely quantum terms are given by a rational function with values in Steinberg correspondences

  $$\text{purely quantum} \in \hbar\, H_{\mathrm{top}}(X \times_{X_0} X) \otimes \mathbb{Q}(z)\,, \tag{13}$$

  where

  $$\hbar = -(t_1+t_2)$$

  is the equivariant weight of the symplectic form.

- the shift $z \mapsto -z$ is an example of the shift by a canonical element of $H^2(X,\mathbb{Z}/2)$, called the theta-characteristic in Maulik and Okounkov [ibid.].

- there is a certain Lie algebra $\mathfrak{g}_Q$ associated to an arbitrary quiver Q in Maulik and Okounkov [ibid.], whose positive roots are represented by effective curve classes $d \in H_2(X,\mathbb{Z})$. Among these, there is a finite set of *Kähler roots* of X such that

  $$\lambda\star = \lambda \cup -\hbar \sum_{d\in\{\text{positive roots}\}} (\lambda, d)\, \frac{z^d}{1-z^d}\, C_d\,, \tag{14}$$

  where $C_d \in \mathfrak{g}_{-d}\, \mathfrak{g}_d$ is the corresponding root components of the Casimir element, that is, the image of the invariant bilinear form on $\mathfrak{g}_{-d} \otimes \mathfrak{g}_d$. For $X = \mathrm{Hilb}(\mathbb{C}^2, n)$,

[6] If one interprets (5) as a quantum version of the Benjamin–Ono equation then the new terms deform it to a quantization of the intermediate long wave (ILW) equation. This observation has been rediscovered by many authors.

the quiver is the quiver with one vertex and one loop with $\mathfrak{g}_Q = \widehat{\mathfrak{gl}}(1)$, $C_d = \alpha_{-d}\,\alpha_d$, and

$$\text{positive Kähler roots} = \{1, \dots, n\} \subset \mathbb{Z}\,. \tag{15}$$

- the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}_Q$ acts by central elements and by the ranks of the tautological bundles. The operator of cup products by other characteristic classes of the tautological bundles, together with $\mathfrak{g}$, generate a Hopf algebra deformation $\mathcal{Y}(\mathfrak{g}_Q)$ of $\mathcal{U}(\mathfrak{g}_Q[t])$ known as the Yangian.

  The operators of quantum multiplication form a remarkable family of maximal commutative subalgebras of $\mathcal{Y}(\mathfrak{g}_Q)$ known as the Baxter subalgebras in the theory of quantum integrable systems, see Section 2.1. They are parametrized by z and, as $z \to 0$, they become the algebra $\mathcal{Y}(\mathfrak{h}) \subset \mathcal{Y}(\mathfrak{g}_Q)$ of cup products by tautological classes.

  The identification between the $\star$-product ring and Baxter's quantum integrals of motion was predicted by Nekrasov and Shatashvili based on their computation of the spectra of the operators. This served as very important inspiration for Maulik and Okounkov [ibid.].

- The Yangian description identifies the quantum differential equation with the *Casimir connection* for the Lie algebra $\mathfrak{g}_Q$, as studied (in the finite-dimensional case) in Toledano Laredo [2011]. This fits very nicely with the conjecture of Bezrukavnikov and collaborators about the monodromy ∇_X, see below, and was another important inspiration for Maulik and Okounkov [2012], see the historical notes there.

For general symplectic resolutions, there is a definite gap between what is known abstractly, and what can be seen in known examples. I expect that a complete classification of the equivariant symplectic resolutions is within the reach of the current generation of algebraic geometers, and we will see how representative the known examples are. For general symplectic resolutions, the Steinberg correspondence in (13) are constructed in Braverman, Maulik, and Okounkov [2011], while the rationality in z remains abstractly a conjecture that can be checked in all known cases.

A generalization of these structures appears In enumerative K-theory. There, instead of pairing virtual cycles with cohomology classes, we compute the Euler characteristics of natural sheaves, including the virtual structure sheaf $\mathcal{O}_{\mathrm{vir}}$, on the moduli spaces in question (here, the Hilbert scheme of curves in Y).

From its very beginning, K-theory has been inseparable from the indices of differential operators and related questions in mathematical physics. Equivariant K-theoretic DT counts represent a Hamiltonian approach to supersymmetric indices in a certain physical theory (namely, the theory on a D6 brane), in which the space is Y and the time is periodic[7]. Morally, what one computes is the index of a certain infinite-dimensional Dirac operator as a representation of $\mathrm{Aut}(Y)$ which is additionally graded by (β, n).

[7] Or, more precisely, quasiperiodic, with a twist by an element of $\mathsf{T} \subset \mathrm{Aut}(Y)$ after a full circle of time.

Because this is the index of a Dirac operator, the right analog of the virtual cycle is the symmetrized virtual structure sheaf

$$\widehat{\mathcal{O}}_{\mathrm{vir}} = \mathcal{O}_{\mathrm{vir}} \otimes \mathcal{K}_{\mathrm{vir}}^{1/2} \otimes \dots \tag{16}$$

where $\mathcal{K}_{\mathrm{vir}}^{1/2}$ is a square root of the virtual canonical bundle $\mathcal{K}_{\mathrm{vir}}$, the importance of which was emphasized by N. Nekrasov [2005], and the existence of which is shown in N. Nekrasov and Okounkov [2016]. The dots in (16) denote a certain further twist by a tautological line bundle of lesser importance, see N. Nekrasov and Okounkov [ibid.].

While it is not uncommon for different moduli spaces to give the same or equivalent cohomological counts, the K-theoretic counts really feel every point in the moduli space and are very sensitive to the exact enumerative setup. In particular, for both the existence of (16) and the computations with this sheaf, certain self-duality features of the DT deformation theory are crucial. It remains to be seen whether computations with moduli spaces like $\overline{\mathcal{M}}_{0,3}(X)$, that lack such self-duality, can really reproduce the K-theoretic DT counts.

In K-theory, the best setup for counting curves in $X = \mathrm{Hilb}(\mathbb{C}^2, n)$ is the moduli space of stable *quasimaps* to X, see Ciocan-Fontanine, Kim, and Maulik [2014]. Recall that

$$X = \left\{x_1, x_2 \in \mathrm{End}(\mathbb{C}^n), v \in \mathbb{C}^n \middle| [x_1, x_2] = 0\right\} /\!\!/ GL(n)\,,$$

where the stability condition in the GIT quotient is equivalent to $\mathbb{C}[x_1, x_2]v$ spanning $\mathbb{C}^n$. By definition Ciocan-Fontanine, Kim, and Maulik [ibid.], a stable quasimap from B to a GIT quotient is a map to the quotient stack[8] that evaluates to a stable point away from a finite set of point in B. In return for allowing such singularities, quasimaps offer many technical advantages.

If Y in (8) is a fibration in $\mathcal{S} = \mathbb{C}^2$, one can consider quasimap sections of the corresponding X-bundle over B, and these are easily seen to be identical to the Pandharipande–Thomas stable pairs for Y. Recall that by definition Pandharipande and Thomas [2009], a stable pair is a complex of the form

$$\mathcal{O}_Y \xrightarrow{s} \mathcal{F}$$

where $\mathcal{F}$ is a pure 1-dimensional sheaf and $\dim \mathrm{Coker}\, s = 0$. For our Y, $g_*\mathcal{F}$ is a vector bundle on B, the section s gives v, while x_1, x_2 come from multiplication by the coordinates in the fiber. If the fiber $\mathcal{S}$ contains curve, the picture becomes modified and the PT spaces for the A_n-fibrations, $n > 0$, are related to quasimaps via a certain sequence of wall crossings.

The K-theoretic quasimap counts to $\mathrm{Hilb}(\mathbb{C}^2)$ and, in fact, to all Nakajima varieties have been computed in Okounkov [2017a] and Okounkov and Smirnov [2016], and their structure is a certain q-difference deformation of what we have seen for the cohomological counts. In particular, the Yangian $\mathcal{Y}(\mathfrak{g})$ is replaced by a quantum loop algebra $\mathcal{U}_\hbar(\widehat{\mathfrak{g}})$ formed by K-theoretic analogs of the correspondences that define the action of $\mathcal{Y}(\mathfrak{g})$, see Sections 2.2 and 3.1.

[8] i.e. a principal G-bundle on B together with a section of the associated bundle of prequotients, where G is the group by which we quotient. For $G = GL(n)$, a principal G-bundle is the same as a vector bundle on B of rank n.

2 Geometric actions of quantum groups

Geometric representation theory in the sense of making interesting algebras act by correspondences is a mature subject and its exposition in Chriss and Ginzburg [2010] is a classic. Geometric construction of representation of quantum groups has been a very important stimulus in the development of the theory of Nakajima quiver varieties Nakajima [1998, 2001].

Below we discuss a complementary approach of Maulik and Okounkov [2012], which mixes geometry and algebra in a different proportion. It has certain convenient hands-off features, in the sense that it constructs a certain category of representations without, for example, a complete description of the algebra by generators and relations. In algebraic geometry, one certainly prefers having a handle on the category $\mathrm{Coh}(X)$ to a complete list of equation that cut out X inside some ambient variety, so the construction should be of some appeal to algebraic geometers. It also interact very nicely with the enumerative question, as it has certain basic compatibilities built in by design.

2.1 Braiding. If G is a group then the category of G-modules over a field has a tensor product — the usual tensor product $M_1 \otimes M_2$ of vector spaces in which an element $g \in G$ acts by $g \otimes g$. There is also a trivial representation $g \mapsto 1 \in \mathrm{End}()$, which is the identity for $\otimes$. This reflects the existence of a coproduct, that is, of an algebra homomorphism

$$G \ni g \xrightarrow{\Delta} g \otimes g \in G \otimes G\,, \tag{17}$$

where G is the group algebra of G, with the counit

$$G \ni g \xrightarrow{\varepsilon} 1 \in \ .$$

There are also dual module $M^* = \mathrm{Hom}(\ M,\ )$ in which g acts by $(g^{-1})^T$, reflecting the antiautomorphism

$$G \ni g \xrightarrow{\text{antipode}} g^{-1} \in G\,.$$

Just like the inverse in the group, the antipode is unique if it exist, and so it will be outside of our focus in what follows, see Etingof and Schiffmann [2002].

An infinitesimal version of this for a Lie or algebraic group G is to replace G by the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$, $\mathfrak{g} = \mathrm{Lie}\, G$, with the coproduct obtained from (17) by Leibniz rule

$$\Delta(\xi) = \xi \otimes 1 + 1 \otimes \xi\,, \quad \xi \in \mathfrak{g}\,.$$

The multiplication, comultiplication, unit, counit, and the antipode form an beautiful algebraic structure known as a Hopf algebra, see e.g. Etingof and Schiffmann [ibid.]. Another classical example is the algebra $[G]$ of regular functions on an algebraic group. Remarkably, the axioms of a Hopf algebra are self-dual under taking duals and reversing all arrows. Observe that all of the above examples are either commutative, or cocommutative.

Broadly, a quantum group is a deformation of the above examples in the class of Hopf algebras. Our main interests is in Yangians and quantum loop algebras that are deformations of

$$\mathcal{U}(\mathfrak{g}[t]) \rightsquigarrow \mathcal{Y}(\mathfrak{g}), \quad \mathcal{U}(\mathfrak{g}[t^{\pm 1}]) \rightsquigarrow \mathcal{U}_\hbar(\widehat{\mathfrak{g}}), \tag{18}$$

respectively. Their main feature is the loss of cocommutativity. In other words, the order of tensor factors now matters and

$$M_1 \otimes M_2 \ncong M_2 \otimes M_1,$$

in general, or at least the permutation of the tensor factors is no longer an intertwining operator. While tensor categories are very familiar to algebraic geometers, this may be an unfamiliar feature. But, as the representation-theorists know, a mild noncommutativity of the tensor product makes the theory richer and more constrained.

The Lie algebras $\mathfrak{g}[t]$ and $\mathfrak{g}[t^{\pm 1}]$ in (18) are $\mathfrak{g}$-valued functions on the additive, respectively multiplicative, group of the field and they have natural automorphisms

$$t \mapsto t + a, \quad \text{resp.} \quad t \mapsto at, \quad a \in \mathbb{G},$$

where we use $\mathbb{G}$ as a generic symbol for either an additive or multiplicative group. The action of $\mathbb{G}$ will deform to an automorphism of the quantum group[9] and we denote by $M(a)$ the module M, with the action precomposed by an automorphism from $\mathbb{G}$.

The main feature of the theory is the existence of intertwiner (known as the braiding, or the R-matrix)

$$R^\vee(a_1 - a_2) : M_1(a_1) \otimes M_2(a_2) \to M_2(a_2) \otimes M_1(a_1) \tag{19}$$

which is invertible as a rational function of $a_1 - a_2 \in \mathbb{G}$ and develops a kernel and cokernel for those values of the parameters where the two tensor products are really not isomorphic. One often works with the operator $R = (12) \circ R^\vee$ that intertwines two different actions on the same vector space

$$R(a_1 - a_2) : M_1(a_1) \otimes M_2(a_2) \to M_1(a_1) \otimes_{\text{opp}} M_2(a_2). \tag{20}$$

As the word *braiding* suggest, there is a constraint on the R-matrices coming from two different ways to put three tensor factors in the opposite order. In our situation, the corresponding products of intertwiners will be simply be equal, corresponding to the Yang–Baxter equation

$$\begin{aligned} R_{M_1,M_2}(a_1 - a_2)\, R_{M_1,M_3}(a_1 - a_3)\, R_{M_2,M_3}(a_2 - a_3) &= \\ = R_{M_2,M_3}(a_2 - a_3)\, R_{M_1,M_3}(a_1 - a_3)\, R_{M_1,M_2}(a_1 - a_2), \end{aligned} \tag{21}$$

satisfied by the R-matrices.

There exists general reconstruction theorem that describe tensor categories of a certain shape and equipped with a fiber functor to vector spaces as representation categories

[9] In fact, for $\mathcal{U}_\hbar(\widehat{\mathfrak{g}})$, it is natural to view this *loop rotation* automorphism as part of the Cartan torus.

of quantum groups, see Etingof, Gelaki, Nikshych, and Ostrik [2015] and Etingof and Schiffmann [2002]. For practical purposes, however, one may be satisfied by the following simple-minded approach that may be traced back to the work of the Leningrad school of quantum integrable systems N. Y. Reshetikhin [1989].

Let $\{M_i\}$ be a collection of vector spaces over a field and

$$R_{M_i,M_j}(a) \in GL(M_1 \otimes M_2, (a))$$

a collection of operators satisfying the YB equation (21). From this data, one constructs a certain category $\mathcal{C}$ of representation of a Hopf algebra $\mathcal{Y}$ as follows. We first extend the R-matrices to tensor products by the rule

$$R_{M_1(a_1),M_2(a_2)\otimes M_3(a_3)} = R_{M_1,M_3}(a_1 - a_3)\, R_{M_1,M_2}(a_1 - a_2)\,, \quad \text{etc.}$$

It is clear that these also satisfy the YB equation. One further extends R-matrices to dual vector spaces using inversions and transpositions, see N. Y. Reshetikhin [ibid.]. Note that in the noncocommutative situation, one has to distinguish between left and right dual modules. For simplicity, we may assume that the set $\{M_i\}$ is already closed under duals.

The objects of the category $\mathcal{C}$ are thus $M = \bigotimes M_i(a_i)$ and we define the quantum group operators in M as the as the matrix coefficients of the R-matrices. Concretely, we have

$$\mathsf{T}_{M_0,m_0}(u) \stackrel{\text{def}}{=} \operatorname{tr}_{M_0}(m_0 \otimes 1) R_{M_0(u),M} \in \operatorname{End}(M) \otimes (u) \tag{22}$$

for any operator m_0 of finite rank in an auxiliary space $M_0 \in \operatorname{Ob}(\mathcal{C})$. The coefficients of u in (22) give us a supply of operators

$$\mathcal{Y} \subset \prod_M \operatorname{End}(M)\,.$$

The YB equation (21) can now be read in two different ways, depending on whether we designate one or two factors as auxiliary. With these two interpretations, it gives either:

- a commutation relation between the generators (22) of $\mathcal{Y}$, or
- a braiding of two $\mathcal{Y}$ modules.

Further, any morphism in $\mathcal{C}$, that is, any operator that commutes with $\mathcal{Y}$ gives us relations in $\mathcal{Y}$ when used in the auxiliary space. This is a generalization of the following classical fact: if $G \subset \prod GL(M_i)$ is a reductive algebraic group, then to know the equations of G is equivalent to knowing how $\otimes M_{k_i}$ decompose as G-modules.

The construction explained below gives geometric R-matrices, that is, geometric solutions of the YB equation acting in spaces like (3) in the Yangian situation, or in the corresponding equivariant K-theories for $\mathcal{U}_\hbar(\widehat{\mathfrak{g}})$. This gives a quantum group $\mathcal{Y}$ which is precisely of the right size for the enumerative application. With certain care, see Maulik and Okounkov [2012], the above construction works over a ring like $= H_{\mathsf{T}}^\bullet(\mathrm{pt})$.

The construction of R-matrices uses *stable envelopes*, which is a certain technical notion that continues to find applications in both in enumerative and representation-theoretic contexts.

Let z be an operator in each M_i such that

$$[z \otimes z, R_{M_1,M_2}] = 0\,.$$

A supply of such is provided by the Cartan torus $\exp(\mathfrak{h})$ where, in geometric situations, $\mathfrak{h}$ acts by the ranks of the universal bundles. A classical observation of Baxter then implies that

$$\left[\mathsf{T}_{M_0,z}(u), \mathsf{T}_{M_0',z}(u')\right] = 0$$

for fixed z and any $M_0(u)$ and $M_0'(u')$. In general, these depend rationally on the entries of z as it is not, usually, an operator of finite rank. The corresponding commutative subalgebras of $\mathcal{Y}$ are known as the Baxter, or Bethe subalgebras. They have a direct geometric interpretation as the operators of quantum multiplication, see below.

2.2 Stable envelopes. There are two ways in which a symplectic resolution $X \xrightarrow{\pi} X_0$ may break up into simpler symplectic resolutions. One of them is deformation. There is a stratification of the deformation space $\mathrm{Def}(X,\omega) = \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{C}$ by the different singularities that occur, the open stratum corresponding to smooth X_0 or, equivalently, affine X, see Kaledin [2009]. In codimension 1, one sees the simplest singularities into which X can break, and this is related to the decompositions (13) and (14), see Braverman, Maulik, and Okounkov [2011], and so to the notion of the Kähler roots of X introduced above. In the context of Section 3.2, the hyperplanes of the quantum dynamical Weyl group may be interpreted as a further refinement of this stratification that records singular *noncommutative* deformations of X_0.

A different way to break up X into simpler pieces is to consider the fixed points X^{A} of a symplectic torus $\mathsf{A} \subset \mathrm{Aut}(X,\omega)$. The first order information about the geometry of X around X^{A} is given by the A-weights in the normal bundle $N_{X/X^{\mathsf{A}}}$. These are called the equivariant roots for the action of A, or just the equivariant roots of X if A is a maximal torus in $\mathrm{Aut}(X,\omega)$.

There is a certain deep Langlands-like (partial) duality for equivariant symplectic resolutions that interchanges the roles of equivariant and Kähler variables. The origin of this duality, sometimes called the 3-dimensional mirror symmetry, is in 3-dimensional supersymmetric gauge theories, see Intriligator and Seiberg [1996] and de Boer, Hori, Ooguri, and Oz [1997] and also e.g. Bullimore, Dimofte, and Gaiotto [2017], Bullimore, Dimofte, Gaiotto, and Hilburn [2016], and Braden, Licata, Proudfoot, and Webster [2010] for a thin sample of references. As this duality interchanges K-theoretic enumerative information, the quantum difference equation for X becomes the shift operators for its dual $X^\vee$, see Okounkov [2017a] for an introduction to these notions. Among other things, the Kähler and equivariant roots control the poles in these difference equations, which makes it clear that they should be exchanges by the duality.

For the development of the theory, it is very important to be able to both break X into simpler piece and also to find X as such a piece in a more complex ambient geometry.

For instance for $X = \mathrm{Hilb}(\mathbb{C}^2, n)$, we can take the maximal torus $\mathsf{A} \subset SL(2)$, in which case X^{A} is a finite set of monomial ideals

$$I_\lambda \subset \mathbb{C}[x_1, x_2]\,, \quad |\lambda| = n\,,$$

indexed by partitions λ of the number n. At I_λ, the normal weights are $\{\pm\mathrm{hook}(\square)\}_{\square\in\lambda}$, with the result that

$$\text{equivariant roots of } \mathrm{Hilb}(\mathbb{C}^2, n) = \{\pm 1, \dots, \pm n\}\,. \tag{23}$$

Notice the parallel with (15). It just happens that $\mathrm{Hilb}(\mathbb{C}^2, n)$ is self-dual, we will see other manifestations of this below.

More importantly, products $\prod_{i=1}^r \mathrm{Hilb}(\mathbb{C}^2, n_i)$ may be realized in a very nontrivial way as fixed loci of a certain torus A on an ambient variety $\mathcal{M}(r, \sum n_i)$. Here $\mathcal{M}(r,n)$ is the moduli space of framed torsion free sheaves $\mathcal{F}$ on $\mathbb{P}^2$ of rank r and $c_2(\mathcal{F}) = n$. A framing is a choice of an isomorphism

$$\phi : \mathcal{F}\big|_L \xrightarrow{\sim} \mathcal{O}_L^{\oplus r}\,, \quad L = \mathbb{P}^2 \setminus \mathbb{C}^2\,,$$

on which the automorphism group $GL(r)$ acts by postcomposition. The spaces $\mathcal{M}(r,n)$ are the general Nakajima varieties associated to the quiver with one vertex and one loop. They play a central role in supersymmetric gauge theories as symplectic resolutions of the Uhlenbeck spaces of framed instantons, see Nakajima [1999]. It is easy to see that $\mathcal{M}(1,n) = \mathrm{Hilb}(\mathbb{C}^2, n)$ and

$$\mathcal{M}(r,n)^{\mathsf{A}} = \bigsqcup_{\sum n_i = n} \prod_{i=1}^r \mathrm{Hilb}(\mathbb{C}^2, n_i)\,,$$

where $\mathsf{A} \subset GL(r)$ is the maximal torus. For general Nakajima varieties, there is a similar decomposition for the maximal torus A of framing automorphisms.

The cohomological stable envelope is a certain Lagrangian correspondence

$$\mathrm{Stab} \subset X \times X^{\mathsf{A}}\,, \tag{24}$$

which may be seen as an improved version of the attracting manifold

$$\mathrm{Attr} = \Big\{(x,y)\,\Big|\, \lim_{a\to 0} a\cdot x = y\Big\} \subset X \times X^{\mathsf{A}}\,.$$

The support of (24) is the full attracting set $\mathrm{Attr}_f \subset X \times X^{\mathsf{A}}$, which is the smallest closed subset that contains the diagonal and is closed under taking $\mathrm{Attr}(\,\cdot\,)$.

To define attracting and repelling manifolds, we need to separate the roots for A-weights into positive and negative, that is, we need to choose a chamber $\mathfrak{C} \subset \mathrm{Lie}\,\mathsf{A}$ in the complement of the root hyperplanes. Note this gives an ordering on the set of components $\bigsqcup F_i = X^{\mathsf{A}}$ of the fixed locus: $F_1 > F_2$ if $\mathrm{Attr}_f(F_1)$ meets F_2.

For

$$\mathrm{Lie}\,\mathsf{A} = \{\mathrm{diag}(a_1, \dots, a_r)\} \subset \mathfrak{gl}(r) \tag{25}$$

as in Section 2.2, the roots are $\{a_i - a_j\}$, and so a choice of $\mathfrak{C}$ is the usual choice of a Weyl chamber. As we will see, it will correspond to an ordering of tensor factors as in Section 2.1.

The stable envelope is characterized by:

- it is supported on Attr_f,
- it equals[10] $\pm\,\mathrm{Attr}$ near the diagonal in $X^{\mathsf{A}} \times X^{\mathsf{A}} \subset X \times X^{\mathsf{A}}$,
- for an off-diagonal component $F_2 \times F_1$ of $X^{\mathsf{A}} \times X^{\mathsf{A}}$, we have

$$\deg_{\mathrm{Lie}(\mathsf{A})} \mathrm{Stab}\Big|_{F_2\times F_1} < \tfrac{1}{2}\operatorname{codim} F_2 = \deg_{\mathrm{Lie}(\mathsf{A})} \mathrm{Attr}\Big|_{F_2\times F_2}, \tag{26}$$

where the degree is the usual degree of polynomials for

$$H^\bullet_{\mathsf{A}}(X^{\mathsf{A}}, \mathbb{Z}) \cong H^\bullet(X^{\mathsf{A}}) \otimes \mathbb{Z}[\mathrm{Lie}\,\mathsf{A}].$$

Condition (26) is a way to quantify the idea that $\mathrm{Stab}\big|_{F_2\times F_1}$ is smaller than $\mathrm{Attr}(F_2)$. This makes the stable envelope a canonical representative of $\overline{\mathrm{Attr}(F_1)}$ modulo cycles supported on the lower strata of Attr_f.

The existence and uniqueness of stable envelopes are proven, under very general assumptions on X in Maulik and Okounkov [2012]. As these correspondences are canonical, they are invariant under the centralizer of A and, in particular, act in T-equivariant cohomology for any ambient torus T.

To put ourselves in the situation of Section 2.1, we define a category in which the objects are

$$\mathsf{F}(a_1, \dots, a_r) = \bigoplus_{n\geq 0} H^\bullet_{\mathsf{T}}(\mathcal{M}(r,n)) \tag{27}$$

and the maps defined by stable envelopes, like

$$\begin{tikzcd} \mathsf{F}(a_1) \otimes \mathsf{F}(a_2) \arrow[r, bend left, "\mathrm{Stab}_+"] \arrow[r, bend right, "\mathrm{Stab}_-"'] & \mathsf{F}(a_1, a_2) \end{tikzcd}, \tag{28}$$

where $\mathfrak{C}_\pm = \{a_1 \gtrless a_2\}$, are declared to be morphisms. Since both maps in (28) are isomorphisms after A-equivariant localization, we get a rational matrix

$$R(a_1 - a_2) = \mathrm{Stab}_-^{-1} \circ \mathrm{Stab}_+ \in \mathrm{End}(\mathsf{F}(a_1) \otimes \mathsf{F}(a_2)) \otimes \mathbb{Q}(a_1 - a_2)\,. \tag{29}$$

Its basic properties are summarized in the following

Theorem 1 (Maulik and Okounkov [ibid.]). *The R-matrix (29) satisfies the YB equation and defines, as in Section 2.1, an action of $\mathcal{Y}(\widehat{\mathfrak{gl}}(1))$ in equivariant cohomology of $\mathcal{M}(r,n)$. The Baxter subalgebras in $\mathcal{Y}(\widehat{\mathfrak{gl}}(1))$ are the algebras of operators of quantum multiplication. In particular, the vacuum-vacuum elements of the R-matrix are the operators of classical multiplication in $\mathcal{M}(r,n)$.*

[10] The $\pm$ a choice here, reflecting a choice of a *polarization* of X, which is a certain auxiliary piece of data that one needs to fix in the full development of the theory.

Here z in the Cartan torus of $\widehat{\mathfrak{gl}}(1)$ acts by z^n in the nth term of (27), which clearly commutes with R-matrices. The vacuum in (27) is the $n = 0$ term.

For a general Nakajima variety, it is proven in Maulik and Okounkov [ibid.] that the corresponding R-matrices define an action of $\mathcal{Y}(\mathfrak{g})$ for a certain Borcherds–Kac–Moody Lie algebra

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \neq 0} \mathfrak{g}_\alpha ,$$

with finite-dimensional root spaces $\mathfrak{g}_\alpha$. This Lie algebra is additionally graded by the cohomological degree and it has been conjectured in Okounkov [n.d.] that graded dimensions of $\mathfrak{g}_\alpha$ are given by the Kac polynomial for the dimension vector α. A slightly weaker version of this conjecture is proven in Schiffmann [2008a,b].

Again, the operators of classical multiplication are given by the vacuum-vacuum matrix elements of the R-matrix, while for the quantum multiplication we have the formula already announced in Section 1.2

Theorem 2 (Maulik and Okounkov [2012]). *For a general Nakajima variety, quantum multiplication by divisors is given by the formula* (14) *and hence the quantum differential equation is the Casimir connection for* $\mathcal{Y}(\mathfrak{g})$.

Back to the Hilbert scheme case, the R-matrix (29) acting in the tensor product of two Fock spaces is a very important object for which various formulas and descriptions are available. The following description was obtained in Maulik and Okounkov [ibid.].

The operators

$$\alpha_n^{\pm} = \alpha_n \otimes 1 \pm 1 \otimes \alpha_n$$

act in the tensor product of two Fock spaces, and form two commuting Heisenberg subalgebras. They are analogous to the center of mass and separation coordinates in a system of two bosons, and we can similarly decompose

$$\mathsf{F}(a_1) \otimes \mathsf{F}(a_2) = \mathsf{F}_+(a_1 + a_2) \otimes \mathsf{F}_-(a_1 - a_2) , \tag{30}$$

where to justify the labels we introduce the zero modes $\alpha_0(\gamma)$ that act on $\mathsf{F}(a_i)$ by $-a_i \int \gamma$. Here integral denotes the equivariant integration of $\gamma \in H_{\mathsf{T}}^\bullet(\mathbb{C}^2)$. Consider the operators L_n defined by

$$\sum L_n \zeta^{-n} = \frac{1}{4} :\boldsymbol{\alpha}_-(\zeta)^2: \pm \frac{1}{2} \hbar\, \partial \boldsymbol{\alpha}_-(\zeta) - \frac{1}{4} \int \hbar^2 , \tag{31}$$

where $:\boldsymbol{\alpha}_-(\zeta)^2:$ is the normally ordered square of the operator $\boldsymbol{\alpha}_-(\zeta) = \sum \alpha_n^- \zeta^n$, which now has a constant term in ζ, and ∂ stands for $\zeta \frac{\partial}{\partial \zeta}$. For the cohomology arguments of α_n^-, we use the conventions of Section 1.1. For either choice of sign, (31) form the Virasoro algebra

$$[L_n, L_m] = (m - n) L_{n+m} + \frac{(n^3 - n)}{12} \left(1 + 6 \int \hbar^2\right) \delta_{n+m} , \tag{32}$$

in a particular free field realization that is very familiar from the work of B. Feigin and Fuchs [1990] and from CFT. Note that, for generic $a_1 - a_2$, $\mathsf{F}_-(a_1 - a_2)$ is irreducible

with highest weight

$$L_0 \left|\right\rangle = \frac{1}{4} \int ((a_1 - a_2)^2 - \hbar^2) \left|\right\rangle ,$$

which, like (32) is invariant under $a_1 \leftrightarrow a_2$ and $\hbar \mapsto -\hbar$

Theorem 3 (Maulik and Okounkov [2012]). *The R-matrix is the unique operator in* F_- *in* (30) *that preserves the vacuum* $\left|\right\rangle$ *and interchanges the two signs in* (31).

This is the technical basis of numerous fruitful connection between $\mathcal{Y}(\widehat{\mathfrak{gl}}(1))$ and CFT. For instance, the operator $R^\vee = (12) \circ R$ is related to the reflection operator in Liouville CFT, see Zamolodchikov and Zamolodchikov [1996], and the YB equation satisfied by R reveals new unexpected features of affine Toda field theories.

3 Some further directions

3.1 K-theoretic counts. K-theoretic counts require a definite technical investment to be done properly, but offer an ample return producing deeper and more symmetric theories. For example, the dualities already mentioned in Section 2.2 reveal their full power only in equivariant K-theory.

Focusing on $\mathrm{Hilb}(\mathbb{C}^2, n)$, its self-duality may be put into an even deeper and, not surprisingly, widely conjectural framework of M-theory, which is a certain unique 11-dimensional supergravity theory with the power to unify many plots in modern theoretical physics as well as pure mathematics. Its basic actors are membranes with a 3-dimensional world volume that may appear as strings or even point particles to a low-resolution observer, see Bagger, Lambert, Mukhi, and Papageorgakis [2013] for a recent review. The supersymmetric index for membranes of the form

$$C \times S^1 \subset Z \times S^1 ,$$

where C is a complex curve in a complex Calabi–Yau 5-fold Z, should be a virtual representation of the automorphisms $\mathrm{Aut}(Z, \Omega^5_Z)$ that preserve the 5-form, so a certain K-theoretic curve count in Z. See N. Nekrasov and Okounkov [2016] for what it might look like and a conjectural equivalence with K-theoretic DT counts for

$$Y = Z^{\mathbb{C}^\times_z} ,$$

for any $\mathbb{C}^\times_z \subset \mathrm{Aut}(Z, \Omega^5_Z)$ with a purely 3-dimensional fixed locus. The most striking feature of this equivalence is the interpretation of the variable z

$$\begin{aligned} z &= \text{degree-counting variable in (12)} \\ &= \text{the } \chi(\mathcal{O}_C)\text{-counting variable in DT theory, see Section 1.2} \\ &\overset{!}{=} \text{equivariant variable } z \in \mathbb{C}^\times_z \text{ in M-theory} . \end{aligned} \tag{33}$$

Note that in cohomology equivariant variables take values in a Lie algebra, and so cannot be literally on the same footing as Kähler variables.

As a local model, one can take the total space

$$Z = \begin{array}{c} \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3 \oplus \mathcal{L}_4 \\ \downarrow \\ B \end{array}, \quad \bigotimes_{i=1}^{4} \mathcal{L}_i = \mathcal{K}_B\,, \tag{34}$$

of 4 line bundles over a smooth curve B. In this geometry, one can designate any two $\mathcal{L}_i$ to form Y by making z scale the other two line bundles with opposite weights. Thus the counting of Section 1.2 with $\mathcal{S} = \mathbb{C}^2$, when properly set up in K-theory, has a conjectural $S(4)$-symmetry that permutes the weights

$$(t_1, t_2, \frac{z}{\sqrt{qt_1t_2}}, \frac{1}{z\sqrt{qt_1t_2}})\,, \quad \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} \in \mathrm{Aut}(\mathbb{C}^2)\,,$$

where q^{-1} is the Chern root of $\mathcal{K}_B$.

This theory is described by certain q-difference equation that describes the change of the counts as the degrees of $\{\mathcal{L}_i\}$ change, and correspondingly the variables t_1, t_2, z are shifted by powers of q. In fact, similar q-difference equations can be defined for any Nakajima variety including $\mathcal{M}(r,n)$ from Section 2.2. For those, there are also difference equations in the framing equivariant variables a_i.

Theorem 4 (Okounkov [2017a] and Okounkov and Smirnov [2016]). *The q-difference equations in variables a_i are the quantum Knizhnik–Zamolodchikov (qKZ) equations for $\mathcal{U}_\hbar(\widehat{\widehat{\mathfrak{gl}}}(1))$ and the q-difference equation in z is the lattice part of the dynamical Weyl group of this quantum group. Same is true for a general Nakajima variety and the corresponding quantum loop algebra $\mathcal{U}_\hbar(\mathfrak{g})$.*

The qKZ equations, introduced in Frenkel and N. Reshetikhin [1992], have the form

$$\Psi(qa_1, a_2, \dots, a_n) = (z \otimes 1 \otimes \cdots \otimes 1)\, R_{1,n}(a_1/a_n) \dots R_{1,2}(a_1/a_2)\, \Psi \tag{35}$$

with similar equations in other variables a_i, where z is as in (25). For the R-matrices of $\mathcal{U}_\hbar(\widehat{\mathfrak{g}})$, where $\dim \mathfrak{g} < \infty$, these play the same role in integrable 2-dimensional lattice models as the classical KZ equations play for their conformal limits, see e.g. Jimbo and Miwa [1995].

For $\mathcal{U}_\hbar(\widehat{\widehat{\mathfrak{gl}}}(1))$, the R-matrix is a generalization of the R-matrix of Theorem 3 constructed using the K-theoretic stable envelopes. The algebra $\mathcal{U}_\hbar(\widehat{\widehat{\mathfrak{gl}}}(1))$ is constructed from this R-matrix using the general procedure of Section 2.1. For this particular geometry, it also coincides with the algebras constructed by many authors by different means, including explicit presentations see e.g. Burban and Schiffmann [2012], B. Feigin, E. Feigin, Jimbo, Miwa, and Mukhin [2011], Negut [2015], and Schiffmann and Vasserot [2013] for a sample of references where the same algebra appears.

It takes a certain development of the theory to define and make concrete the operators from the quantum dynamical Weyl group of an algebra like $\mathcal{U}_\hbar(\widehat{\mathfrak{g}})$. They crucially depend on certain features of K-theoretic stable envelopes that are not visible in cohomology.

In the search for the right K-theoretic generalization of (26), one should keep in mind that the correct notion of the degree of a multivariate Laurent polynomial is its Newton polytope, that is, the convex hull of its exponents, considered *up to translation*. Translations correspond to multiplication by monomials, which are invertible functions. The right generalization of (26) is then

$$\text{Newton}\left(\text{Stab}\Big|_{F_2\times F_1}\right) \subset \text{Newton}\left(\text{Attr}\Big|_{F_2\times F_2}\right) + \text{shift}_{F_1,F_2}\,, \tag{36}$$

for a certain collection of shifts in $\mathsf{A}^\vee \otimes \mathbb{R}$, where $\mathsf{A}^\vee$ is the character lattice of the torus A. Shifts come from weights of fractional line bundles $\mathcal{L} \in \text{Pic}(X) \otimes \mathbb{R}$ at the fixed points, see Okounkov [2017a] for a survey. This fractional line bundle, called the *slope* of the stable envelope is a new parameter, the dependence on which is locally constant and quasiperiodic — if the slopes differ by integral line bundle $\mathcal{L}$, then the corresponding stable envelopes differ by a twist by $\mathcal{L}$.

Stable envelopes change across certain hyperplanes in $\text{Pic}(X)\otimes\mathbb{R}$ that form a $\text{Pic}(X)$-periodic hyperplane arrangement closely related to the Kähler roots of X. For example, for $\widehat{\mathfrak{gl}}(1)$ the roots are $\{\alpha\} = \mathbb{N}\setminus\{0\}$ and so the affine root hyperplanes are all rationals

$$\{x \mid \exists\alpha\ \langle\alpha, x\rangle \in \mathbb{Z}\} = \mathbb{Q} \subset \mathbb{R}\,. \tag{37}$$

The dynamical Weyl group element $\mathsf{B}_{a/b}$ in (39) corresponding to a wall $\frac{a}{b} \in \mathbb{Q}$ acts trivially in $K_{\mathsf{T}}(\text{Hilb}(\mathbb{C}^2, n)$ if $b > n$, reflecting the fact that the Kähler roots of $\text{Hilb}(\mathbb{C}^2, n)$ form a finite subset $\{\pm 1, \dots, \pm n\}$ of the roots of $\widehat{\mathfrak{gl}}(1)$.

The change across a particular wall is recorded by a certain wall R-matrix R_{wall} and the R-matrices corresponding to a change of chamber $\mathfrak{C}$ as in (29) factor into an infinite product of those, see Okounkov [2017a] and Okounkov and Smirnov [2016]. Each term in such factorization corresponds to a certain root subalgebra

$$\mathcal{U}_\hbar(\mathfrak{g}_{\text{wall}}) \subset \mathcal{U}_\hbar(\widehat{\mathfrak{g}})$$

stable under the action of

$$\widehat{z} \in \text{Cartan torus of } \mathfrak{g} \times \mathbb{C}_q^\times\,, \tag{38}$$

where $\mathbb{C}_q^\times$ acts by loop rotation automorphisms of $\mathcal{U}_\hbar(\widehat{\mathfrak{g}})$. One defines Okounkov and Smirnov [2016] the dynamical Weyl group by taking certain specific (z, q)-dependent elements

$$\mathsf{B}_{\text{wall}} \in \mathcal{U}_\hbar(\mathfrak{g}_{\text{wall}})\,. \tag{39}$$

For finite-dimensional $\mathfrak{g}$, we have $\mathfrak{g}_{\text{wall}} = \mathfrak{sl}(2)$ for every wall and the construction specializes to the classical construction of Etingof and Varchenko [2002]. The operators B_{wall} satisfy the braid relations of the wall arrangement. Because each B_{wall} depends on (38) through the equation of the corresponding wall, these relations look like the Yang–Baxter equations (21), in which each term depends on $a = (a_1, a_2, a_3)$ through the equation $a_i - a_j = 0$ of the hyperplane being crossed in Lie A.

3.2 Monodromy and derived equivalences. As a special $\widehat{z}$-independent case, the dynamical Weyl group contains the so-called:

(w) quantum Weyl group of $\mathcal{U}_\hbar(\widehat{\mathfrak{g}})$, which plays many roles, including

(m) this is the monodromy group of the quantum differential equation (11), and

(p) this group describes the action on $K_{\mathsf{T}}(X)$ of the derived automorphisms of X constructed by Bezrukavnikov and Kaledin using quantizations $\widehat{X}_c$ of X in characteristic $p \gg 0$, see Bezrukavnikov and Finkelberg [2014], Bezrukavnikov and Kaledin [2008], Bezrukavnikov and Losev [2013], Kaledin [2008], and Losev [2014, 2017a, 2016, 2017b]. It thus plays the same role in modular representation theory of $\widehat{X}_c$ as the Hecke algebra plays in the classical Kazhdan–Lusztig theory.

Theorem 5 (Bezrukavnikov and Okounkov [n.d.]). *We have* (w) = (m) = (p) *for all Nakajima varieties.*

Other known infinite series of equivariant symplectic resolutions are also considered in Bezrukavnikov and Okounkov [ibid.]. For finite-dimensional $\mathfrak{g}$, the description of the monodromy of the Casimir connection via the quantum Weyl group is a conjecture of V. Toledano Laredo [2011]. The equality (m) = (p) for all equivariant symplectic resolutions is a conjecture of Bezrukavnikov and the author, see the discussion in Anno, Bezrukavnikov, and Mirković [2015] and Okounkov [2017b].

Here $\widehat{X}_c$ is an associative algebra deformation of the Poisson algebra of functions on X or X_0. While it may be studied abstractly, quantizations of Nakajima varieties may be described concretely as quantum Hamiltonian reductions, see Etingof [2007] and Gan and Ginzburg [2006]. For $X = \mathrm{Hilb}(\mathbb{C}^2, n)$, $\widehat{X}_c$ is the algebra generated by symmetric polynomials in $\{w_1, \dots, w_n\}$, and the operators of the rational Calogero quantum integrable systems — a commutative algebra of differential operators that includes the following rational analog of (6)

$$H_{\mathrm{C,rat}} = \tfrac{1}{2} \sum_{i=1}^{N} \frac{\partial^2}{\partial w_i^2} - c(c+1) \sum_{i<j\le N} \frac{1}{(w_i - w_j)^2}\,. \tag{40}$$

The kinship between (6) and (40) is closer than normal because of the self-duality of $\mathrm{Hilb}(\mathbb{C}^2, n)$. Recall that duality swaps equivariant and Kähler variables and instead of an equivariant variable θ in (6) we have a Kähler variable c in (40). It parametrizes deformations of $\widehat{X}_c$ in the same way as $\mathrm{Pic}(X)\otimes_{\mathbb{Z}}$ parametrizes deformations of X over a field .

For $p \gg 0$, the BK theory produces derived equivalences

$$\mathrm{D^b\,Coh}\, X^{(1)} \xleftrightarrow{\sim} \mathrm{D^b}\, \widehat{X}_c\text{-mod} \xleftrightarrow{\sim} \mathrm{D^b\,Coh}\, X^{(1)}_{\mathrm{flop}} \tag{41}$$

for every nonsingular value of $c \in \mathbb{Z}$. A shift $c \mapsto c + p$ twists (41) by $\mathcal{O}(1)$. Here $X^{(1)}$ denotes the Frobenius twist of X and X_{flop} refers to a change of stability condition in the GIT construction of X.

In the $(\mathsf{m}) = (\mathsf{p})$ interpretation, the composed equivalence in (41) becomes the transport of the QDE from the point $z = 0$, that corresponds to X, to the point $z = \infty$, that corresponds to X_{flop}, along the ray with

$$\arg z = -2\pi \frac{c}{p}.$$

In particular, this identifies the singularities of the QDE, given by the Kähler roots of X, with the limit $\lim_{p\to\infty} \frac{1}{p}\{c_{\text{sing}}\}$ of the singular parameters of the quantization. See Bezrukavnikov and Okounkov [n.d.] for details.

Monodromy of a flat connection with regular singularities is an analytic map between algebraic varieties that may be seen as a generalization of the exponential map of a Lie group. There is a long tradition, going back to at least the work of Kohno and Drinfeld of computing the monodromy of connections of representation-theoretic origin in terms of closely related algebraic structures. Just like for the exponential map, there is a certain progression in this, as one goes from additive variables to multiplicative, and also from multiplicative — to elliptic. E.g. in the case at hand the QDE (= the Casimir connection) is defined for modules M over the Yangian $\mathcal{Y}(\mathfrak{g})$, and is computed in terms of the action of $\mathcal{U}_\hbar(\widehat{\mathfrak{g}})$ in a closely related representation[11], see in particular Gautam and Toledano Laredo [2016].

A key step in capturing the monodromy algebraically is typically a certain compatibility constraint between the monodromy for $M = M_1 \otimes M_2$ and the monodromy for the tensor factors. The framework introduced above gives a very conceptual and powerful way to prove such statements. Recall that, geometrically, $\otimes$ arises as a special correspondence between X^{A} and X, where A is a torus that acts on X preserving the symplectic form. Therefore, it is natural to ask, more generally, for a compatibility between the monodromy of the QDE for X and X^{A}.

In fact, one can ask a more general question about the compatibility of the corresponding q-difference equations as in Theorem 4. Let

$$\mathsf{Z} = \operatorname{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{C}^\times$$

be the Kähler torus of X and $\overline{\mathsf{Z}}$ be the toric compactification of Z corresponding to the fan of ample cones of flops of X. Its torus-fixed points $0_{X_{\text{flop}}} \in \overline{\mathsf{Z}}$ correspond to all possible flops of X. A regular q-difference connection on a smooth toric variety is an action of the the cocharacter lattice

$$\operatorname{Pic}(X) \ni \lambda \to q^\lambda \in \mathsf{Z}, \quad q \in \mathbb{C}^\times,$$

on a (trivial) vector bundle over $\overline{\mathsf{Z}}$. Shift operators define a commuting regular q-difference connections in the variables $a \in \mathsf{A} \subset \overline{\mathsf{A}}$, where $\overline{\mathsf{A}}$ is the toric variety given by the fan of the chambers $\mathfrak{C}$. The q-difference connection for X^{A} sits overs the torus fixed points $0_{\mathfrak{C}} \in \overline{\mathsf{A}}$.

[11] geometrically, the right relation between $H_{\mathsf{T}}(X)$, where the Yangian acts, and $K_{\mathsf{T}}(X)$ of X, where $\mathcal{U}_\hbar(\widehat{\mathfrak{g}})$ acts, is given by a certain Γ-function analog of the Mukai vector, as in the work of Iritani [2009]

The most interesting analytic feature here is that the connections in z and a, while compatible and separately regular, are *not* regular jointly. This can never happen for differential equations, see Deligne [1970], but is commonplace for q-difference equations as illustrated by the system:

$$f(qz,a) = af(z,a)\,, \quad f(z,qa) = zf(z,a)\,.$$

As a result, near any point $(0_X, 0_{\mathfrak{C}}) \in \bar{\mathsf{Z}} \times \bar{\mathsf{A}}$, we get two kinds of solutions. Those naturally arising enumeratively are holomorphic in z in a punctured neighborhood of 0_X and meromorphic in a with poles accumulating to $0_{\mathfrak{C}}$. These may be called the z-solutions. For a-solutions, the roles of z and a are exchanged. These naturally appear in the Langlands dual setup and the initial conditions at $a = 0_{\mathfrak{C}}$ for them are the z-solutions for X^{A}.

Transition matrices between the a-solutions and the z-solutions, which is by construction elliptic, intertwine the monodromy for X^{A} and X, and vice versa. Note these transition matrices may, in principle, be computed from the series expansions near $(0_X, 0_{\mathfrak{C}})$, which differentiates them from more analytic objects like monodromy or Stokes matrices.

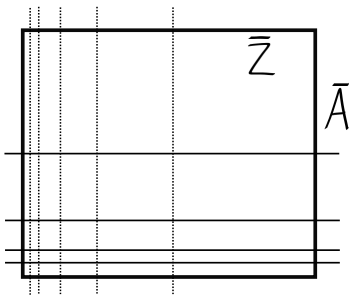

Figure 3: z-solutions are convergent power series in z with coefficients in $\mathbb{Q}(a)$, and the poles (solid lines in the picture) of these coefficients accumulate to $a = 0_{\mathfrak{C}}$. The poles of a-solutions, the dashed lines in the figure, similarly accumulate to $z = 0_X$.

Theorem 6 (Aganagic and Okounkov [2016]). *The transformation from the a-solutions to z-solutions is given by elliptic stable envelopes, a certain elliptic analog of* Stab *constructed in Aganagic and Okounkov [ibid.].*

To complete the picture, one can give Mellin–Barnes-type integral solutions to the quantum q-difference equations, and so, in particular to the qKZ and dynamical equations for $\mathcal{U}_\hbar(\widehat{\mathfrak{g}})$ in tensor products of evaluation representations Aganagic and Okounkov [2017]. It is well-known that the stationary phase $q \to 1$ limit in such integrals diagonalizes the Baxter subalgebra and hence generalizes the classical ideas of Bethe Ansatz.

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COHOMOLOGICAL FIELD THEORY CALCULATIONS

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Abstract

Cohomological field theories (CohFTs) were defined in the mid 1990s by Kontsevich and Manin to capture the formal properties of the virtual fundamental class in Gromov–Witten theory. A beautiful classification result for semisimple CohFTs (via the action of the Givental group) was proven by Teleman in 2012. The Givental–Teleman classification can be used to explicitly calculate the full CohFT in many interesting cases not approachable by earlier methods.

My goal here is to present an introduction to these ideas together with a survey of the calculations of the CohFTs obtained from

- Witten's classes on the moduli spaces of r-spin curves,
- Chern characters of the Verlinde bundles on the moduli of curves,
- Gromov–Witten classes of Hilbert schemes of points of $\mathbb{C}^2$.

The subject is full of basic open questions.

Introduction

0.1 Moduli of curves. The moduli space $\mathfrak{M}_g$ of complete, nonsingular, irreducible, algebraic curves over $\mathbb{C}$ of genus g has been a central object in mathematics since Riemann's work in the middle of the 19th century. The Deligne–Mumford compactification

$$\mathfrak{M}_g \subset \overline{\mathfrak{M}}_g$$

by nodal curves was defined almost 50 years ago Deligne and Mumford [1969].

We will be concerned here with the moduli space of curves with marked points,

$$\mathfrak{M}_{g,n} \subset \overline{\mathfrak{M}}_{g,n}\,,$$

in the stable range $2g-2+n>0$. As a Deligne–Mumford stack (or orbifold), $\overline{\mathfrak{M}}_{g,n}$ is nonsingular, irreducible, and of (complex) dimension $3g-3+n$. There are natural forgetful morphisms

$$p : \overline{\mathfrak{M}}_{g,n+1} \to \overline{\mathfrak{M}}_{g,n}$$

dropping the last marking.

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The *boundary*[1] of the Deligne–Mumford compactification is the closed locus parameterizing curves with a least one node,

$$\partial\overline{\mathfrak{M}}_{g,n} = \overline{\mathfrak{M}}_{g,n} \setminus \mathfrak{M}_{g,n} .$$

By identifying the last two markings of a single $(n+2)$-pointed curve of genus $g-1$, we obtain a morphism

$$q : \overline{\mathfrak{M}}_{g-1,n+2} \to \overline{\mathfrak{M}}_{g,n} .$$

Similarly, by identifying the last markings of separate pointed curves, we obtain

$$r : \overline{\mathfrak{M}}_{g_1,n_1+1} \times \overline{\mathfrak{M}}_{g_2,n_2+1} \to \overline{\mathfrak{M}}_{g,n} ,$$

where $n = n_1 + n_2$ and $g = g_1 + g_2$. The images of both q and r lie in the boundary $\partial\overline{\mathfrak{M}}_{g,n} \subset \overline{\mathfrak{M}}_{g,n}$.

The cohomology and Chow groups of the moduli space of curves are

$$H^*(\overline{\mathfrak{M}}_{g,n}, \mathbb{Q}) \quad \text{and} \quad A^*(\overline{\mathfrak{M}}_{g,n}, \mathbb{Q}) .$$

While there has been considerable progress in recent years, many basic questions about the cohomology and algebraic cycle theory remain open.[2]

0.2 Gromov–Witten classes. Let X be a nonsingular projective variety over $\mathbb{C}$, and let

$$\overline{\mathfrak{M}}_{g,n}(X, \beta)$$

be the moduli space of genus g, n-pointed stable maps to X representing the class $\beta \in H_2(X, \mathbb{Z})$. The basic structures carried by $\overline{\mathfrak{M}}_{g,n}(X, \beta)$ are forgetful maps,

$$\pi : \overline{\mathfrak{M}}_{g,n}(X, \beta) \to \overline{\mathfrak{M}}_{g,n} ,$$

to the moduli space of curves via the domain (in case $2g - 2 + n > 0$) and evaluation maps,

$$\mathrm{ev}_i : \overline{\mathfrak{M}}_{g,n}(X, \beta) \to X ,$$

for each marking $1 \le i \le n$.

Given cohomology classes $v_1, \ldots, v_n \in H^*(X, \mathbb{Q})$, the associated *Gromov–Witten class* is defined by

$$\Omega^X_{g,n,\beta}(v_1, \ldots, v_n) = \pi_* \left(\prod_{i=1}^{n} \mathrm{ev}_i^*(v_i) \cap [\overline{\mathfrak{M}}_{g,n}(X, \beta)]^{\mathrm{vir}} \right) \in H^*(\overline{\mathfrak{M}}_{g,n}, \mathbb{Q}) .$$

Central to the construction is the *virtual fundamental class* of the moduli space of stable maps,

$$[\overline{\mathfrak{M}}_{g,n}(X, \beta)]^{\mathrm{vir}} \in H_{2\cdot\mathrm{virdim}}(\overline{\mathfrak{M}}_{g,n}(X, \beta), \mathbb{Q}) ,$$

[1] Since $\overline{\mathfrak{M}}_{g,n}$ is a closed nonsingular orbifold, the boundary here is *not* in the sense of orbifold with boundary. If $g = 0$, there is no boundary map q.

[2] See Pandharipande [2018] for a survey of results and open questions.

of *virtual dimension*

$$\text{virdim} = \int_\beta c_1(X) + (1-g)\cdot(\dim_{\mathbb{C}}(X) - 3) + n\,.$$

Gromov–Witten classes contain much more information than the *Gromov–Witten invariants* defined by integration,

$$\langle v_1, \ldots, v_n \rangle^X_{g,n,\beta} = \int_{\overline{\mathfrak{M}}_{g,n}} \Omega^X_{g,n,\beta}(v_1, \ldots, v_n)\,.$$

We refer the reader to Behrend [1997], Behrend and Fantechi [1997], Cox and Katz [1999], and Fulton and Pandharipande [1997] for a detailed treatment of stable maps, virtual fundamental classes, and Gromov–Witten invariants in algebraic geometry.

The Gromov–Witten classes satisfy formal properties with respect to the natural forgetful and boundary maps p, q, and r discussed in Section 0.1. The idea of a cohomological field theory was introduced by Kontsevich and Manin Kontsevich and Manin [1994] to fully capture these formal properties.

0.3 Cohomological field theories. The starting point for defining a cohomological field theory is a triple of data $(V, \eta, \mathbf{1})$ where

- V is a finite dimensional $\mathbb{Q}$-vector space,
- η is a non-degenerate symmetric 2-form on V,
- $\mathbf{1} \in V$ is a distinguished element.

Given a $\mathbb{Q}$-basis $\{e_i\}$ of V, the symmetric form η can be written as a matrix

$$\eta_{jk} = \eta(e_j, e_k)\,.$$

The inverse matrix is denoted, as usual, by η^{jk}.

A *cohomological field theory* consists of a system $\Omega = (\Omega_{g,n})_{2g-2+n>0}$ of tensors

$$\Omega_{g,n} \in H^*(\overline{\mathfrak{M}}_{g,n}, \mathbb{Q}) \otimes (V^*)^{\otimes n}.$$

The tensor $\Omega_{g,n}$ associates a cohomology class in $H^*(\overline{\mathfrak{M}}_{g,n}, \mathbb{Q})$ to vectors

$$v_1, \ldots, v_n \in V$$

assigned to the n markings. We will use both

$$\Omega_{g,n}(v_1 \otimes \cdots \otimes v_n) \quad \text{and} \quad \Omega_{g,n}(v_1, \ldots, v_n)$$

to denote the associated cohomology class in $H^*(\overline{\mathfrak{M}}_{g,n}, \mathbb{Q})$.

In order to define a cohomological field theory, the system $\Omega = (\Omega_{g,n})_{2g-2+n>0}$ must satisfy the CohFT axioms:

(i) Each tensor $\Omega_{g,n}$ is Σ_n-invariant for the natural action of the symmetric group Σ_n on

$$H^*(\overline{\mathfrak{M}}_{g,n}, \mathbb{Q}) \otimes (V^*)^{\otimes n}$$

obtained by simultaneously permuting the n marked points of $\overline{\mathfrak{M}}_{g,n}$ and the n factors of V^*.

(ii) The tensor $q^*(\Omega_{g,n}) \in H^*(\overline{\mathfrak{M}}_{g-1,n+2}, \mathbb{Q}) \otimes (V^*)^{\otimes n}$, obtained via pull-back by the boundary morphism

$$q : \overline{\mathfrak{M}}_{g-1,n+2} \to \overline{\mathfrak{M}}_{g,n} \ ,$$

is required to equal the contraction of $\Omega_{g-1,n+2}$ by the bi-vector

$$\sum_{j,k} \eta^{jk} e_j \otimes e_k$$

inserted at the two identified points:

$$q^*(\Omega_{g,n}(v_1, \ldots, v_n)) = \sum_{j,k} \eta^{jk} \, \Omega_{g-1,n+2}(v_1, \ldots, v_n, e_j, e_k)$$

in $H^*(\overline{\mathfrak{M}}_{g-1,n+2}, \mathbb{Q})$ for all $v_i \in V$.

The tensor $r^*(\Omega_{g,n})$, obtained via pull-back by the boundary morphism

$$r : \overline{\mathfrak{M}}_{g_1,n_1+1} \times \overline{\mathfrak{M}}_{g_2,n_2+1} \to \overline{\mathfrak{M}}_{g,n} \ ,$$

is similarly required to equal the contraction of $\Omega_{g_1,n_1+1} \otimes \Omega_{g_2,n_2+1}$ by the same bi-vector:

$$r^*(\Omega_{g,n}(v_1, \ldots, v_n)) = \sum_{j,k} \eta^{jk} \, \Omega_{g_1,n_1+1}(v_1, \ldots, v_{n_1}, e_j) \otimes \Omega_{g_2,n_2+1}(v_{n_1+1}, \ldots, v_n, e_k)$$

in $H^*(\overline{\mathfrak{M}}_{g_1,n_1+1}, \mathbb{Q}) \otimes H^*(\overline{\mathfrak{M}}_{g_2,n_2+1}, \mathbb{Q})$ for all $v_i \in V$.

(iii) The tensor $p^*(\Omega_{g,n})$, obtained via pull-back by the forgetful map

$$p : \overline{\mathfrak{M}}_{g,n+1} \to \overline{\mathfrak{M}}_{g,n} \ ,$$

is required to satisfy

$$\Omega_{g,n+1}(v_1, \ldots, v_n, \mathbf{1}) = p^* \Omega_{g,n}(v_1, \ldots, v_n)$$

for all $v_i \in V$. In addition, the equality

$$\Omega_{0,3}(v_1, v_2, \mathbf{1}) = \eta(v_1, v_2)$$

is required for all $v_i \in V$.

Definition 1. *A system* $\Omega = (\Omega_{g,n})_{2g-2+n>0}$ *of tensors*

$$\Omega_{g,n} \in H^*(\overline{\mathfrak{M}}_{g,n}, \mathbb{Q}) \otimes (V^*)^{\otimes n}$$

satisfying (i) and (ii) is a cohomological field theory *or a* CohFT. *If (iii) is also satisfied,* Ω *is a* CohFT with unit.

The simplest example of a cohomological field theory with unit is given by the *trivial CohFT*,

$$V = \mathbb{Q}\,, \quad \eta(1,1) = 1\,, \quad \mathbf{1} = 1\,, \quad \Omega_{g,n}(1,\ldots,1) = 1 \in H^0(\overline{\mathfrak{M}}_{g,n}, \mathbb{Q})\,.$$

A more interesting example is given by the total Chern class

$$c(\mathbb{E}) = 1 + \lambda_1 + \ldots + \lambda_g \;\in\; H^*(\overline{\mathfrak{M}}_{g,n}, \mathbb{Q})$$

of the rank g Hodge bundle $\mathbb{E} \to \overline{\mathfrak{M}}_{g,n}$,

$$V = \mathbb{Q}\,, \quad \eta(1,1) = 1\,, \quad \mathbf{1} = 1\,, \quad \Omega_{g,n}(1,\ldots,1) = c(\mathbb{E}) \in H^*(\overline{\mathfrak{M}}_{g,n}, \mathbb{Q})\,.$$

Definition 2. *For a CohFT* $\Omega = (\Omega_{g,n})_{2g-2+n>0}$, *the* topological part ω *of* Ω *is defined by*

$$\omega_{g,n} = [\Omega_{g,n}]^0 \;\in\; H^0(\overline{\mathfrak{M}}_{g,n}, \mathbb{Q}) \otimes (V^*)^{\otimes n}\,.$$

The degree 0 part $[\ ]^0$ of Ω is simply obtained from the canonical summand projection

$$[\ ]^0 : H^*(\overline{\mathfrak{M}}_{g,n}, \mathbb{Q}) \to H^0(\overline{\mathfrak{M}}_{g,n}, \mathbb{Q})\,.$$

If Ω is a CohFT with unit, then ω is also a CohFT with unit. The topological part of the CohFT obtained from the total Chern class of the Hodge bundle is the trivial CohFT.

The motivating example of a CohFT with unit is obtained from the Gromov–Witten theory of a nonsingular projective variety X. Here,

$$V = H^*(X, \mathbb{Q})\,, \quad \eta(v_1, v_2) = \int_X v_1 \cup v_2\,, \quad \mathbf{1} = 1\,.$$

Of course, the Poincaré pairing on $H^*(X, \mathbb{Q})$ is symmetric only if X has no odd cohomology.[3] The tensor $\Omega_{g,n}$ is defined using the Gromov–Witten classes $\Omega^X_{g,n,\beta}$ of Section 0.2 (together with a Novikov[4] parameter q),

$$\Omega_{g,n}(v_1,\ldots,v_n) = \sum_{\beta \in H_2(X,\mathbb{Q})} \Omega^X_{g,n,\beta}\, q^\beta\,.$$

The CohFT axioms here coincide exactly with the axioms[5] of Gromov–Witten theory related to the morphisms p, q, and r. For example, axiom (ii) of a CohFT here is the *splitting axiom* of Gromov–Witten theory, see Kontsevich and Manin [1994].

[3] To accommodate the case of arbitrary X, the definition of a CohFT can be formulated with signs and $\mathbb{Z}/2\mathbb{Z}$-gradings. We do not take the super vector space path here.

[4] Formally, we must extend scalars in the definition of a CohFT from $\mathbb{Q}$ to the Novikov ring to capture the Gromov–Witten theory of X.

[5] The divisor axiom of Gromov–Witten theory (which concerns divisor and curve classes on X) is not part of the CohFT axioms.

0.4 Semisimplicity. A CohFT with unit Ω defines a *quantum product* $\bullet$ on V by[6]

$$\eta(v_1 \bullet v_2, v_3) = \Omega_{0,3}(v_1 \otimes v_2 \otimes v_3)\,.$$

The quantum product $\bullet$ is commutative by CohFT axiom (i). The associativity of $\bullet$ follows from CohFT axiom (ii). The element $\mathbf{1} \in V$ is the identity for $\bullet$ by the second clause of CohFT axiom (iii). Hence,

$$(V, \bullet, \mathbf{1})$$

is a commutative $\mathbb{Q}$-algebra.

Lemma 3. *The topological part ω of Ω is uniquely and effectively determined by the coefficients*

$$\Omega_{0,3}(v_1, v_2, v_3) \in H^*(\overline{\mathfrak{M}}_{0,3}, \mathbb{Q})$$

of the quantum product $\bullet$.

Proof. Let the moduli point $[C, p_1, \ldots, p_n] \in \overline{\mathfrak{M}}_{g,n}$ correspond to a maximally degenerate curve (with every component isomorphic to $\mathbb{P}^1$ with exactly 3 special points). Since

$$\omega_{g,n}(v_1, \ldots, v_n) \in H^0(\overline{\mathfrak{M}}_{g,n}, \mathbb{Q})\,,$$

is a multiple of the identity class, $\omega_{g,n}(v_1, \ldots, v_n)$ is determined by the pull-back to the point $[C, p_1, \ldots, p_n]$. The equality

$$\omega_{g,n}(v_1, \ldots, v_n)\big|_{[C,p_1,\ldots,p_n]} = \Omega_{g,n}(v_1, \ldots, v_n)\big|_{[C,p_1,\ldots,p_n]}$$

holds, and the latter restriction is determined by 3-point values $\Omega_{0,3}(w_1, w_2, w_3)$ from repeated application of CohFT axiom (ii). $\diamond$

A finite dimensional $\mathbb{Q}$-algebra is *semisimple* if there exists a basis $\{e_i\}$ of idempotents,

$$e_i e_j = \delta_{ij} e_i\,,$$

after an extension of scalars to $\mathbb{C}$.

Definition 4. *A CohFT with unit $\Omega = (\Omega_{g,n})_{2g-2+n>0}$ is* semisimple *if $(V, \bullet, \mathbf{1})$ is a semisimple algebra.*

0.5 Classification and calculation. The Givental–Teleman classification concerns semisimple CohFTs with unit.[7] The form of the classification result is as follows: *a semisimple CohFT with unit Ω is uniquely determined by the following two structures:*

- *the topological part ω of Ω,*

[6] Since $\overline{\mathfrak{M}}_{0,3}$ is a point, we canonically identify $H^*(\overline{\mathfrak{M}}_{0,3}, \mathbb{Q}) \cong \mathbb{Q}$, so $\Omega_{0,3}(v_1, v_2, v_3) \in \mathbb{Q}$.

[7] Semisimple CohFTs without unit are also covered, but we are interested here in the unital case. Semisimplicity is an essential condition.

- *an R-matrix*

$$R(z) = \mathsf{Id} + R_1 z + R_2 z^2 + R_3 z^3 + \ldots, \quad R_k \in \mathrm{End}(V)$$

satisfying the symplectic property

$$R(z) \cdot R^\star(-z) = \mathsf{Id}\,,$$

where $\star$ denotes the adjoint with respect to the metric η.

The precise statement of the Givental–Teleman classification will be discussed in Section 1.

Via the Givental–Teleman classification, a semisimple CohFT with unit Ω can be calculated in three steps:

(i) determine the ring $(V, \bullet, \mathbf{1})$ as explicitly as possible,

(ii) find a closed formula for the topological part ω of Ω via Lemma 3,

(iii) calculate the R-matrix of the theory.

In the language of Gromov–Witten theory, step (i) is the determination of the *small quantum cohomology* ring $QH^*(X, \mathbb{Q})$ via the 3-pointed genus 0 Gromov–Witten invariants. Step (ii) is then to calculate the Gromov–Witten invariants where the domain has a *fixed* complex structure of higher genus. New ideas are often required for the leap to higher genus moduli in step (iii). Finding a closed formula for the R-matrix requires a certain amount of luck.

Explaining how the above path to calculation plays out in three important CohFTs is my goal here. The three theories are:

- Witten's class on the moduli of r-spin curves,
- the Chern character of the Verlinde bundle on the moduli of curves,
- the Gromov–Witten theory of the Hilbert scheme of points of $\mathbb{C}^2$.

While each theory has geometric interest and the calculations have consequences in several directions, the focus of the paper will be on the CohFT determination. The paths to calculation pursued here are applicable in many other cases.

0.6 Past and future directions. The roots of the classification of semisimple CohFTs can be found in Givental's analysis Givental [2001a,b] and Lee and Pandharipande [2004] of the torus localization formula Graber and Pandharipande [1999] for the higher genus Gromov–Witten theory of toric varieties. The three CohFTs treated here are not directly accessible via the older torus localization methods. Givental's approach to the R-matrix via oscillating integrals (used often in the study of toric geometries) is not covered in the paper.

Many interesting CohFTs are *not* semisimple. For example, the Gromov–Witten theory of the famous Calabi–Yau quintic 3-fold,

$$X_5 \subset \mathbb{P}^4\,,$$

does *not* define a semisimple CohFT. However, in the past year, an approach to the quintic via the semisimple *formal quintic* theory Guo, Janda, and Ruan [2017] and Lho and Pandharipande [2017] appears possible. These developments are not surveyed here.

0.7 Acknowledgments. Much of what I know about the Givental–Teleman classification was learned through writing Lee and Pandharipande [2004] with Y.-P. Lee and Pandharipande, Pixton, and Zvonkine [2015] with A. Pixton and D. Zvonkine. For the study of the three CohFTs discussed in the paper, my collaborators have been J. Bryan, F. Janda, A. Marian, A. Okounkov, D. Oprea, A. Pixton, H.-H. Tseng, and D. Zvonkine. More specifically,

- Sections 1-2 are based on the papers Pandharipande, Pixton, and Zvonkine [2015] and Pandharipande, Pixton, and Zvonkine [2016] and the Appendix of Pandharipande, Pixton, and Zvonkine [2016],
- Section 3 is based on the paper Marian, Oprea, Pandharipande, Pixton, and Zvonkine [2017],
- Section 4 is based on the papers Bryan and Pandharipande [2008] and Okounkov and Pandharipande [2010a] and especially Pandharipande and Tseng [2017].

Discussions with A. Givental, T. Graber, H. Lho, and Y. Ruan have played an important role in my view of the subject. I was partially supported by SNF grant 200021-143274, ERC grant AdG-320368-MCSK, SwissMAP, and the Einstein Stiftung.

1 Givental–Teleman classification

1.1 Stable graphs. The boundary strata of the moduli space of curves correspond to *stable graphs*

$$\Gamma = (\mathrm{V}, \mathrm{H}, \mathrm{L},\ \mathrm{g} : \mathrm{V} \to \mathbb{Z}_{\geq 0},\ v : \mathrm{H} \to \mathrm{V},\ \mathrm{i} : \mathrm{H} \to \mathrm{H})$$

satisfying the following properties:

(i) V is a vertex set with a genus function $\mathrm{g} : V \to \mathbb{Z}_{\geq 0}$,

(ii) H is a half-edge set equipped with a vertex assignment $v : H \to V$ and an involution i,

(iii) E, the edge set, is defined by the 2-cycles of i in H (self-edges at vertices are permitted),

(iv) L, the set of legs, is defined by the fixed points of i and endowed with a bijective correspondence with a set of markings,

(v) the pair (V, E) defines a *connected* graph,

(vi) for each vertex v, the stability condition holds:

$$2\mathrm{g}(v) - 2 + \mathrm{n}(v) > 0,$$

where $\mathrm{n}(v)$ is the valence of Γ at v including both half-edges and legs.

An automorphism of Γ consists of automorphisms of the sets V and H which leave invariant the structures g, v, and i (and hence respect E and L). Let $\mathrm{Aut}(\Gamma)$ denote the automorphism group of Γ.

The genus of a stable graph Γ is defined by:

$$g(\Gamma) = \sum_{v \in \mathrm{V}} \mathrm{g}(v) + h^1(\Gamma)\,.$$

Let $\mathsf{G}_{g,n}$ denote the set of all stable graphs (up to isomorphism) of genus g with n legs. The strata[8] of the moduli space $\overline{\mathfrak{M}}_{g,n}$ of Deligne–Mumford stable curves are in bijective correspondence to $\mathsf{G}_{g,n}$ by considering the dual graph of a generic pointed curve parameterized by the stratum.

To each stable graph Γ, we associate the moduli space

$$\overline{\mathfrak{M}}_\Gamma = \prod_{v \in \mathrm{V}} \overline{\mathfrak{M}}_{\mathrm{g}(v),\mathrm{n}(v)}\,.$$

Let π_v denote the projection from $\overline{\mathfrak{M}}_\Gamma$ to $\overline{\mathfrak{M}}_{\mathrm{g}(v),\mathrm{n}(v)}$ associated to the vertex v. There is a canonical morphism

$$\iota_\Gamma : \overline{\mathfrak{M}}_\Gamma \to \overline{\mathfrak{M}}_{g,n} \tag{1}$$

with image[9] equal to the boundary stratum associated to the graph Γ.

1.2 R-matrix action.

1.2.1 First action. Let $\Omega = (\Omega_{g,n})_{2g-2+n>0}$ be a CohFT[10] on the vector space (V, η). Let R be a matrix series

$$R(z) = \sum_{k=0}^{\infty} R_k z^k \in \mathsf{Id} + z \cdot \mathrm{End}(V)[[z]]$$

which satisfies the symplectic condition

$$R(z) \cdot R^\star(-z) = \mathsf{Id}\,.$$

We define a new CohFT $R\Omega$ on the vector space (V, η) by summing over stable graphs Γ with summands given by a product of vertex, edge, and leg contributions,

$$(R\Omega)_{g,n} = \sum_{\Gamma \in \mathsf{G}_{g,n}} \frac{1}{|\mathrm{Aut}\,(\Gamma)|} \iota_{\Gamma\star} \left(\prod_{v \in \mathrm{V}} \mathsf{Cont}(v) \prod_{e \in \mathrm{E}} \mathsf{Cont}(e) \prod_{l \in \mathrm{L}} \mathsf{Cont}(l) \right), \tag{2}$$

where

[8] We consider here the standard stratification by topological type of the pointed curve

[9] The degree of ι_Γ is $|\mathrm{Aut}(\Gamma)|$.

[10] Ω is not assumed here to be unital – only CohFT axioms (i) and (ii) are imposed.

(i) the vertex contribution is

$$\mathsf{Cont}(v) = \Omega_{g(v),n(v)},$$

where $g(v)$ and $n(v)$ denote the genus and number of half-edges and legs of the vertex,

(ii) the leg contribution is the $\mathrm{End}(V)$-valued cohomology class

$$\mathsf{Cont}(l) = R(\psi_l)\,,$$

where $\psi_l \in H^2(\overline{\mathfrak{M}}_{g(v),n(v)}, \mathbb{Q})$ is the cotangent class at the marking corresponding to the leg,

(iii) the edge contribution is

$$\mathsf{Cont}(e) = \frac{\eta^{-1} - R(\psi_e')\eta^{-1}R(\psi_e'')^{\top}}{\psi_e' + \psi_e''}\,,$$

where ψ_e' and ψ_e'' are the cotangent classes at the node which represents the edge e. The symplectic condition guarantees that the edge contribution is well-defined.

We clarify the meaning of the edge contribution (iii),

$$\mathsf{Cont}(e) \in V^{\otimes 2} \otimes H^{\star}(\overline{\mathfrak{M}}_{g',n'}) \otimes H^{\star}(\overline{\mathfrak{M}}_{g'',n''}),$$

where (g', n') and (g'', n'') are the labels of the vertices adjacent to the edge e by writing the formula explicitly in coordinates.

Let $\{e_\mu\}$ be a $\mathbb{Q}$-basis of V. The components of the R-matrix in the basis are $R^\nu_\mu(z)$,

$$R(z)(e_\mu) = \sum_\nu R^\nu_\mu(z) \cdot e_\nu\,.$$

The components of $\mathsf{Cont}(e)$ are

$$\mathsf{Cont}(e)^{\mu\nu} = \frac{\eta^{\mu\nu} - \sum_{\rho,\sigma} R^\mu_\rho(\psi_e') \cdot \eta^{\rho\sigma} \cdot R^\nu_\sigma(\psi_e'')}{\psi_e' + \psi_e''} \in H^{\star}(\overline{\mathfrak{M}}_{g',n'}) \otimes H^{\star}(\overline{\mathfrak{M}}_{g'',n''}).$$

The fraction

$$\frac{\eta^{\mu\nu} - \sum_{\rho,\sigma} R^\mu_\rho(z) \cdot \eta^{\rho\sigma} \cdot R^\nu_\sigma(w)}{z + w}$$

is a power series in z and w since the numerator vanishes when $z = -w$ as a consequence of the symplectic condition which, in coordinates, takes the form

$$\sum_{\rho,\sigma} R^\mu_\rho(z) \cdot \eta^{\rho\sigma} \cdot R^\nu_\sigma(-z) = \eta^{\mu\nu}.$$

The substitution $z = \psi_e'$ and $w = \psi_e''$ is therefore unambiguously defined.

Definition 5. *Let $R\Omega$ be the CohFT obtained from Ω by the R-action* (2).

The above R-action was first defined[11] on Gromov–Witten potentials by Givental [2001a]. An abbreviated treatment of the lift to CohFTs appears in papers by Teleman [2012] and Shadrin [2009]. A careful proof that $R\Omega$ satisfies CohFT axioms (i) and (ii) can be found in Pandharipande, Pixton, and Zvonkine [2015, Section 2].

If Ω is a CohFT with unit on $(V, \eta, \mathbf{1})$, then $R\Omega$ may not respect the unit $\mathbf{1}$. To handle the unit, a second action is required.

1.2.2 Second action. A second action on the CohFT[12] Ω on (V, η) is given by translations. Let $T \in V[[z]]$ be a series with *no terms of degree* 0 *or* 1,

$$T(z) = T_2 z^2 + T_3 z^3 + \dots, \quad T_k \in V .$$

Definition 6. *Let $T\Omega$ be the CohFT obtained from Ω by the formula*

$$(T\Omega)_{g,n}(v_1, \dots, v_n) = \sum_{m=0}^{\infty} \frac{1}{m!} p_{m\star}\Big(\Omega_{g,n+m}(v_1, \dots, v_n, T(\psi_{n+1}), \dots, T(\psi_{n+m}))\Big),$$

where $p_m : \overline{\mathfrak{M}}_{g,n+m} \to \overline{\mathfrak{M}}_{g,n}$ is the morphism forgetting the last m markings.

The right side of the formula in Definition 6 is a formal expansion by distributing the powers of the ψ classes as follows:

$$\Omega_{g,n+m}(\cdots, T(\psi_\bullet), \cdots) = \sum_{k=2}^{\infty} \psi_\bullet^k \cdot \Omega_{g,n+m}(\cdots, T_k, \cdots).$$

The summation is finite because T has no terms of degree 0 or 1.

1.3 Reconstruction. We can now state the Givental–Teleman classification result Teleman [2012]. Let Ω be a semisimple CohFT with unit on $(V, \eta, \mathbf{1})$, and let ω be the topological part of Ω. For a symplectic matrix R, define

$$R.\omega = R(T(\omega)) \text{ with } T(z) = z((\mathsf{Id} - R(z)) \cdot \mathbf{1}) \in V[[z]].$$

By Pandharipande, Pixton, and Zvonkine [2015, Proposition 2.12], $R.\omega$ is a CohFT with unit on $(V, \eta, \mathbf{1})$. The Givental–Teleman classification asserts the *existence* of a unique R-matrix which exactly recovers Ω.

Theorem 7. *There exists a unique symplectic matrix*

$$R \in \mathsf{Id} + z \cdot \mathrm{End}(V)[[z]]$$

which reconstructs Ω from ω,

$$\Omega = R.\omega,$$

as a CohFT with unit.

[11] To simplify our formulas, we have changed Givental's and Teleman's conventions by replacing R with R^{-1}. Equation (2) above then determines a right group action on CohFTs rather than a left group action as in Givental's and Teleman's papers.

[12] To define the translation action, Ω is required only to be CohFT and not necessarily a CohFT with unit.

The first example concerns the total Chern class CohFT of Section 0.3,

$$V = \mathbb{Q}\,, \quad \eta(1,1) = 1\,, \quad \mathbf{1} = 1\,, \quad \Omega_{g,n}(1,\ldots,1) = c(\mathbb{E}) \in H^*(\overline{\mathfrak{M}}_{g,n}, \mathbb{Q})\,.$$

The topological part is the trivial CohFT, and the R-matrix is

$$R(z) = \exp\left(-\sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)(2k-1)} z^{2k-1}\right).$$

That the above R-matrix reconstructs the total Chern class CohFT is a consequence of Mumford's calculation Mumford [1983] of the Chern character of the Hodge bundle by Grothendieck–Riemann–Roch.

1.4 Chow field theories. Let $(V, \eta, \mathbf{1})$ be a $\mathbb{Q}$-vector space with a non-degenerate symmetric 2-form and a distinguished element. Let $\Omega = (\Omega_{g,n})_{2g-2+n>0}$ be a system of tensors

$$\Omega_{g,n} \in A^*(\overline{\mathfrak{M}}_{g,n}, \mathbb{Q}) \otimes (V^*)^{\otimes n}$$

where A^* is the Chow group of algebraic cycles modulo rational equivalence. In order to define a Chow field theory, the system Ω must satisfy the CohFT axioms of Section 0.3 with cohomology H^* replaced everywhere by Chow A^*.

Definition 8. *A system $\Omega = (\Omega_{g,n})_{2g-2+n>0}$ of elements*

$$\Omega_{g,n} \in A^*(\overline{\mathfrak{M}}_{g,n}, \mathbb{Q}) \otimes (V^*)^{\otimes n}$$

satisfying (i) and (ii) is a Chow field theory *or a* ChowFT. *If (iii) is also satisfied, Ω is a* ChowFT with unit.

For ChowFTs, the quantum product $(V, \bullet, \mathbf{1})$ and semisimplicity are defined just as for CohFTs. The R- and T-actions of Sections 1.2 also lift immediately to ChowFTs. However, the classification of semisimple ChowFTs is an open question.

Question 9. *Does the Givental–Teleman classification of Theorem 7 hold for a semisimple Chow field theory Ω with unit?*

2 Witten's r-spin class

2.1 r-spin CohFT. Let $r \geq 2$ be an integer. Let $(V_r, \eta, \mathbf{1})$ be the following triple:

- V_r is an $(r-1)$-dimensional $\mathbb{Q}$-vector space with basis $e_0, \ldots, e_{r-2}$,
- η is the non-degenerate symmetric 2-form

$$\eta_{ab} = \langle e_a, e_b \rangle = \delta_{a+b,r-2}\,,$$

- $\mathbf{1} = e_0$.

Witten's r-spin theory provides a family of classes

$$\mathfrak{W}^r_{g,n}(a_1,\ldots,a_n) \in H^*(\overline{\mathfrak{M}}_{g,n},\mathbb{Q})$$

for $a_1,\ldots,a_n \in \{0,\ldots,r-2\}$ which define a CohFT $\mathsf{W}^r = (\mathsf{W}^r_{g,n})_{2g-2+n>0}$ by

$$\mathsf{W}^r_{g,n} : V_r^{\otimes n} \to H^*(\overline{\mathfrak{M}}_{g,n},\mathbb{Q}), \quad \mathsf{W}^r_{g,n}(e_{a_1}\otimes\cdots\otimes e_{a_n}) = \mathfrak{W}^r_{g,n}(a_1,\ldots,a_n).$$

The class $\mathfrak{W}^r_{g,n}(a_1,\ldots,a_n)$ has (complex) degree[13]

$$\begin{aligned}\deg_{\mathbb{C}}\, \mathfrak{W}^r_{g,n}(a_1,\ldots,a_n) &= \mathsf{D}^r_{g,n}(a_1,\ldots,a_n)\\ &= \frac{(r-2)(g-1)+\sum_{i=1}^n a_i}{r}\,.\end{aligned} \tag{3}$$

If $\mathsf{D}^r_{g,n}(a_1,\ldots,a_n)$ is not an integer, the corresponding Witten's class vanishes.

The construction of $\mathfrak{W}^r_{0,n}(a_1,\ldots,a_n)$ in genus 0 was carried out by Witten [1993] using r-spin structures. Let $\overline{\mathfrak{M}}^r_{0,n}(a_1,\ldots,a_n)$ be the Deligne–Mumford moduli space parameterizing r^{th} roots,

$$\mathfrak{L}^{\otimes r} \cong \omega_C\Big(-\sum_{i=1}^n a_i\, p_i\Big) \quad \text{where} \quad [C,p_1,\ldots,p_n] \in \overline{\mathfrak{M}}_{0,n}\,.$$

The class $\frac{1}{r}\mathfrak{W}^r_{0,n}(a_1,\ldots,a_n)$ is defined to be the push-forward to $\overline{\mathfrak{M}}_{0,n}$ of the top Chern class of the bundle on $\overline{\mathfrak{M}}^r_{0,n}(a_1,\ldots,a_n)$ with fiber $H^1(C,\mathfrak{L})^*$.

The existence of Witten's class in higher genus is both remarkable and highly non-trivial. Polishchuk [2004] and Polishchuk and Vaintrob [2001] constructed

$$\mathfrak{W}^r_{g,n}(a_1,\ldots,a_n) \in A^*(\overline{\mathfrak{M}}_{g,n},\mathbb{Q})$$

as an algebraic cycle class and proved and the CohFT axioms (i-iii) for a Chow field theory hold. The algebraic approach was later simplified in Chang, J. Li, and W.-P. Li [2015] and Chiodo [2006]. Analytic constructions appear in Fan, Jarvis, and Ruan [2013] and Mochizuki [2006].

2.2 Genus 0.

2.2.1 3 and 4 markings. Witten [1993] determined the following initial conditions in genus 0 with $n = 3, 4$:

$$\int_{\overline{\mathfrak{M}}_{0,3}} \mathfrak{W}^r_{0,3}(a_1,a_2,a_3) = \begin{cases} 1 & \text{if } a_1+a_2+a_3 = r-2\,,\\ 0 & \text{otherwise,}\end{cases} \tag{4}$$

$$\int_{\overline{\mathfrak{M}}_{0,4}} \mathfrak{W}^r_{0,4}(1,1,r-2,r-2) = \frac{1}{r}\,.$$

[13] So $\mathfrak{W}^r_{g,n}(a_1,\ldots,a_n) \in H^{2\cdot\mathsf{D}^r_{g,n}(a_1,\ldots,a_n)}(\overline{\mathfrak{M}}_{g,n},\mathbb{Q})$.

Uniqueness of the r-spin CohFT in genus 0 follows easily from the initial conditions (4) and the axioms of a CohFT with unit.

The genus 0 sector of the CohFT W^r defines a quantum product[14] $\bullet$ on V_r. The resulting algebra $(V_r, \bullet, \mathbf{1})$, even after extension to $\mathbb{C}$, is *not* semisimple. Therefore, the Givental–Teleman classification can not be directly applied.

2.2.2 Witten's r-spin class and representations of $\mathsf{sl}_2(\mathbb{C})$. Consider the Lie algebra $\mathsf{sl}_2 = \mathsf{sl}_2(\mathbb{C})$. Denote by ρ_k the kth symmetric power of the standard 2-dimensional representation of sl_2,

$$\rho_k = \mathrm{Sym}^k(\rho_1)\,, \quad \dim \rho_k = k+1\,.$$

The following complete solution of the genus 0 part of the CohFT W^r (after integration) was found by Pixton, see Pandharipande, Pixton, and Zvonkine [2016] for a proof.

Theorem 10. *Let* $\mathbf{a} = (a_1, \ldots, a_{n\geq 3})$ *with* $a_i \in \{0, \ldots, r-2\}$ *satisfy the degree constraint* $\mathsf{D}^r_{0,n}(\mathbf{a}) = n-3$. *Then,*

$$\int_{\overline{\mathfrak{M}}_{0,n}} \mathfrak{W}^r_{0,n}(\mathbf{a}) = \frac{(n-3)!}{r^{n-3}} \dim\Big[\rho_{r-2-a_1} \otimes \cdots \otimes \rho_{r-2-a_n}\Big]^{\mathsf{sl}_2},$$

where the superscript sl_2 *denotes the* sl_2*-invariant subspace.*

2.2.3 Shifted Witten class.

Definition 11. *For* $\gamma \in V_r$, *the* shifted r*-spin CohFT* $\mathsf{W}^{r,\gamma}$ *is defined by*

$$\mathsf{W}^{r,\gamma}_{g,n}(v_1 \otimes \cdots \otimes v_n) = \sum_{m\geq 0} \frac{1}{m!}\, p_{m\star}\, \mathsf{W}^r_{g,n+m}(v_1 \otimes \cdots \otimes v_n \otimes \gamma^{\otimes m}),$$

where $p_m \colon \overline{\mathfrak{M}}_{g,n+m} \to \overline{\mathfrak{M}}_{g,n}$ *is the map forgetting the last* m *markings.*

Using degree formula $\mathsf{D}^r_{g,n}$, the summation in the definition of the shift is easily seen to be finite. The shifted Witten class $\mathsf{W}^{r,\gamma}$ determines a CohFT with unit, see Pandharipande, Pixton, and Zvonkine [2015, Section 1.1].

Definition 12. *Define the CohFT* $\widehat{\mathsf{W}}^r$ *with unit on* $(V_r, \eta, \mathbf{1})$ *by the shift*

$$\widehat{\mathsf{W}}^r = \mathsf{W}^{r,(0,\ldots,0,r)}$$

along the special vector $re_{r-2} \in V_r$. *Let* $(V_r, \widehat{\bullet}, \mathbf{1})$ *be the* $\mathbb{Q}$*-algebra determined by the quantum product defined by* $\widehat{\mathsf{W}}^r$.

The *Verlinde algebra* of level r for sl_2 is spanned by the weights of sl_2 from 0 to $r-2$. The coefficient of c in the product $a \bullet b$ is equal to the dimension of the sl_2-invariant subspace of the representation $\rho_a \otimes \rho_b \otimes \rho_c$ provided the inequality

$$a + b + c \leq 2r - 4$$

is satisfied. Using Theorem 10 for the integral r-spin theory in genus 0, the following basic result is proven in Pandharipande, Pixton, and Zvonkine [2016].

[14] See Section 0.4.

Theorem 13. *The algebra* $(V_r, \widehat{\bullet}, \mathbf{1})$ *is isomorphic to the Verlinde algebra of level r for* sl_2.

Since the Verlinde algebra is well-known to be semisimple[15], the Givental–Teleman classification of Theorem 7 can be applied to the CohFT $\widehat{\mathsf{W}}^r$. Using the degree formula (3) and Definition 12, we see the (complex) degree $\mathsf{D}^r_{g,n}(a_1,\ldots,a_n)$ part of $\widehat{\mathsf{W}}^r$ equals W^r,

$$\left[\widehat{\mathsf{W}}^r_{g,n}(a_1,\ldots,a_n)\right]^{\mathsf{D}^r_{g,n}(a_1,\ldots,a_n)} = \mathsf{W}^r_{g,n}(a_1,\ldots,a_n)\,.$$

Hence, a complete computation of $\widehat{\mathsf{W}}^r$ also provides a computation of W^r.

2.3 The topological field theory. After the studying genus 0 theory, we turn our attention to the topological part $\widehat{\omega}^r$ of $\widehat{\mathsf{W}}^r$. The following two results of Pandharipande, Pixton, and Zvonkine [ibid.] provide a complete calculation.

Proposition 14. *The basis of normalized idempotents of* $(V_r, \widehat{\bullet}, \mathbf{1})$ *is given by*

$$v_k = \sqrt{\frac{2}{r}} \sum_{a=0}^{r-2} \sin\left(\frac{(a+1)k\pi}{r}\right) e_a, \qquad k \in \{1,\ldots,r-1\}\,.$$

More precisely, we have

$$\eta(v_k, v_l) = (-1)^{k-1}\delta_{k,l}\,, \qquad v_k \,\widehat{\bullet}\, v_l = \frac{\sqrt{r/2}}{\sin(\frac{k\pi}{r})} v_k\, \delta_{k,l}\,.$$

Once the normalized idempotents are found, the computation of $\widehat{\omega}^r$ is straightforward by Lemma 3 and elementary trigonometric identities.

Proposition 15. *For* $a_1,\ldots,a_n \in \{0,\ldots,r-2\}$, *we have*

$$\widehat{\omega}^r_{g,n}(e_{a_1},\ldots,e_{a_n}) = \left(\frac{r}{2}\right)^{g-1} \sum_{k=1}^{r-1} \frac{(-1)^{(k-1)(g-1)} \prod_{i=1}^{n} \sin\left(\frac{(a_i+1)k\pi}{r}\right)}{\left(\sin(\frac{k\pi}{r})\right)^{2g-2+n}}\,.$$

In Proposition 15, the CohFT $\widehat{\omega}^r$ is viewed as taking values in $\mathbb{Q}$ via the canonical identification

$$H^0(\overline{\mathfrak{M}}_{g,n}, \mathbb{Q}) \cong \mathbb{Q}\,.$$

2.4 The R-matrix. The last (and often hardest) step in the computation of a semisimple CohFT via the Givental–Teleman classification is to find the unique R-matrix. Remarkably, there exists a closed formula in hypergeometric series for the R-matrix of the CohFT $\widehat{\mathsf{W}}^r$. The precise shift in Definition 12 of the CohFT $\widehat{\mathsf{W}}^r$ is crucial: the shift re_{r-2} is (up to scale) the *only* shift of W^r for which closed formulas for the R-matrix are known.

[15] An explicit normalized idempotent basis is given in Proposition 14 below.

The method of finding the unique R-matrix for W^r uses the *Euler field* e_{r-2} at the shift re_{r-2}. The operator of quantum multiplication $\widehat{\bullet}$ by the Euler field in the basis $e_0, \ldots, e_{r-2}$ is

$$\xi = \begin{pmatrix} 0 & \cdots & \cdots & 0 & 2 \\ 0 & & 0 & 2 & 0 \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ 0 & 2 & 0 & & 0 \\ 2 & 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

In the same frame, the *shifted degree operator* is

$$\mu = \frac{1}{2r} \begin{pmatrix} -(r-2) & 0 & \cdots & \cdots & 0 \\ 0 & -(r-4) & 0 & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & & 0 & r-4 & 0 \\ 0 & \cdots & \cdots & 0 & r-2 \end{pmatrix}.$$

Since $\widehat{\mathsf{W}}^r$ has an Euler field with an associated degree operator, the unique R-matrix for the classification is given by the solution of

$$[R_{m+1}, \xi] = (m - \mu) R_m \tag{5}$$

with the initial condition $R_0 = \mathsf{Id}$, see Teleman [2012].

Definition 16. *For each integer $a \in \{0, \ldots, r-2\}$, define the hypergeometric series*

$$\boldsymbol{B}_{r,a}(z) = \sum_{m=0}^{\infty} \left[\prod_{i=1}^{m} \frac{\big((2i-1)r - 2(a+1)\big)\big((2i-1)r + 2(a+1)\big)}{i} \right] \left(-\frac{z}{16r^2}\right)^m.$$

Let $\boldsymbol{B}_{r,a}^{\mathrm{even}}$ and $\boldsymbol{B}_{r,a}^{\mathrm{odd}}$ the even and odd summands[16] *of the series $\boldsymbol{B}_{r,a}$.*

The unique solution to (5) is computed in Pandharipande, Pixton, and Zvonkine [2016]. The R-matrix of $\widehat{\mathsf{W}}^r$ has a surprisingly simple form.

Theorem 17. *The unique R-matrix classifying $\widehat{\mathsf{W}}^r$ has coefficients*

$$R_a^a = \boldsymbol{B}_{r,a}^{\mathrm{even}}(z), \quad a \in \{0, \ldots, r-2\}$$

on the main diagonal, and

$$R_a^{r-2-a} = \boldsymbol{B}_{r,a}^{\mathrm{odd}}(z), \quad a \in \{0, \ldots, r-2\}$$

[16] The even summand consists of all the even powers of z (and likewise for the odd summand).

on the antidiagonal (if r is even, the coefficient at the intersection of both diagonals is 1), and 0 *everywhere else.*

In case $r = 2$, the matrix is trivial $R(z) = \mathsf{Id}$. For $r = 3$ and 4 respectively, the R-matrices[17] are

$$R(z) = \begin{pmatrix} \boldsymbol{B}^{\text{even}}_{3,0}(z) & \boldsymbol{B}^{\text{odd}}_{3,1}(z) \\ \boldsymbol{B}^{\text{odd}}_{3,0}(z) & \boldsymbol{B}^{\text{even}}_{3,1}(z) \end{pmatrix},$$

$$R(z) = \begin{pmatrix} \boldsymbol{B}^{\text{even}}_{4,0}(z) & 0 & \boldsymbol{B}^{\text{odd}}_{4,2}(z) \\ 0 & 1 & 0 \\ \boldsymbol{B}^{\text{odd}}_{4,0}(z) & 0 & \boldsymbol{B}^{\text{even}}_{4,2}(z) \end{pmatrix}.$$

2.5 Calculation of W^r. The analysis of Sections 2.2-2.4 together complete the calculation of $\widehat{\mathsf{W}}^r$,

$$\widehat{\mathsf{W}}^r = R.\widehat{\omega}^r,$$

in exactly the steps (i)-(iii) proposed in Section 0.5 of the Introduction. Then, as we have seen,

$$\mathsf{W}^r_{g,n}(a_1,\ldots,a_n) = \left[\widehat{\mathsf{W}}^r_{g,n}(a_1,\ldots,a_n)\right]^{\mathsf{D}^r_{g,n}(a_1,\ldots,a_n)}.$$

The calculation has an immediate consequence Pandharipande, Pixton, and Zvonkine [ibid.].

Corollary 18. *Witten's r-spin class on $\overline{\mathfrak{M}}_{g,n}$ lies in the tautological ring (in cohomology),*

$$\mathfrak{W}^r_{g,n}(a_1,\ldots,a_n) \in RH^*(\overline{\mathfrak{M}}_{g,n},\mathbb{Q}).$$

We refer the reader to Pandharipande [2018] for a discussion of tautological classes on the moduli space of curves. In fact, the first proof of Pixton's relations in $RH^*(\overline{\mathfrak{M}}_{g,n},\mathbb{Q})$ was obtained via the calculation of $\widehat{\mathsf{W}}^3$ in Pandharipande, Pixton, and Zvonkine [2015].

2.6 Questions. Whether Corollary 18 also holds in Chow is an interesting question: is

$$\mathfrak{W}^r_{g,n}(a_1,\ldots,a_n) \in R^*(\overline{\mathfrak{M}}_{g,n},\mathbb{Q})\,? \tag{6}$$

A positive answer to Question 9 about the classification of Chow field theories would imply a positive answer here. The following question may be viewed as a refinement of (6).

Question 19. *Find a formula in algebraic cycles for Witten's r-spin class on* $\overline{\mathfrak{M}}^r_{g,n}(a_1,\ldots,a_n)$ before *push-forward to* $\overline{\mathfrak{M}}_{g,n}$.

[17] R here is R^{-1} in Pandharipande, Pixton, and Zvonkine [2015] and Pandharipande, Pixton, and Zvonkine [2016] because of a change of conventions.

Another open direction concerns the moduli spaces of holomorphic differentials Bainbridge, Chen, Gendron, Grushevsky, and Moeller [2016] and Farkas and Pandharipande [2018]. Let $(a_1, \ldots, a_n)$ be a partition of $2g-2$ with non-negative parts. Let

$$\overline{\mathcal{H}}_g(a_1, \ldots, a_n) \subset \overline{\mathfrak{M}}_{g,n}$$

be the closure of the locus of moduli points

$$[C, p_1, \ldots, p_n] \in \mathfrak{M}_{g,n} \quad \text{where} \quad \omega_C \cong \mathcal{O}_C\Big(\sum_{i=1}^{n} a_i\, p_i\Big).$$

For $r-2 \geq \mathrm{Max}\{a_1, \ldots, a_n\}$, Witten's r-spin class $\mathcal{W}^r_{g,n}(a_1, \ldots, a_n)$ is well-defined and of degree *independent* of r,

$$\mathsf{D}^r_{g,n}(a_1, \ldots, a_n) = \frac{(r-2)(g-1) + \sum_{i=1}^n a_i}{r} = g-1\,.$$

By Pandharipande, Pixton, and Zvonkine [2016, Theorem 7], after scaling by r^{g-1},

$$\mathcal{W}_{g,n}(a_1, \ldots, a_n)[r] = r^{g-1} \cdot \mathcal{W}^r_{g,n}(a_1, \ldots, a_n) \in RH^{g-1}(\overline{\mathfrak{M}}_{g,n}, \mathbb{Q})$$

is a *polynomial* in r for all sufficiently large r.

Question 20. *Prove the following conjecture of Pandharipande, Pixton, and Zvonkine [ibid., Appendix]:*

$$(-1)^g\, \mathcal{W}_{g,n}(a_1, \ldots, a_n)[0] = [\overline{\mathcal{H}}_g(a_1, \ldots, a_n)] \in H^{2(g-1)}(\overline{\mathfrak{M}}_{g,n}, \mathbb{Q})\,.$$

3 Chern character of the Verlinde bundle

3.1 Verlinde CohFT. Let G be a complex, simple, simply connected Lie group with Lie algebra $\mathfrak{g}$. Fix an integer *level* $\ell > 0$. Let $(V_\ell, \eta, \mathbf{1})$ be the following triple:

- V_ℓ is the $\mathbb{Q}$-vector space with basis indexed by the irreducible representations of $\mathfrak{g}$ at level ℓ,
- η is the non-degenerate symmetric 2-form

  $$\eta(\mu, \nu) = \delta_{\mu, \nu^\star}$$

  where $\nu^\star$ denotes the dual representation,
- $\mathbf{1}$ is the basis element corresponding to the trivial representation.

Let $\mu_1, \ldots, \mu_n$ be n irreducible representations of $\mathfrak{g}$ at level ℓ. A vector bundle

$$\mathbb{V}_g(\mu_1, \ldots, \mu_n) \to \overline{\mathfrak{M}}_{g,n}$$

is constructed in Tsuchiya, Ueno, and Yamada [1989]. Over nonsingular curves, the fibers of $\mathbb{V}_g(\mu_1, \ldots, \mu_n)$ are the spaces of non-abelian theta functions – spaces of global

sections of the determinant line bundles over the moduli of parabolic G-bundles. To extend $\mathbb{V}_g(\mu_1, \ldots, \mu_n)$ over the boundary

$$\partial \mathfrak{M}_{g,n} \subset \overline{\mathfrak{M}}_{g,n} ,$$

the theory of *conformal blocks* is required Tsuchiya, Ueno, and Yamada [ibid.]. The vector bundle $\mathbb{V}_g(\mu_1, \ldots, \mu_n)$ has various names in the literature: the Verlinde bundle, the bundle of conformal blocks, and the bundle of vacua. A study in genus 0 and 1 can be found in Fakhruddin [2012].

A CohFT Ω^ℓ is defined via the Chern character[18] of the Verlinde bundle:

$$\Omega^\ell_{g,n}(\mu_1, \ldots, \mu_n) = \mathsf{ch}_t(\mathbb{V}_g(\mu_1, \ldots, \mu_n)) \in H^\star(\overline{\mathfrak{M}}_{g,n}, \mathbb{Q}) .$$

CohFT axiom (i) for Ω^ℓ is trivial. Axiom (ii) follows from the fusion rules Tsuchiya, Ueno, and Yamada [1989]. Axiom (iii) for the unit $\mathbf{1}$ is the *propagation of vacua* Fakhruddin [2012, Proposition 2.4(i)].

3.2 Genus 0 and the topological part. Since the variable t carries the degree grading, the topological part ω^ℓ of Ω^ℓ is obtained by setting $t = 0$,

$$\omega^\ell_{g,n} = \Omega^\ell_{g,n}\Big|_{t=0} .$$

The result is the just the rank of the Verlinde bundle,

$$\omega^\ell_{g,n}(\mu_1, \ldots, \mu_n) = \mathrm{rk}\ \mathbb{V}_g(\mu_1, \ldots, \mu_n) = d_g(\mu_1, \ldots, \mu_n) .$$

With the quantum product obtained[19] from ω^ℓ, $(V_\ell, \bullet, \mathbf{1})$ is the *fusion algebra*.

Since the fusion algebra is well-known[20] to be semisimple, the CohFT with unit Ω^ℓ is also semisimple, The subject has a history starting in the mid 80s with the discovery and in 90s with several proofs of the *Verlinde formula* for the rank $d_g(\mu_1, \ldots, \mu_n)$, see Beauville [1996] for an overview.

Hence, steps (i) and (ii) of the computational strategy of Section 0.5 for Ω^ℓ are complete (and have been for many years). Step (iii) is the jump to moduli.

3.3 Path to the R-matrix. The shifted r-spin CohFT $\widehat{\mathsf{W}}^r$ has an Euler field obtained from the pure dimensionality of Witten's r-spin class which was used to find the unique R-matrix in Section 2. The CohFT Ω^ℓ is not of pure dimension and has no Euler field. A different path to the R-matrix is required here.

[18] For a vector bundle $\mathbb{V}$ with Chern roots $r_1, \ldots, r_k$,

$$\mathsf{ch}_t(\mathbb{V}) = \sum_{j=1}^{k} e^{t r_j} .$$

The parameter t may be treated either as a formal variable, in which case the CohFT is defined over the ring $\mathbb{Q}[[t]]$ instead of $\mathbb{Q}$, or as a rational number $t \in \mathbb{Q}$.

[19] Since the quantum product depends only upon the tensors of genus 0 with 3 markings, the quantum products of Ω^ℓ and ω^ℓ are equal.

[20] See, for example, Beauville [1996, Proposition 6.1].

The restriction of the tensor $\Omega^{\ell}_{g,n}$ to the open set of nonsingular curves

$$\mathfrak{M}_{g,n} \subset \overline{\mathfrak{M}}_{g,n}$$

forgets a lot of the data of the CohFT. However, by Marian, Oprea, Pandharipande, Pixton, and Zvonkine [2017, Lemma 2.2], the restriction is enough to uniquely determine the R-matrix of Ω^{ℓ}. Fortunately, the restriction is calculable in closed form:

- the first Chern class of the Verlinde bundle over $\mathfrak{M}_{g,n}$ is found in Tsuchimoto [1993],
- the existence of a projectively flat connection[21] Tsuchiya, Ueno, and Yamada [1989] on the Verlinde bundle over $\mathfrak{M}_{g,n}$ then determines the full Chern character over $\mathfrak{M}_{g,n}$.

As should be expected, the computation of Ω^{ℓ} relies significantly upon the past study of the Verlinde bundles.

3.4 The R-matrix. For a simple Lie algebra $\mathfrak{g}$ and a level ℓ, the conformal anomaly is

$$c = c(\mathfrak{g}, \ell) = \frac{\ell \dim \mathfrak{g}}{\check{h} + \ell},$$

where $\check{h}$ is the dual Coxeter number. For each representation with highest weight μ of level ℓ, define

$$\mathsf{w}(\mu) = \frac{(\mu, \mu + 2\rho)}{2(\check{h} + \ell)}.$$

Here, ρ is half of the sum of the positive roots, and the Cartan–Killing form $(,)$ is normalized so that the longest root θ satisfies

$$(\theta, \theta) = 2.$$

Example 21. *For $\mathfrak{g} = \mathfrak{sl}(r, \mathbb{C})$, the highest weight of a representation of level ℓ is given by an r-tuple of integers*

$$\mu = (\mu^1, \ldots, \mu^r), \ \ell \geq \mu^1 \geq \cdots \geq \mu^r \geq 0,$$

defined up to shifting the vector components by the same integer. Furthermore, we have

$$c(\mathfrak{g}, \ell) = \frac{\ell(r^2 - 1)}{\ell + r},$$

$$\mathsf{w}(\mu) = \frac{1}{2(\ell + r)} \left(\sum_{i=1}^{r} (\mu^i)^2 - \frac{1}{r} \left(\sum_{i=1}^{r} \mu^i \right)^2 + \sum_{i=1}^{r} (r - 2i + 1)\mu^i \right).$$

Via the path to the R-matrix discussed in Section 3.3, a simple closed form for the R-matrix of Ω^{ℓ} is found in Marian, Oprea, Pandharipande, Pixton, and Zvonkine [2017] using the constants $c(\mathfrak{g}, \ell)$ and $\mathsf{w}(\mu)$ from representation theory.

[21] Often called the Hitchin connection.

Theorem 22. *The CohFT Ω^ℓ is reconstructed from the topological part ω^ℓ by the diagonal R-matrix*

$$R(z)^\mu_\mu = \exp\left(tz \cdot \left(-\mathsf{w}(\mu) + \frac{c(\mathfrak{g},\ell)}{24}\right)\right).$$

For the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2$ at level $\ell = 1$, there are only two representations $\{\varnothing, \square\}$ to consider[22],

$$c(\mathfrak{sl}_2, 1) = 1, \quad \mathsf{w}(\varnothing) = 0, \quad \mathsf{w}(\square) = \frac{1}{4}.$$

As an example of the reconstruction result of Theorem 22,

$$\Omega^\ell = R.\omega^\ell,$$

the total Chern character $\operatorname{ch} \mathbb{V}_g(\square, \ldots, \square)$ at $t = 1$ is

$$\exp\left(-\frac{\lambda_1}{2}\right) \cdot \sum_{\Gamma \in \mathsf{G}^{\mathsf{even}}_{g,n}} \frac{2^{g-h^1(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} \cdot \iota_{\Gamma\star}\left(\prod_{e\in \mathrm{E}} \frac{1-\exp\left(-\frac{1}{4}(\psi'_e + \psi''_e)\right)}{\psi'_e + \psi''_e} \cdot \prod_{l \in \mathrm{L}} e^{-\psi_l/4}\right)$$

in $H^*(\overline{\mathfrak{M}}_{g,n}, \mathbb{Q})$. A few remarks about the above formula are required:

- The classes λ_1 and ψ are the first Chern classes of the Hodge bundle and the cotangent line bundle respectively.
- The sum is over the set of *even* stable graphs,

  $$\mathsf{G}^{\mathsf{even}}_{g,n} \subset \mathsf{G}_{g,n},$$

  defined by requiring the valence $\mathsf{n}(v)$ to be even for every vertex v of the graph.
- The Verlinde rank $d_{\mathsf{g}(v)}(\square, \ldots, \square)$ with $\mathsf{n}(v)$ insertions equals 2^g in the even case, see Beauville [1996]. The product of $2^{\mathsf{g}(v)}$ over the vertices of Γ yields $2^{g-h^1(\Gamma)}$, where h^1 denotes the first Betti number.

3.5 Questions. A different approach to the calculation of the Chern character of the Verlinde bundle in the $\mathfrak{sl}_2$ case (for every level) was pursued in Faber, Marian, and Pandharipande [n.d.] using the geometry introduced by Thaddeus [1994] to prove the Verlinde formula. The outcome of Faber, Marian, and Pandharipande [n.d.] is a more difficult calculation (with a much more complex answer), but with one advantage: the projective flatness of the Hitchin connection over $\mathfrak{M}_{g,n}$ is *not* used. When the flatness is introduced, the method of Faber, Marian, and Pandharipande [ibid.] yields tautological relations. Unfortunately, no such relations are obtained by the above R-matrix calculation of Ω^ℓ since the projective flatness is an input.

Question 23. *Is there an alternative computation of Ω^ℓ which does* not *use the projective flatness of the Hitchin connection and which systematically produces tautological relations in $RH^*(\mathfrak{M}_{g,n}, \mathbb{Q})$?*

Of course, if the answer to Question 23 is yes, then the next question is whether *all* tautological relations are produced.[23]

[22] Here, $\varnothing$ is the trivial representation (corresponding to **1**) and $\square$ is the standard representation.

[23] I first heard an early version of this question from R. Bott at Harvard in the 90s.

4 Gromov–Witten theory of $\mathsf{Hilb}^m(\mathbb{C}^2)$

4.1 T-equivariant cohomology of $\mathsf{Hilb}^m(\mathbb{C}^2)$. The *Hilbert scheme* $\mathsf{Hilb}^m(\mathbb{C}^2)$ of m points in the plane $\mathbb{C}^2$ parameterizes ideals $\mathfrak{I} \subset \mathbb{C}[x, y]$ of colength m,

$$\dim_{\mathbb{C}} \mathbb{C}[x, y]/\mathfrak{I} = m\,.$$

The Hilbert scheme $\mathsf{Hilb}^m(\mathbb{C}^2)$ is a nonsingular, irreducible, quasi-projective variety of dimension $2m$, see Nakajima [1999] for an introduction. An open dense set of $\mathsf{Hilb}^m(\mathbb{C}^2)$ parameterizes ideals associated to configurations of m distinct points.

The symmetries of $\mathbb{C}^2$ lift to the Hilbert scheme. The algebraic torus

$$\mathsf{T} = (\mathbb{C}^*)^2$$

acts diagonally on $\mathbb{C}^2$ by scaling coordinates,

$$(z_1, z_2) \cdot (x, y) = (z_1 x, z_2 y)\,.$$

We review the Fock space description of the T-equivariant cohomology of the Hilbert scheme of points of $\mathbb{C}^2$ following the notation of Okounkov and Pandharipande [2010a, Section 2.1].

By definition, the *Fock space* $\mathcal{F}$ is freely generated over $\mathbb{Q}$ by commuting creation operators α_{-k}, $k \in \mathbb{Z}_{>0}$, acting on the vacuum vector $v_\varnothing$. The annihilation operators α_k, $k \in \mathbb{Z}_{>0}$, kill the vacuum

$$\alpha_k \cdot v_\varnothing = 0, \quad k > 0\,,$$

and satisfy the commutation relations $[\alpha_k, \alpha_l] = k\,\delta_{k+l}$.

A natural basis of $\mathcal{F}$ is given by the vectors

$$|\mu\rangle = \frac{1}{\mathfrak{z}(\mu)} \prod_i \alpha_{-\mu_i}\, v_\varnothing \tag{7}$$

indexed by partitions μ. Here, $\mathfrak{z}(\mu) = |\mathrm{Aut}(\mu)| \prod_i \mu_i$ is the usual normalization factor. Let the length $\ell(\mu)$ denote the number of parts of the partition μ.

The *Nakajima basis* defines a canonical isomorphism,

$$\mathcal{F} \otimes_{\mathbb{Q}} \mathbb{Q}[t_1, t_2] \stackrel{\sim}{=} \bigoplus_{n \geq 0} H^*_{\mathsf{T}}(\mathsf{Hilb}^m(\mathbb{C}^2), \mathbb{Q}). \tag{8}$$

The Nakajima basis element corresponding to $|\mu\rangle$ is

$$\frac{1}{\Pi_i \mu_i}[\mathcal{V}_\mu]$$

where $[\mathcal{V}_\mu]$ is (the cohomological dual of) the class of the subvariety of $\mathsf{Hilb}^{|\mu|}(\mathbb{C}^2)$ with generic element given by a union of schemes of lengths

$$\mu_1, \ldots, \mu_{\ell(\mu)}$$

supported at $\ell(\mu)$ distinct points[24] of $\mathbb{C}^2$. The vacuum vector $v_\varnothing$ corresponds to the unit in

$$\mathsf{1} \in H^*_{\mathsf{T}}(\mathsf{Hilb}^0(\mathbb{C}^2), \mathbb{Q})\,.$$

The variables t_1 and t_2 are the equivariant parameters corresponding to the weights of the T-action on the tangent space $\mathrm{Tan}_0(\mathbb{C}^2)$ at the origin of $\mathbb{C}^2$.

The subspace $\mathcal{F}_m \subset \mathcal{F} \otimes_{\mathbb{Q}} \mathbb{Q}[t_1, t_2]$ corresponding to $H^*_{\mathsf{T}}(\mathsf{Hilb}^m(\mathbb{C}^2), \mathbb{Q})$ is spanned by the vectors (7) with $|\mu| = n$. The subspace can also be described as the n-eigenspace of the *energy operator*:

$$|\cdot| = \sum_{k>0} \alpha_{-k}\, \alpha_k\,.$$

The vector $|1^n\rangle$ corresponds to the unit

$$\mathsf{1} \in H^*_{\mathsf{T}}(\mathsf{Hilb}^m(\mathbb{C}^2), \mathbb{Q})\,.$$

The standard inner product on the T-equivariant cohomology of $\mathsf{Hilb}^m(\mathbb{C}^2)$ induces the following *nonstandard* inner product on Fock space after an extension of scalars:

$$\langle \mu | \nu \rangle = \frac{(-1)^{|\mu|-\ell(\mu)}}{(t_1 t_2)^{\ell(\mu)}} \frac{\delta_{\mu\nu}}{\mathfrak{z}(\mu)}\,. \tag{9}$$

4.2 Gromov–Witten CohFT of $\mathsf{Hilb}^m(\mathbb{C}^2)$. Let $m > 0$ be a colength. Let $(V_m, \eta, \mathbf{1})$ be the following triple:

- V_m is the free $\mathbb{Q}(t_1, t_2)[[q]]$-module $\mathcal{F}_m \otimes_{\mathbb{Q}[t_1,t_2]} \mathbb{Q}(t_1, t_2)[[q]]$,
- η is the non-degenerate symmetric 2-form (9),
- $\mathbf{1}$ is the basis element $|1^n\rangle$.

Since $\mathsf{Hilb}^m(\mathbb{C}^2)$ is *not* proper, the Gromov–Witten theory is only defined after localization by T. The CohFT with unit

$$\Omega^{\mathsf{Hilb}^m(\mathbb{C}^2)} = (\Omega^{\mathsf{Hilb}^m(\mathbb{C}^2)}_{g,n})_{2g-2+n>0}$$

is defined via the localized T-equivariant Gromov–Witten classes of $\mathsf{Hilb}^m(\mathbb{C}^2)$,

$$\Omega^{\mathsf{Hilb}^m(\mathbb{C}^2)}_{g,n} \in H^*(\overline{\mathfrak{M}}_{g,n}, \mathbb{Q}(t_1, t_2)[[q]]) \otimes (V_m^*)^n\,.$$

Here, q is the Novikov parameter. Curves of degree d are counted with weight q^d, where the curve degree is defined by the pairing with the divisor

$$D = -\left|2, 1^{m-2}\right\rangle, \quad d = \int_\beta D\,.$$

Formally, $\Omega^{\mathsf{Hilb}^m(\mathbb{C}^2)}$ is a CohFT *not* over the field $\mathbb{Q}$ as in the r-spin and Verlinde cases, but over the ring $\mathbb{Q}(t_1, t_2)[[q]]$. To simplify notation, let

$$\Omega^m = \Omega^{\mathsf{Hilb}^m(\mathbb{C}^2)}\,.$$

[24] The points and parts of μ are considered here to be unordered.

4.3 Genus 0. Since the T-action on $\mathsf{Hilb}^m(\mathbb{C}^2)$ has finitely many T-fixed points, the localized T-equivariant cohomology

$$H^*_{\mathsf{T}}(\mathsf{Hilb}^m(\mathbb{C}^2),\mathbb{Q}) \otimes_{\mathbb{Q}[t_1,t_2]} \mathbb{Q}(t_1,t_2)$$

is semisimple. At $q=0$, the quantum cohomology ring,

$$(V_m, \bullet, \mathbf{1}), \tag{10}$$

defined by Ω^m specializes to the localized T-equivariant cohomology of $\mathsf{Hilb}^m(\mathbb{C}^2)$. Hence, the quantum cohomology (10) is semisimple over the ring $\mathbb{Q}(t_1,t_2)[[q]]$, see Lee and Pandharipande [2004].

Let M_D denote the operator of T-equivariant quantum multiplication by the divisor D. A central result of Okounkov and Pandharipande [2010a] is the following explicit formula for M_D an as operator on Fock space:

$$\mathsf{M}_D(q,t_1,t_2) = (t_1+t_2)\sum_{k>0}\frac{k}{2}\frac{(-q)^k+1}{(-q)^k-1}\,\alpha_{-k}\,\alpha_k \;-\; \frac{t_1+t_2}{2}\frac{(-q)+1}{(-q)-1}|\cdot| \\ + \frac{1}{2}\sum_{k,l>0}\Big[t_1t_2\,\alpha_{k+l}\,\alpha_{-k}\,\alpha_{-l} - \alpha_{-k-l}\,\alpha_k\,\alpha_l\Big].$$

The q-dependence of M_D occurs only in the first two terms (which act diagonally in the basis (7)).

Let μ^1 and μ^2 be partitions of m. The T-equivariant Gromov–Witten invariants of $\mathsf{Hilb}^m(\mathbb{C}^2)$ in genus 0 with 3 cohomology insertions given (in the Nakajima basis) by μ^1, D, and μ^2 are determined by M_D:

$$\sum_{d=0}^{\infty}\Omega^m_{0,3,d}(\mu^1,D,\mu^2)\,q^d \;=\; \langle\mu^1\,|\,\mathsf{M}_D\,|\,\mu^2\rangle.$$

The following result is proven in Okounkov and Pandharipande [ibid.].

Theorem 24. *The restriction of Ω^m to genus 0 is uniquely and effectively determined from the calculation of M_D.*

While Theorem 24 in principle completes the genus 0 study of Ω^m, the result is not as strong as the genus 0 determinations in the r-spin and Verlinde cases. The proof of Theorem 24 provides an effective linear algebraic procedure, but not a formula, for calculating the genus 0 part of Ω^m from M_D.

4.4 The topological part. Let ω^m be the topological part of the CohFT with unit Ω^m. A closed formula for ω^m can not be expected since closed formulas are already missing in the genus 0 study.

The CohFT with unit ω^m has been considered earlier from another perspective. Using fundamental correspondences Maulik, Nekrasov, Okounkov, and Pandharipande [2006], ω^m is equivalent to the *local GW/DT theory* of 3-folds of the form

$$\mathbb{C}^2 \times C\,, \tag{11}$$

where C is a curve or arbitrary genus. Such local theories have been studied extensively Bryan and Pandharipande [2008] in the investigation of the GW/DT theory of 3-folds[25].

4.5 The R-matrix. Since Ω^m is not of pure dimension (and does not carry an Euler field), the R-matrix is *not* determined by the T-equivariant genus 0 theory alone. As in the Verlinde case, a different method is required. Fortunately, together with the divisor equation, an evaluation of the T-equivariant higher genus theory in degree 0 is enough to uniquely determine the R-matrix.

Let $\mathrm{Part}(m)$ be the set of of partitions of m corresponding to the T-fixed points of $\mathsf{Hilb}^m(\mathbb{C}^2)$. For each $\eta \in \mathrm{Part}(m)$, let $\mathrm{Tan}_\eta(\mathsf{Hilb}^m(\mathbb{C}^2))$ be the T-representation on the tangent space at the T-fixed point corresponding to η. As before, let

$$\mathbb{E} \to \overline{\mathfrak{M}}_{g,n}$$

be the Hodge bundle. The follow result is proven in Pandharipande and Tseng [2017].

Theorem 25. *The R-matrix of Ω^m is uniquely determined by the divisor equation and the degree 0 invariants*

$$\langle \mu \rangle_{1,0}^{\mathsf{Hilb}^m(\mathbb{C}^2)} = \sum_{\eta \in \mathrm{Part}(m)} \mu|_\eta \int_{\overline{\mathfrak{M}}_{1,1}} \frac{e\left(\mathbb{E}^* \otimes \mathrm{Tan}_\eta(\mathsf{Hilb}^m(\mathbb{C}^2))\right)}{e\left(\mathrm{Tan}_\eta(\mathsf{Hilb}^m(\mathbb{C}^2))\right)},$$

$$\langle \rangle_{g\geq 2,0}^{\mathsf{Hilb}^m(\mathbb{C}^2)} = \sum_{\eta \in \mathrm{Part}(m)} \int_{\overline{\mathfrak{M}}_{g}} \frac{e\left(\mathbb{E}^* \otimes \mathrm{Tan}_\eta(\mathsf{Hilb}^m(\mathbb{C}^2))\right)}{e\left(\mathrm{Tan}_\eta(\mathsf{Hilb}^m(\mathbb{C}^2))\right)}.$$

While Theorem 25 is weaker than the explicit R-matrix solutions in the r-spin and Verlinde cases, the result nevertheless has several consequences. The first is a rationality result Pandharipande and Tseng [ibid.].

Theorem 26. *For all genera $g \geq 0$ and $\mu^1, \ldots, \mu^n \in \mathrm{Part}(m)$, the series*[26]

$$\int_{\overline{\mathfrak{M}}_{g,n}} \Omega^m_{g,n}(\mu^1, \ldots, \mu^n) \in \mathbb{Q}(t_1, t_2)[[q]]$$

is the Taylor expansion in q of a rational function in $\mathbb{Q}(t_1, t_1, q)$.

The statement of Theorem 26 can be strengthened (with an R-matrix argument using Theorem 25) to prove that the CohFT with unit Ω^m can be defined over the field $\mathbb{Q}(t_1, t_2, q)$.

[25] A natural generalization of the geometry (11) is to consider the 3-fold total space,

$$L_1 \oplus L_2 \to C,$$

of a sum of line bundles $L_1, L_2 \to C$. For particular pairs L_1 and L_2, simple closed form solutions were found Bryan and Pandharipande [2008] and have later played a role in the study of the structure of the Gromov–Witten theory of Calabi–Yau 3-folds by Ionel and Parker [2018].

[26] As always, g and n are required to be in the stable range $2g - 2 + n > 0$.

4.6 Crepant resolution. The Hilbert scheme of points of $\mathbb{C}^2$ is well-known to be a crepant resolution of the symmetric product,

$$\epsilon : \mathsf{Hilb}^m(\mathbb{C}^2) \to \mathsf{Sym}^m(\mathbb{C}^2) = (\mathbb{C}^2)^m/S_m\,.$$

Viewed as an *orbifold*, the symmetric product $\mathsf{Sym}^m(\mathbb{C}^2)$ has a T-equivariant Gromov–Witten theory with insertions indexed by partitions of m and an associated CohFT with unit $\Omega^{\mathsf{Sym}^m(\mathbb{C}^2)}$ determined by the Gromov–Witten classes. The CohFT $\Omega^{\mathsf{Sym}^m(\mathbb{C}^2)}$ is defined over the ring $\mathbb{Q}(t_1,t_2)[[u]]$, where u is variable associated to the free ramification points, see Pandharipande and Tseng [2017] for a detailed treatment.

In genus 0, the equivalence of the T-equivariant Gromov–Witten theories of $\mathsf{Hilb}^m(\mathbb{C}^2)$ and the orbifold $\mathsf{Sym}^m(\mathbb{C}^2)$ was proven[27] in Bryan and Graber [2009]. Another consequence of the R-matrix study of $\Omega^{\mathsf{Hilb}^m(\mathbb{C}^2)}$ is the proof in Pandharipande and Tseng [2017] of the *crepant resolution conjecture* here.

Theorem 27. *For all genera $g \geq 0$ and $\mu^1,\ldots,\mu^n \in \mathrm{Part}(m)$, we have*

$$\Omega_{g,n}^{\mathsf{Hilb}^m(\mathbb{C}^2)}(\mu^1,\ldots,\mu^n) = (-i)^{\sum_{i=1}^n \ell(\mu^i)-|\mu^i|}\,\Omega_{g,n}^{\mathsf{Sym}^m(\mathbb{C}^2)}(\mu^1,\ldots,\mu^n)$$

after the variable change $-q = e^{iu}$.

The variable change of Theorem 27 is well-defined by the rationality of Theorem 26. The analysis Okounkov and Pandharipande [2010b] of the quantum differential equation of $\mathsf{Hilb}^m(\mathbb{C}^2)$ plays an important role in the proof. Theorem 27 is closely related to the GW/DT correspondence for local curves (11) in families, see Pandharipande and Tseng [2017].

4.7 Questions. The most basic open question is to find an expression for the R-matrix of Ω^m in terms of natural operators on Fock space.

Question 28. *Is there a representation theoretic formula for the R-matrix of the CohFT with unit Ω^m?*

The difficulty in attacking Question 28 starts with the lack of higher genus calculations in closed form. The first nontrivial example Pandharipande and Tseng [ibid.] occurs in genus 1 for the Hilbert scheme of 2 points:

$$\int_{\overline{\mathfrak{M}}_{1,1}} \Omega_{1,1}^{\mathsf{Hilb}^2(\mathbb{C}^2)}\big((2)\big) = -\frac{1}{24}\frac{(t_1+t_2)^2}{t_1t_2}\cdot\frac{1+q}{1-q}\,. \tag{12}$$

While there are numerous calculations to do, the higher m analogue of (12) surely has a simple answer.

Question 29. *Calculate the series*

$$\int_{\overline{\mathfrak{M}}_{1,1}} \Omega_{1,1}^{\mathsf{Hilb}^m(\mathbb{C}^2)}\big((2,1^{n-2})\big) \in \mathbb{Q}(t_1,t_2,q)\,,$$

in closed form for all m.

[27] The prefactor $(-i)^{\sum_{i=1}^n \ell(\mu^i)-|\mu^i|}$ was treated incorrectly in Bryan and Graber [2009] because of an arithmetical error. The prefactor here is correct.

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p-ADIC GEOMETRY

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Abstract

We discuss recent developments in p-adic geometry, ranging from foundational results such as the degeneration of the Hodge-to-de Rham spectral sequence for "compact p-adic manifolds" over new period maps on moduli spaces of abelian varieties to applications to the local and global Langlands conjectures, and the construction of "universal" p-adic cohomology theories. We finish with some speculations on how a theory that combines all primes p, including the archimedean prime, might look like.

1 Introduction

In this survey paper, we want to give an introduction to the world of ideas which the author has explored in the past few years, and indicate some possible future directions. The two general themes that dominate this work are the cohomology of algebraic varieties, and the local and global Langlands correspondences. These two topics are classically intertwined ever since the cohomology of the moduli space of elliptic curves and more general Shimura varieties has been used for the construction of Langlands correspondences. Most of our work so far is over p-adic fields, where we have established analogues of the basic results of Hodge theory for "compact p-adic manifolds", have constructed a "universal" p-adic cohomology theory, and have made progress towards establishing the local Langlands correspondence for a general p-adic reductive group by using a theory of p-adic shtukas, and we will recall these results below.

However, here we wish to relay another, deeper, relation between the cohomology of algebraic varieties and the structures underlying the Langlands corresondence, a relation that pertains not to the cohomology of specific algebraic varieties, but to the very notion of what "the" cohomology of an algebraic variety is. Classically, the study of the latter is the paradigm of "motives" envisioned by Grothendieck; however, that vision has still only been partially realized, by Voevodsky [2000], and others. Basically, Grothendieck's idea was to find the "universal" cohomology as the universal solution to a few basic axioms; in order to see that this has the desired properties, one however needs to know the existence of "enough" algebraic cycles as encoded in the standard conjectures, and more generally the Hodge and Tate conjectures. However, little progress has been made on these questions. We propose to approach the subject from the other side

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and construct an *explicit* cohomology theory that practically behaves like a universal cohomology theory (so that, for example, it specializes to all other known cohomology theories); whether or not it is universal in the technical sense of being the universal solution to certain axioms will then be a secondary question.

This deeper relation builds on the realization of Drinfeld [1980], that in the function field case, at the heart of the Langlands correspondence lie moduli spaces of *shtukas*. Anderson [1986], Goss [1996], and others have since studied the notion of t-motives, which is a special kind of shtuka, and is a remarkable function field analogue of motives, however without any relation to the cohomology of algebraic varieties. What we are proposing here is that, despite extreme difficulties in making sense of this, there should exist a theory of shtukas in the number field case, and that the cohomology of an algebraic variety, i.e. a motive, should be an example of such a shtuka.

This picture has been essentially fully realized in the p-adic case. In the first sections of this survey, we will explain these results in the p-adic case; towards the end, we will then speculate on how the full picture over $\operatorname{Spec}\mathbb{Z}$ should look like, and give some evidence that this is a reasonable picture.

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2 p-adic Hodge theory

Let us fix a complete algebraically closed extension C of $\mathbb{Q}_p$. The analogue of a compact complex manifold in this setting is a proper smooth rigid-analytic variety X over C.[1] Basic examples include the analytification of proper smooth algebraic varieties over C, or the generic fibres of proper smooth formal schemes over the ring of integers $\mathcal{O}_C$ of C. More exotic examples with no direct relation to algebraic varieties are given by the Hopf surfaces

$$X = ((\mathbb{A}^2_C)^{\mathrm{rig}} \setminus \{(0,0)\})/q^{\mathbb{Z}} ,$$

where $q \in C$ is an element with $0 < |q| < 1$ acting via diagonal multiplication on $\mathbb{A}^2$. Recall that their complex analogues are non-Kähler, as follows from the asymmetry in Hodge numbers, $H^0(X, \Omega^1_X) = 0$ while $H^1(X, \mathcal{O}_X) = C$. On the other hand, some other non-Kähler manifolds such as the Iwasawa manifolds do not have a p-adic analogue.

[1] Rigid-analytic varieties were first defined by Tate [1971], and alternative and more general foundations have been proposed by various authors, including Raynaud [1974], Berkovich [1993], Fujiwara and Kato [2006], and Huber [1996]. We have found Huber's setup to be the most natural, and consequently we will often implicitly regard all schemes, formal schemes and rigid spaces that appear in the following as adic spaces in the sense of Huber; his category naturally contains all of these categories as full subcategories.

The basic cohomological invariants of a compact complex manifold also exist in this setting. The analogue of singular cohomology is étale cohomology $H^i_{\text{ét}}(X, \mathbb{Z}_\ell)$, which is defined for any prime ℓ, including $\ell = p$. If $\ell \neq p$, it follows for example from work of Huber [1996, Proposition 0.5.3], that this is a finitely generated $\mathbb{Z}_\ell$-module. For $\ell = p$, this is also true by Scholze [2013a, Theorem 1.1], but the argument is significantly harder.

Moreover, one has de Rham cohomology groups $H^i_{\text{dR}}(X/C)$ and Hodge cohomology groups $H^i(X, \Omega^j_X)$, exactly as for compact complex manifolds. These are finite-dimensional by a theorem of Kiehl [1967]. By the definition of $H^i_{\text{dR}}(X/C)$ as the hypercohomology of the de Rham complex, one finds an E_1-spectral sequence

$$E_1^{ij} = H^j(X, \Omega^i_X) \Rightarrow H^{i+j}_{\text{dR}}(X/C)$$

called the Hodge-to-de Rham spectral sequence. In complex geometry, a basic consequence of Hodge theory is that this spectral sequence degenerates at E_1 if X admits a Kähler metric. This assumption is not necessary in p-adic geometry:

Theorem 2.1 (Scholze [2013a, Corollary 1.8], Bhatt, Morrow, and Scholze [2016, Theorem 13.12])**.** *For any proper smooth rigid-analytic space X over C, the Hodge-to-de Rham spectral sequence*

$$E_1^{ij} = H^j(X, \Omega^i_X) \Rightarrow H^{i+j}_{\text{dR}}(X/C)$$

degenerates at E_1. Moreover, for all $i \geq 0$,

$$\sum_{j=0}^{i} \dim_C H^{i-j}(X, \Omega^j_X) = \dim_C H^i_{\text{dR}}(X/C) = \dim_{\mathbb{Q}_p} H^i_{\text{ét}}(X, \mathbb{Q}_p)\,.$$

Fortunately, the Hodge-to-de Rham spectral sequence does degenerate for the Hopf surface – and the examples of nondegeneration such as the Iwasawa manifolds do not have p-adic analogues.

Over the complex numbers, the analogue of the equality

$$\dim_C H^i_{\text{dR}}(X/C) = \dim_{\mathbb{Q}_p} H^i_{\text{ét}}(X, \mathbb{Q}_p)$$

follows from the comparison isomorphism between singular and de Rham cohomology. In the p-adic case, the situation is slightly more complicated, and the comparison isomorphism only exists after extending scalars to Fontaine's field of p-adic periods B_{dR}. If X is only defined over C, it is nontrivial to formulate the correct statement, as there is no natural map $C \to B_{\text{dR}}$ along which one can extend scalars; the correct statement is Theorem 6.3 below.

There is however a different way to obtain the desired equality of dimensions. This relies on the Hodge–Tate spectral sequence, a form of which is implicit in Faltings's proof of the Hodge–Tate decomposition, Faltings [1988].

Theorem 2.2 (Scholze [2013b, Theorem 3.20], Bhatt, Morrow, and Scholze [2016, Theorem 13.12])**.** *For any proper smooth rigid-analytic space X over C, there is a Hodge–Tate spectral sequence*

$$E_2^{ij} = H^i(X, \Omega^j_X)(-j) \Rightarrow H^{i+j}_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} C$$

that degenerates at E_2.

Here, $(-j)$ denotes a Tate twist, which becomes important when one wants to make everything Galois-equivariant. Note that the Hodge cohomology groups appear in the other order than in the Hodge-to-de Rham spectral sequence.

Remark 2.3. If X is the base change of a proper smooth rigid space defined over a discretely valued field $K \subset C$, then everything in sight carries a Galois action, and it follows from the results of Tate [1967], that there is a unique Galois-equivariant splitting of the abutment filtration, leading to a Galois-equivariant isomorphism

$$H^i_{\text{ét}}(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} C \cong \bigoplus_{j=0}^{i} H^{i-j}(X, \Omega^j_X)(-j)\,,$$

answering a question of Tate [ibid., Section 4.1, Remark]. However, this isomorphism does not exist in families. This is analogous to the Hodge decomposition over the complex numbers that does not vary holomorphically in families.

An interesting question is whether Hodge symmetry could still hold under some condition on X. Such an analogue of the Kähler condition has recently been proposed by Li [2017]. In joint work with Hansen and Li [2017], they state the following conjecture that they prove in the case $i + j = 1$.

Conjecture 2.4. *Let X be a proper smooth rigid-analytic variety that admits a formal model $\mathfrak{X}$ whose special fibre is projective. Then for all $i, j \geq 0$, one has*

$$\dim_C H^i(X, \Omega^j_X) = \dim_C H^j(X, \Omega^i_X).$$

The condition is indeed analogous to the Kähler condition; in Arakelov theory, the analogue of a metric is a formal model, and the positivity condition on the Kähler metric finds a reasonable analogue in the condition that the special fibre is projective.

Note that in particular it follows from the results of Hansen–Li that for any proper formal model $\mathfrak{X}$ of the Hopf surface (which exist by Raynaud's theory of formal models), the special fibre is a non-projective (singular) proper surface.

From the Hodge-to-de Rham and the Hodge–Tate spectral sequence, one obtains abutments filtrations that we call the Hodge–de Rham filtration and the Hodge–Tate filtration. Their variation in families defines interesting period maps as we will recall in the next sections.

3 Period maps from de Rham cohomology

First, we recall the more classical case of the period maps arising from the variation of the Hodge filtration on de Rham cohomology. For simplicity, we will discuss the moduli space $\mathfrak{M}/\mathbb{Z}$ of elliptic curves. If E is a an elliptic curve over the complex numbers, then $H^0(E, \Omega^1_E) = (\operatorname{Lie} E)^*$ is the dual of the Lie algebra, $H^1(E, \mathcal{O}_E) = \operatorname{Lie} E^*$ is the Lie algebra of the dual elliptic curve E^* (which, for elliptic curves, is canonically isomorphic to E itself), and the Hodge-de Rham filtration is a short exact sequence

$$0 \to (\operatorname{Lie} E)^* \to H^1_{\text{dR}}(E/\mathbb{C}) \to \operatorname{Lie} E^* \to 0\,,$$

where $H^1_{\mathrm{dR}}(E/\mathbb{C}) \cong H^1_{\mathrm{sing}}(E,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$. The classical period map takes the form

$$\begin{tikzcd} & \widetilde{\mathfrak{M}} \arrow[dl] \arrow[dr] & \\ \mathfrak{M}(\mathbb{C}) & & \mathbb{H}^{\pm} = \mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R}) \end{tikzcd}$$

where $\widetilde{\mathfrak{M}} \to \mathfrak{M}(\mathbb{C})$ is the $\mathrm{GL}_2(\mathbb{Z})$-torsor parametrizing trivializations of the first singular cohomology of the elliptic curve. Given an elliptic curve $E/\mathbb{C}$ with such a trivialization $H^1(E,\mathbb{Z}) \cong \mathbb{Z}^2$, the Hodge filtration $(\mathrm{Lie}\, E)^* \subset H^1_{\mathrm{sing}}(E,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathbb{C}^2$ defines a point of $\mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R})$ as

$$H^1_{\mathrm{sing}}(E,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \cong H^1_{\mathrm{dR}}(E/\mathbb{C}) \cong (\mathrm{Lie}\, E)^* \oplus \overline{(\mathrm{Lie}\, E)^*}\,.$$

If now E is an elliptic curve over the algebraically closed p-adic field C, then we still have the identifications $H^0(E,\Omega^1_E) = (\mathrm{Lie}\, E)^*$, $H^1(E,\mathcal{O}_E) = \mathrm{Lie}\, E^*$ and the Hodge-de Rham filtration

$$0 \to (\mathrm{Lie}\, E)^* \to H^1_{\mathrm{dR}}(E/C) \to \mathrm{Lie}\, E^* \to 0\,.$$

To construct period maps, we need to fix a point $x \in \mathfrak{M}(\overline{\mathbb{F}}_p)$. Let $\breve{\mathbb{Q}}_p$ be the completion of the maximal unramified extension of $\mathbb{Q}_p$, and consider the rigid space

$$\breve{\mathfrak{M}} = \mathfrak{M}^{\mathrm{rig}}_{\breve{\mathbb{Q}}_p}\,.$$

Let $U_x \subset \breve{\mathfrak{M}}$ be the tube of x, i.e. the open subspace of all points specializing to x. This is isomorphic to an open unit ball $\mathbb{D} = \{z \mid |z| < 1\}$. Then the crystalline nature of de Rham cohomology (in particular, the connection) imply that the de Rham cohomology $H^i_{\mathrm{dR}}(E_{\tilde{x}})$ is canonically identified for all $\tilde{x} \in U_x$. Now the variation of the Hodge filtration defines a period map

$$\pi_x : U_x \cong \mathbb{D} \to \breve{\mathbb{P}}^1 = (\mathbb{P}^1_{\breve{\mathbb{Q}}_p})^{\mathrm{rig}}\,.$$

There are two cases to consider here. If x corresponds to an ordinary elliptic curve, then there is an identification $U_x \cong \mathbb{D}$ such that the map π_x is given by the logarithm map

$$z \mapsto \log(1+z) : U_x = \mathbb{D} = \{z \mid |z| < 1\} \to \breve{\mathbb{A}}^1 \subset \breve{\mathbb{P}}^1\,.$$

This map is an étale covering map onto $\breve{\mathbb{A}}^1$, and the geometric fibres are given by copies of $\mathbb{Q}_p/\mathbb{Z}_p$; for example, the fibre over 0 is given by all p-power roots of unity minus 1, i.e. $z = \zeta_{p^r} - 1$. Note that this implies in particular that $(\mathbb{A}^1_{\mathbb{C}_p})^{\mathrm{rig}}$ has interesting (non-finite) étale coverings, contrary both to the scheme case and the case over the complex numbers.

If x corresponds to a supersingular elliptic curve, then π_x becomes a map

$$\pi_x : U_x \cong \mathbb{D} \to \breve{\mathbb{P}}^1\,.$$

In this case, the map is known as the Gross–Hopkins period map, Hopkins and Gross [1994]. It is an étale covering map of $\breve{\mathbb{P}}^1$ whose geometric fibres are given by copies

of $\mathrm{GL}_2(\mathbb{Q}_p)_1/\mathrm{GL}_2(\mathbb{Z}_p)$, where $\mathrm{GL}_2(\mathbb{Q}_p)_1 \subset \mathrm{GL}_2(\mathbb{Q}_p)$ is the open subgroup of all $g \in \mathrm{GL}_2(\mathbb{Q}_p)$ with $\det g \in \mathbb{Z}_p^\times$.

It is important to note that these maps π_x for varying x cannot be assembled into a single map from $\breve{\mathfrak{M}}$ towards $\breve{\mathbb{P}}^1$, contrary to the case over the complex numbers. We will however see a global period map in the next section.

These examples are the basic examples of *local Shimura varieties*, which are associated with local Shimura data, Rapoport and Viehmann [2014]:

(i) A reductive group G over $\mathbb{Q}_p$.

(ii) A conjugacy class of minuscule cocharacters $\mu : \mathbb{G}_m \to G_{\overline{\mathbb{Q}}_p}$, defined over the reflex field E (a finite extension of $\mathbb{Q}_p$).

(iii) A σ-conjugacy class $b \in B(G, \mu)$.

We will say something about datum (iii) in Theorem 6.5 below. In our example, we have $G = \mathrm{GL}_2$, and μ is the conjugacy class of $t \mapsto \mathrm{diag}(t, 1)$. In this case, $B(G, \mu)$ contains exactly two elements, corresponding to the cases of ordinary and supersingular elliptic curves, respectively.

Let $\breve{E}$ be the completion of the maximal unramified extension of E. Corresponding to G and μ, one gets a flag variety $\breve{\mathcal{F}\ell}_{G,\mu}$ over $\breve{E}$, which we consider as an adic space; in our example, this is $\breve{\mathbb{P}}^1$. The following theorem on the existence of local Shimura varieties proves a conjecture of Rapoport and Viehmann [ibid.].

Theorem 3.1 (Scholze and Weinstein [2017]). *There is a natural open subspace $\mathcal{F}\ell^a_{G,\mu} \subset \mathcal{F}\ell_{G,\mu}$, called the admissible locus, and an étale covering map*

$$\mathfrak{M}_{(G,b,\mu),K} \to \mathcal{F}\ell^a_{G,\mu} \subset \mathcal{F}\ell_{G,\mu}$$

with geometric fibres $G(\mathbb{Q}_p)/K$ for any compact open subgroup $K \subset G(\mathbb{Q}_p)$.

Remark 3.2. The geometric fibres differ from the examples above. In the examples, $\mathfrak{M}_{(G,b,\mu),\mathrm{GL}_2(\mathbb{Z}_p)}$ is a certain disjoint union of various U_x for x ranging through a p-power-isogeny class of elliptic curves.

The (towers of) spaces $(\mathfrak{M}_{(G,b,\mu),K})_{K \subset G(\mathbb{Q}_p)}$ are known as local Shimura varieties; their cohomology is expected to realize local Langlands correspondences, cf. Rapoport and Viehmann [2014]. This is one of our primary motivations for constructing these spaces.

In hindsight, one can say that in Rapoport and Zink [1996], Rapoport–Zink constructed these local Shimura varieties in many examples, by explicitly constructing $\mathfrak{M}_{(G,b,\mu),K}$ as a moduli space of p-divisible groups with extra structures such as endomorphisms and polarization, together with a quasi-isogeny to a fixed p-divisible group. An advantage of this approach is that, at least for special choices of K such as parahoric subgroups, one actually constructs formal schemes whose generic fibre is the local Shimura variety, and it is often easier to understand the formal scheme.

We will explain our construction of local Shimura varieties in Section 6, and in Section 7 we will deal with integral models of local Shimura varieties.

4 Period maps from étale cohomology

A different period map known as the Hodge–Tate period map parametrizes the variation of the Hodge–Tate filtration in families, and has been defined for general Shimura varieties of Hodge type in Scholze [2015].

If E is an elliptic curve over the algebraically closed p-adic field C, the Hodge–Tate filtration is given by a short exact sequence

$$0 \to \operatorname{Lie} E^* \to H^1_{\text{ét}}(E, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} C \to (\operatorname{Lie} E)^*(-1) \to 0 .$$

Note that the Lie algebra terms here appear in opposite order when compared to the situation over $\mathbb{C}$. To obtain the period map, we need to trivialize the middle term. Thus, using the space $\breve{\mathfrak{M}} = (\mathfrak{M}_{\breve{\mathbb{Q}}_p})^{\text{rig}}$ as in the last section, we consider the diagram

$$\begin{tikzcd} & \widetilde{\breve{\mathfrak{M}}} \arrow[dl] \arrow[dr, "\pi_{HT}"] & \\ \breve{\mathfrak{M}} & & \breve{\mathbb{P}}^1 . \end{tikzcd}$$

Here $\widetilde{\breve{\mathfrak{M}}} \to \breve{\mathfrak{M}}$ parametrizes isomorphisms $H^1_{\text{ét}}(E, \mathbb{Z}_p) \cong \mathbb{Z}_p^2$; this defines a $\mathrm{GL}_2(\mathbb{Z}_p)$-torsor. An essential difficulty here is that this space will be very big, and in particular highly nonnoetherian. By Scholze [ibid.], it gives a basic example of a perfectoid space in the sense of Scholze [2012].

Now $\widetilde{\breve{\mathfrak{M}}}$ admits the Hodge–Tate period map

$$\pi_{HT} : \widetilde{\breve{\mathfrak{M}}} \to \breve{\mathbb{P}}^1$$

sending a pair of an elliptic curve E/C with an isomorphism $H^1_{\text{ét}}(E, \mathbb{Z}_p) \cong \mathbb{Z}_p^2$ to the filtration $\operatorname{Lie} E^* \subset H^1_{\text{ét}}(E, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} C \cong C^2$.

The geometry of this map is very interesting. When restricted to Drinfeld's upper half-plane $\breve{\Omega}^2 = \breve{\mathbb{P}}^1 \setminus \mathbb{P}^1(\mathbb{Q}_p)$, the map is a pro-finite étale cover, while the fibres over points in $\mathbb{P}^1(\mathbb{Q}_p)$ are curves, so that the fibre dimension jumps. Let us discuss these two situations in turn.

If we fix a supersingular elliptic curve $x \in \mathfrak{M}(\overline{\mathbb{F}}_p)$, then we can restrict the Hodge–Tate period map to

$$\widetilde{U}_x = U_x \times_{\breve{\mathfrak{M}}} \widetilde{\breve{\mathfrak{M}}} ,$$

and arrive at the following picture:

$$\begin{tikzcd} & & \widetilde{U}_x \arrow[dl] \arrow[dr, "\pi_{HT}"] & \\ & U_x \arrow[dl, "\pi_x"'] & & \\ \breve{\mathbb{P}}^1 & & & \breve{\Omega}^2 . \end{tikzcd}$$

Here, the map $\widetilde{U}_x \to U_x$ is a $\mathrm{GL}_2(\mathbb{Z}_p)$-torsor, while $U_x \to \breve{\mathbb{P}}^1$ has geometric fibers $\mathrm{GL}_2(\mathbb{Q}_p)_1/\mathrm{GL}_2(\mathbb{Z}_p)$; in fact, in total $\widetilde{U}_x \to \breve{\mathbb{P}}^1$ is a $\mathrm{GL}_2(\mathbb{Q}_p)_1$-torsor. To restore full $\mathrm{GL}_2(\mathbb{Q}_p)$-equivarience, we consider the perfectoid space

$$\mathfrak{M}_{LT,\infty} = \widetilde{U}_x \times^{\mathrm{GL}_2(\mathbb{Q}_p)_1} \mathrm{GL}_2(\mathbb{Q}_p) \cong \bigsqcup_{\mathbb{Z}} \widetilde{U}_x ,$$

which is known as the Lubin–Tate tower at infinite level, cf. Scholze and Weinstein [2013], Weinstein [2016]. On the other hand, $\widetilde{U}_x \to \breve{\Omega}^2$ turns out to be an $\mathcal{O}_D^\times$-torsor, where $D/\mathbb{Q}_p$ is the quaternion algebra. Here, the $\mathcal{O}_D$-action arises from the identification $\mathcal{O}_D = \mathrm{End}(E_x) \otimes_{\mathbb{Z}} \mathbb{Z}_p$, where E_x is the supersingular elliptic curve corresponding to x; by functoriality, this acts on the deformation space of x, and thus also on U_x and $\widetilde{U}_x$. In terms of the Lubin–Tate tower,

$$\pi_{HT} : \mathfrak{M}_{LT,\infty} \to \breve{\Omega}^2$$

is a $D^\times$-torsor.

We see that $\mathfrak{M}_{LT,\infty}$ has two different period morphisms, corresponding to the Hodge-de Rham filtration and the Hodge–Tate filtration. This gives the isomorphism between the Lubin–Tate and Drinfeld tower at infinite level, Faltings [2002], Fargues, Genestier, and V. Lafforgue [2008],

$$\begin{tikzcd} & \mathfrak{M}_{LT,\infty} \arrow[dl, "\mathrm{GL}_2(\mathbb{Q}_p)"'] \arrow[r, equal] & \mathfrak{M}_{Dr,\infty} \arrow[dr, "D^\times"] & \\ \breve{\mathbb{P}}^1 & & & \breve{\Omega}^2 . \end{tikzcd}$$

A similar duality theorem holds true for any local Shimura variety for which b is basic. The local Shimura datum (G, b, μ) then has a dual datum $(\widehat{G}, \widehat{b}, \widehat{\mu})$, where $\widehat{G} = J_b$ is the σ-centralizer of b, $\widehat{b} = b^{-1} \in J_b$, and $\widehat{\mu} = \mu^{-1}$ under the identification $G_{\overline{\mathbb{Q}}_p} \cong \widehat{G}_{\overline{\mathbb{Q}}_p}$.

Theorem 4.1 (Scholze and Weinstein [2013], Scholze and Weinstein [2017]). *There is a natural isomorphism*

$$\varprojlim_{K \subset G(\mathbb{Q}_p)} \mathfrak{M}_{(G,b,\mu),K} \cong \varprojlim_{\widehat{K} \subset \widehat{G}(\mathbb{Q}_p)} \mathfrak{M}_{(\widehat{G},\widehat{b},\widehat{\mu}),\widehat{K}} .$$

Remark 4.2. This proves a conjecture of Rapoport and Zink [1996, Section 5.54]. One has to be careful with the notion of inverse limits here, as inverse limits in adic spaces do not exist in general. One interpretation is to take the inverse limit in the category of diamonds discussed below.

On the other hand, we can restrict the Hodge–Tate period map to $\mathbb{P}^1(\mathbb{Q}_p)$. In this case, the fibres of π_{HT} are curves. More precisely, consider the Igusa curve Ig parametrizing ordinary elliptic curves E in characteristic p together with an isomorphism $E[p^\infty]_{\text{ét}} \cong \mathbb{Q}_p/\mathbb{Z}_p$ of the of the étale quotient $E[p^\infty]_{\text{ét}}$ of the p-divisible group $E[p^\infty]$. Its perfection $\mathrm{Ig}^{\mathrm{perf}}$ lifts uniquely to a flat formal scheme $\mathfrak{Ig}$ over $\breve{\mathbb{Z}}_p$, and then, modulo boundary issues, $\pi_{HT}^{-1}(x)$ is the generic fibre of $\mathfrak{Ig}$.

Let us state these results about the Hodge–Tate period map for a general Shimura variety of Hodge type.

Theorem 4.3 (Scholze [2015], Caraiani and Scholze [2017]). *Consider a Shimura variety* Sh_K, $K \subset G(\mathbb{A}_f)$, *of Hodge type, associated with some reductive group* $G/\mathbb{Q}$ *and Shimura data, including the conjugacy class of cocharacters* μ *with field of definition* E. *Let* $\mathcal{F}\ell_{G,\mu} = G/P_\mu$ *be the corresponding flag variety.*[2] *Fix a prime* p *and a prime* $\mathfrak{p}$ *of* E *dividing* p.

[2] There are actually two choices for this flag variety; we refer to Caraiani and Scholze [2017] for a discussion of which one to choose.

(i) *For any compact open subgroup $K^p \subset G(\mathbb{A}_f^p)$, there is a unique perfectoid space Sh_{K^p} over $E_{\mathfrak{p}}$ such that*

$$\mathrm{Sh}_{K^p} \sim \varprojlim_{K_p \subset G(\mathbb{Q}_p)} (\mathrm{Sh}_{K_p K^p} \otimes E_{\mathfrak{p}})^{\mathrm{rig}} .$$

(ii) *There is a $G(\mathbb{Q}_p)$-equivariant Hodge–Tate period map*

$$\pi_{HT} : \mathrm{Sh}_{K^p} \to \mathcal{F}\ell_{G,\mu} ,$$

where we consider the right-hand side as an adic space over $E_{\mathfrak{p}}$.

(iii) *There is a Newton stratification*

$$\mathcal{F}\ell_{G,\mu} = \bigsqcup_{b \in B(G,\mu)} \mathcal{F}\ell^b_{G,\mu}$$

into locally closed strata.

(iv) *If the Shimura variety is compact and of PEL type, then if $\overline{x} \in \mathcal{F}\ell^b_{G,\mu}$ is a geometric point, the fibre $\pi_{HT}^{-1}(\overline{x})$ is the canonical lift of the perfection of the Igusa variety associated with b.*

We note that the Newton strata in (iii) are only defined on the adic space: sometimes they are nonempty but have no classical points!

5 Applications to Langlands reciprocity

The geometry of the Hodge–Tate period map has been used to obtain new results on the Langlands conjectures relating automorphic forms and Galois representations, especially in the case of torsion coefficients.

Let us first recall the results obtained in Scholze [2015]. For any reductive group G over $\mathbb{Q}$ and a congruence subgroup $\Gamma \subset G(\mathbb{Q})$, one can look at the locally symmetric $X_\Gamma = \Gamma \backslash X$, where X is the symmetric space for $G(\mathbb{R})$. To study Hecke operators, it is more convenient to switch to the adelic formalism, and consider

$$X_K = G(\mathbb{Q}) \backslash (X \times G(\mathbb{A}_f)/K)$$

for a compact open subgroup $K \subset G(\mathbb{A}_f)$, assumed sufficiently small from now on. The cohomology $H^i(X_K, \mathbb{C})$ with complex coefficients can be computed in terms of automorphic forms. By the Langlands correspondence, one expects associated Galois representations. Much progress was made on these questions in case X is a hermitian symmetric space, so that the X_K are algebraic varieties over number fields, and their étale cohomology gives Galois representations. However, the groups $G = \mathrm{GL}_n$ for $n > 2$ are not of this type.

It was conjectured by Grunewald in the 70's and later more precisely Ash [1992], that this relation between cohomology and Galois representations extends to the full integral cohomology groups $H^i(X_K, \mathbb{Z})$, including their torsion subgroups, which can be enormous, especially in the case of hyperbolic 3-manifolds, cf. Bergeron and Venkatesh [2013].

Theorem 5.1 (Scholze [2015]). *Assume that $G = \mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_n$ for some totally real or CM field F. Consider the abstract Hecke algebra $\mathbb{T}$ acting on X_K, generated by Hecke operators at good primes, and let $\mathbb{T}_K \subset \mathrm{End}(\bigoplus_i H^i(X_K, \mathbb{Z}))$ be the image of $\mathbb{T}$. For any maximal ideal $\mathfrak{m} \subset \mathbb{T}_K$, there is a continuous semisimple representation*

$$\overline{\rho}_{\mathfrak{m}} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\mathbb{T}_K/\mathfrak{m})$$

which is unramified at good primes, with Frobenius eigenvalues determined in terms of Hecke operators. If $\overline{\rho}_{\mathfrak{m}}$ is absolutely irreducible, then there is a nilpotent ideal $I_{\mathfrak{m}} \subset \mathbb{T}_{K,\mathfrak{m}}$ in the $\mathfrak{m}$-adic completion $\mathbb{T}_{K,\mathfrak{m}}$ of $\mathbb{T}_K$ whose nilpotence degree is bounded in terms of n and $[F : \mathbb{Q}]$, such that there is a continuous representation

$$\rho_{\mathfrak{m}} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\mathbb{T}_{K,\mathfrak{m}}/I_{\mathfrak{m}})$$

which is unramified at good primes, with the characteristic polynomials of Frobenius elements determined in terms of Hecke operators.

In particular, for all cohomological automorphic representations π of G, there exists a corresponding continuous semisimple Galois representation $\rho_\pi : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ unramified at good primes.

The final part of this theorem was proved previously by Harris, Lan, Taylor, and Thorne [2016]. The general strategy is to realize X_K as a boundary component of the Borel–Serre compactification of a Shimura variety $\tilde{X}_{\tilde{K}}$ associated with a quasisplit symplectic or unitary group, and use the known existence of Galois representations for cusp forms on that space. The key part of the argument then is to show that all torsion cohomology classes on $\tilde{X}_{\tilde{K}}$ can be lifted to characteristic 0, which is done using certain subtle results from p-adic Hodge theory in Scholze [2015].

For applications, one needs a precise understanding of the behaviour of the Galois representations at bad primes as well, in particular for primes dividing ℓ. To attack such questions, a better understanding of the torsion in $\tilde{X}_{\tilde{K}}$ is necessary. This has been obtained recently in joint work with Caraiani.

Theorem 5.2 (Caraiani and Scholze [2015]). *Let $\widetilde{\mathbb{T}}$ be the Hecke algebra acting on the cohomology of $\tilde{X}_{\tilde{K}}$. Assume that the maximal ideal $\widetilde{\mathfrak{m}} \subset \widetilde{\mathbb{T}}$ is generic. Then*

$$H^i_c(\tilde{X}_{\tilde{K}}, \mathbb{Z}_\ell)_{\widetilde{\mathfrak{m}}} = 0$$

for $i > d = \dim_{\mathbb{C}} \tilde{X}_{\tilde{K}}$.

Essentially by Poincaré duality, this implies that also $H^i(\tilde{X}_{\tilde{K}}, \mathbb{Z}_\ell)_{\widetilde{\mathfrak{m}}} = 0$ for $i < d$, and that $H^d(\tilde{X}_{\tilde{K}}, \mathbb{Z}_\ell)_{\widetilde{\mathfrak{m}}}$ is ℓ-torsion free. From the realization of X_K in the boundary of $\tilde{X}_{\tilde{K}}$, one gets a long exact sequence

$$\ldots \to H^d(\tilde{X}_{\tilde{K}}, \mathbb{Z}_\ell)_{\widetilde{\mathfrak{m}}} \to H^d((X_K)', \mathbb{Z}_\ell)_{\widetilde{\mathfrak{m}}} \to H^{d+1}_c(\tilde{X}_{\tilde{K}}, \mathbb{Z}_\ell)_{\widetilde{\mathfrak{m}}} = 0 \to \ldots,$$

where $(X_K)'$ is a torus bundle over X_K, and the source is ℓ-torsion free. Now the general strategy is to move any cohomology class in $H^i(X_K, \mathbb{Z}_\ell)_{\mathfrak{m}}$ to $H^d((X_K)', \mathbb{Z}_\ell)_{\widetilde{\mathfrak{m}}}$ by using the torus bundle to shift cohomological degrees, lift it to $H^d(\tilde{X}_{\tilde{K}}, \mathbb{Z}_\ell)_{\widetilde{\mathfrak{m}}}$, and

then use that this group injects into its rationalization, which can be expressed in terms of automorphic forms on $\tilde{G}$.

Let us comment on the proof of Theorem 5.2. The result works for more general Shimura varieties,[3] so let us change the notation. Let $(\mathrm{Sh}_K)_{K\subset G(\mathbb{A}_f)}$ be a Shimura variety corresponding to a reductive group G over $\mathbb{Q}$ together with some extra data, including a conjugacy class of minuscule cocharacters $\mu : \mathbb{G}_m \to G$, defined over the reflex field E. Fix a prime $p \neq \ell$ so that a certain genericity condition on the fixed maximal ideal of the Hecke algebra holds true at p, and a (sufficiently small) tame level $K^p \subset G(\mathbb{A}_f^p)$. We get (the minimal compactification of) the perfectoid Shimura variety at infinite level,

$$\mathrm{Sh}^*_{K^p} \sim \varprojlim_{K_p} (\mathrm{Sh}^*_{K_pK^p} \otimes E_{\mathfrak{p}})^{\mathrm{rig}} ,$$

where $\mathfrak{p}$ is a prime of E dividing p. We have the Hodge–Tate period map

$$\pi_{HT} : \mathrm{Sh}^*_{K^p} \to \mathcal{F}\ell_{G,\mu} .$$

The strategy now is to rewrite $R\Gamma_c(\mathrm{Sh}_{K^p}, \mathbb{Z}_\ell)$ as $R\Gamma(\mathcal{F}\ell_{G,\mu}, R\pi_{HT*}j_!\mathbb{Z}_\ell)$, where $j : \mathrm{Sh}_{K^p} \to \mathrm{Sh}^*_{K^p}$ is the open immersion. The Hecke operators away from p act trivially on the flag variety, so one can also rewrite

$$R\Gamma_c(\mathrm{Sh}_{K^p}, \mathbb{Z}_\ell)_{\mathfrak{m}} = R\Gamma(\mathcal{F}\ell_{G,\mu}, (R\pi_{HT*}j_!\mathbb{Z}_\ell)_{\mathfrak{m}}) .$$

The task is now to understand the sheaf $(R\pi_{HT*}j_!\mathbb{Z}_\ell)_{\mathfrak{m}}$. The first observation is that, with a suitable definition, it lies in ${}^pD^{\leq d}$ for the perverse t-structure; this uses that $R\pi_{HT*}$ is simultaneously affine and partially proper (but still has fibres of positive dimension – a phenomenon only possible in this highly nonnoetherian setup). The other observation is that its fibres are given by the cohomology of Igusa varieties, by using Theorem 4.3. The cohomology of Igusa varieties has been computed by Shin [2010], but only with $\overline{\mathbb{Q}}_\ell$-coefficients and in the Grothendieck group. Under the genericity condition, one finds that this always gives zero except if the point lies in $\mathcal{F}\ell_{G,\mu}(\mathbb{Q}_p) \subset \mathcal{F}\ell_{G,\mu}$. One can now play off these observations, which gives the conclusion that $(R\pi_{HT*}j_!\mathbb{Z}_\ell)_{\mathfrak{m}}$ is concentrated on the 0-dimensional space $\mathcal{F}\ell_{G,\mu}(\mathbb{Q}_p)$. Thus, there is no higher cohomology, and the bound $(R\pi_{HT*}j_!\mathbb{Z}_\ell)_{\mathfrak{m}} \in {}^pD^{\leq d}$ implies that $R\Gamma_c(\mathrm{Sh}_{K^p}, \mathbb{Z}_\ell)_{\mathfrak{m}} = R\Gamma(\mathcal{F}\ell_{G,\mu}, (R\pi_{HT*}j_!\mathbb{Z}_\ell)_{\mathfrak{m}})$ is in degrees $\leq d$.

One nice aspect of this strategy is that it describes the cohomology of Shimura varieties in terms of the cohomology of certain sheaves on the flag varieties $\mathcal{F}\ell_{G,\mu}$, and it becomes an interesting question to understand those sheaves themselves. This leads to a relation to the geometrization of the local Langlands correspondence conjectured by Fargues, cf. Section 6.

Finally, let us mention that these results have led to the following applications.

Theorem 5.3 (Allen et al. [2018]). *Let F be a CM field.*

1. *For any elliptic curve E over F, the L-function $L(E, s)$ has meromorphic continuation to $\mathbb{C}$. Moreover, E satisfies the Sato–Tate conjecture.*

[3] The compact case has appeared in Caraiani and Scholze [2017].

2. *For any cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_F)$ whose archimedean component is of parallel weight 2, the Ramanujan–Petersson conjecture holds true at all places.*

Theorem 5.3 is proved by establishing the potential automorphy of E and of all symmetric powers of E resp. π; here "potential" means after base change to (many) extensions $\tilde{F}/F$. Thus, the second part follows Langlands's strategy for establishing the Ramanujan–Petersson conjecture in general, and it is the first time that this conjecture has been proved in a case where the associated motive (or some closely related motive) is not known to exist, and in particular the proof does not invoke Deligne's theorem on the Weil conjectures.

For the proof of Theorem 5.3, one wants to see that "all" Galois representations arise via Theorem 5.1. This follows Wiles' strategy Wiles [1995]; more precisely, we use the variant proposed by Calegari and Geraghty [2017].

6 p-adic twistor theory

Further developments in p-adic geometry arose from the realization that the structures arising from the cohomology of proper smooth rigid-analytic varieties over the algebraically closed p-adic field C can be naturally organized into a modification of vector bundles on the Fargues–Fontaine curve in a way closely resembling a reinterpretation of Hodge theory in terms of vector bundles on the twistor-$\mathbb{P}^1$.

Let us first recall the statements over $\mathbb{C}$. Consider the twistor-$\mathbb{P}^1$, i.e. the nonsplit real form $\widetilde{\mathbb{P}^1_{\mathbb{R}}}$ of $\mathbb{P}^1$, which we will take to be given as the descent of $\mathbb{P}^1_{\mathbb{C}}$ to $\mathbb{R}$ via $z \mapsto -\frac{1}{z}$. We fix the point $\infty \in \widetilde{\mathbb{P}^1_{\mathbb{R}}}(\mathbb{C})$ corresponding to $\{0, \infty\} \subset \mathbb{P}^1_{\mathbb{C}}$. There is an action of the nonsplit real torus $U(1)$ (that we consider as an algebraic group) on $\widetilde{\mathbb{P}^1_{\mathbb{R}}}$ fixing ∞.

Proposition 6.1 (Simpson [1997]). *The category of $U(1)$-equivariant semistable vector bundles on $\widetilde{\mathbb{P}^1_{\mathbb{R}}}$ is equivalent to the category of pure $\mathbb{R}$-Hodge structures.*

Let us briefly recall the proof of this result. As the action of the algebraic group $U(1)$ on $\widetilde{\mathbb{P}^1_{\mathbb{R}}} \setminus \{\infty\}$ is simply transitive (as after base change to $\mathbb{C}$, it is the simply transitive action of $\mathbb{G}_m$ on itself), $U(1)$-equivariant vector bundles on $\widetilde{\mathbb{P}^1_{\mathbb{R}}} \setminus \{\infty\}$ are equivalent to $\mathbb{R}$-vector spaces V. Thus, given a $U(1)$-equivariant vector bundle $\mathcal{E}$ on $\widetilde{\mathbb{P}^1_{\mathbb{R}}}$, we get an $\mathbb{R}$-vector space V such that

$$\mathcal{E}|_{\widetilde{\mathbb{P}^1_{\mathbb{R}}} \setminus \{\infty\}} \cong V \otimes_{\mathbb{R}} \mathcal{O}_{\widetilde{\mathbb{P}^1_{\mathbb{R}}} \setminus \{\infty\}}$$

equivariantly for the $U(1)$-action. Identifying the completion of $\widetilde{\mathbb{P}^1_{\mathbb{R}}}$ at ∞ with $\operatorname{Spf} \mathbb{C}[[t]]$, we get at the completion at ∞ a $U(1)$-equivariant $\mathbb{C}[[t]]$-lattice $\Lambda \subset V \otimes_{\mathbb{R}} \mathbb{C}((t))$. However, this is equivalent to a decreasing filtration $\mathrm{Fil}^i\, V_{\mathbb{C}} \subset V_{\mathbb{C}}$, via the Rees construction

$$\Lambda = \sum_{i \in \mathbb{Z}} t^{-i} (\mathrm{Fil}^i\, V_{\mathbb{C}})[[t]] \subset V \otimes_{\mathbb{R}} \mathbb{C}((t)) .$$

One then checks that $\mathcal{E}$ is semistable precisely when $(V, \mathrm{Fil}^i\, V_{\mathbb{C}})$ defines a pure Hodge structure.

Thus, a description of the inverse functor is that one starts with a pure $\mathbb{R}$-Hodge structure $(V, \mathrm{Fil}^i V_{\mathbb{C}})$ and the trivial vector bundle $\mathcal{E}' = V \otimes_{\mathbb{R}} \mathcal{O}_{\widetilde{\mathbb{P}^1_{\mathbb{R}}}}$, and modifies it at ∞ via the lattice Λ defined above to obtain a new vector bundle $\mathcal{E}$ which is still $U(1)$-equivariant.

A very similar formalism exists in the p-adic case. Associated with the algebraically closed p-adic field C, there is the Fargues–Fontaine curve FF_C, which is a regular noetherian scheme over $\mathbb{Q}_p$ of Krull dimension 1 with a distinguished point $\infty \in \mathrm{FF}_C$. The completed local ring of FF_C at ∞ is Fontaine's period ring B^+_{dR}, a complete discrete valuation ring with residue field C and fraction field B_{dR}.

Theorem 6.2 (Fargues and Fontaine [2017]). *All residue fields of* FF_C *at closed points are algebraically closed nonarchimedean extensions of* $\mathbb{Q}_p$. *Any vector bundle on* FF_C *is a direct sum of stable vector bundles, and there is a unique stable vector bundle* $\mathcal{O}_{\mathrm{FF}_C}(\lambda)$ *for every rational slope* $\lambda = \frac{s}{r} \in \mathbb{Q}$, *which is of rank* r *and degree* s *(if* r *and* s *are chosen coprime with* $r > 0$*).*

Note that a similar result holds true for $\widetilde{\mathbb{P}^1_{\mathbb{R}}}$, but only half-integral slopes $\lambda \in \frac{1}{2}\mathbb{Z}$ occur in that case.

Given a proper smooth rigid-analytic space X over C, one can form the trivial vector bundle

$$\mathcal{E} = H^i_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathrm{FF}_C} .$$

Theorem 6.3 (Bhatt, Morrow, and Scholze [2016, Theorem 13.1, Theorem 13.8]). *There is a functorial* B^+_{dR}*-lattice*

$$\Xi = H^i_{\mathrm{crys}}(X/B^+_{\mathrm{dR}}) \subset H^i_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}} .$$

If $X = X_0 \hat{\otimes}_K C$ *for some discretely valued subfield* $K \subset C$, *then* $H^i_{\mathrm{crys}}(X/B^+_{\mathrm{dR}}) = H^i_{\mathrm{dR}}(X_0/K) \otimes_K B^+_{\mathrm{dR}}$ *and the inclusion comes from the de Rham comparison isomorphism.*

Thus, one can form a modification $\mathcal{E}'$ of $\mathcal{E}$ along ∞. The new vector bundle will in general be related to the (log-)crystalline cohomology of a (log-)smooth formal model. In other words, the comparison isomorphism between étale and crystalline cohomology can be understood in terms of a modification of vector bundles on the Fargues–Fontaine curve.

In fact, contrary to étale cohomology that is ill-behaved for spaces like the unit disc, it should be possible to define this modified vector bundle much more generally, without properness and without using formal models. The following conjecture arose from discussions of the author with Arthur-César le Bras.

Conjecture 6.4. *There is a cohomology theory* $H^i_{\mathrm{FF}_C}(X)$ *for quasicompact separated smooth rigid spaces* X *over* C *taking values in vector bundles on* FF_C. *If* X *has an overconvergent model* $X^\dagger$ *(for example, if* X *is affinoid), then the fibre of* $H^i_{\mathrm{FF}_C}(X)$ *at* ∞ *is the overconvergent de Rham cohomology* $H^i_{\mathrm{dR}}(X^\dagger/C)$ *in the sense of Große-Klönne [2004], and the completion of* $H^i_{\mathrm{FF}_C}(X)$ *at* ∞ *is the overconvergent crystalline cohomology* $H^i_{\mathrm{crys}}(X^\dagger/B^+_{\mathrm{dR}})$ *defined following Bhatt, Morrow, and Scholze [2016, Section 13].*

In other words, not only does de Rham cohomology lift naturally to B_{dR}^+ along the surjection $B_{\mathrm{dR}}^+ \to C$ as can be explained in terms of a version of crystalline cohomology, but it does actually deform into a vector bundle on all of FF_C.

Now we define for any local Shimura datum (G, b, μ) the local Shimura variety $\mathfrak{M}_{(G,b,\mu),K}$ in terms of modifications of G-torsors on the Fargues–Fontaine curve.

Theorem 6.5 (Fargues [2017]). *For any reductive group G over $\mathbb{Q}_p$, there is a natural bijection $b \mapsto \mathcal{E}_b$ between Kottwitz' set $B(G)$ of σ-conjugacy classes and the set of G-torsors on FF_C up to isomorphism. Moreover, $b \in B(G, \mu) \subset B(G)$ if and only if $\mathcal{E}_b$ can be written as a modification at ∞ of type μ of the trivial G-torsor $\mathcal{E}_1$.*

The final statement appears in Rapoport [2017]. Now we define

$$\mathfrak{M}_{(G,b,\mu),\infty}(C) = \varprojlim_K \mathfrak{M}_{(G,b,\mu),K}(C)$$

to be the set of all modifications $\mathcal{E}_1 \dashrightarrow \mathcal{E}_b$ at ∞ of type μ. The group $G(\mathbb{Q}_p)$ is the group of automorphisms of $\mathcal{E}_1$, and thus acts on this inverse limit; then for all $K \subset G(\mathbb{Q}_p)$,

$$\mathfrak{M}_{(G,b,\mu),K}(C) = \mathfrak{M}_{(G,b,\mu),\infty}(C)/K\,.$$

In order to make $\mathfrak{M}_{(G,b,\mu),K}$ into a rigid space, we need to define it as a moduli problem. For the rest of this section, we make extensive use of the theory of perfectoid spaces, and assume that the reader is familiar with it, cf. Scholze [2014]; some of the structures that appear now will however be motivated in the next section. We use that for any perfectoid space S of characteristic p together with an untilt $S^\sharp$ over $\mathbb{Q}_p$, one can construct a relative Fargues–Fontaine curve FF_S that is an adic space over $\mathbb{Q}_p$ and comes with a section $\infty : \operatorname{Spa} S^\sharp \to \mathrm{FF}_S$. Then $\mathfrak{M}_{(G,b,\mu),\infty}(S)$ parametrizes modifications $\mathcal{E}_1|_{\mathrm{FF}_S} \dashrightarrow \mathcal{E}_b|_{\mathrm{FF}_S}$ at ∞ of type μ as before. This defines a structure of a diamond, Scholze [2017b]:

Definition 6.6. *Let* Perf *be the category of perfectoid spaces of characteristic p. A diamond is a pro-étale sheaf Y on* Perf *that can be written as a quotient $Y = X/R$ of a perfectoid space X by a pro-étale equivalence relation $R \subset X \times X$.*

An example of a diamond is given by the sheaf $\operatorname{Spd} \mathbb{Q}_p$ that attaches to any perfectoid space S of characteristic p the set of all untilts $S^\sharp$ over $\mathbb{Q}_p$. More generally, if X is an adic space over $\mathbb{Q}_p$, one can define a diamond $X^\diamond$ whose S-valued points are given by an untilt $S^\sharp$ over $\mathbb{Q}_p$ together with a map $S^\sharp \to X$.

Theorem 6.7 (Scholze and Weinstein [2017]). *For any nonarchimedean field $L/\mathbb{Q}_p$, the functor $X \mapsto X^\diamond$ defines a fully faithful functor from the category of seminormal rigid spaces over L to the category of diamonds over* $\operatorname{Spd} L = (\operatorname{Spa} L)^\diamond$.

The diamond $\mathfrak{M}_{(G,b,\mu),K} = \mathfrak{M}_{(G,b,\mu),\infty}/\underline{K}$ over $\operatorname{Spd} \breve{E}$ *is the image of a smooth rigid space over $\breve{E}$ under this functor.*

The proof makes use of the results of Kedlaya [2010], and Kedlaya and Liu [2015], on families of vector bundles on the Fargues–Fontaine curve.

From here, it becomes natural to consider much more general spaces, parametrizing modifications of arbitrary G-bundles at several points with modifications of arbitrary type, not necessarily minuscule. Such spaces live over the base space $\operatorname{Spd}\mathbb{Q}_p \times \ldots \times \operatorname{Spd}\mathbb{Q}_p$ parametrizing the points of modification: These products give good meaning to the non-existent products $\operatorname{Spec}\mathbb{Q}_p \times_{\operatorname{Spec}\mathbb{F}_1} \ldots \times_{\operatorname{Spec}\mathbb{F}_1} \operatorname{Spec}\mathbb{Q}_p$. The resulting spaces are now in general just diamonds. The purpose of the foundational manuscript Scholze [2017b] is to develop a solid theory of étale cohomology for diamonds with the aim of using these spaces to obtain a general local Langlands correspondence.

These spaces are naturally organized into Hecke stacks acting on the stack Bun_G of G-bundles on the Fargues–Fontaine curve. Fargues realized that this gives rise to a picture perfectly resembling the geometric Langlands correspondence for classical smooth projective curves, Fargues [2016], Fargues [2018], Fargues and Scholze [n.d.]. This in particular involves perverse ℓ-adic sheaves on Bun_G whose pullback to the flag varieties $\mathcal{F}\ell_{G,\mu}$ should arise globally from the construction $R\pi_{HT*}\mathbb{Z}_\ell$ sketched in the previous section.

It is too early to state the expected theorems about the applications to the local Langlands correspondence, but let us mention that recently Kaletha and Weinstein [2017], have used the étale cohomology of diamonds as developed in Scholze [2017b] to prove Kottwitz' conjecture, Rapoport [1995], about the realization of the Jacquet–Langlands correspondence in the cohomology of local Shimura varieties.

Remark 6.8. As the theory of diamonds is critical to the foundations, as is the possibility of defining $\operatorname{Spd}\mathbb{Q}_p \times \operatorname{Spd}\mathbb{Q}_p$, let us quickly give a description of the product $\operatorname{Spd}\mathbb{Q}_p \times \operatorname{Spd}\mathbb{Q}_p$. Consider the open unit disc $\mathbb{D}_{\mathbb{Q}_p} = \{z \mid |z| < 1\}$ as embedded in the multiplicative group via $z \mapsto 1 + z$; then $\mathbb{D}_{\mathbb{Q}_p}$ itself is a group object. Let $\widetilde{\mathbb{D}}_{\mathbb{Q}_p}$ be the inverse limit of $\mathbb{D}_{\mathbb{Q}_p}$ along $z \mapsto (1+z)^p - 1$. Then $\widetilde{\mathbb{D}}_{\mathbb{Q}_p}$ is (pre)perfectoid, and admits a natural $\mathbb{Q}_p$-action. Now

$$\operatorname{Spd}\mathbb{Q}_p \times \operatorname{Spd}\mathbb{Q}_p = (\widetilde{\mathbb{D}}_{\mathbb{Q}_p} \setminus \{0\})^\diamond / \underline{\mathbb{Z}_p^\times}\,.$$

On the right-hand side, one factor of $\operatorname{Spd}\mathbb{Q}_p$ arises from the structure map $\widetilde{\mathbb{D}}_{\mathbb{Q}_p} \to \operatorname{Spa}\mathbb{Q}_p$, but the other factor gets realized in terms of the perfectoid punctured unit disc, and thus has become geometric.

These constructions in fact lead to a description of the absolute Galois group of $\mathbb{Q}_p$ as a geometric fundamental group:

Theorem 6.9 (Weinstein [2017])**.** *For any algebraically closed nonarchimedean field $C/\mathbb{Q}_p$, the étale fundamental group of $(\widetilde{\mathbb{D}}_C \setminus \{0\})/\underline{\mathbb{Q}_p^\times}$ agrees with the absolute Galois group of $\mathbb{Q}_p$.*

A search for a hypothetical space $\operatorname{Spec}\mathbb{Z} \times \operatorname{Spec}\mathbb{Z}$ thus seems closely related to a realization of the absolute Galois group of $\mathbb{Q}$ as a geometric fundamental group. For a step in this direction, let us mention the following result, which uses the ring of rational Witt vectors $W_{\mathrm{rat}}(R) \subset W_{\mathrm{big}}(R)$. For all $r \in R$, there is the Teichmüller lift $[r] \in W_{\mathrm{rat}}(R)$.

Theorem 6.10 (Kucharczyk and Scholze [2016]). *Let L be a field of characteristic 0 that contains all roots of unity, and fix an embedding $\mathbb{Q}/\mathbb{Z} \hookrightarrow L^\times$, $1/n \mapsto \zeta_n$. Then the category of finite extensions of L is equivalent to the category of connected finite coverings of the topological space*

$$X(L) \subset (\operatorname{Spec} W_{\mathrm{rat}}(L))(\mathbb{C})$$

that is the connected component singled out by the condition that $[\zeta_n]$ maps to $e^{2\pi i/n} \in \mathbb{C}$ for all $n \geq 1$.

In fact, $X(L)$ has a deformation retract to a compact Hausdorff space, which gives a realization of the absolute Galois group of L as the profinite fundamental group of a compact Hausdorff space! This formally implies that the absolute Galois group of L is torsion-free, cf. Kucharczyk and Scholze [ibid., Proposition 7.10].

The space $X(L)$ gives rise to certain non-profinitely complete structures on natural arithmetic invariants. If L is the cyclotomic extension of $\mathbb{Q}$, one can show that the actual fundamental group of $X(L)$ given by topological loops is a proper dense subgroup of the absolute Galois group. Moreover, it acts naturally on the set $\log \overline{\mathbb{Q}} \subset \mathbb{C}$ of logarithms of algebraic numbers, compatibly with $\exp : \log \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}^\times$. Similarly, for general L as in the theorem, the Čech cohomology groups $H^i(X(L), \mathbb{Z})$ give certain torsion-free non-profinitely completed abelian groups with

$$H^i(X(L), \mathbb{Z})/n = H^i(\mathrm{Gal}(\overline{L}/L), \mathbb{Z}/n\mathbb{Z})$$

for all $n \geq 1$, cf. Kucharczyk and Scholze [ibid., Theorem 1.8].

7 p-adic shtukas

Going further, one can reinterpret the linear-algebraic structures that appeared in the last section in terms of a p-adic analogue of shtukas. Let us first recall the basic definition of a shtuka from the function field case. There is a version of the following definitions for any reductive group G obtained by replacing vector bundles by G-torsors.

Definition 7.1. *Let X be a smooth projective curve over $\mathbb{F}_p$ and let S be a scheme over $\mathbb{F}_p$. A shtuka over S relative to X with legs at $x_1, \dots, x_n : S \to X$ is a vector bundle $\mathcal{E}$ over $S \times_{\mathbb{F}_p} X$ together with an isomorphism*

$$(\mathrm{Frob}_S^* \mathcal{E})|_{S \times_{\mathbb{F}_p} X \setminus \bigcup_{i=1}^n \Gamma_{x_i}} \cong \mathcal{E}|_{S \times_{\mathbb{F}_p} X \setminus \bigcup_{i=1}^n \Gamma_{x_i}},$$

where $\Gamma_{x_i} : S \hookrightarrow S \times_{\mathbb{F}_p} X$ is the graph of x_i.

Drinfeld used moduli spaces of shtukas with two legs to prove the global Langlands correspondence for GL_2, Drinfeld [1980]. This was generalized to GL_n, still using moduli spaces with two legs, by L. Lafforgue [2002]. Recently, V. Lafforgue has used all moduli spaces with an arbitrary number of legs simultaneously to prove the automorphic to Galois direction of the global Langlands correspondence for any reductive group G, V. Lafforgue [2012].

There is a corresponding notion of local shtuka, where one works with the local curve $X = \operatorname{Spf} \mathbb{F}_p[[t]]$. The legs are now parametrized by maps $x_1, \dots, x_n : S \to X = \operatorname{Spf} \mathbb{F}_p[[t]]$, i.e. locally nilpotent elements $t_i \in \mathcal{O}_S(S)$. As this contains no topological information, we pass to the world of rigid geometry. For simplicity, we discuss the case of geometric points, and so assume that $S = \operatorname{Spa} C$ for some complete algebraically closed nonarchimedean field C of characteristic p; we will later allow more general base spaces again. The legs are maps $S \to X$, which are parametrized by topologically nilpotent elements $t_i \in C$, i.e. $|t_i| < 1$. The fibre product $S \times_{\mathbb{F}_p} X$ becomes the open unit ball $\mathbb{D}_C = \{t \in C \mid |t| < 1\}$, and the legs give rise to points $x_i \in \mathbb{D}_C$. There is a Frobenius Frob_C acting on $\mathbb{D}_C$, coming from the Frobenius on C. This is not a map of rigid spaces over C, but it does exist in the category of adic spaces.

Definition 7.2. *A (local) shtuka over $S = \operatorname{Spa} C$ relative to $X = \operatorname{Spf} \mathbb{F}_p[[t]]$ with legs at $x_1, \dots, x_n : S \to X$ given by elements $t_i \in C$, $|t_i| < 1$, is a vector bundle $\mathcal{E}$ over $\mathbb{D}_C = S \times_{\mathbb{F}_p} X$ together with an isomorphism*

$$(\mathrm{Frob}_C^* \mathcal{E})|_{\mathbb{D}_C \setminus \{t_1, \dots, t_n\}} \cong \mathcal{E}|_{\mathbb{D}_C \setminus \{t_1, \dots, t_n\}}$$

that is meromorphic along the t_i.

The main observation of Scholze and Weinstein [2017] is that it is possible to give a mixed-characteristic version of this definition, where $X = \operatorname{Spf} \mathbb{Z}_p$. As a geometric point, one still takes $S = \operatorname{Spa} C$ where C is a complete algebraically closed nonarchimedean field of characteristic p. Naively, there are now no interesting maps $S = \operatorname{Spa} C \to X = \operatorname{Spf} \mathbb{Z}_p$; there is exactly one, which factors over $\operatorname{Spec} \mathbb{F}_p$. We will see momentarily how to solve this problem. But let us first face the other problem of defining a reasonable fibre product $S \times X$", where the product should be over $\operatorname{Spec} \mathbb{F}_p$. The basic insight is that for any perfect ring R, $\operatorname{Spec} R \times \operatorname{Spf} \mathbb{Z}_p$" should be given by $\operatorname{Spf} W(R)$ with the p-adic topology on $W(R)$. If R is itself an adic ring with the I-adic topology, then $\operatorname{Spf} R \times \operatorname{Spf} \mathbb{Z}_p = \operatorname{Spf} W(R)$", where $W(R)$ is equipped with the $(p, [I])$-adic topology.

In particular, we should set $\operatorname{Spf} \mathcal{O}_C \times \operatorname{Spf} \mathbb{Z}_p" = \operatorname{Spf}(W(\mathcal{O}_C))$. The ring $W(\mathcal{O}_C)$ plays an important role in p-adic Hodge theory, and is traditionally called A_{inf} there. If $\varpi \in C$ is a topologically nilpotent unit, then passing from $\operatorname{Spf} \mathcal{O}_C$ to its generic fibre $\operatorname{Spa} C$ amounts to inverting ϖ. Thus, we define the open subspace

$$\operatorname{Spa} C \times \operatorname{Spa} \mathbb{Z}_p" = \{[\varpi] \neq 0\} \subset \operatorname{Spa}(W(\mathcal{O}_C)) ,$$

which is a well-defined adic space. The following proposition shows that this behaves very much like a classical smooth rigid curve, except that it does not live over any base field.

Proposition 7.3 (Fargues and Fontaine [2017], Kedlaya [2016])**.** *For any connected open affinoid subspace*

$$U = \operatorname{Spa}(R, R^+) \subset \operatorname{Spa} C \times \operatorname{Spa} \mathbb{Z}_p" ,$$

the ring R is a principal ideal domain. For any maximal ideal $\mathfrak{m} \subset R$, the quotient $C_{\mathfrak{m}} = R/\mathfrak{m}$ is a complete algebraically closed nonarchimedean field. Moreover, there

is a canonical isomorphism

$$\varprojlim_{x \mapsto x^p} \mathcal{O}_{C_{\mathfrak{m}}}/p \cong \mathcal{O}_C ,$$

i.e. $C \cong C_{\mathfrak{m}}^\flat$ *is the tilt of* $C_{\mathfrak{m}}$ *in the terminology of Scholze [2012]. Conversely, for any complete algebraically closed nonarchimedean field* $C^\sharp$ *with an isomorphism* $(C^\sharp)^\flat \cong C$*, there is a unique maximal ideal* $\mathfrak{m} \subset R$ *(for any large enough open subset* U*) such that* $C^\sharp = R/\mathfrak{m}$*, compatibly with the identification of* $(C^\sharp)^\flat$ *with* C*.*

In other words, "classical points" of $\operatorname{Spa} C \times \operatorname{Spa} \mathbb{Z}_p$" parametrize untilts of C. This shows that we should reinterpret the data of the maps $x_i : S = \operatorname{Spa} C \to X = \operatorname{Spa} \mathbb{Z}_p$ as the data of untilts $C_1^\sharp, \dots, C_n^\sharp$ of C.

Definition 7.4. *A (local/p-adic) shtuka over* $S = \operatorname{Spa} C$ *relative to* $X = \operatorname{Spf} \mathbb{Z}_p$ *with legs at* $x_1, \dots, x_n$ *given by untilts* $C_1^\sharp, \dots, C_n^\sharp$ *of* C*, is a vector bundle* $\mathcal{E}$ *over* $\operatorname{Spa} C \times \operatorname{Spa} \mathbb{Z}_p$" *together with an isomorphism*

$$(\operatorname{Frob}_C^* \mathcal{E})|_{\operatorname{Spa} C \times \operatorname{Spa} \mathbb{Z}_p"\setminus\{x_1,\dots,x_n\}} \cong \mathcal{E}|_{\operatorname{Spa} C \times \operatorname{Spa} \mathbb{Z}_p"\setminus\{x_1,\dots,x_n\}}$$

that is meromorphic along the x_i*.*

In the case of one leg, these structures are closely related to the structures that appeared in the last section, by the following theorem of Fargues.

Theorem 7.5 (Scholze and Weinstein [2017])**.** *Assume that* $C^\sharp$ *is an untilt over* $\mathbb{Q}_p$*, let* $\infty : \operatorname{Spec} C^\sharp \to \mathrm{FF}_C$ *be the corresponding point of the Fargues–Fontaine curve,*[4] *and let* $B_{\mathrm{dR}}^+ = B_{\mathrm{dR}}^+(C^\sharp)$ *be the complete local ring at* ∞ *with quotient field* B_{dR}*. The following categories are equivalent.*

(i) *Shtukas over* $S = \operatorname{Spa} C$ *relative to* $X = \operatorname{Spf} \mathbb{Z}_p$ *with one leg at* $C^\sharp$*.*

(ii) *Pairs* (T, Ξ)*, where* T *is a finite free* $\mathbb{Z}_p$*-module and* $\Xi \subset T \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}$ *is a* B_{dR}^+*-lattice.*

(iii) *Quadruples* $(\mathcal{F}_1, \mathcal{F}_2, \beta, T)$*, where* $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *are two vector bundles on the Fargues–Fontaine curve* FF_C*,* $\beta : \mathcal{F}_1|_{\mathrm{FF}_C \setminus \{\infty\}} \cong \mathcal{F}_2|_{\mathrm{FF}_C \setminus \{\infty\}}$ *is an isomorphism, and* T *is a finite free* $\mathbb{Z}_p$*-module such that* $\mathcal{F}_1 = T \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathrm{FF}_C}$ *is the corresponding trivial vector bundle.*

(iv) *Breuil–Kisin–Fargues modules, i.e. finite free* $A_{\mathrm{inf}} = W(\mathcal{O}_C)$*-modules* M *together with a* φ*-linear isomorphism* $M[\frac{1}{\varphi^{-1}(\xi)}] \cong M[\frac{1}{\xi}]$*, where* $\xi \in A_{\mathrm{inf}}$ *is a generator of* $\ker(A_{\mathrm{inf}} \to \mathcal{O}_{C^\sharp})$*.*

Note that local Shimura varieties are parametrizing data of type (iii). The equivalence with (i) shows that one can regard local Shimura varieties as moduli spaces of shtukas, and it gives a possibility of formulating an integral model for the local Shimura varieties that we will now discuss.

[4] The Fargues–Fontaine curve depends only on the tilt of $C^\sharp$, but the point ∞ depends on $C^\sharp$.

In Scholze and Weinstein [ibid.], we construct moduli spaces of shtukas with any number of legs. However, already the case of one leg has important applications, so let us focus on this case. In addition to the group G over $\mathbb{Q}_p$, we need to fix a model $\mathcal{G}$ of G over $\mathbb{Z}_p$, and the most important case is when $\mathcal{G}$ is parahoric. To define the moduli problem, we need to bound the modification at the leg; this leads to a version of the affine Grassmannian. Here, $\operatorname{Spd}\mathbb{Z}_p$ denotes the functor on Perf taking any perfectoid space S of characteristic p to the set of untilts $S^\sharp$ over $\mathbb{Z}_p$. If $S = \operatorname{Spa}(R, R^+)$ is affinoid, then $S^\sharp = \operatorname{Spa}(R^\sharp, R^{\sharp+})$, and there is a natural ring $B_{\mathrm{dR}}^+(R^\sharp)$ that surjects onto $R^\sharp$ with kernel generated by some nonzerodivisor $\xi \in B_{\mathrm{dR}}^+(R^\sharp)$ such that $B_{\mathrm{dR}}^+(R^\sharp)$ is ξ-adically complete. This interpolates between the following cases:

1. If $R^\sharp = C^\sharp$ is an untilt over $\mathbb{Q}_p$ of an algebraically closed nonarchimedean field $R = C$ of characteristic p, then $B_{\mathrm{dR}}^+(C^\sharp)$ is Fontaine's ring considered previously.

2. If $R^\sharp = R$ is of characteristic p, then $B_{\mathrm{dR}}^+(R^\sharp) = W(R)$ is the ring of p-typical Witt vectors.

Definition 7.6. *Let $\mathcal{G}$ be a parahoric group scheme over $\mathbb{Z}_p$. The Beilinson–Drinfeld Grassmannian*

$$\mathrm{Gr}_{\mathcal{G},\operatorname{Spd}\mathbb{Z}_p} \to \operatorname{Spd}\mathbb{Z}_p$$

is the moduli problem on Perf *taking an affinoid perfectoid space $S = \operatorname{Spa}(R, R^+)$ of characteristic p to the set of untilts $S^\sharp = \operatorname{Spa}(R^\sharp, R^{\sharp+})$ over $\mathbb{Z}_p$ together with a $\mathcal{G}$-torsor over $B_{\mathrm{dR}}^+(R^\sharp)$ that is trivialized over $B_{\mathrm{dR}}(R^\sharp)$.*

In the following, we make use of the following proposition to identify perfect schemes or formal schemes as certain pro-étale sheaves. In both cases, one can define a functor $X \mapsto X^\diamond$, where $X^\diamond$ parametrizes untilts over X.

Proposition 7.7. *The functor $X \mapsto X^\diamond$ defines a fully faithful functor from the category of perfect schemes to the category of pro-étale sheaves on* Perf. *Similarly, for any nonarchimedean field L over $\mathbb{Q}_p$, the functor $X \mapsto X^\diamond$ from normal and flat formal schemes locally formally of finite type over $\operatorname{Spf}\mathcal{O}_L$ to pro-étale sheaves over $\operatorname{Spd}\mathcal{O}_L$ is fully faithful.*

The proof of the second part relies on a result of Lourenço [2017], that recovers formal schemes as in the proposition from their generic fibre and the perfection of their special fibre, together with the specialization map.

Theorem 7.8 (Zhu [2017], Bhatt and Scholze [2017], Scholze and Weinstein [2017])**.** *The special fibre of $\mathrm{Gr}_{\mathcal{G},\operatorname{Spd}\mathbb{Z}_p}$ is given by the Witt vector affine Grassmannian $\mathrm{Gr}_{\mathcal{G}}^W$ constructed in Zhu [2017] and Bhatt and Scholze [2017] that can be written as an increasing union of perfections of projective varieties along closed immersions. The generic fibre of $\mathrm{Gr}_{\mathcal{G},\operatorname{Spd}\mathbb{Z}_p}$ is the B_{dR}^+-affine Grassmannian of G, and can be written as an increasing union of proper diamonds along closed immersions.*

For a conjugacy class of minuscule cocharacters μ defined over E, there is a natural closed immersion

$$\mathcal{F}\ell_{G,\mu}^\diamond \hookrightarrow \mathrm{Gr}_{\mathcal{G},\operatorname{Spd}\mathbb{Z}_p} \times_{\operatorname{Spd}\mathbb{Z}_p} \operatorname{Spd} E\ .$$

We let

$$\mathbb{M}^{\mathrm{loc}}_{(\mathcal{G},\mu)} \subset \mathrm{Gr}_{\mathcal{G},\mathrm{Spd}\,\mathbb{Z}_p} \times_{\mathrm{Spd}\,\mathbb{Z}_p} \mathrm{Spd}\,\mathcal{O}_E$$

be the closure of $\mathcal{F}\ell^{\diamond}_{G,\mu}$. We conjecture that it comes from a normal and flat projective scheme over $\mathrm{Spec}\,\mathcal{O}_E$ under Proposition 7.7. This would give a group-theoretic definition of the local model in the theory of Shimura varieties, cf. e.g. Pappas [2018].

Definition 7.9. *Given local Shimura data (G, b, μ) and a parahoric model $\mathcal{G}$, let*

$$\mathfrak{M}^{\mathrm{int}}_{(\mathcal{G},b,\mu)} \to \mathrm{Spd}\,\breve{\mathcal{O}}_E$$

be the moduli problem taking $S \in \mathrm{Perf}$ to the set of untilts $S^\sharp$ over $\breve{\mathcal{O}}_E$ together with a $\mathcal{G}$-shtuka over $S \times \mathrm{Spa}\,\mathbb{Z}_p$" with one leg at $S^\sharp$ that is bounded by $\mathbb{M}^{\mathrm{loc}}_{(\mathcal{G},\mu)}$, and a trivialization of the shtuka at the boundary of $S \times \mathrm{Spa}\,\mathbb{Z}_p$" by $\mathcal{E}_b$.

For a precise formulation of the following result, we refer to Scholze and Weinstein [2017].

Theorem 7.10 (Scholze and Weinstein [ibid.])**.** *This definition recovers Rapoport–Zink spaces.*

Using this group-theoretic characterization of Rapoport–Zink spaces, we can obtain new isomorphisms between different Rapoport–Zink spaces.

Corollary 7.11. *The conjectures of Rapoport and Zink [2017], and Kudla, Rapoport, and Zink [n.d.], on alternative descriptions of the Drinfeld moduli problem hold true.*

In particular, by Kudla, Rapoport, and Zink [ibid.], one gets an integral version and moduli-theoretic proof of Čerednik's p-adic uniformization theorem, Cerednik [1976].

To prove Theorem 7.10, one needs to see that p-adic shtukas are related to the cohomology of algebraic varieties or p-divisible groups. This is the subject of integral p-adic Hodge theory that we will discuss next.

8 Integral p-adic Hodge theory

The following question arises naturally from Theorem 7.5. As it is more natural in this section, let us change notation, and start with an algebraically closed p-adic field C and consider shtukas over its tilt $C^\flat$ with one leg at the untilt C of $C^\flat$. Given a proper smooth rigid-analytic space X over C, consider $T = H^i_{\text{ét}}(X, \mathbb{Z}_p)/(\text{torsion})$ with the finite free B^+_{dR}-module $\Xi = H^i_{\mathrm{crys}}(X/B^+_{\mathrm{dR}}) \subset T \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}$ from Theorem 6.3. By Theorem 7.5, there is a corresponding Breuil–Kisin–Fargues module $H^i_{A_{\mathrm{inf}}}(X)$; we normalize it here so that the Frobenius is an isomorphism after inverting ξ resp. $\varphi(\xi)$. Can one give a direct cohomological construction of this?

We expect that without further input, the answer is no; in fact, Fargues's equivalence is not exact, and does manifestly not pass to the derived category. However, in joint work with Bhatt and Morrow, Bhatt, Morrow, and Scholze [2016], we show that it is possible once a proper smooth formal model $\mathfrak{X}$ of X is given.[5]

[5] All of these results were recently extended to the case of semistable reduction by Cesnavičius and Koshikawa [2017].

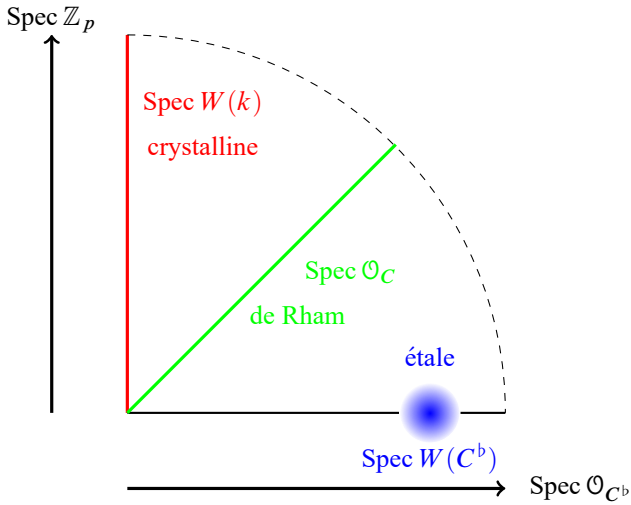

Figure 1: A picture of some parts of Spec $A_{\mathrm{inf}} = \mathrm{Spec}\,\mathcal{O}_{C^\flat} \times \mathrm{Spec}\,\mathbb{Z}_p$".

Theorem 8.1 (Bhatt, Morrow, and Scholze [ibid.]). *Let $\mathfrak{X}$ be a proper smooth formal scheme over $\mathrm{Spf}\,\mathcal{O}_C$ with generic fibre X. There is a perfect complex $R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X})$ of A_{inf}-modules together with a φ-linear map $\varphi : R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X}) \to R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X})$ that becomes an isomorphism after inverting ξ resp. $\varphi(\xi)$. Each $H^i_{A_{\mathrm{inf}}}(\mathfrak{X})$ is a finitely presented A_{inf}-module that becomes free over $A_{\mathrm{inf}}[\frac{1}{p}]$ after inverting p. Moreover, one has the following comparison results.*

(i) *Crystalline comparison: $R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X}) \overset{\mathbb{L}}{\otimes}_{A_{\mathrm{inf}}} W(k) \cong R\Gamma_{\mathrm{crys}}(\mathfrak{X}_k/W(k))$, φ-equivariantly, where k is the residue field of $\mathcal{O}_C$.*

(ii) *De Rham comparison: $R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X}) \overset{\mathbb{L}}{\otimes}_{A_{\mathrm{inf}}} \mathcal{O}_C \cong R\Gamma_{\mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C)$.*

(iii) *Étale comparison: $R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X}) \otimes_{A_{\mathrm{inf}}} W(C^\flat) \cong R\Gamma_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(C^\flat)$, φ-equivariantly.*

Moreover, if $H^i_{\mathrm{crys}}(\mathfrak{X}_k/W(k))$ is p-torsion free[6]*, then $H^i_{A_{\mathrm{inf}}}(\mathfrak{X})$ is finite free over A_{inf}, and agrees with $H^i_{A_{\mathrm{inf}}}(X)$ as defined above.*[7]

The theorem implies a similar result for p-divisible groups, which is the key input into the proof of Theorem 7.10. On the other hand, the theorem has direct consequences for the behaviour of torsion under specialization from characteristic 0 to charateristic p.

Corollary 8.2. *For any $i \geq 0$, one has $\dim_k H^i_{\mathrm{dR}}(\mathfrak{X}_k/k) \geq \dim_{\mathbb{F}_p} H^i_{\text{ét}}(X, \mathbb{F}_p)$, and*

$$\mathrm{length}_{W(k)} H^i_{\mathrm{crys}}(\mathfrak{X}_k/W(k))_{\mathrm{tor}} \geq \mathrm{length}_{\mathbb{Z}_p} H^i_{\text{ét}}(X, \mathbb{Z}_p)_{\mathrm{tor}} .$$

[6] Equivalently, $H^i_{\mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C)$ is p-torsion free, cf. Bhatt, Morrow, and Scholze [2016, Remark 14.4].

[7] In that situation, part (iii) implies that also $H^i_{\text{ét}}(X, \mathbb{Z}_p)$ is p-torsion free.

For example, for an Enriques surface X over C, so that it has a double cover by a K3 surface, there is 2-torsion in the second étale cohomology group. By the theorem, this implies that for all Enriques surfaces in characteristic 2 (all of which deform to characteristic 0), one has 2-torsion in the second crystalline cohomology, or equivalently $H^1_{\mathrm{dR}} \neq 0$, contrary to the situation in any other characteristic. This is a well-known "pathology" of Enriques surfaces in characteristic 2 that finds a natural explanation here. In this case, one actually has equality in the corollary; however, in Bhatt, Morrow, and Scholze [2016] we give an example of a projective smooth surface over $\mathbb{Z}_2$ whose étale cohomology is torsionfree while there is torsion in crystalline cohomology.

The construction of the A_{inf}-cohomology given in Bhatt, Morrow, and Scholze [ibid.] was the end result of a long detour that led the author to study topological Hochschild and cyclic homology. This started with the paper Hesselholt [2006] of Lars Hesselholt that computes $\mathrm{THH}(\mathcal{O}_C)$. This is an E_∞-ring spectrum with an action by the circle group $\mathbb{T} = S^1$. In particular, one can form the homotopy fixed points $\mathrm{TC}^-(\mathcal{O}_C) = \mathrm{THH}(\mathcal{O}_C)^{h\mathbb{T}}$ to get another E_∞-ring spectrum.

Theorem 8.3 (Hesselholt [ibid.]). *The homotopy groups of the p-completion of $\mathrm{TC}^-(\mathcal{O}_C) = \mathrm{THH}(\mathcal{O}_C)^{h\mathbb{T}}$ are given by A_{inf} in all even degrees, and 0 in all odd degrees. The generators in degree 2 and -2 multiply to $\xi \in A_{\mathrm{inf}}$.*

If now X is a proper smooth (formal) scheme over $\mathcal{O}_C$, it follows that the p-completion of $\mathrm{TC}^-(X)$ is a perfect module over the p-completion of $\mathrm{TC}^-(\mathcal{O}_C)$, and in particular its homotopy groups are A_{inf}-modules. Moreover, topological Hochschild homology comes with a $\mathbb{T}$-equivariant Frobenius operator $\mathrm{THH}(X) \to \mathrm{THH}(X)^{tC_p}$, where $C_p \subset \mathbb{T}$ is the cyclic group of order p and $S^{tC_p} = \mathrm{cone}(\mathrm{Nm} : S_{hC_p} \to S^{hC_p})$ denotes the Tate construction. This gives rise to the desired Frobenius on TC^- by passing to homotopy $\mathbb{T}$-fixed points. In the classical formulation, THH has even more structure as encoded in the structure of a cyclotomic spectrum. However, in joint work with Nikolaus, we proved that this extra structure is actually redundant.

Theorem 8.4 (Nikolaus and Scholze [2017]). *On respective subcategories of bounded below objects, the ∞-category of cyclotomic spectra (in the sense of Hesselholt and Madsen [1997], and made more precise by Blumberg and Mandell [2015]) is equivalent to the ∞-category of naïvely $\mathbb{T}$-equivariant spectra S together with a $\mathbb{T}$-equivariant map $\varphi_p : S \to S^{tC_p}$ for all primes p.*[8]

The only issue with this construction of the A_{inf}-cohomology theory (besides its heavy formalism) is that one gets an essentially 2-periodic cohomology theory. To get the desired cohomology theory itself, one needs to find a filtration on $\mathrm{TC}^-(X)$ and then pass to graded pieces; this is similar to the "motivic" filtration on algebraic K-theory with graded pieces given by motivic cohomology, and the existence of such a filtration was first conjectured by Hesselholt. Using perfectoid techniques, it is possible to construct the desired filtration on TC^-; this is the content of Bhatt, Morrow, and Scholze [n.d.]. On the other hand, in the original paper Bhatt, Morrow, and Scholze [2016], we were able to build the theory independently of THH. We refer to Bhatt, Morrow,

[8] No compatibility between the different φ_p is required.

and Scholze [ibid.] and the surveys Bhatt [2016], Morrow [2016] for details on this construction.

9 Shtukas for Spec $\mathbb{Z}$

Roughly, the upshot of the A_{inf}-cohomology theory is that the "universal" p-adic cohomology theory is given by a shtuka relative to $\operatorname{Spf}\mathbb{Z}_p$. Currently, the author is trying to understand to what extent it might be true that the "universal" cohomology theory is given by a shtuka relative to Spec $\mathbb{Z}$. It seems that this is a very fruitful philosophy.

Let us phrase the question more precisely. Given a proper smooth scheme X over a base scheme S, what is "the" cohomology of X? Experience in arithmetic and algebraic geometry shows that there is no simple answer, and that in fact there are many cohomology theories: singular, de Rham, étale, crystalline, etc.[9] It helps to organize these cohomology theories according to two parameters: First, they may only defined for certain (geometric) points $s \in S$, and their coefficients may not be $\mathbb{Z}$ but only $\mathbb{Z}/n\mathbb{Z}$ or $\mathbb{Z}_\ell$ or $\mathbb{R}$.

This makes it natural to draw a picture of a space $S \times \operatorname{Spec}\mathbb{Z}$", where at a point (s, ℓ) we put $H^i(X_s, \mathbb{Z}/\ell\mathbb{Z})$", whatever we want this to mean. Note that inside this space, we have the graph of S, taking any $s \in S$ to $(s, \operatorname{char}(s))$ where $\operatorname{char}(s)$ is the characteristic of s. For simplicity, we assume that $S = \operatorname{Spec}\mathbb{Z}$ in the following; then the graph is given by the diagonal $\operatorname{Spec}\mathbb{Z} \subset \operatorname{Spec}\mathbb{Z} \times \operatorname{Spec}\mathbb{Z}$". We will use p to denote a point of $p \in S = \operatorname{Spec}\mathbb{Z}$, and ℓ to denote a point of the vertical $\operatorname{Spec}\mathbb{Z}$.

Thus, let X be a proper smooth scheme over $S = \operatorname{Spec}\mathbb{Z}$.[10] We include the following cohomology theories in the picture:

(i) Singular cohomology $H^i_{\mathrm{sing}}(X(\mathbb{C}), \mathbb{Z})$. This gives a vertical line over the infinite point of Spec $\mathbb{Z}$.

(ii) De Rham cohomology $H^i_{\mathrm{dR}}(X/\mathbb{Z})$. Note that (up to torsion problems)

$$H^i_{\mathrm{dR}}(X/\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}/\ell\mathbb{Z} = H^i_{\mathrm{dR}}(X_{\mathbb{F}_p}/\mathbb{F}_p)$$

for $p = \ell$, so a horizontal fibre of $H^i_{\mathrm{dR}}(X/\mathbb{Z})$ agrees with a vertical fibre. Thus, it sits on the diagonal. It meets singular cohomology, via the comparison isomorphism

$$H^i_{\mathrm{dR}}(X/\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \cong H^i_{\mathrm{sing}}(X(\mathbb{C}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}\ .$$

(iii) Étale cohomology $H^i_{\text{ét}}(X_{\bar{s}}, \mathbb{Z}/\ell^m\mathbb{Z})$, which is defined for all geometric points $\bar{s} \in \operatorname{Spec}\mathbb{Z} \setminus \{\ell\}$. This gives a horizontal line with a hole at ℓ, and in fact an

[9] We use the word "cohomology theory" in a loose sense, but it should be of Weil type and compatible with base change, so for example H^1 of an elliptic curve should be of rank 2 over some ring, and base changing the elliptic curve should amount to a corresponding base change of its H^1 along an associated map of rings. In particular, "absolute" cohomologies such as motivic cohomology or syntomic cohomology are disregarded.

[10] The known examples are not too interesting, so the reader may prefer to take $S = \operatorname{Spec}\mathbb{Z}[\frac{1}{N}]$ for some N to get more interesting examples. He should then remove the vertical fibres over $p|N$ from the picture.

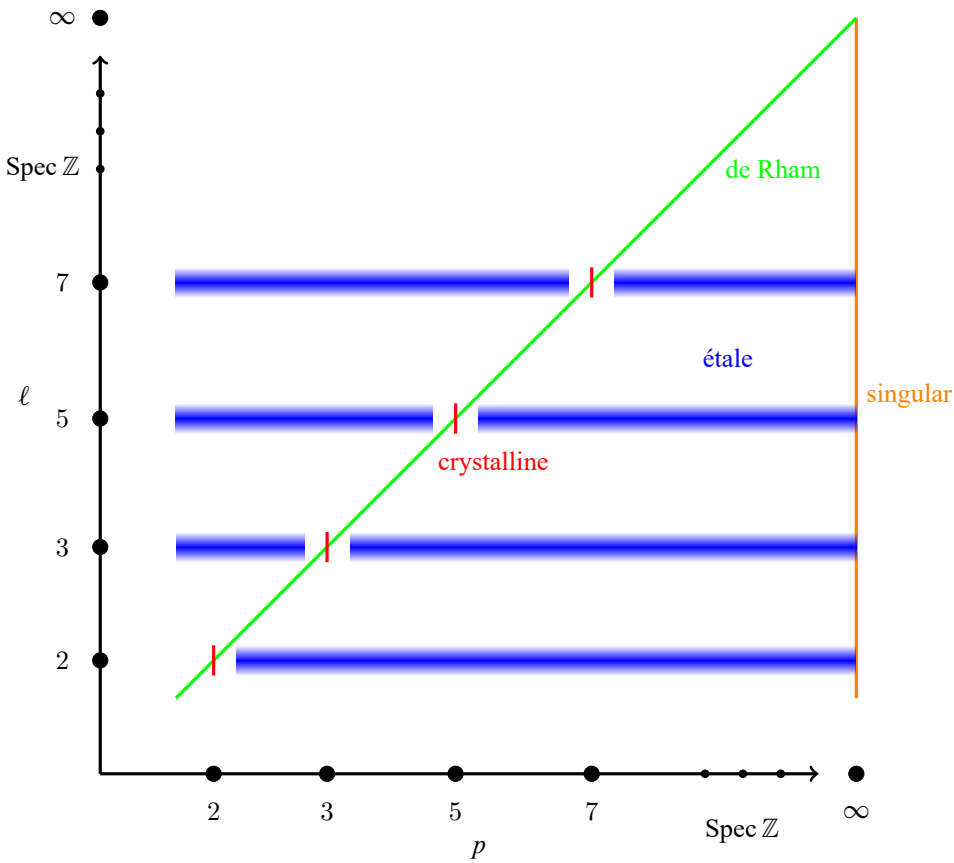

Figure 2: A picture of known cohomology theories.

infinitesimal neighborhood of the horizontal line. It meets singular cohomology via the comparison isomorphism

$$H^i_{\text{ét}}(X_{\mathbb{C}}, \mathbb{Z}/\ell^m\mathbb{Z}) \cong H^i_{\text{sing}}(X(\mathbb{C}), \mathbb{Z}/\ell^m\mathbb{Z}) .$$

(iv) Crystalline cohomology $H^i_{\text{crys}}(X_{\mathbb{F}_p}/\mathbb{Z}_p)$. This sits in a vertical fibre over p, and extends de Rham cohomology infinitesimally.

If we zoom in near a point (p, p) in this picture, we arrive at a picture that looks exactly like Figure 1 depicting the A_{inf}-cohomology theory! And indeed, one can use the A_{inf}-cohomology theory to fill in this part of the picture.

The picture bears a remarkable similarity with the following equal-characteristic structure.

Definition 9.1 (Anderson [1986] and Goss [1996]). *For a scheme $S = \operatorname{Spec} R$ over $\operatorname{Spec} \mathbb{F}_p[T]$ sending T to $t \in R$, a t-motive is a finite projective $R[T]$-module M together with an isomorphism*

$$\varphi_M : \operatorname{Frob}_R^* M[\tfrac{1}{t-T}] \cong M[\tfrac{1}{t-T}] .$$

Let us briefly discuss the similar specializations.

(i) If $R = K$ is an algebraically closed field mapping to the infinite point of $\operatorname{Spec} \mathbb{F}_p[T]$ (which, strictly speaking, is not allowed – but we can compactify $\operatorname{Spec} \mathbb{F}_p[T]$ into $\mathbb{P}^1_{\mathbb{F}_p}$) we get a vector bundle on $\mathbb{P}^1_K$ together with an isomorphism with its Frobenius pullback.[11] These are equivalent to vector bundles on $\mathbb{P}^1_{\mathbb{F}_p}$, and by restricting back to $\operatorname{Spec} \mathbb{F}_p[T]$, we get an $\mathbb{F}_p[T]$-module.

(ii) Restricting along $R[T] \to R$ sending T to t we get a finite projective R-module. Moreover, φ_M gives rise to a $R[[t-T]]$-lattice in $M \otimes_{R[T]} R((t-T))$, which gives rise to a filtration of $M \otimes_{R[T]} R$ similarly to the discussion in the case of B^+_{dR}-lattices or $\mathbb{C}[[t]]$-lattices in Section 6.[12]

(iii) Restricting modulo a power of T (or $T - a$, $a \in \mathbb{F}_p$), the pair

$$(M_n, \varphi_{M_n}) = (M \otimes_{R[T]} R[t^{-1}, T]/T^n, \varphi_M)$$

is equivalent to an étale $\mathbb{F}_p[T]/T^n$-local system on $\operatorname{Spec} R[t^{-1}]$ by taking the étale sheaf $M_n^{\varphi_{M_n}=1}$.

(iv) Restricting to the fibre $t = 0$ (or $t = a$, $a \in \mathbb{F}_p$) and taking the T-adic completion, $M \otimes_{R[T]} (R/t)[[T]]$ with φ_M defines a structure resembling crystalline cohomology.

[11] One might think that one has to allow a pole at ∞, but the analogy seems to suggest that it is in fact not there.

[12] A subtle point is that $M \otimes_{R[T], T \mapsto t} R$ is the analogue of Hodge(-Tate) cohomology, and de Rham cohomology corresponds to the specialization $M \otimes_{R[T], T \mapsto t^p} R$.

(v) Similarly, by the analogy between local shtukas and local p-adic shtukas, the picture near $(t, T) = (0, 0)$ resembles the picture of the A_{inf}-cohomology theory.

An interesting question is what happens in vertical fibres, i.e. for $S = \operatorname{Spec} \overline{\mathbb{F}}_p$. In the function field case, we get generically the following structure.

Definition 9.2. *A (generic) shtuka over* $\operatorname{Spec} \overline{\mathbb{F}}_p$ *relative to* $\operatorname{Spec} \mathbb{F}_p(T)$ *is a finite-dimensional* $\overline{\mathbb{F}}_p(T)$*-vector space* V *together with a* $\mathrm{Frob}_{\overline{\mathbb{F}}_p}$*-semilinear isomorphism* $\varphi_V : V \cong V$.

Amazingly, Kottwitz [2014] was able to define an analogue of this category for number fields.

Construction 9.3 (Kottwitz)**.** *For any local or global field* F*, there is an* F*-linear* $\otimes$*-category* Kt_F*, constructed as the category of representations of a gerbe banded by an explicit (pro-)torus which is constructed using (local or global) class field theory. Moreover, one has the following identifications.*

(i) *If* $F = \mathbb{F}_p((T))$*, then* Kt_F *is the category of finite-dimensional* $\overline{\mathbb{F}}_p((T))$*-vector spaces* V *with a semilinear isomorphism* $V \cong V$.

(ii) *If* $F = \mathbb{F}_p(T)$*, then* Kt_F *is the category of finite-dimensional* $\overline{\mathbb{F}}_p(T)$*-vector spaces* V *with a semilinear isomorphism* $V \cong V$.

(iii) *If* $F = \mathbb{Q}_p$*, then* Kt_F *is the category of finite-dimensional* $W(\overline{\mathbb{F}}_p)[\frac{1}{p}]$*-vector spaces* V *with a semilinear isomorphism* $V \cong V$.

(iv) *If* $F = \mathbb{R}$*, then* Kt_F *is the category of finite-dimensional* $\mathbb{C}$*-vector spaces* V *together with a grading* $V = \bigoplus_{i \in \mathbb{Z}} V_i$ *and a graded antiholomorphic isomorphism* $\alpha : V \cong V$ *such that* $\alpha^2 = (-1)^i$ *on* V_i.

Remark 9.4. In the local cases (i) and (iii), one can replace $\overline{\mathbb{F}}_p$ with any algebraically closed field of characteristic p without changing the category. However, in the global case (ii), the equivalence holds only for $\overline{\mathbb{F}}_p$; for this reason, we fix this choice.

It is an important problem to find a linear-algebraic description of $\mathrm{Kt}_{\mathbb{Q}}$. There are natural global-to-local maps, so there are functors $\mathrm{Kt}_{\mathbb{Q}} \to \mathrm{Kt}_{\mathbb{Q}_p}$ and $\mathrm{Kt}_{\mathbb{Q}} \to \mathrm{Kt}_{\mathbb{R}}$. The analogy between cohomology theories and shtukas suggests the following conjecture.

Conjecture 9.5. *There is a Weil cohomology theory* $H^i_{\mathrm{Kt}_{\mathbb{Q}}}(X)$ *for varieties* X *over* $\overline{\mathbb{F}}_p$ *taking values in* $\mathrm{Kt}_{\mathbb{Q}}$*. Under the functor* $\mathrm{Kt}_{\mathbb{Q}} \to \mathrm{Kt}_{\mathbb{Q}_p}$*, this maps to crystalline cohomology, and under the functor* $\mathrm{Kt}_{\mathbb{Q}} \to \mathrm{Kt}_{\mathbb{Q}_\ell}$ *for* $\ell \neq p$*, this maps to étale cohomology (considered as an object of* $\mathrm{Kt}_{\mathbb{Q}_\ell}$ *via the fully faithful embedding of finite-dimensional* $\mathbb{Q}_\ell$*-vector spaces into* $\mathrm{Kt}_{\mathbb{Q}_\ell}$*). Under the restriction* $\mathrm{Kt}_{\mathbb{Q}} \to \mathrm{Kt}_{\mathbb{R}}$*, it gives a Weil cohomology theory* $H^i_{\mathrm{Kt}_{\mathbb{R}}}(X)$ *with values in* $\mathrm{Kt}_{\mathbb{R}}$.

This conjecture is known to follow from the conjunction of the standard conjectures and the Tate conjecture (over $\overline{\mathbb{F}}_p$) and the Hodge conjecture (for CM abelian varieties) by the work of Langlands and Rapoport [1987]. In fact, Langlands–Rapoport show that

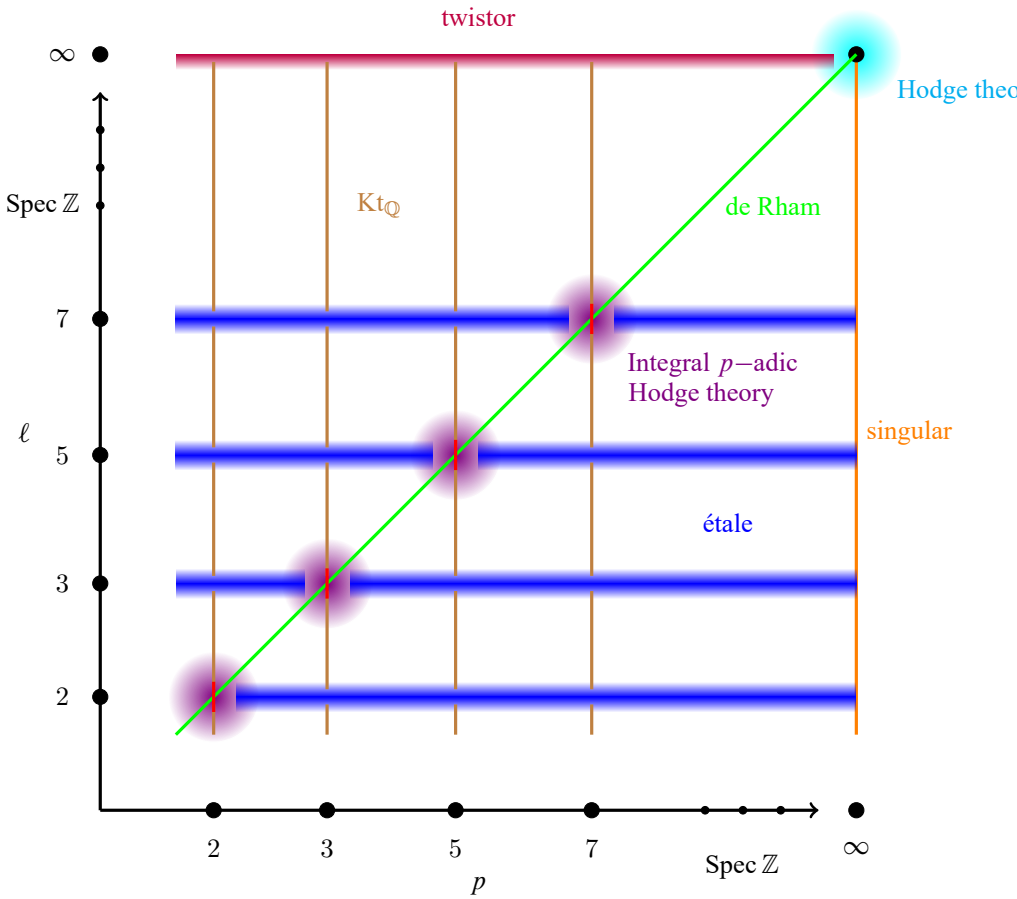

Figure 3: A picture of the known, and some unknown, cohomology theories.

these conjectures imply that the category of motives over $\overline{\mathbb{F}}_p$ can be described as the category of representations of an explicit gerbe, and it is clear by inspection that there is a surjective map from Kottwitz' gerbe (which is independent of p!) to Langlands–Rapoport's gerbe. In particular, $\mathrm{Kt}_{\mathbb{Q}}$ conjecturally contains the category of motives over $\overline{\mathbb{F}}_p$ as a full subcategory, for all p.

It may be interesting to note how Serre's objection to a 2-dimensional $\mathbb{R}$-cohomology theory is overcome by the $\mathbb{R}$-linear cohomology theory $H^i_{\mathrm{Kt}_{\mathbb{R}}}$, in an essentially minimal way: For a supersingular elliptic curve $E/\overline{\mathbb{F}}_p$, its associated $H^1_{\mathrm{Kt}_{\mathbb{R}}}(E)$ will be given by a 2-dimensional $\mathbb{C}$-vector space $V = V_1$ together with an antiholomorphic isomorphism $\alpha : V \cong V$ such that $\alpha^2 = -1$, so α does not give rise to a real structure; instead, it gives rise to a quaternionic structure. But the endomorphism algebra of E, which is a quaternion algebra over $\mathbb{Q}$ that is nonsplit over $\mathbb{R}$, can of course act on the Hamilton quaternions!

It is also interesting to note that there are functors from the category of isocrystals $\mathrm{Kt}_{\mathbb{Q}_p}$ to the category of vector bundles on the Fargues–Fontaine curve, and similarly from the category $\mathrm{Kt}_{\mathbb{R}}$ to the category of vector bundles on the twistor-$\mathbb{P}^1$, which in both cases induce a bijection on isomorphism classes. This gives another strong indication of the parallel between the Fargues–Fontaine curve and the twistor-$\mathbb{P}^1$, and how Kottwitz' categories play naturally into them.

In particular, one can also draw a horizontal line at $\ell = \infty$, where for all $s \in S$ one gets a vector bundle on the twistor-$\mathbb{P}^1$ (via the previous conjecture in finite characteristic, and via the twistor interpretation of Hodge theory in characteristic 0). In other words, the complex variation of twistor structures must, in a suitable sense, be defined over the scheme S (which is a general scheme over $\operatorname{Spec}\mathbb{Z}$, not necessarily over $\mathbb{C}$).

10 q-de Rham cohomology

One may wonder what the completion of $\operatorname{Spec}\mathbb{Z} \times \operatorname{Spec}\mathbb{Z}$" along the diagonal looks like; in the picture of cohomology theories, this should combine p-adic Hodge theory for all primes p with usual Hodge theory. One proposal for how this might happen was made in Scholze [2017a]. This paper builds on the following observation on the A_{inf}-cohomology theory. Namely, one can write down explicit complexes computing this cohomology on affines. For example, if $R = \mathcal{O}_C\langle T\rangle$ is the p-adic completion of $\mathcal{O}_C[T]$, then one looks at the following complex:

$$A_{\mathrm{inf}}\langle T\rangle \xrightarrow{\nabla_q} A_{\mathrm{inf}}\langle T\rangle \ : \ T^n \mapsto \nabla_q(T^n) = [n]_q T^{n-1} \ ,$$

where $[n]_q = 1 + q + \ldots + q^{n-1} = \frac{q^n-1}{q-1}$ is Gauß's q-integer, and $q = [\epsilon] \in A_{\mathrm{inf}}$ is a certain element of A_{inf}. This is precisely the q-de Rham complex studied by Aomoto [1990]; it sends a general function $f(T)$ to the Jackson q-derivative

$$(\nabla_q f)(T) = \frac{f(qT) - f(T)}{qT - T} \ .$$

Unfortunately, the q-de Rham complex depends heavily on the choice of coordinates, and it is not clear how to see the independence of the A_{inf}-cohomology from the choice of coordinates using this picture.

However, in recent work with Bhatt, Bhatt and Scholze [n.d.], we were able to prove the following theorem, proving a conjecture of Scholze [2017a].

Theorem 10.1 (Bhatt and Scholze [n.d.]). *There is a $\mathbb{Z}[[q-1]]$-linear q-de Rham cohomology theory for smooth schemes over* $\operatorname{Spec}\mathbb{Z}$ *that for any smooth $\mathbb{Z}$-algebra R with a choice of an étale map $\mathbb{Z}[T_1,\dots,T_n]\to R$ is computed by a q-deformation $q\Omega^\bullet_R$ of the de Rham complex $\Omega^\bullet_R$. For example,*

$$q\Omega^\bullet_{\mathbb{Z}[T]} = \mathbb{Z}[T][[q-1]] \xrightarrow{\nabla_q} \mathbb{Z}[T][[q-1]] \ : T^n \mapsto \nabla_q(T^n) = [n]_q T^{n-1}\ .$$

After base change along $\mathbb{Z}[[q-1]]\to A_{\mathrm{inf}}$ via $q\mapsto[\epsilon]$, this recovers the A_{inf}-cohomology theory.

In particular, this suggests that the completion of $\operatorname{Spec}\mathbb{Z}\times\operatorname{Spec}\mathbb{Z}$" along the diagonal is related to $\operatorname{Spf}\mathbb{Z}[[q-1]]$.

Previous progress towards this result was made by Pridham [2016], and Chatzistamatiou [n.d.], and announced by Masullo. In particular, Chatzistamatiou was able to write down explicitly the quasi-automorphism of $q\Omega^\bullet_{\mathbb{Z}[T]}$ for an automorphism of $\mathbb{Z}[T]$, like $T\mapsto T+1$; this is a highly nontrivial result![13]

Unfortunately, the proof of Theorem 10.1 proceeds by first constructing such complexes after p-adic completion, and then patching them together in some slightly artificial way, so we believe that there is still much more to be understood about what exactly happens along the diagonal.

In fact, the prismatic cohomology defined in Bhatt and Scholze [n.d.] gives in the p-adic case a Breuil–Kisin variant[14] of the A_{inf}-cohomology theory for varieties over $\mathcal{O}_K$ where K is a finite extension of $\mathbb{Q}_p$. This Breuil–Kisin module contains some slightly finer information than the q-de Rham cohomology and is in some sense a Frobenius descent of it, but it is not clear how to combine them for varying p. This is related to the observation that in the analogy with t-motives, the diagonal $T=t$ corresponds to Hodge(-Tate) cohomology, while the Frobenius-twisted diagonal $T=t^p$ corresponds to de Rham cohomology. In this picture, the q-de Rham cohomology would sit at the completion at $T=t^p$, while the prismatic picture seems more adapted to a picture corresponding to the completion at the diagonal $T=t$.

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[14] Taking values in Breuil–Kisin modules as defined in Kisin [2006].

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SYSTEMS OF POINTS WITH COULOMB INTERACTIONS

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Abstract

Large ensembles of points with Coulomb interactions arise in various settings of condensed matter physics, classical and quantum mechanics, statistical mechanics, random matrices and even approximation theory, and give rise to a variety of questions pertaining to calculus of variations, Partial Differential Equations and probability. We will review these as well as "the mean-field limit" results that allow to derive effective models and equations describing the system at the macroscopic scale. We then explain how to analyze the next order beyond the mean-field limit, giving information on the system at the microscopic level. In the setting of statistical mechanics, this allows for instance to observe the effect of the temperature and to connect with crystallization questions.

1 General setups

We are interested in large systems of points with Coulomb-type interactions, described through an energy of the form

$$\mathcal{H}_N(x_1,\dots,x_N) = \frac{1}{2}\sum_{i\neq j} \mathsf{g}(x_i - x_j) + N\sum_{i=1}^{N} V(x_i). \tag{1-1}$$

Here the points x_i belong to the Euclidean space $\mathbb{R}^{\mathsf{d}}$, although it is also interesting to consider points on manifolds. The interaction kernel $\mathsf{g}(x)$ is taken to be

$$(\mathsf{Log2}\text{ case})\quad \mathsf{g}(x) = -\log|x|, \quad \text{in dimension } \mathsf{d}=2, \tag{1-2}$$

$$(\mathsf{Coul}\text{ case})\quad \mathsf{g}(x) = \frac{1}{|x|^{\mathsf{d}-2}}, \quad \text{in dimension } \mathsf{d}\geq 3. \tag{1-3}$$

This is (up to a multiplicative constant) the Coulomb kernel in dimension $\mathsf{d}\geq 2$, i.e. the fundamental solution to the Laplace operator, solving

$$-\Delta \mathsf{g} = \mathsf{c}_{\mathsf{d}}\delta_0 \tag{1-4}$$

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where δ_0 is the Dirac mass at the origin and c_d is an explicit constant depending only on the dimension. It is also interesting to broaden the study to the one-dimensional logarithmic case

$$\text{(1-5)} \qquad (\mathsf{Log1}\text{ case}) \quad \mathsf{g}(x) = -\log|x|, \quad \text{in dimension } \mathsf{d} = 1,$$

which is not Coulombic, and to more general Riesz interaction kernels of the form

$$\text{(1-6)} \qquad \mathsf{g}(x) = \frac{1}{|x|^{\mathsf{s}}} \quad \mathsf{s} > 0.$$

The one-dimensional Coulomb interaction with kernel $-|x|$ is also of interest, but we will not consider it as it has been extensively studied and understood, see Lenard [1961], Lenard [1963], and Kunz [1974].

Finally, we have included a possible external field or confining potential V, which is assumed to be regular enough and tending to ∞ fast enough at ∞. The factor N in front of V makes the total confinement energy of the same order as the total repulsion energy, effectively balancing them and confining the system to a subset of $\mathbb{R}^{\mathsf{d}}$ of fixed size. Other choices of scaling would lead to systems of very large or very small size as $N \to \infty$.

The Coulomb interaction and the Laplace operator are obviously extremely important and ubiquitous in physics as the fundamental interactions of nature (gravitational and electromagnetic) are Coulombic. Coulomb was a French engineer and physicist working in the late 18th century, who did a lot of work on applied mechanics (such as modeling friction and torsion) and is most famous for his theory of electrostatics and magnetism. He is the first one who postulated that the force exerted by charged particles is proportional to the inverse distance squared, which corresponds in dimension $\mathsf{d} = 3$ to the gradient of the Coulomb potential energy $\mathsf{g}(x)$ as above. More precisely he wrote in Coulomb [1785] " It follows therefore from these three tests, that the repulsive force that the two balls [which were] electrified with the same kind of electricity exert on each other, follows the inverse proportion of the square of the distance." He developed a method based on systematic use of mathematical calculus (with the help of suitable approximations) and mathematical modeling (in contemporary terms) to predict physical behavior, systematically comparing the results with the measurements of the experiments he was designing and conducting himself. As such, he is considered as a pioneer of the "mathematization" of physics and in trusting fully the capacities of mathematics to transcribe physical phenomena Blondel and Wolff [2015].

Here we are more specifically focusing on Coulomb interactions between points, or in physics terms, discrete point charges. There are several mathematical problems that are interesting to study, all in the asymptotics of $N \to \infty$:

(1) understand minimizers and possibly critical points of (1-1) ;

(2) understand the statistical mechanics of systems with energy $\mathcal{H}_N$ and inverse temperature $\beta > 0$, governed by the so-called Gibbs measure

$$\text{(1-7)} \qquad d\mathbb{P}_{N,\beta}(x_1, \dots, x_N) = \frac{1}{Z_{N,\beta}} e^{-\beta \mathcal{H}_N(x_1,\dots,x_N)} dx_1 \dots dx_N.$$

Here, as postulated by statistical mechanics, $\mathbb{P}_{N,\beta}$ is the density of probability of observing the system in the configuration $(x_1, \dots, x_N)$ if the number of particles is fixed to N and the inverse of the temperature is $\beta > 0$. The constant $Z_{N,\beta}$ is called the "partition function" in physics, it is the normalization constant that makes $\mathbb{P}_{N,\beta}$ a probability measure, [1] i.e.

$$Z_{N,\beta} = \int_{(\mathbb{R}^d)^N} e^{-\beta \mathcal{H}_N(x_1,\dots,x_N)} dx_1 \dots dx_N, \tag{1-8}$$

where the inverse temperature $\beta = \beta_N$ can be taken to depend on N, as there are several interesting scalings of β relative to N;

(3) understand dynamic evolutions associated to (1-1), such as the gradient flow of $\mathcal{H}_N$ given by the system of coupled ODEs

$$\dot{x_i} = -\frac{1}{N} \nabla_i \mathcal{H}_N(x_1, \dots, x_N), \tag{1-9}$$

conservative dynamics given by the systems of ODEs

$$\dot{x_i} = \frac{1}{N} \mathbb{J} \nabla_i \mathcal{H}_N(x_1, \dots, x_N) \tag{1-10}$$

where $\mathbb{J}$ is an antisymmetric matrix (for example a rotation by $\pi/2$ in dimension 2), or the Hamiltonian dynamics given by Newton's law

$$\ddot{x_i} = -\frac{1}{N} \nabla_i \mathcal{H}_N(x_1, \dots, x_N); \tag{1-11}$$

(4) understand the previous dynamic evolutions with temperature β^{-1} in the form of an added noise (Langevin-type equations) such as

$$dx_i = -\frac{1}{N} \nabla_i \mathcal{H}_N(x_1, \dots, x_N) dt + \sqrt{\beta^{-1}} dW_i \tag{1-12}$$

with W_i independent Brownian motions, or

$$dx_i = \frac{1}{N} \mathbb{J} \nabla_i \mathcal{H}_N(x_1, \dots, x_N) dt + \sqrt{\beta^{-1}} dW_i \tag{1-13}$$

with $\mathbb{J}$ as above, or

$$dx_i = v_i dt \qquad dv_i = -\frac{1}{N} \nabla_i \mathcal{H}_N(x_1, \dots, x_N) dt + \sqrt{\beta^{-1}} dW_i. \tag{1-14}$$

From a mathematical point of view, the study of such systems touches on the fields of analysis (Partial Differential Equations and calculus of variations, approximation theory) particularly for **(1)-(3)-(4)**, probability (particularly for **(2)-(4)**), mathematical physics, and even geometry (when one considers such systems on manifolds or with

[1] One does not know how to explicitly compute the integrals (1-8) except in the particular case of (1-5) for specific V's where they are called Selberg integrals (cf. Mehta [2004] and Forrester [2010])

curved geometries). Some of the crystallization questions they lead to also overlap with number theory as we will see below.

In the sequel we will mostly focus on the stationary settings **(1)** and **(2)**, while mentioning more briefly some results about **(3)** and **(4)**, for which many questions remain open. Of course these various points are not unrelated, as for instance the Gibbs measure (1-7) can also be seen as an invariant measure for dynamics of the form (1-11) or (1-12).

The plan of the discussion is as follows: in the next section we review various motivations for studying such questions, whether from physics or within mathematics. In Section 3 we turn to the so-called "mean-field" or leading order description of systems **(1)** to **(4)** and review the standard questions and known results. We emphasize that this part can be extended to general interaction kernels g, starting with regular (smooth) interactions which are in fact the easiest to treat. In Section 4, we discuss questions that can be asked and results that can be obtained at the next order level of expansion of the energy. This has only been tackled for problems **(1)** and **(2)**, and the specificity of the Coulomb interaction becomes important then.

2 Motivations

It is in fact impossible to list all possible topics in which such systems arise, as they are really numerous. We will attempt to give a short, necessarily biased, list of examples, with possible pointers to the relevant literature.

2.1 Vortices in condensed matter physics and fluids. In superconductors with applied magnetic fields, and in rotating superfluids and Bose–Einstein condensates, one observes the occurrence of quantized "vortices" (which are local point defects of superconductivity or superfluidity, surrounded by a current loop). The vortices repel each other, while being confined together by the effect of the magnetic field or rotation, and the result of the competition between these two effects is that, as predicted by Abrikosov [1957], they arrange themselves in a particular *triangular lattice* pattern, called *Abrikosov lattice*, cf. Fig. 1[2]. Superconductors and superfluids are modelled by the celebrated Ginzburg–Landau energy Landau and Ginzburg [1965], which in simplified form [3] can be written

$$\int |\nabla\psi|^2 + \frac{(1-|\psi|^2)^2}{2\varepsilon^2} \tag{2-1}$$

where ψ is a complex-valued unknown function (the "order parameter" in physics) and ε is a small parameter, and gives rise to the associated Ginzburg–Landau equation

$$\Delta\psi + \frac{1}{\varepsilon^2}\psi(1-|\psi|^2) = 0 \tag{2-2}$$

[2]For more pictures, see `www.fys.uio.no/super/vortex/`

[3]The complete form for superconductivity contains a gauge-field, but we omit it here for simplicity.

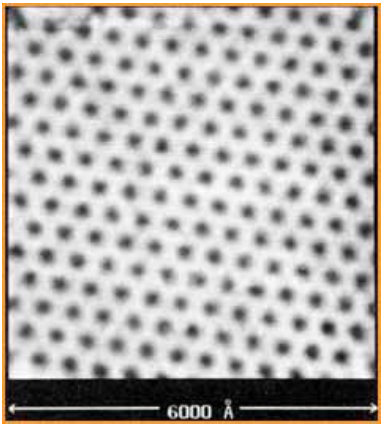

Figure 1: Abrikosov lattice, H. F. Hess et al. Bell Labs *Phys. Rev. Lett.* 62, 214 (1989)

and its dynamical versions, the heat flow

$$\partial_t \psi = \Delta\psi + \frac{1}{\varepsilon^2}\psi(1-|\psi|^2) \tag{2-3}$$

and Schrödinger-type flow (also called the Gross–Pitaevskii equation)

$$i\,\partial_t \psi = \Delta\psi + \frac{1}{\varepsilon^2}\psi(1-|\psi|^2). \tag{2-4}$$

When restricting to a two-dimensional situation, it can be shown rigorously (this was pioneered by Bethuel, Brezis, and Hélein [1994] for (2-1) and extended to the full gauged model Bethuel and Rivière [1995] and Sandier and Serfaty [2007, 2012]) that the minimization of (2-1) can be reduced, in terms of the vortices and as $\varepsilon \to 0$, to the minimization of an energy of the form (1-1) in the case (1-2) (for a formal derivation, see also Serfaty [2015, Chap. 1]) and this naturally leads to the question of understanding the connection between minimizers of (1-1) + (1-2) and the Abrikosov triangular lattice. Similarly, the dynamics of vortices under (2-3) can be formally reduced to (1-9), respectively under (2-4) to (1-10). This was established formally for instance in L. Peres and Rubinstein [1993] and E [1994a] and proven for a fixed number of vortices N and in the limit $\varepsilon \to 0$ in F. H. Lin [1996], Jerrard and Soner [1998], Colliander and Jerrard [1998, 1999], F.-H. Lin and Xin [1999a,b], and Bethuel, Jerrard, and Smets [2008] until the first collision time and in Bethuel, Orlandi, and Smets [2005], Smets, Bethuel, and Orlandi [2007], Bethuel, Orlandi, and Smets [2007], and Serfaty [2007] including after collision.

Vortices also arise in classical fluids, where in contrast with what happens in superconductors and superfluids, their charge is not quantized. In that context the energy (1-1)+(1-2) is sometimes called the Kirchhoff energy and the system (1-10) with $\mathbb{J}$ taken to be a rotation by $\pi/2$, known as the point-vortex system, corresponds to the dynamics of idealized vortices in an incompressible fluid whose statistical mechanics analysis

was initiated by Onsager, cf. Eyink and Sreenivasan [2006] (one of the motivations for studying (1-13) is precisely to understand fluid turbulence as he conceived). It has thus been quite studied as such, see Marchioro and Pulvirenti [1994] for further reference. The study of evolutions like (1-11) is also motivated by plasma physics in which the interaction between ions is Coulombic, cf. Jabin [2014].

2.2 Fekete points and approximation theory. Fekete points arise in interpolation theory as the points minimizing interpolation errors for numerical integration Saff and Totik [1997]. More precisely, if one is looking for N interpolation points $\{x_1, \dots, x_N\}$ in K such that the relation

$$\int_K f(x)dx = \sum_{j=1}^{N} w_j f(x_j)$$

is exact when f is any polynomial of degree $\leq N-1$, one sees that one needs to compute the coefficients w_j such that $\int_K x^k = \sum_{j=1}^{N} w_j x_j^k$ for $0 \leq k \leq N-1$, and this computation is easy if one knows to invert the Vandermonde matrix of the $\{x_j\}_{j=1\dots N}$. The numerical stability of this operation is as large as the *condition number* of the matrix, i.e. as the Vandermonde determinant of the $(x_1, \dots, x_N)$. The points that minimize the maximal interpolation error for general functions are easily shown to be the Fekete points, defined as those that maximize

$$\prod_{i \neq j} |x_i - x_j|$$

or equivalently minimize

$$-\sum_{i \neq j} \log |x_i - x_j|.$$

They are often studied on manifolds, such as the d-dimensional sphere. In Euclidean space, one also considers "weighted Fekete points" which maximize

$$\prod_{i<j} |x_i - x_j| e^{-N \sum_i V(x_i)}$$

or equivalently minimize

$$-\frac{1}{2} \sum_{i \neq j} \log |x_i - x_j| + N \sum_{i=1}^{N} V(x_i)$$

which in dimension 2 corresponds exactly to the minimization of $\mathcal{H}_N$ in the particular case $\mathsf{Log2}$. They also happen to be zeroes of orthogonal polynomials, see Simon [2008].

Since $-\log|x|$ can be obtained as $\lim_{s\to 0} \frac{1}{s}(|x|^{-s} - 1)$, there is also interest in studying "Riesz s-energies", i.e. the minimization of

$$\sum_{i \neq j} \frac{1}{|x_i - x_j|^{\mathsf{s}}} \tag{2-5}$$

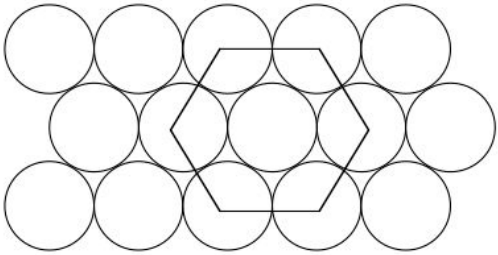

Figure 2: The triangular lattice solves the sphere packing problem in dimension 2

for all possible s, hence a motivation for (1-6). For these aspects, we refer to the the review papers Saff and Kuijlaars [1997] and Brauchart, Hardin, and Saff [2012] and references therein.

Varying s from 0 to ∞ connects Fekete points to the optimal sphere packing problem, which formally corresponds to the minimization of (2-5) with $\mathsf{s} = \infty$.

The optimal sphere packing problem has been solved in 1, 2 and 3 dimensions, as well as in dimensions 8 and 24 in the recent breakthrough Viazovska [2017] and Cohn, Kumar, Miller, Radchenko, and Viazovska [2017] (we refer the reader to the nice presentation in Cohn [2017] and the review Sloane [1998]). The solution in dimension 2 is the triangular lattice Fejes [1940] (i.e. the same as the Abrikosov lattice, see Figure 2), in dimension 3 it is the FCC (face-centered cubic) lattice Hales [2005], in dimension 8 the E_8 lattice Viazovska [2017], and in dimension 24 the Leech lattice Cohn, Kumar, Miller, Radchenko, and Viazovska [2017].

In other dimensions, the solution is in general not known and it is expected that in high dimension, where the problem is important for error-correcting codes, it is *not* a lattice (in dimension 10 already, the so-called "Best lattice", a non-lattice competitor, is known to beat the lattices), see Conway and Sloane [1999] for these aspects.

2.3 Statistical mechanics and quantum mechanics. The ensemble given by (1-7) in the $\mathsf{Log2}$ case is called in physics a two-dimensional *Coulomb gas* or *one-component plasma* and is a classical ensemble of statistical mechanics (see e.g. Alastuey and Jancovici [1981], Jancovici, Lebowitz, and Manificat [1993], Jancovici [1995], Sari and Merlini [1976], Kiessling [1993], and Kiessling and Spohn [1999]). The Coulomb case with $\mathsf{d} = 3$ can be seen as a toy (classical) model for matter (see e.g. Penrose and Smith [1972], Jancovici, Lebowitz, and Manificat [1993], Lieb and Lebowitz [1972], and Lieb and Narnhofer [1975]). Several additional motivations come from quantum mechanics. Indeed, the Gibbs measure of the two-dimensional Coulomb gas happens to be directly related to the Laughlin wave-function in the fractional quantum Hall effect Girvin [2004] and Stormer [1999]: this is the "plasma analogy", cf. Laughlin [1983], Girvin [1999], and Laughlin [1999], and for recent mathematical progress using this correspondence, cf. Rougerie, Serfaty, and Yngvason [2014], Rougerie and Yngvason [2015], and Lieb, Rougerie, and Yngvason [2018]. For $\beta = 2$ it also arises as the wave-function density of the ground state for the system of N non-interacting fermions

confined to a plane with a perpendicular magnetic field Forrester [2010, Chap. 15]. The 1-dimensional log gas $\mathsf{Log1}$ also arises as the wave-function density in several exactly solvable quantum mechanics systems: examples are the Tonks–Girardeau model of "impenetrable" Bosons Girardeau, Wright, and Triscari [2001] and Forrester, Frankel, Garoni, and Witte [2003], the Calogero–Sutherland quantum many-body Hamiltonian Forrester, Jancovici, and McAnally [2001] and Forrester [2010] and finally the density of the many-body wave function of non-interacting fermions in a harmonic trap Dean, Le Doussal, Majumdar, and Schehr [2016]. It also arises in several non-intersecting paths models from probability, cf. Forrester [2010].

The general Riesz case (1-6) can be seen as a generalization of the Coulomb case, motivations for studying Riesz gases are numerous in the physics literature (in solid state physics, ferrofluids, elasticity), see for instance Mazars [2011], Barré, Bouchet, Dauxois, and Ruffo [2005], Campa, Dauxois, and Ruffo [2009], and Torquato [2016], they can also correspond to systems with Coulomb interaction constrained to a lower-dimensional subspace: for instance in the quantum Hall effect, electrons confined to a two-dimensional plane interact via the three-dimension Coulomb kernel.

In all cases of interactions, the systems governed by the Gibbs measure $\mathbb{P}_{N,\beta}$ are considered as difficult systems of statistical mechanics because the interactions are truly long-range, singular, and the points are not constrained to live on a lattice.

As always in statistical mechanics K. Huang [1987], one would like to understand if there are phase-transitions for particular values of the (inverse) temperature β in the large volume limit. For the systems studied here, one may expect, after a suitable blow-up of the system, what physicists call a liquid for small β, and a crystal for large β. The meaning of crystal in this instance is not to be taken literally as a lattice, but rather as a system of points whose 2-point correlation function $\rho^{(2)}(x, y)$ defined as the probability to have jointly one point at x and one point at y (see Section 3.1) does not decay too fast as $x - y \to \infty$. A phase-transition at finite β has been conjectured in the physics literature for the $\mathsf{Log2}$ case (see e.g. Brush, Sahlin, and Teller [1966], Caillol, Levesque, Weis, and Hansen [1982], and Choquard and Clerouin [1983]) but its precise nature is still unclear (see e.g. Stishov [1998] for a discussion).

2.4 Two component plasma case. The two-dimensional "one component plasma", consisting of positively charged particles, has a "two-component" counterpart which consists in N particles $x_1, \dots, x_N$ of charge $+1$ and N particles $y_1, \dots, y_N$ of charge -1 interacting logarithmically, with energy

$$\mathcal{H}_N(x_1, \dots, x_N, y_1, \dots, y_N) = -\sum_{i \neq j} \log|x_i - x_j| - \sum_{i \neq j} \log|y_i - y_j| + \sum_{i,j} \log|x_i - y_j|$$

and the Gibbs measure

$$\frac{1}{Z_{N,\beta}} e^{-\beta \mathcal{H}_N(x_1,\dots,x_N,y_1,\dots,y_N)} dx_1 \dots dx_N \, dy_1 \dots dy_N.$$

Although the energy is unbounded below (positive and negative points attract), the Gibbs measure is well defined for β small enough, more precisely the partition function converges for $\beta < 2$. The system is then seen to form dipoles of oppositely

charged particles which attract but do not collapse, thanks to the thermal agitation. The two-component plasma is interesting due to its close relation to two important theoretical physics models: the XY model and the sine-Gordon model (cf. the review Spencer [1997]), which exhibit a Berezinski–Kosterlitz–Thouless phase transition Bietenholz and Gerber [2016] consisting in the binding of these "vortex-antivortex" dipoles. For further reference, see Fröhlich [1976], Deutsch and Lavaud [1974], Fröhlich and Spencer [1981], and Gunson and Panta [1977].

2.5 Random matrix theory. The study of (1-7) has attracted a lot of attention due to its connection with random matrix theory (we refer to Forrester [2010] for a comprehensive treatment). Random matrix theory (RMT) is a relatively old theory, pioneered by statisticians and physicists such as Wishart, Wigner and Dyson, and originally motivated by the study of sample covariance matrices for the former and the understanding of the spectrum of heavy atoms for the two latter, see Mehta [2004]. For more recent mathematical reference see Anderson, Guionnet, and Zeitouni [2010], Deift [1999], and Forrester [2010]. The main question asked by RMT is : what is the law of the spectrum of a large random matrix ? As first noticed in the foundational papers of Wigner [1955] and Dyson [1962], in the particular cases (1-5) and (1-2) the Gibbs measure (1-7) corresponds in some particular instances to the joint law of the eigenvalues (which can be computed algebraically) of some famous random matrix ensembles:

- for $\mathsf{Log2}$, $\beta = 2$ and $V(x) = |x|^2$, (1-7) is the law of the (complex) eigenvalues of an $N \times N$ matrix where the entries are chosen to be normal Gaussian i.i.d. This is called the Ginibre ensemble Ginibre [1965].
- for $\mathsf{Log1}$, $\beta = 2$ and $V(x) = x^2/2$, (1-7) is the law of the (real) eigenvalues of an $N \times N$ Hermitian matrix with complex normal Gaussian iid entries. This is called the Gaussian Unitary Ensemble.
- for $\mathsf{Log1}$, $\beta = 1$ and $V(x) = x^2/2$, (1-7) is the law of the (real) eigenvalues of an $N \times N$ real symmetric matrix with normal Gaussian iid entries. This is called the Gaussian Orthogonal Ensemble.
- for $\mathsf{Log1}$, $\beta = 4$ and $V(x) = x^2/2$, (1-7) is the law of the eigenvalues of an $N \times N$ quaternionic symmetric matrix with normal Gaussian iid entries. This is called the Gaussian Symplectic Ensemble.
- the general-β case of $\mathsf{Log1}$ can also be represented, in a slightly more complicated way, as a random matrix ensemble Dumitriu and Edelman [2002] and Killip and Nenciu [2004].

One thus observes in these ensembles the phenomenon of "repulsion of eigenvalues": they repel each other logarithmically, i.e. like two-dimensional Coulomb particles.

The stochastic evolution (1-12) in the case $\mathsf{Log1}$ is (up to proper scaling) the Dyson Brownian motion, which is of particular importance in random matrices since the GUE process is the invariant measure for this evolution, it has served to prove universality for the statistics of eigenvalues of general Wigner matrices, i.e. those with iid but not

necessarily Gaussian entries, see Erdős and Yau [2017] (and Tao and Vu [2011] for another approach).

For the Log1 and Log2 cases, at the specific temperature $\beta = 2$, the law (1-7) acquires a special algebraic feature: it becomes a *determinantal* process, part of a wider class of processes (see Hough, Krishnapur, Y. Peres, and Virág [2009] and Borodin [2011]) for which the correlation functions are explicitly given by certain determinants. This allows for many explicit algebraic computations, and is part of *integrable probability* on which there is a large literature Borodin and Gorin [2016].

2.6 Complex geometry and theoretical physics. Coulomb systems and higher-dimensional analogues involving powers of determinantal densities are also of interest to geometers as a way to construct Kähler–Einstein metrics with negative Ricci curvature on complex manifolds, cf. Berman [2017] and Berman, Boucksom, and Witt Nyström [2011].

Another important motivation is the construction of Laughlin states for the Fractional Quantum Hall effect on complex manifolds, which effectively reduces to the study of a two-dimensional Coulomb gas on a manifold. The coefficients in the expansion of the (logarithm of the) partition function have interpretations as geometric invariants, cf. for instance Klevtsov [2016].

3 The mean field limits and macroscopic behavior

3.1 Questions. The first question that naturally arises is to understand the limit as $N \to \infty$ of the *empirical measure* defined by [4]

$$\mu_N := \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i} \tag{3-1}$$

for configurations of points that minimize the energy (1-1), critical points, solutions of the evolution problems, or typical configurations under the Gibbs measure (1-7), thus hoping to derive effective equations or minimization problems that describe the average or mean-field behavior of the system. The term mean-field refers to the fact that, from the physics perspective, each particle feels the collective field generated by all the other particles, averaged by dividing it by the number of particles. That collective field is $\mathsf{g} * \mu_N$, except that it is singular at each particle, so to evaluate it at x_i one first has to remove the contribution of x_i itself.

Another point of view is that of correlation functions. One may denote by

$$\rho_N^{(k)}(x_1, \dots, x_k) \tag{3-2}$$

the k-point correlation function, which is the probability density (for each specific problem) of observing a particle at x_1, a particle at $x_2, \dots,$ and a particle at x_k (these functions should of course be symmetric with respect to permutation of the labels). For

[4] Note that the configurations contain N points which also implicitly depend on N themselves, but we do not keep track of this dependence for the sake of lightness of notation.

instance, in the case (1-7), $\rho_N^{(N)}$ is simply $\mathbb{P}_{N,\beta}$ itself, and the $\rho_N^{(k)}$ are its marginals (obtained by integrating $\mathbb{P}_{N,\beta}$ with respect to all its variables but k). One then wants to understand the limit as $N \to \infty$ of each $\rho_N^{(k)}$, with fixed k. Mean-field results will typically imply that the limiting $\rho^{(k)}$'s have a factorized form

$$\rho^{(k)}(x_1, \dots, x_k) = \mu(x_1) \dots \mu(x_k) \tag{3-3}$$

for the appropriate μ which is also equal to $\rho^{(1)}$. This is called *molecular chaos* according to the terminology introduced by Boltzmann, and can be interpreted as the particles becoming independent in the limit. When looking at the dynamic evolutions of problems **(3)** and **(4)**, starting from initial data for which $\rho^{(k)}(0, \cdot)$ are in such a factorized form, one asks whether this remains true for $\rho^{(k)}(t, \cdot)$ for $t > 0$, if so this is called *propagation of (molecular) chaos*. It turns out that the convergence of the empirical measure (3-1) to a limit μ and the fact that each $\rho^{(k)}$ can be put in factorized form are essentially equivalent, see Hauray and Mischler [2014] and Golse [2016] and references therein — ideally, one would also like to find quantitative rates of convergences in N, and they will typically deteriorate as k gets large. In the following we will focus on the mean-field convergence approach, via the empirical measure.

In the case of minimizers **(1)**, a major question is to obtain an expansion as $N \to \infty$ for $\min \mathcal{H}_N$. In the setting of manifolds, the coefficients in such an expansion have geometric interpretations. In the same way, in the statistical mechanics setting **(2)**, one searches for expansions as $N \to \infty$ of the so-called *free energy* $-\beta^{-1} \log Z_{N,\beta}$. The free energy encodes a lot of the physical quantities of the system. For instance, points of non-differentiability of $\log Z_{N,\beta}$ as a function of β are interpreted as phase-transitions.

We will see below that understanding the mean-field behavior of the system essentially amounts to understanding the leading order term in the large N expansion of the minimal energy or respectively the free energy, while understanding the next order term in the expansion essentially amounts to understanding the next order (or fluctuations) of the system.

3.2 The equilibrium measure. The leading order behavior of $\mathcal{H}_N$ is related to the functional

$$\mathcal{I}_V(\mu) := \frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \mathsf{g}(x-y) d\mu(x) d\mu(y) + \int_{\mathbb{R}^d} V(x) d\mu(x) \tag{3-4}$$

defined over the space $\mathcal{P}(\mathbb{R}^d)$ of probability measures on $\mathbb{R}^d$ (which may also take the value $+\infty$). This is something one may naturally expect since $\mathcal{I}_V(\mu)$ appears as the continuum version of the discrete energy $\mathcal{H}_N$. From the point of view of statistical mechanics, $\mathcal{I}_V$ is the mean-field limit energy of $\mathcal{H}_N$, while from the point of view of probability, $\mathcal{I}_V$ plays the role of a *rate function*.

Assuming some lower semi-continuity of V and that it grows faster than g at ∞, it was shown in Frostman [1935] that the minimum of $\mathcal{I}_V$ over $\mathcal{P}(\mathbb{R}^d)$ exists, is finite and is achieved by a unique μ_V (unique by strict convexity of $\mathcal{I}_V$), which has compact support and a density, and is uniquely characterized by the fact that there exists a constant

c such that

$$\begin{cases} h^{\mu_V} + V \geq c & \text{in } \mathbb{R}^{\mathsf{d}} \\ h^{\mu_V} + V = c & \text{in the support of } \mu_V \end{cases} \tag{3-5}$$

where

$$h^{\mu_V}(x) := \int_{\mathbb{R}^{\mathsf{d}}} \mathsf{g}(x-y)d\mu_V(y) = \mathsf{g} * \mu_V \tag{3-6}$$

is the "electrostatic" potential generated by μ_V.

This measure μ_V is called the (Frostman) *equilibrium measure*, and the result is true for more general repulsive kernels than Coulomb, for instance for all regular kernels or inverse powers of the distance which are integrable.

Example 3.1. When g is the Coulomb kernel, applying the Laplacian on both sides of (3-5) gives that, in the interior of the support of the equilibrium measure, if $V \in C^2$,

$$\mathsf{c}_{\mathsf{d}}\mu_V = \Delta V \tag{3-7}$$

i.e. the density of the measure on the interior of its support is given by $\frac{\Delta V}{\mathsf{c}_{\mathsf{d}}}$. For example if V is quadratic, this density is constant on the interior of its support. If $V(x) = |x|^2$ then by symmetry μ_V is the indicator function of a ball (up to a multiplicative factor), this is known as the *circle law* for the Ginibre ensemble in the context of Random Matrix Theory. An illustration of the convergence to this circle law can be found in Figure 3. In dimension $\mathsf{d} = 1$, with $\mathsf{g} = -\log|\cdot|$ and $V(x) = x^2$, the equilibrium measure is $\mu_V(x) = \frac{1}{2\pi}\sqrt{4-x^2}\mathbf{1}_{|x|\leq 2}$, which corresponds in the context of RMT (GUE and GOE ensembles) to the famous *Wigner semi-circle law*, cf. Wigner [1955] and Mehta [2004].

In the Coulomb case, the equilibrium measure μ_V can also be interpreted in terms of the solution to a classical *obstacle problem* (and in the Riesz case (1-6) with $\mathsf{d}-2 \leq \mathsf{s} < \mathsf{d}$ a "fractional obstacle problem"), which is essentially dual to the minimization of $\mathcal{I}_V$, and better studied from the PDE point of view (in particular the regularity of μ_V and of the boundary of its support). For this aspect, see Serfaty [2015, Chap. 2] and references therein.

Frostman's theorem is the basic result of potential theory. The relations (3-5) can be seen as the Euler–Lagrange equations associated to the minimization of $\mathcal{I}_V$. They state that in the static situation, the total potential, sum of the potential generated by μ_V and the external potential V must be constant in the support of μ_V, i.e. in the set where the "charges" are present.

More generally $\nabla(h^\mu + V)$ can be seen as the total mean-field force acting on charges with density μ (i.e. each particle feels the average collective force generated by the other particles), and for the particle to be at rest one needs that force to vanish. Thus $\nabla(h^\mu + V)$ should vanish on the support of μ, in fact the stationarity condition that formally emerges as the limit for critical points of $\mathcal{H}_N$ is

$$\mu\nabla(h^\mu + V) = 0. \tag{3-8}$$

The problem with this relation is that the product $\mu\nabla h^\mu$ does not always make sense, since a priori μ is only a probability measure and ∇h^μ is not necessarily continuous, however, in dimension 2, one can give a weak form of the equation which always makes sense, inspired by Delort's work in fluid mechanics Delort [1991], cf. Sandier and Serfaty [2007, Chap. 13].

3.3 Convergence of minimizers.

Theorem 1. *We have*

$$\lim_{N\to\infty} \frac{\min \mathcal{H}_N}{N^2} = \min \mathcal{E}_V = \mathcal{E}_V(\mu_V) \tag{3-9}$$

and if $(x_1, \dots, x_N)$ *minimize* $\mathcal{H}_N$ *then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i} \rightharpoonup \mu_V \tag{3-10}$$

in the weak sense of probability measures.

This result is usually attributed to Choquet [1958], one may see the proof in Saff and Totik [1997] for the logarithmic cases, the general case can be treated exactly in the same way Serfaty [2015, Chap. 2], and is valid for very general interactions $\mathcal{G}$ (for instance radial decreasing and integrable near 0). In modern language it can be phrased as a Γ-convergence result. It can also easily be expressed in terms of convergence of marginals, as a molecular chaos result.

3.4 Parallel results for Ginzburg–Landau vortices. The analogue mean field result and leading order asymptotic expansion of the minimal energy has also been obtained for the two-dimensional Ginzburg–Landau functional of superconductivity (2-1), see Sandier and Serfaty [2007, Chap. 7]. It is phrased as the convergence of the vorticity $\nabla \times \langle i\psi, \nabla\psi\rangle$, normalized by the proper number of vortices, to an equilibrium measure, or the solution to an obstacle problem. The analogue of (3-8) is also derived for critical points in Sandier and Serfaty [ibid., Chap. 13].

3.5 Deterministic dynamics results - problems (3). For general reference on problems of the form **(3)** and **(4)**, we refer to Spohn [2004]. In view of the above discussion, in the dynamical cases (1-9) or (1-10), one expects as analogue results the convergences of the (time-dependent) empirical measures $\frac{1}{N}\sum_{i=1}^{N}\delta_{x_i}$ to probability densities μ that satisfy the limiting mean-field evolutions

$$\partial_t \mu = \operatorname{div}(\nabla(h^\mu + V)\mu) \tag{3-11}$$

respectively

$$\partial_t \mu = -\operatorname{div}(\mathbb{J}\nabla(h^\mu + V)\mu) \tag{3-12}$$

where again $h^\mu = \mathsf{g} * \mu$ as in (3-6). These are nonlocal transport equations where the density μ is transported along the velocity field $-\nabla(h^\mu + V)$ (respectively $\mathbb{J}\nabla(h^\mu + V)$) i.e. advected by the mean-field force that the distribution generates.

In the two-dimensional Coulomb case (1-2) with $V = 0$, (3-12) with $\mathbb{J}$ chosen as the rotation by $\pi/2$ is also well-known as the vorticity form of the incompressible Euler equation, describing the evolution of the vorticity in an ideal fluid, with velocity given by the Biot–Savart law. As such, this equation is well-studied in this context, and the convergence of solutions of (1-10) to (3-12), also known as the point-vortex approximation to Euler, has been rigorously proven, see Schochet [1996] and Goodman, Hou, and Lowengrub [1990].

As for (3-11), it is a dissipative equation, that can be seen as a gradient flow on the space of probability measures equipped with the so-called Wasserstein W_2 (or Monge–Kantorovitch) metric. In the dimension 2 logarithmic case, it was first introduced by Chapman, Rubinstein, and Schatzman [1996] and E [1994b] as a formal model for superconductivity, and in that setting the gradient flow description has been made rigorous (see Ambrosio and Serfaty [2008]) using the theory of gradient flows in metric spaces of Otto [2001] and Ambrosio, Gigli, and Savaré [2005]. The equation can also be studied by PDE methods F. Lin and Zhang [2000] and Serfaty and Vázquez [2014], which generalize to the Coulomb interaction in any dimension. The derivation of this gradient flow equation (3-11) from (2-3) can be guessed by variational arguments, i.e. "Γ-convergence of gradient flows", see Serfaty [2011]. In the non Coulombic case, i.e. for (1-6), (3-11) is a "fractional porous medium equation", analyzed in Caffarelli and Vazquez [2011], Caffarelli, Soria, and Vázquez [2013], and Zhou and Xiao [2017].

We have the following result which states in slightly informal terms that the desired convergence holds provided the limiting solution is regular enough.

Theorem 2 (Serfaty and Duerinckx [2018]). *For any d, any case (1-2), (1-5) or (1-6) with $\mathsf{d}-2 \le s < \mathsf{d}$, let $\{x_i\}$ solve (1-9), respectively (1-10) with initial data $x_i(0) = x_i^0$. Then if the limit μ^0 of the initial empirical measure $\frac{1}{N}\sum_{i=1}^N \delta_{x_i^0}$ is regular enough so that the solution μ_t of (3-11), resp. (3-12), with initial data μ_0 exists until time $T > 0$ and is regular enough, and if the initial condition is well-prepared in the sense that*

$$\lim_{N\to\infty} \frac{1}{N^2}\mathcal{H}_N(\{x_i^0\}) = \iint_{\mathbb{R}^\mathsf{d}\times\mathbb{R}^\mathsf{d}} \mathsf{g}(x-y)\,d\mu^0(x)\,d\mu^0(y),$$

then we have that for all $t \in [0, T)$,

$$\frac{1}{N}\sum_{i=1}^N \delta_{x_i(t)} \to \mu_t \quad \text{as } N \to \infty.$$

Note that the existence of regular enough solutions that exist for all time, provided the initial data is regular enough, is known to hold for all Coulomb cases and all Riesz cases with $\mathsf{s} < \mathsf{d} - 1$ Zhou and Xiao [2017].

The difficulty in proving this convergence result is due to the singularity of the Coulomb interaction combined with the nonlinear character of the product $h^\mu\mu$ (and its discrete analogue) which prevents from directly taking limits in the equation.

Prior results existed for less singular interactions Hauray [2009], Carrillo, Choi, and Hauray [2014], and Jabin and Wang [2017] or in dimension 1 Berman and Onnheim [2016]. Theorem 2 was first proven in the dissipative case in dimensions 1 and 2 in Duerinckx [2016], then in all dimensions and in the conservative case in Serfaty and Duerinckx [2018]. Both proofs rely on a "modulated energy" approach inspired from Serfaty [2017]. It consists in considering a *Coulomb-based (or Riesz-based) distance* between probability densities, more precisely the distance defined by

$$d_{\mathsf{g}}(\mu,\nu)^2 = \iint_{\mathbb{R}^d\times\mathbb{R}^d} \mathsf{g}(x-y)d(\mu-\nu)(x)d(\mu-\nu)(y)$$

which is a good metric thanks to the particular properties of the Coulomb and Riesz kernels. One can prove a "weak-strong" stability result for the limiting equations (3-11), (3-12) in that metric: if μ is a smooth enough solution to (3-11), resp. (3-12), and if ν is any solution to the same equation, then

$$d_{\mathsf{g}}(\mu(t),\nu(t)) \le e^{Ct} d_{\mathsf{g}}(\mu(0),\nu(0)), \tag{3-13}$$

which is proved by showing a Gronwall inequality. One may then exploit this stability property by taking μ to be the smooth enough expected limiting solution, and ν to be the empirical measure of the solution to the discrete evolution (1-9) or (1-10), after giving an appropriate *renormalized* meaning to the Coulomb distance (which is otherwise infinite) in that setting. We are able to prove that a relation similar to (3-13) holds, thus proving the desired convergence.

The analogue of the rigorous passage from (1-9) or (1-10) to (3-11) or (3-12) was accomplished at the level of the full parabolic and Schrödinger Ginzburg–Landau PDEs (2-3) and (2-4) Kurzke and Spirn [2014], Jerrard and Spirn [2015], and Serfaty [2017]. The method in Serfaty [2017] relies as above on a modulated energy argument which consists in finding a suitable energy, modelled on the Ginzburg–Landau energy, which measures the distance to the desired limiting solution, and for which a Gronwall inequality can be shown to hold.

As far as (1-11) is concerned, the limiting equation is formally found to be the Vlasov–Poisson equation

$$\partial_t\rho + v\cdot\nabla_x\rho + \nabla(h^\mu + V)\cdot\nabla_v\rho = 0 \tag{3-14}$$

where $\rho(t,x,v)$ is the density of particles at time t with position x and velocity v, and $\mu(t,x) = \int \rho(t,x,v)dv$ is the density of particles. The rigorous convergence of (1-11) to (3-14) and propagation of chaos are not proven in all generality (i.e. for all initial data) but it has been established in a statistical sense (i.e. randomizing the initial condition) and often truncating the interactions, see Kiessling [2014], Boers and Pickl [2016], Hauray and Jabin [2015], Lazarovici [2016], Lazarovici and Pickl [2017], and Jabin and Wang [2016] and also the reviews on the topic Jabin [2014] and Golse [2016].

3.6 Noisy dynamics - problems (4). The noise terms in these equations gives rise to an additive Laplacian term in the limiting equations. For instance the limiting equation

for (1-12) is expected to be the McKean equation

$$\partial_t \mu = \frac{1}{\beta}\Delta\mu - \operatorname{div}(\nabla(h^\mu + V)\mu) \tag{3-15}$$

and the convergence is known for regular interactions since the seminal work of McKean [1967], see also the reviews Sznitman [1991] and Jabin [2014].

For singular interactions, the situation has been understood for the one-dimensional logarithmic case Cépa and Lépingle [1997], then for all Riesz interactions (1-6) Berman and Onnheim [2018]. Higher dimensions with singular interactions is largely open, but recent progress of Jabin and Wang [2017] allows to treat possibly rough but bounded interactions, as well as some Coulomb interactions, and prove convergence in an appropriate statistical sense.

For the conservative case (1-13) the limiting equation is a viscous conservative equation of the form

$$\partial_t \mu = \frac{1}{\beta}\Delta\mu - \operatorname{div}(\nabla^\perp(h^\mu + V)\mu) \tag{3-16}$$

which in the two-dimensional logarithmic case (1-2) is the Navier–Stokes equation in vorticity form. The convergence in that particular case was established in Fournier, Hauray, and Mischler [2014], while the most general available result is that of Jabin and Wang [2017].

For the case of (1-14), the limiting equation is the McKean–Vlasov equation

$$\partial_t \rho + v \cdot \nabla_x \rho + \nabla(h^\mu + V) \cdot \nabla_v \rho - \frac{1}{\beta}\Delta\rho = 0 \tag{3-17}$$

with the same notation as for (3-14), and convergence in the case of bounded-gradient kernels is proven in Jabin and Wang [2016], see also references therein.

3.7 With temperature: statistical mechanics. Let us now turn to problem **(2)** and consider the situation with temperature as described via the Gibbs measure (1-7). One can determine that two temperature scaling choices are interesting: the first is taking β independent of N, the second is taking $\beta_N = \frac{\beta}{N}$ with some fixed β. In the former, which can be considered a "low temperature" regime, the behavior of the system is still governed by the equilibrium measure μ_V. The result can be phrased using the language of Large Deviations Principles (LDP), cf. Dembo and Zeitouni [2010] for definitions and reference.

Theorem 3. *The sequence $\{\mathbb{P}_{N,\beta}\}_N$ of probability measures on $\mathcal{P}(\mathbb{R}^d)$ satisfies a large deviations principle at speed N^2 with good rate function $\beta\hat{\mathcal{I}}_V$ where $\hat{\mathcal{I}}_V = \mathcal{I}_V - \min_{\mathcal{P}(\mathbb{R}^d)} \mathcal{I}_V = \mathcal{I}_V - \mathcal{I}_V(\mu_V)$. Moreover*

$$\lim_{N\to+\infty} \frac{1}{N^2} \log Z_{N,\beta} = -\beta\mathcal{I}_V(\mu_V) = -\beta \min_{\mathcal{P}(\mathbb{R}^d)} \mathcal{I}_V. \tag{3-18}$$

The concrete meaning of the LDP is that if E is a subset of the space of probability measures $\mathcal{P}(\mathbb{R}^d)$, after identifying configurations $(x_1, \dots, x_N)$ in $(\mathbb{R}^d)^N$ with their

empirical measures $\frac{1}{N}\sum_{i=1}^{N}\delta_{x_i}$, we may write

$$\mathbb{P}_{N,\beta}(E) \approx e^{-\beta N^2(\min_E \mathcal{I}_V - \min \mathcal{I}_V)}, \tag{3-19}$$

which in view of the uniqueness of the minimizer of $\mathcal{I}_V$ implies that configurations whose empirical measure does not converge to μ_V as $N \to \infty$ have exponentially decaying probability. In other words the Gibbs measure concentrates as $N \to \infty$ on configurations for which the empirical measure is very close to μ_V, i.e. the temperature has no effect on the mean-field behavior.

This result was proven in the logarithmic cases in Petz and Hiai [1998] (in dimension 2), Ben Arous and Guionnet [1997] (in dimension 1) and Ben Arous and Zeitouni [1998] (in dimension 2) for the particular case of a quadratic potential (and $\beta = 2$), see also Berman [2014] with results for general powers of the determinant in the setting of multidimensional complex manifolds, or Chafaï, Gozlan, and Zitt [2014] which recently treated more general singular $\mathcal{G}$'s and V's. This result is actually valid in any dimension, and is not at all specific to the Coulomb interaction (the proof works as well for more general interaction potentials, see Serfaty [2015]).

In the high-temperature regime $\beta_N = \frac{\beta}{N}$, the temperature is felt at leading order and brings an entropy term. More precisely there is a temperature-dependent equilibrium measure $\mu_{V,\beta}$ which is the unique minimizer of

$$I_{V,\beta}(\mu) = \beta\mathcal{I}_V(\mu) + \int \mu \log \mu. \tag{3-20}$$

Contrarily to the equilibrium measure, $\mu_{V,\beta}$ is not compactly supported, but decays exponentially fast at infinity. This mean-field behavior and convergence of marginals was first established for logarithmic interactions Kiessling [1993] and Caglioti, Lions, Marchioro, and Pulvirenti [1992] (see Messer and Spohn [1982] for the case of regular interactions) using an approach based on de Finetti's theorem. In the language of Large Deviations, the same LDP as above then holds with rate function $I_{V,\beta} - \min I_{V,\beta}$, and the Gibbs measure now concentrates as $N \to \infty$ on a neighborhood of $\mu_{V,\beta}$, for a proof see García-Zelada [2017]. Again the Coulomb nature of the interaction is not really needed. One can also refer to Rougerie [2014, 2016] for the mean-field and chaos aspects with a particular focus on their adaptation to the quantum setting.

4 Beyond the mean field limit: next order study

We have seen that studying systems with Coulomb (or more general) interactions at leading order leads to a good understanding of their limiting macroscopic behavior. One would like to go further and describe their microscopic behavior, at the scale of the typical inter-distance between the points, $N^{-1/\mathsf{d}}$. This in fact comes as a by-product of a next-to-leading order description of the energy $\mathcal{H}_N$, which also comes together with a next-to-leading order expansion of the free energy in the case (1-7).

Thinking of energy minimizers or of typical configurations under (1-7), since one already knows that $\sum_{i=1}^{N}\delta_{x_i} - N\mu_V$ is small, one knows that the so-called *discrepancy*

in balls $B_r(x)$ for instance, defined as

$$D(x,r) := \int_{B_r(x)} \sum_{i=1}^N \delta_{x_i} - N\, d\mu_V$$

is $o(r^{\mathsf{d}}N)$ as long as $r > 0$ is fixed. Is this still true at the mesoscopic scales for r of the order $N^{-\alpha}$ with $\alpha < 1/\mathsf{d}$? Is it true down to the microscopic scale, i.e. for $r = RN^{-1/\mathsf{d}}$ with $R \gg 1$? Does it hold regardless of the temperature? This would correspond to a *rigidity result*. Note that point processes with discrepancies growing like the perimeter of the ball have been called *hyperuniform* and are of interest to physicists for a variety of applications, cf. Torquato [2016], see also Ghosh and Lebowitz [2017] for a review of the link between rigidity and hyperuniformity. An addition question is: how much of the microscopic behavior depends on V or in another words is there a form of universality in this behavior? Such questions had only been answered in details in the one-dimensional case (1-5) as we will see below.

4.1 Expanding the energy to next order. The first step that we will describe is how to expand the energy $\mathcal{H}_N$ around the measure $N\mu_V$, following the approach initiated in Sandier and Serfaty [2015b] and continued in Sandier and Serfaty [2015a], Rougerie and Serfaty [2016], Petrache and Serfaty [2017], and Leblé and Serfaty [2017]. It relies on a splitting of the energy into a fixed leading order term and a next order term expressed in terms of the charge fluctuations, and on a rewriting of this next order term via the "electric potential" generated by the points. More precisely, exploiting the quadratic nature of the interaction, and letting $\triangle$ denote the diagonal in $\mathbb{R}^{\mathsf{d}} \times \mathbb{R}^{\mathsf{d}}$, let us expand

$$\begin{aligned}
\mathcal{H}_N(x_1,\dots,x_N) &= \frac{1}{2}\sum_{i\neq j} \mathsf{g}(x_i - x_j) + N\sum_{i=1}^N V(x_i)\\
&= \frac{1}{2}\iint_{\triangle^c} \mathsf{g}(x-y)d\Big(\sum_{i=1}^N \delta_{x_i}\Big)(x)d\Big(\sum_{i=1}^N \delta_{x_i}\Big)(y) + N\int_{\mathbb{R}^{\mathsf{d}}} Vd\Big(\sum_{i=1}^N \delta_{x_i}\Big)(x)\\
&= \frac{N^2}{2}\iint_{\triangle^c} \mathsf{g}(x-y)d\mu_V(x)d\mu_V(y) + N^2\int_{\mathbb{R}^{\mathsf{d}}} Vd\mu_V\\
&\quad + N\iint_{\triangle^c} \mathsf{g}(x-y)d\mu_V(x)\Big(\sum_{i=1}^N \delta_{x_i} - N\mu_V\Big)(y) + N\int_{\mathbb{R}^{\mathsf{d}}} Vd\Big(\sum_{i=1}^N \delta_{x_i} - N\mu_V\Big)\\
&\quad + \frac{1}{2}\iint_{\triangle^c} \mathsf{g}(x-y)d\Big(\sum_{i=1}^N \delta_{x_i} - N\mu_V\Big)(x)d\Big(\sum_{i=1}^N \delta_{x_i} - N\mu_V\Big)(y).
\end{aligned} \tag{4-1}$$

Recalling that μ_V is characterized by (3-5), we see that the middle term

$$\begin{aligned}
N\iint \mathsf{g}(x-y)d\mu_V(x)d(\sum_{i=1}^N \delta_{x_i} - N\mu_V)(y) + N\int_{\mathbb{R}^{\mathsf{d}}} Vd(\sum_{i=1}^N \delta_{x_i} - N\mu_V)\\
= N\int_{\mathbb{R}^{\mathsf{d}}} (h^{\mu_V} + V)d(\sum_{i=1}^N \delta_{x_i} - N\mu_V)
\end{aligned} \tag{4-2}$$

can be considered as vanishing (at least it does if all the points x_i fall in the support of μ_V). We are then left with

$$\mathcal{H}_N(x_1,\dots,x_N) = N^2\mathcal{I}_V(\mu_V) + F_N^{\mu_V}(x_1,\dots,x_N) \tag{4-3}$$

with

(4-4)
$$F_N^{\mu_V}(x_1,\dots,x_N) = \frac{1}{2}\iint_{\Delta^c} \mathsf{g}(x-y)d\Big(\sum_{i=1}^N \delta_{x_i} - N\mu_V\Big)(x)d\Big(\sum_{i=1}^N \delta_{x_i} - N\mu_V\Big)(y).$$

The relation (4-3) is a next-order expansion of $\mathcal{H}_N$ (recall (3-9)), valid for arbitrary configurations. The "next-order energy" $F_N^{\mu_V}$ can be seen as the Coulomb energy of the neutral system formed by the N positive point charges at the x_i's and the diffuse negative charge $-N\mu_V$ of same mass. To further understand $F_N^{\mu_V}$ let us introduce the potential generated by this system, i.e.

$$H_N(x) = \int_{\mathbb{R}^d} \mathsf{g}(x-y)d\Big(\sum_{i=1}^N \delta_{x_i} - N\mu_V\Big)(y) \tag{4-5}$$

(compare with (3-6)) which solves the linear elliptic PDE (in the sense of distributions)

$$-\Delta H_N = \mathsf{c_d}\Big(\sum_{i=1}^N \delta_{x_i} - N\mu_V\Big) \tag{4-6}$$

and use for the first time crucially the Coulomb nature of the interaction to write

$$\begin{aligned}\iint_{\Delta^c} \mathsf{g}(x-y)d\Big(\sum_{i=1}^N \delta_{x_i} - N\mu_V\Big)(x)d\Big(\sum_{i=1}^N \delta_{x_i} - N\mu_V\Big)(y) \\ \simeq -\frac{1}{\mathsf{c_d}}\int_{\mathbb{R}^d} H_N\Delta H_N = \frac{1}{\mathsf{c_d}}\int_{\mathbb{R}^d} |\nabla H_N|^2\end{aligned} \tag{4-7}$$

after integrating by parts by Green's formula. This computation is in fact incorrect because it ignores the diagonal terms which must be removed from the integral, and yields a divergent integral $\int |\nabla H_N|^2$ (it diverges near each point x_i of the configuration). However, this computation can be done properly by removing the infinite diagonal terms and "renormalizing" the infinite integral, replacing $\int |\nabla H_N|^2$ by

$$\int_{\mathbb{R}^d} |\nabla H_{N,\eta}|^2 - N\mathsf{c_d}g(\eta)$$

where we replace H_N by $H_{N,\eta}$, its "truncation" at level η (here $\eta = \alpha N^{-1/d}$ with α a small fixed number) — more precisely $H_{N,\eta}$ is obtained by replacing the Dirac masses in (4-5) by uniform measures of total mass 1 supported on the sphere $\partial B(x_i,\eta)$ — and then removing the appropriate divergent part $\mathsf{c_d}\mathsf{g}(\eta)$. The name *renormalized energy* originates in the work of Bethuel, Brezis, and Hélein [1994] in the context of two-dimensional Ginzburg–Landau vortices, where a similar (although different) renormalization procedure was introduced. Such a computation allows to replace the double integral, or sum of pairwise interactions of all the charges and "background", by a single integral, which is local in the potential H_N. This transformation is very useful, and uses crucially the fact that g is the kernel of a local operator (the Laplacian).

This electric energy $\int_{\mathbb{R}^{\mathsf{d}}} |\nabla H_{N,\eta}|^2$ is coercive and can thus serve to control the "fluctuations" $\sum_{i=1}^N \delta_{x_i} - N\mu_V$, in fact it is formally $\frac{1}{\mathsf{c}_{\mathsf{d}}}\|\nabla\Delta^{-1}(\sum_{i=1}^N \delta_{x_i} - N\mu_V)\|_{L^2}^2$. The relations (4-3)–(4-7) can be inserted into the Gibbs measure (1-7) to yield so-called "concentration results" in the case with temperature, see Serfaty [2014] (for prior such concentration results, see Maïda and Maurel-Segala [2014], Borot and Guionnet [2013a], and Chafai, Hardy, and Maida [2018]).

4.2 Blow-up and limiting energy. As we have seen, the configurations we are interested in are concentrated on (or near) the support of μ_V which is a set of macroscopic size and dimension d, and the typical distance between neighboring points is $N^{-1/\mathsf{d}}$. The next step is then to blow-up the configurations by $N^{1/\mathsf{d}}$ and take the $N \to \infty$ limit in $F_N^{\mu_V}$. This leads us to a renormalized energy that we define just below. It allows to compute a total Coulomb interaction for an infinite system of discrete point charges in a constant neutralizing background of fixed density 1. Such a system is often called a *jellium* in physics, and is sometimes considered as a toy model for matter, with a uniform electron sea and ions whose positions remain to be optimized.

From now on, we assume that Σ, the support of μ_V is a set with a regular boundary and $\mu_V(x)$ is a regular density function in Σ. Centering at some point x in Σ, we may blow-up the configuration by setting $x_i' = (N\mu_V(x))^{1/\mathsf{d}}(x_i - x)$ for each i. This way we expect to have a density of points equal to 1 after rescaling. Rescaling and taking $N \to \infty$ in (4-6), we are led to $H_N \to H$ with H solving an equation of the form

$$-\Delta H = \mathsf{c}_{\mathsf{d}}(\mathcal{C} - 1) \tag{4-8}$$

where $\mathcal{C}$ is a locally finite sum of Dirac masses.

Definition 4.1 (Sandier and Serfaty [2015b,a], Rougerie and Serfaty [2016], and Petrache and Serfaty [2017])**.** The (Coulomb) renormalized energy of H is

$$\mathcal{W}(H) := \lim_{\eta\to 0} \mathcal{W}_\eta(H) \tag{4-9}$$

where we let

$$\mathcal{W}_\eta(H) := \limsup_{R\to\infty} \frac{1}{R^{\mathsf{d}}} \int_{[-\frac{R}{2},\frac{R}{2}]^{\mathsf{d}}} |\nabla H_\eta|^2 - \mathsf{c}_{\mathsf{d}}\mathsf{g}(\eta) \tag{4-10}$$

and H_η is a truncation of H performed similarly as above.
We define the renormalized energy of a point configuration $\mathcal{C}$ as

$$\mathbb{W}(\mathcal{C}) := \inf\{\mathcal{W}(H) \mid -\Delta H = \mathsf{c}_{\mathsf{d}}(\mathcal{C} - 1)\} \tag{4-11}$$

with the convention $\inf(\varnothing) = +\infty$.

It is not a priori clear how to define a total Coulomb interaction of such a jellium system, because of the infinite size of the system and because of its lack of local charge neutrality. The definitions we presented avoid having to go through computing the sum of pairwise interactions between particles (it would not even be clear how to sum them), but instead replace it with (renormalized variants of) the extensive quantity $\int |\nabla H|^2$.

The energy $\mathbb{W}$ can be proven to be bounded below and to have a minimizer; moreover, its minimum can be achieved as the limit of energies of periodic configurations (with larger and larger period), for all these aspects see for instance Serfaty [2015].

4.3 Crystallization questions for minimizers. Determining the value of min $\mathbb{W}$ is an open question, with the exception of the one-dimensional analogues for which the minimum is achieved at the lattice $\mathbb{Z}$ Sandier and Serfaty [2015a] and Leblé [2016].

The only question that we can completely answer so far is that of the minimization over the restricted class of lattice configurations in dimension $\mathsf{d} = 2$, i.e. configurations which are exactly a lattice $\mathbb{Z}\vec{u} + \mathbb{Z}\vec{v}$ with $det(\vec{u}, \vec{v}) = 1$.

Theorem 4. *The minimum of $\mathbb{W}$ over lattices of volume 1 in dimension 2 is achieved uniquely by the triangular lattice.*

Here the triangular lattice means $\mathbb{Z} + \mathbb{Z}e^{i\pi/3}$, properly scaled, i.e. what is called the Abrikosov lattice in the context of superconductivity. This result is essentially equivalent (see Osgood, Phillips, and Sarnak [1988] and Chiu [1997]) to a result on the minimization of the Epstein ζ function of the lattice

$$\zeta_s(\Lambda) := \sum_{p \in \Lambda \setminus \{0\}} \frac{1}{|p|^s}$$

proven in the 50's by Cassels, Rankin, Ennola, Diananda, cf. Montgomery [1988] and references therein. It corresponds to the minimization of the "height" of flat tori, in the sense of Arakelov geometry. In dimension $\mathsf{d} \geq 3$ the minimization of $\mathbb{W}$ restricted to the class of lattices is an open question, except in dimensions 4, 8 and 24 where a strict local minimizer is known Sarnak and Strömbergsson [2006] (it is the E_8 lattice in dimension 8 and the Leech lattice in dimension 24, which were already mentioned before).

One may ask whether the triangular lattice does achieve the global minimum of $\mathbb{W}$ in dimension 2. The fact that the Abrikosov lattice is observed in superconductors, combined with the fact that $\mathbb{W}$ can be derived as the limiting minimization problem of Ginzburg–Landau, see Sandier and Serfaty [2012], justify conjecturing this.

Conjecture 4.2. The triangular lattice is a global minimizer of $\mathbb{W}$ in dimension 2.

It was also recently proven in Bétermin and Sandier [2018] that this conjecture is equivalent to a conjecture of Brauchart, Hardin, and Saff [2012] on the next order term in the asymptotic expansion of the minimal logarithmic energy on the sphere (an important problem in approximation theory, also related to Smale's "7th problem for the 21st century"), which is obtained by formal analytic continuation, hence by very different arguments. In addition, the result of Coulangeon and Schürmann [2012] essentially yields the local minimality of the triangular lattice within all periodic (with possibly large period) configurations.

Note that the triangular lattice, the E_8 lattice in dimension 8 and Leech lattice in dimension 24, mentioned above, are also conjectured by Cohn and Kumar [2007] to have

universally minimizing properties i.e. to be the minimizer for a broad class of interactions. The proof of this conjecture in dimensions 8 and 24 was recently announced by the authors of Cohn, Kumar, Miller, Radchenko, and Viazovska [2017], and it should imply that these lattices also minimize $\mathbb{W}$.

One may expect that in general low dimensions, the minimum of $\mathbb{W}$ is achieved by some particular lattice. Folklore knowledge is that lattices are not minimizing in large enough dimensions, as indicated by the situation for the sphere packing problem mentioned above.

These questions belongs to the more general family of crystallization problems, see Blanc and Lewin [2015] for a review. A typical such question is, given an interaction kernel g in any dimension, to determine the point positions that minimize

$$\sum_{i\neq j} \mathsf{g}(x_i - x_j)$$

(with some kind of boundary condition), or rather

$$\lim_{R\to\infty} \frac{1}{|B_R|} \sum_{i\neq j, x_i, x_j \in B_R} \mathsf{g}(x_i - x_j),$$

and to determine whether the minimizing configurations are lattices. Such questions are fundamental in order to understand the crystalline structure of matter. There are very few positive results in that direction in the literature, with the exception of Theil [2006] generalizing Radin [1981] for a class of very short range Lennard–Jones potentials, which is why the resolution of the sphere packing problem and the Cohn–Kumar conjecture are such breakthroughs.

4.4 Convergence results for minimizers. Given a (sequence of) configuration(s) $(x_1, \dots, x_N)$, we examine as mentioned before the blow-up point configurations

$$\{(\mu_V(x)N)^{1/d}(x_i - x)\}$$

and their infinite limits $\mathcal{C}$. We also need to let the blow-up center x vary over Σ, the support of μ_V. Averaging near the blow-up center x yields a "point process" P_N^x: a point process is precisely defined as a probability distribution on the space of possibly infinite point configurations, denoted Config. Here the point process P_N^x is essentially the Dirac mass at the blown-up configuration $\{(\mu_V(x)N)^{1/d}(x_i - x)\}$. This way, we form a "tagged point process" P_N (where the tag is the memory of the blow-up center), probability on $\Sigma \times$ Config, whose "slices" are the P_N^x. Taking limits $N \to \infty$ (up to subsequences), we obtain limiting tagged point processes P, which are all stationary, i.e. translation-invariant. We may also define the renormalized Coulomb energy at the level of tagged point processes as

$$\overline{\mathbb{W}}(P) := \frac{1}{2\mathsf{c}_\mathsf{d}} \int_\Sigma \int \mathbb{W}(\mathcal{C}) dP^x(\mathcal{C}) dx.$$

In view of (4-3) and the previous discussion, we may expect the following informally stated result (which we state only in the Coulomb cases, for extensions to (1-5) see Sandier and Serfaty [2015a] and to (1-6) see Petrache and Serfaty [2017]).

Theorem 5 (Sandier and Serfaty [2015b] and Rougerie and Serfaty [2016]). *Consider configurations such that*

$$\mathcal{H}_N(x_1,\dots,x_N) - N^2\mathcal{I}_V(\mu_V) \le CN^{2-\frac{2}{\mathsf{d}}}.$$

Then up to extraction P_N converges to some P and

$$\mathcal{H}_N(x_1,\dots,x_N) \simeq N^2\mathcal{I}_V(\mu_V) + N^{2-\frac{2}{\mathsf{d}}}\overline{\mathbb{W}}(P) + o(N^{2-\frac{2}{\mathsf{d}}}) \tag{4-12}$$

[5] *and in particular*

$$\min \mathcal{H}_N = N^2\mathcal{I}_V(\mu_V) + N^{2-\frac{2}{\mathsf{d}}}\min\overline{\mathbb{W}} + o(N^{2-\frac{2}{\mathsf{d}}}). \tag{4-13}$$

Since $\overline{\mathbb{W}}$ is an average of $\mathbb{W}$, the result (4-13) can be read as: after suitable blow-up around a point x, for a.e. $x \in \Sigma$, the minimizing configurations converge to minimizers of $\mathbb{W}$. If one believes minimizers of $\mathbb{W}$ to ressemble lattices, then it means that minimizers of $\mathcal{H}_N$ should do so as well. In any case, $\mathbb{W}$ can distinguish between different lattices (in dimension 2, the triangular lattice has less energy than the square lattice) and we expect $\mathbb{W}$ to be a good quantitative measure of disorder of a configuration (see Borodin and Serfaty [2013]).

The analogous result was proven in Sandier and Serfaty [2012] for the vortices in minimizers of the Ginzburg–Landau energy (2-1): they also converge after blow-up to minimizers of $\mathbb{W}$, providing a first rigorous justification of the Abrikosov lattice observed in experiments, modulo Conjecture 4.2. The same result was also obtained in Goldman, Muratov, and Serfaty [2014] for a two-dimensional model of small charged droplets interacting logarithmically called the Ohta–Kawasaki model – a sort of variant of Gamov's liquid drop model, after the corresponding mean-field limit results was established in Goldman, Muratov, and Serfaty [2013].

One advantage of the above theorem is that it is valid for generic configurations and not just for minimizers. When using the minimality, better "rigidity results" (as alluded to above) of minimizers can be proven: points are separated by $\frac{C}{(N\|\mu_V\|_\infty)^{1/d}}$ for some fixed $C > 0$ and there is uniform distribution of points and energy, down to the microscopic scale, see Petrache and Serfaty [2017], Nodari and Serfaty [2015], and Petrache and Rota Nodari [2018].

Theorem 5 relies on two ingredients which serve to prove respectively a lower bound and an upper bound for the next-order energy. The first is a general method for proving lower bounds for energies which have two intrinsic scales (here the macroscopic scale 1 and the microscopic scale $N^{-1/\mathsf{d}}$) and which is handled via the introduction of the probability measures on point patterns P_N described above. This method (see Sandier and Serfaty [2015b] and Serfaty [2015]), inspired by Varadhan, is reminiscent of Young measures and of Alberti and Müller [2001]. The second is a "screening procedure" which allows to exploit the local nature of the next-order energy expressed in terms of H_N, to paste together configurations given over large microscopic cubes and compute their next-order energy additively. To do so, we need to modify the configuration in a neighborhood of the boundary of the cube so as to make the cube neutral in charge and to

[5] In dimension $\mathsf{d} = 2$, there is an additional additive term $\frac{N}{4}\log N$ in both relations

make ∇H_N tangent to the boundary. This effectively screens the configuration in each cube in the sense that it makes the interaction between the different cubes vanish, so that the energy $\int |\nabla H_N|^2$ becomes proportional to the volume. One needs to show that this modification can be made while altering only a negligible fraction of the points and a negligible amount of the energy. This construction is reminiscent of Alberti, Choksi, and Otto [2009]. It is here crucial that the interaction is Coulomb so that the energy is expressed by a local function of H_N, which itself solves an elliptic PDE, making it possible to use the toolbox on estimates for such PDEs.

The next order study has not at all been touched in the case of dynamics, but it has been tackled in the statistical mechanics setting of (1-7).

4.5 Next-order with temperature. Here the interesting temperature regime (to see nontrivial temperature effects) turns out to be $\beta_N = \beta N^{\frac{2}{d}-1}$.

In contrast to the macroscopic result, several observations (e.g. by numerical simulation, see Figure 3) suggest that the behavior of the system at the microscopic scale depends heavily on β, and one would like to describe this more precisely. In the particular case of (1-5) or (1-2) with $\beta = 2$, which both arise in Random Matrix Theory, many things can be computed explicitly, and expansions of $\log Z_{N,\beta}$ as $N \to \infty$, Central Limit Theorems for linear statistics, universality in V (after suitable rescaling) of the microscopic behavior and local statistics of the points, are known Johansson [1998], Shcherbina [2013], Borot and Guionnet [2013a], Borot and Guionnet [2013b], Bourgade, Erdős, and Yau [2014, 2012], Bekerman, Figalli, and Guionnet [2015], and Bekerman and Lodhia [2016]. Generalizing such results to higher dimensions and all β's is a significant challenge.

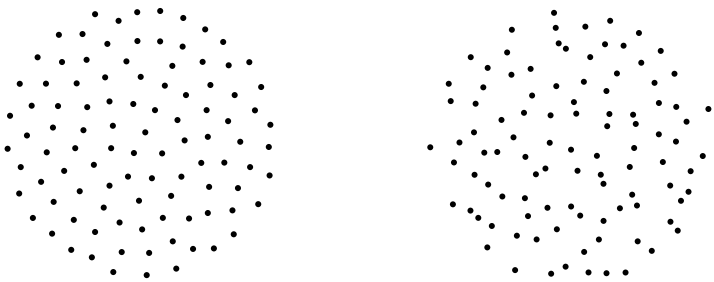

Figure 3: Case Log2 with $N = 100$ and $V(x) = |x|^2$, for $\beta = 400$ (left) and $\beta = 5$ (right).

4.6 Large Deviations Principle. A first approach consists in following the path taken for minimizers and using the next-order expansion of $\mathcal{H}_N$ given in (4-12). This

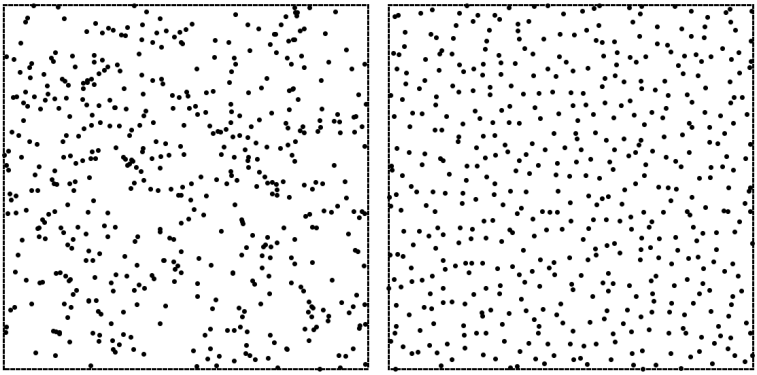

Figure 4: Simulation of the Poisson point process with intensity 1 (left), and the Ginibre point process with intensity 1 (right)

expansion can be formally inserted into (1-7), however this is not sufficient: to get a complete result, one needs to understand precisely how much volume in configuration space $(\mathbb{R}^{\mathsf{d}})^N$ is occupied near a given tagged point process P — this will give rise to an entropy term — and how much error (in both volume and energy) the screening construction creates. At the end we obtain a Large Deviations Principle expressed at the level of the microscopic point processes P, instead of the macroscopic empirical measures μ in Theorem 3. This is sometimes called "type-III large deviations" or large deviations at the level of empirical fields. Such results can be found in Varadhan [1988], Föllmer [1988], the relative specific entropy that we will use is formalized in Föllmer and Orey [1988] (for the non-interacting discrete case), Georgii [1993] (for the interacting discrete case) and Georgii and Zessin [1993] (for the interacting continuous case).

To state the result precisely, we need to introduce the Poisson point process with intensity 1, denoted Π, as the point process characterized by the fact that for any bounded Borel set B in $\mathbb{R}^{\mathsf{d}}$

$$\Pi\left(N(B)=n\right)=\frac{|B|^n}{n!}e^{-|B|}$$

where $N(B)$ denotes the number of points in B. The expectation of the number of points in B can then be computed to be $|B|$, and one also observes that the number of points in two disjoint sets are independent, thus the points "don't interact", see Figure 4 for a picture. The "specific" relative entropy $\overline{\mathsf{ent}}$ with respect to Π refers to the fact that it has to be computed taking an infinite volume limit, see Rassoul-Agha and Seppäläinen [2015] for a precise definition. One can just think that it measures how close the point process is to the Poisson one.

For any $\beta>0$, we then define a free energy functional $\overline{\mathcal{F}}_\beta$ as

$$\overline{\mathcal{F}}_\beta(P):=\frac{\beta}{2}\overline{\mathbb{W}}(P)+\overline{\mathsf{ent}}[P|\Pi]. \tag{4-14}$$

Theorem 6 (Leblé and Serfaty [2017]). *Under suitable assumptions, for any $\beta > 0$ a Large Deviations Principle at speed N with good rate function $\overline{\mathfrak{F}}_\beta - \inf \overline{\mathfrak{F}}_\beta$ holds in the sense that*

$$\mathbb{P}_{N,\beta}(P_N \simeq P) \simeq e^{-N(\overline{\mathfrak{F}}_\beta(P) - \inf \overline{\mathfrak{F}}_\beta)}$$

This way, the the Gibbs measure $\mathbb{P}_{N,\beta}$ concentrates on microscopic point processes which minimize $\overline{\mathfrak{F}}_\beta$. This minimization problem corresponds to some balancing (depending on β) between $\overline{\mathbb{W}}$, which prefers order of the configurations (and expectedly crystallization in low dimensions), and the relative entropy term which measures the distance to the Poisson process, thus prefers microscopic disorder and decorrelation between the points. As $\beta \to 0$, or temperature gets very large, the entropy term dominates and one can prove Leblé [2016] that the minimizer of $\overline{\mathfrak{F}}_\beta$ converges to the Poisson process. On the contrary, when $\beta \to \infty$, the $\mathbb{W}$ term dominates, and prefers regular and rigid configurations. (In the case (1-5) where the minimum of $\mathbb{W}$ is known to be achieved by the lattice, this can be made into a complete proof of crystallization as $\beta \to \infty$, cf. Leblé [ibid.]). When β is intermediate then both terms are important and one does not expect crystallization in that sense nor complete decorrelation. For separation results analogous to those quoted about minimizers, one may see Ameur [2017] and references therein.

The existence of a minimizer to $\overline{\mathfrak{F}}_\beta$ is known, it is certainly nonunique due to the rotational invariance of the problem, but it is not known whether it is unique modulo rotations, nor is the existence of a limiting point process P (independent of the subsequence) in general. The latter is however known to exist in certain ensembles arising in random matrix theory: for (1-5) for any β, it is the so-called sine-β process Kurzke and Spirn [2014] and Valkó and Virág [2009], and for (1-2) for $\beta = 2$ and V quadratic, it is the Ginibre point process Ginibre [1965], shown in Figure 4. It was also shown to exist for the jellium for small β in Imbrie [1982/83]. A consequence of Theorem 6 is to provide a variational interpretation to these point processes. One may hope to understand phase-transitions at the level of these processes, possibly via this variational interpretation, however this is completely open. While in dimension 1, the point process is expected to always be unique, in dimension 2, phase-transitions and symmetry breaking in positional or orientational order may happen. One would also like to understand the decay of the two-point correlation function and its possible change in rate, corresponding to a phase-transition. In the one-dimensional logarithmic case, the limits of the correlation functions are computed for rational β's Forrester [1993] and indicate a phase-transition.

A second corollary obtained as a by-product of Theorem 6 is the existence of a next order expansion of the free energy $-\beta^{-1} \log Z_{N,\beta}$.

Corollary 4.3 (Leblé and Serfaty [2017]).

$$-\beta^{-1} \log Z_{N,\beta} = N^{1+\frac{2}{\mathrm{d}}} \mathfrak{l}_V(\mu_V) + N \min \overline{\mathfrak{F}} + o(N) \tag{4-15}$$

in the cases (1-3)*; and in the cases* (1-2), (1-5),

$$-\beta^{-1} \log Z_{N,\beta} = N^2 \mathfrak{l}_V(\mu_V) - \frac{N}{2\mathrm{d}} \log N + N \min \overline{\mathfrak{F}}_\beta + o(N)$$

or more explicitly

$$-\beta^{-1}\log Z_{N,\beta} = \tag{4-16}$$
$$= N^2\mathcal{I}_V(\mu_V) - \frac{N}{2\mathsf{d}}\log N + NC_\beta + N\left(\frac{1}{\beta}-\frac{1}{2\mathsf{d}}\right)\int_\Sigma \mu_V(x)\log\mu_V(x)\,dx + o(N),$$

where C_β depends only on β, but not on V.

This formulae are to be compared with the results of Shcherbina [2013], Borot and Guionnet [2013a], Borot and Guionnet [2013b], and Bekerman, Figalli, and Guionnet [2015] in the $\mathsf{Log1}$ case, the semi-rigorous formulae in Zabrodin and Wiegmann [2006] in the dimension 2 Coulomb case, and are the best-known information on the free energy otherwise. We recall that understanding the free energy is fundamental for the description of the properties of the system. For instance, the explicit dependence in V exhibited in (4-16) will be the key to proving the result of the next section.

Finally, note that a similar result to the above theorem and corollary can be obtained in the case of the two-dimensional two-component plasma alluded to in Section 2.4, see Leblé, Serfaty, and Zeitouni [2017].

4.7 A Central Limit Theorem for fluctuations. Another approach to understanding the rigidity of configurations and how it depends on the temperature is to examine the behavior of the linear statistics of the fluctuations, i.e. consider, for a regular test function f, the quantity

$$\sum_{i=1}^N f(x_i) - N\int f d\mu_V.$$

Theorem 7 (Leblé and Serfaty [2018])**.** *In the case (1-2), assume $V \in C^4$ and the previous assumptions on μ_V and $\partial\Sigma$, and let $f \in C_c^4(\mathbb{R}^2)$ or $C_c^3(\Sigma)$. If Σ has $m \geq 2$ connected components Σ_i, add $m-1$ conditions $\int_{\partial\Sigma_i}\Delta f^\Sigma = 0$ where f^Σ is the harmonic extension of f outside Σ. Then*

$$\sum_{i=1}^N f(x_i) - N\int_\Sigma f\,d\mu_V$$

converges in law as $N\to\infty$ to a Gaussian distribution with

$$\textit{mean} = \frac{1}{2\pi}\left(\frac{1}{\beta}-\frac{1}{4}\right)\int_{\mathbb{R}^2}\Delta f\,(\mathbf{1}_\Sigma + \log\Delta V)^\Sigma \qquad \textit{variance} = \frac{1}{2\pi\beta}\int_{\mathbb{R}^2}|\nabla f^\Sigma|^2.$$

The result can moreover be localized with f supported on any mesoscale $N^{-\alpha}$, $\alpha < \frac{1}{2}$, and it is true as well for energy minimizers, taking formally $\beta = \infty$.

This result can be interpreted in terms of the convergence of H_N (of (4-5)) to a suitable so-called "Gaussian Free Field", a sort of two-dimensional analogue of Brownian motion. This theorem shows that if f is smooth enough, the fluctuations of linear statistics are typically of order 1, i.e. much smaller than the sum of N iid random variables

which is typically or order $\sqrt{N}$. This a manifestation of rigidity, which even holds down to the mesoscales. Note that the regularity of f is necessary, the result is false if f is discontinuous, however the precise threshold of regularity is not known.

In dimension 1, this theorem was first proven in Johansson [1998] for polynomial V and f analytic. It was later generalized in Shcherbina [2013], Borot and Guionnet [2013a], Borot and Guionnet [2013b], Bekerman and Lodhia [2016], Webb [2016], and Bekerman, Leblé, and Serfaty [2017]. In dimension 2, this result was proven for the determinantal case $\beta = 2$, first in Rider and Virág [2007] (for V quadratic), Berman [n.d.] assuming just $f \in C^1$, and then Ameur, Hedenmalm, and Makarov [2011] under analyticity assumptions. It was then proven for all β simultaneously as Leblé and Serfaty [2018] in Bauerschmidt, Bourgade, Nikula, and Yau [n.d.], with f assumed to be supported in Σ. The approach for proving such results has generally been based on Dyson–Schwinger (or "loop") equations.

If the extra conditions do not hold, then the CLT is not expected to hold. Rather, the limit should be a Gaussian convolved with a discrete Gaussian variable, as shown in the $\mathsf{Log1}$ case in Borot and Guionnet [2013b].

To prove Theorem 7, following the approach pioneered by Johansson [1998], we compute the Laplace transform of these linear statistics and see that it reduces to understanding the ratio of two partition functions, the original one and that of a Coulomb gas with potential V replaced by $V_t = V + tf$ with t small. Thanks to Serfaty and Serra [n.d.] the variation of the equilibrium measure associated to this replacement is well understood. We are then able to leverage on the expansion of the partition function of (4-16) to compute the desired ratio, using also a change of variables which is a transport map between the equilibrium measure μ_V and the perturbed equilibrium measure. Note that the use of changes of variables in this context is not new, cf. Johansson [1998], Borot and Guionnet [2013a], Shcherbina [2013], and Bekerman, Figalli, and Guionnet [2015]. In our approach, it essentially replaces the use of the loop or Dyson–Schwinger equations.

4.8 More general interactions. It remains to understand how much of the behavior we described are really specific to Coulomb interactions. Already Theorems 5 and 6 were shown in Petrache and Serfaty [2017] and Leblé and Serfaty [2017] to hold for the more general Riesz interactions with $\mathsf{d} - 2 \le s < \mathsf{d}$. This is thanks to the fact that the Riesz kernel is the kernel for a fractional Laplacian, which is not a local operator but can be interpreted as a local operator after adding one spatial dimension, according to the procedure of Caffarelli and Silvestre [2007]. The results of Theorem 6 are also valid in the *hypersingular* Riesz interactions $\mathsf{s} > \mathsf{d}$ (see Hardin, Leblé, Saff, and Serfaty [2018]), where the kernel is very singular but also decays very fast. The Gaussian behavior of the fluctuations seen in Theorem 7 is for now proved only in the logarithmic cases, but it remains to show whether it holds for more general Coulomb cases and even possibly more general interactions as well.

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PARITY SHEAVES AND THE HECKE CATEGORY

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Abstract

We survey some applications of parity sheaves and Soergel calculus to representation theory.

Dedicated to Wolfgang Soergel, in admiration.
"…we're still using your imagination…" (L. A. Murray)

Introduction

One of the first theorems of representation theory is Maschke's theorem: any representation of a finite group over a field of characteristic zero is semi-simple. This theorem is ubiquitous throughout mathematics. (We often use it without realising it; for example, when we write a function of one variable as the sum of an odd and an even function.) The next step is Weyl's theorem: any finite-dimensional representation of a compact Lie group is semi-simple[1]. It is likewise fundamental: for the circle group Weyl's theorem is closely tied to the theory of Fourier series.

Beyond the theorems of Maschke and Weyl lies the realm where semi-simplicity fails. Non semi-simple phenomena in representation theory were first encountered when studying the modular (i.e. characteristic p) representations of finite groups. This theory is the next step beyond the classical theory of the character table, and is important in understanding the deeper structure of finite groups. A second example (of fundamental importance throughout mathematics from number theory to mathematical physics) occurs when studying the infinite-dimensional representation theory of semi-simple Lie groups and their p-adic counterparts.

Throughout the history of representation theory, geometric methods have played an important role. Over the last forty years, the theory of intersection cohomology and perverse sheaves has provided powerful new tools. To any complex reductive group is naturally associated several varieties (e.g. unipotent and nilpotent orbits and their closures, the flag variety and its Schubert varieties, the affine Grassmannian and its Schubert varieties …). In contrast to the group itself, these varieties are often singular. The theory of perverse sheaves provides a collection of constructible complexes of sheaves (intersection cohomology sheaves) on such varieties, and the "IC data" associated to

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[1] Weyl first proved his theorem via integration over the group to produce an invariant Hermitian form. To do this he needed the theory of manifolds. One can view his proof as an early appearance of geometric methods in representation theory.

intersection cohomology sheaves (graded dimensions of stalks, total cohomology, …) appears throughout Lie theory.

The first example of the power of this theory is the Kazhdan–Lusztig conjecture (a theorem of Beilinson–Bernstein and Brylinski–Kashiwara), which expresses the character of a simple highest weight module over a complex semi-simple Lie algebra in terms of IC data of Schubert varieties in the flag variety. This theorem is an important first step towards understanding the irreducible representations of semi-simple Lie groups. A second example is Lusztig's theory of character sheaves, which provides a family of conjugation equivariant sheaves on the group which are fundamental to the study of the characters of finite groups of Lie type.[2]

An important aspect of the IC data appearing in representation theory is that it is computable. For example, a key step in the proof of the conjecture of Kazhdan and Lusztig is their theorem that the IC data attached to Schubert varieties in the flag variety is encoded in Kazhdan–Lusztig polynomials, which are given by an explicit combinatorial algorithm involving only the Weyl group. Often this computability of IC data is thanks to the Decomposition Theorem, which asserts the semi-simplicity (with coefficients of characteristic zero) of a direct image sheaf, and implies that one can compute IC data via a resolution of singularities.

One can view the appearance of the Decomposition Theorem throughout representation theory as asserting some form of (perhaps well-hidden) semi-simplicity. A trivial instance of this philosophy is that Maschke's theorem is equivalent to the Decomposition Theorem for a finite morphism. A less trivial example is the tendency of categories in highest weight representation theory to admit Koszul gradings; indeed, according to Beilinson, Ginzburg, and Soergel [1996], a Koszul ring is "as close to semisimple as a $\mathbb{Z}$-graded ring possibly can be". Since the Kazhdan–Lusztig conjecture and its proof, many character formulae have been discovered resembling the Kazhdan–Lusztig conjecture (e.g. for affine Lie algebras, quantum groups at roots of unity, Hecke algebras at roots of unity, …) and these are often accompanied by a Koszul grading.

All of the above character formulae involve representations of objects defined over $\mathbb{C}$. On the other hand modular representation theory has been dominated since 1979 by conjectures (the Lusztig conjecture Lusztig [1980] on simple representations of reductive algebraic groups and the James conjecture James [1990] on simple representations of symmetric groups) which would imply that characteristic p representations of algebraic groups and symmetric groups are controlled by related objects over $\mathbb{C}$ (quantum groups and Hecke algebras at a pth root of unity) where character formulae are given by Kazhdan–Lusztig like formulae.

The Decomposition Theorem fails in general with coefficients in a field of characteristic p, as is already evident from the failure of Maschke's theorem in characteristic p. It was pointed out by Soergel [2000] (and extended by Fiebig [2011] and Achar and Riche [2016b]) that, after passage through deep equivalences, the Lusztig conjecture is equivalent to the Decomposition Theorem holding for Bott–Samelson resolutions of certain complex Schubert varieties, with coefficients in a field of characteristic p. For

[2]The reader is referred to Lusztig's contribution Lusztig [1991] to these proceedings in 1990 for an impressive list of applications of IC techniques in representation theory.

a fixed morphism, the Decomposition Theorem can only fail in finitely many characteristics, which implies that the Lusztig conjecture holds for large primes[3]. More recently, it was discovered that there are many large characteristics for which the Decomposition Theorem fails for Bott–Samelson resolutions Williamson [2017c]. This led to exponentially large counter-examples to the expected bounds in the Lusztig conjecture as well as counter-examples to the James conjecture.

Thus the picture for modular representations is much more complicated than we thought. Recently it has proven useful (see Soergel [2000] and Juteau, Mautner, and Williamson [2014]) to accept the failure of the Decomposition Theorem in characteristic p and consider indecomposable summands of direct image sheaves as interesting objects in their own right. It was pointed out by Juteau, Mautner and the author that, in examples in representation theory, these summands are often characterised by simple cohomology parity vanishing conditions, and are called parity sheaves.

Most questions in representation theory whose answer involves (or is conjectured to involve) Kazhdan–Lusztig polynomials are controlled by the Hecke category, a categorification of the Hecke algebra of a Coxeter system. Thus it seems that the Hecke category is a fundamental object in representation theory, like a group ring or an enveloping algebra. The goal of this survey is to provide a motivated introduction to the Hecke category in both its geometric (via parity sheaves) and diagrammatic (generators and relations) incarnations.

When we consider the Hecke category in characteristic p it gives rise to an interesting new Kazhdan–Lusztig-like basis of the Hecke algebra, called the p-canonical basis. The failure of this basis to agree with the Kazhdan–Lusztig basis measures the failure of the Decomposition Theorem in characteristic p. Conjecturally (and provably in many cases), this basis leads to character formulae for simple modules for algebraic groups and symmetric groups which are valid for all p. Its uniform calculation for affine Weyl groups and large p seems to me to be one of the most interesting problems in representation theory.[4]

If one sees the appearance of the Decomposition Theorem and Koszulity as some form of semi-simplicity, then this semi-simplicity fails in many settings in modular representation theory. However it is tempting to see the appearance of parity sheaves and the p-canonical basis as a deeper and better hidden layer of semi-simplicity, beyond what we have previously encountered. Some evidence for this is the fact that some form of Koszul duality still holds, although here IC sheaves are replaced by parity sheaves and there are no Koszul rings.

[3]The first proof of Lusztig's conjecture for $p \gg 0$ was obtained as a consequence of works by Kazhdan and Lusztig [1993], Lusztig [1994], Kashiwara and Tanisaki [1995] and Andersen, Jantzen, and Soergel [1994].

[4]See Lusztig and Williamson [2018] for a conjecture in a very special case, which gives some idea of (or at least a lower bound on!) the complexity of this problem.

Structure of the paper.

1. In §1 we discuss the Decomposition Theorem, parity sheaves and the role of intersection forms. We conclude with examples of parity sheaves and the failure of the Decomposition Theorem with coefficients of characteristic p.

2. In §2 we introduce the Hecke category. We explain two incarnations of this category (via parity sheaves, and via diagrammatics) and discuss its spherical and anti-spherical modules. We conclude by defining the p-canonical basis, giving some examples, and discussing several open problems.

3. In §3 we give a bird's eye view of Koszul duality for the Hecke category in its classical, monoidal and modular forms.

Although this work is motivated by representation theory, we only touch on applications in remarks. The reader is referred to Williamson [2017a] for a survey of applications of this material to representation theory.

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1 The Decomposition Theorem and Parity Sheaves

The Decomposition Theorem is a beautiful theorem about algebraic maps. However its statement is technical and it takes some effort to understand its geometric content. To motivate the Decomposition Theorem and the definition of parity sheaves, we consider one of the paths that led to its discovery, namely Deligne's proof of the Weil conjectures Deligne [1974]. We must necessarily be brief; for more background on the Decomposition Theorem see Beĭlinson, Bernstein, and Deligne [1982], de Cataldo and Migliorini [2009], and Williamson [2017d].

1.1 Motivation: The Weil conjectures. Suppose that X is a smooth projective variety defined over a finite field $\mathbb{F}_q$. On X one has the Frobenius endomorphism $\mathrm{Fr} : X \to X$ and the deepest of the Weil conjectures ("purity") implies that the eigenvalues of Fr on the étale cohomology $H^i(X)$[5] are of a very special form ("Weil numbers of weight i"). By the Grothendieck–Lefschetz trace formula, we have

$$|X(\mathbb{F}_{q^m})| = \sum_i (-1)^i \operatorname{Tr}((\mathrm{Fr}^*)^m : H^i(X) \to H^i(X))$$

[5] In this section only we will use $H^i(X)$ to denote the étale cohomology group $H^i(X_{\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell)$ of the extension of scalars of X to an algebraic closure $\overline{\mathbb{F}}_q$ of $\mathbb{F}_q$, where ℓ is a fixed prime number coprime to q.

for all $m \geq 1$, where $X(\mathbb{F}_{q^m})$ denotes the (finite) set of $\mathbb{F}_{q^m}$-rational points of X. In this way, the Weil conjectures have remarkable implications for the number of $\mathbb{F}_{q^m}$-points of X.

How should we go about proving purity? We might relate the cohomology of X to that of other varieties, slowly expanding the world where the Weil conjectures hold. A first attempt along these lines might be to consider long exact sequences associated to open or closed subvarieties of X. However this is problematic because purity no longer holds if one drops the "smooth" or "proper" assumption.

For any map $f : X \to Y$ of varieties we have a push-forward functor f_* and its derived functor Rf_* between (derived) categories of sheaves on X and Y. (In this paper we will never consider non-derived functors; we will write f_* instead of Rf_* from now on.) The cohomology of X (with its action of Frobenius) is computed by $p_*\overline{\mathbb{Q}}_{\ell,X}$, where $\overline{\mathbb{Q}}_{\ell,X}$ denotes the constant sheaf on X and $p : X \to \mathrm{pt}$ denotes the projection to a point.

This reinterpretation of what cohomology "means" provides a more promising approach to purity. For any map $f : X \to Y$ we can use the commutative diagram

$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[dr, "p"] & \\ Y \arrow[r, "g"'] & \mathrm{pt} \end{tikzcd}$$

and the isomorphism $p_*\overline{\mathbb{Q}}_{\ell,X} = (g \circ f)_*\overline{\mathbb{Q}}_{\ell,X} = g_*(f_*\overline{\mathbb{Q}}_{\ell,X})$ to factor the calculation of $H^*(X)$ into two steps: we can first understand $f_*\overline{\mathbb{Q}}_{\ell,X}$; then understand the direct image of this complex to a point. One can think of the complex $f_*\overline{\mathbb{Q}}_{\ell,X}$ as a linearisation of the map f. For example, if f is proper and y is a (geometric) point of Y then the stalk at y is

$$(f_*\mathbb{Q}_{\ell,X})_y = H^*(f^{-1}(y)).$$

It turns out that[6] $f_*\overline{\mathbb{Q}}_{\ell,X}$ splits as a direct sum of simple pieces (this is the Decomposition Theorem). Thus, each summand contributes a piece of the cohomology of X, and one can try to understand them separately. This approach provides the skeleton of Deligne's proof of the Weil conjectures: after some harmless modifications to X, the theory of Lefschetz pencils provides a surjective morphism $f : X \to \mathbb{P}^1$, and one has to show purity for the cohomology of each of the summands of $f_*\overline{\mathbb{Q}}_{\ell,X}$ (sheaves on $\mathbb{P}^1$). Showing the purity of the cohomology of each summand is the heart of the proof, which we don't enter into here!

1.2 The Decomposition Theorem. We now change setting slightly: from now on we consider complex algebraic varieties equipped with their classical (metric) topology and sheaves of $\Bbbk$-vector spaces on them, for some field of coefficients $\Bbbk$. For such a variety Y and a stratification

$$Y = \bigsqcup_{\lambda \in \Lambda} Y_\lambda$$

[6] after passage to $\overline{\mathbb{F}}_q$

of Y into finitely many locally-closed, smooth and connected subvarieties we denote by $D^b_\Lambda(Y;\Bbbk)$ the full subcategory of the derived category of complexes of sheaves of $\Bbbk$-vector spaces with Λ-constructible[7] cohomology sheaves. We will always assume that our stratification is such that $D^b_\Lambda(Y;\Bbbk)$ is preserved under Verdier duality (this is the case, for example, if our stratification is given by the orbits of a group). We denote by $D^b_c(Y;\Bbbk)$ the constructible derived category: it consists of those complexes which are Λ-constructible for some Λ as above. Both $D^b_\Lambda(Y;\Bbbk)$ and $D^b_c(Y;\Bbbk)$ are triangulated with shift functor $[1]$. For any morphism $f : X \to Y$ we have functors

$$D^b_c(X;\Bbbk) \underset{f^*,f^!}{\overset{f_*,f_!}{\rightleftarrows}} D^b_c(Y;\Bbbk)$$

satisfying a menagerie of relations (see e.g. de Cataldo and Migliorini [2009]).

Consider a proper morphism $f : X \to Y$ of complex algebraic varieties with X smooth. We consider the constant sheaf $\Bbbk_X$ on X with values in $\Bbbk$ and its (derived) direct image on Y:

$$f_*\Bbbk_X.$$

A fundamental problem (which we tried to motivate in the previous section) is to understand how this complex of sheaves decomposes. The Decomposition Theorem states that, if $\Bbbk$ is a field of characteristic zero, then $f_*\Bbbk_X$ is semi-simple in the sense of perverse sheaves. Roughly speaking, this means that much of the topology of the fibres of f is "forced" by the nature of the singularities of Y. More precisely, if we fix a stratification of Y as above for which $f_*\Bbbk_X$ is constructible, then we have an isomorphism:

$$f_*\Bbbk_X \cong \bigoplus H^*_{\lambda,\mathcal{L}} \otimes_\Bbbk \mathbf{IC}^{\mathcal{L}}_\lambda. \tag{1}$$

Here the (finite) sum is over certain pairs $(\lambda, \mathcal{L})$ where $\mathcal{L}$ is an irreducible local system on Y_λ, $H^*_{\lambda,\mathcal{L}}$ is a graded vector space, and $\mathbf{IC}^{\mathcal{L}}_\lambda$ denotes IC extension of $\mathcal{L}$. (The complex of sheaves $\mathbf{IC}^{\mathcal{L}}_\lambda$ is supported on $\overline{Y_\lambda}$ and extends $\mathcal{L}[\dim_{\mathbb{C}} Y_\lambda]$ in a "minimal" way, taking into account singularities. For example, if $\overline{Y}_\lambda$ is smooth and $\mathcal{L}$ extends to a local system $\overline{\mathcal{L}}$ on $\overline{Y}_\lambda$, then $\mathbf{IC}^{\mathcal{L}}_\lambda = \overline{\mathcal{L}}[\dim_{\mathbb{C}} Y_\lambda]$.) If $\mathcal{L} = \Bbbk$ is the trivial local system we sometimes write $\mathbf{IC}_\lambda$ instead of $\mathbf{IC}^{\Bbbk}_\lambda$.

Below we will often consider coefficient fields $\Bbbk$ of positive characteristic, where in general (1) does not hold. We will say that the Decomposition Theorem *holds* (resp. *fails*) with $\Bbbk$-coefficients if an isomorphism of the form (1) holds (resp. fails).

1.3 Parity Sheaves. Consider $f : X \to Y$, a proper map between complex algebraic varieties, with X smooth. Motivated by the considerations that led to the Decomposition Theorem we ask:

Question 1.1. *Fix a field of coefficients* $\Bbbk$.

1. *What can one say about the indecomposable summands of* $f_*\Bbbk_X$*?*

[7] i.e. those sheaves whose restriction to each Y_λ is a local system

2. *What about the indecomposable summands of $f_*\mathcal{L}$, for $\mathcal{L}$ a local system of $\Bbbk$-vector spaces on X?*

(Recall f_* means derived direct image.) If $\Bbbk$ is of characteristic zero, then (1) has a beautiful answer: by the Decomposition Theorem, any indecomposable summand is a shift of an IC extension of an irreducible local system. The same is true of (2) if $\mathcal{L}$ is irreducible. (This is Kashiwara's conjecture, proved by Mochizuki [2011].)

If the characteristic of $\Bbbk$ is positive this question seems difficult. However it has a nice answer (in terms of "parity sheaves") under restrictions on X, Y and f.

Remark 1.2. It seems unlikely that this question will have a good answer as phrased in general. It is possible that it does have a good answer if one instead works in an appropriate category of motives, perhaps with restrictions on allowable maps f and local systems $\mathcal{L}$.

Assume that Y admits a stratification $Y = \bigsqcup_{\lambda\in\Lambda} Y_\lambda$ as above. For $\lambda \in \Lambda$, let $j_\lambda : Y_\lambda \hookrightarrow Y$ denote the inclusion. A complex $\mathcal{F} \in D^b_\Lambda(Y;\Bbbk)$ is *even* if

$$\mathcal{H}^i(j_\lambda^*\mathcal{F}) = \mathcal{H}^i(j_\lambda^!\mathcal{F}) = 0 \quad \text{for } i \text{ odd, and all } \lambda \in \Lambda. \tag{2}$$

(Here $\mathcal{H}^i$ denotes the ith cohomology sheaf of a complex of sheaves.) A complex $\mathcal{F}$ is *odd* if $\mathcal{F}[1]$ is even; a complex is *parity* if it can be written as a sum $\mathcal{F}_0 \oplus \mathcal{F}_1$ with $\mathcal{F}_0$ (resp. $\mathcal{F}_1$) even (resp. odd).

Example 1.3. The archetypal example of a parity complex is $f_*\Bbbk_X[\dim_{\mathbb{C}} X]$, where f is proper and X is smooth as above, and f is in addition *even*: $f_*\Bbbk_X[\dim_{\mathbb{C}} X]$ is Λ-constructible and the cohomology of the fibres of f with $\Bbbk$-coefficients vanishes in odd degree. (Indeed, in this case, $\Bbbk_X[\dim_{\mathbb{C}} X]$ is Verdier self-dual, hence so is $f_*\Bbbk_X[\dim_{\mathbb{C}} X]$ (by properness) and the conditions (2) follow from our assumptions on the cohomology of the fibres of f.)

We make the following (strong) assumptions on each stratum:

$$Y_\lambda \text{ is simply connected;} \tag{3}$$

$$H^i(Y_\lambda, \Bbbk) = 0 \quad \text{for } i \text{ odd.} \tag{4}$$

Theorem 1.4. *Suppose that $\mathcal{F}$ is indecomposable and parity:*

1. *The support of $\mathcal{F}$ is irreducible, and hence is equal to $\overline{Y}_\lambda$ for some $\lambda \in \Lambda$.*

2. *The restriction of $\mathcal{F}$ to Y_λ is isomorphic to a constant sheaf, up to a shift.*

Moreover, any two indecomposable parity complexes with equal support are isomorphic, up to a shift.

(The proof of this theorem is not difficult, see Juteau, Mautner, and Williamson [2014, §2.2].) If $\mathcal{F}$ is an indecomposable parity complex with support Y_λ then there is a unique shift of $\mathcal{F}$ making it Verdier self-dual. We denote it by $\mathcal{E}_\lambda^{\Bbbk}$ or $\mathcal{E}_\lambda$ and call it a *parity sheaf*.

Remark 1.5. The above theorem is a uniqueness statement. In general, there might be no parity complex with support $\overline{Y}_\lambda$. A condition guaranteeing existence of a parity sheaf with support $\overline{Y}_\lambda$ is that $\overline{Y}_\lambda$ admit an even resolution. In all settings we consider below parity sheaves exist for all strata, and thus are classified in the same way as IC sheaves.

Remark 1.6. In contrast to IC sheaves, parity sheaves are only defined up to non-canonical isomorphism.

Below it will also be important to consider the equivariant setting. We briefly outline the necessary changes. Suppose that a complex algebraic group G acts on Y preserving strata. Let $D^b_{G,\Lambda}(X;\Bbbk)$ denote the Λ-constructible equivariant derived category Bernstein and Lunts [1994]. We have the usual menagerie of functors associated to G-equivariant maps $f : X \to Y$ which commute with the "forget G-equivariance functor" to $D^b_\Lambda(X;\Bbbk)$. In the equivariant setting the definition of even, odd and parity objects remain unchanged. Also Theorem 1.4 holds, if we require "equivariantly simply connected" in (3) and state (4) with equivariant cohomology.

1.4 Intersection Forms. In the previous section we saw that, for any proper even map $f : X \to Y$, the derived direct image $f_*\Bbbk_X$ decomposes into a direct sum of shifts of parity sheaves. In applications it is important to know precisely what form this decomposition takes. It turns out that this is encoded in the ranks of certain intersection forms associated to the strata of Y, as we now explain.

For each stratum Y_λ and point $y \in Y_\lambda$ we can choose a normal slice N to the stratum Y_λ through y. If we set $F := f^{-1}(y)$ and $\widetilde{N} := f^{-1}(N)$ then we have a commutative diagram with Cartesian squares:

$$\begin{tikzcd} F \arrow[r] \arrow[d] & \widetilde{N} \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \{x\} \arrow[r] & N \arrow[r] & Y \end{tikzcd}$$

Set $d := \dim_{\mathbb{C}} \widetilde{N} = \dim_{\mathbb{C}} N = \operatorname{codim}_{\mathbb{C}}(Y_\lambda \subset X)$. The inclusion $F \hookrightarrow \widetilde{N}$ equips the integral homology of F with an intersection form (see Juteau, Mautner, and Williamson [2014, §3.1])

$$IF^j_\lambda : H_{d-j}(F,\mathbb{Z}) \times H_{d+j}(F,\mathbb{Z}) \to H_0(\widetilde{N},\mathbb{Z}) = \mathbb{Z} \quad \text{for } j \in \mathbb{Z}.$$

Remark 1.7. Let us give an intuitive explanation for the intersection form: suppose we wish to pair the classes of submanifolds of real dimension $d-j$ and $d+j$ respectively. We regard our manifolds as sitting in $\widetilde{N}$ and move them until they are transverse. Because $(d-j)+(d+j) = 2d$ (the real dimension of $\widetilde{N}$) they will intersect in a finite number of signed points, which we then count to get the result.

Remark 1.8. The above intersection form depends only on the stratum Y_λ (up to non-unique isomorphism): given any two points $y, y' \in Y_\lambda$ and a (homotopy class of) path from y to y' we get an isometry between the two intersection forms.

Let us assume that our parity assumptions are in force, and that the homology $H_*(F,\mathbb{Z})$ is free for all λ. In this case, for any field $\Bbbk$ the intersection form over $\Bbbk$ is obtained via extension of scalars from IF_λ^j. We denote this form by $IF_\lambda^j \otimes_{\mathbb{Z}} \Bbbk$. The relevance of these forms to the Decomposition Theorem is the following:

Theorem 1.9 (Juteau, Mautner, and Williamson [ibid., Theorem 3.13])**.** *The multiplicity of $\mathcal{E}_\lambda[j]$ as a summand of $f_*\Bbbk_X[\dim_{\mathbb{C}} X]$ is equal to the rank of $IF_\lambda^j \otimes \Bbbk$. Moreover, the Decomposition Theorem holds if and only if $IF_\lambda^j \otimes \Bbbk$ and $IF_\lambda^j \otimes \mathbb{Q}$ have the same rank, for all $\lambda \in \Lambda$ and $j \in \mathbb{Z}$.*

1.5 Examples. Our goal in this section is to give some examples of intersection cohomology sheaves and parity sheaves. Throughout, $\Bbbk$ denotes our field of coefficients and p denotes its characteristic. (The strata in some of examples below do not satisfy our parity conditions. In each example this can be remedied by considering equivariant sheaves for an appropriate group action.)

Example 1.10. (A nilpotent cone) Consider the singular 2-dimensional quadric cone

$$X = \{(x,y,z) \mid x^2 = -yz\} \subset \mathbb{C}^3.$$

Then X is isomorphic to the cone of nilpotent matrices inside $\mathfrak{sl}_2(\mathbb{C}) \cong \mathbb{C}^3$. Let 0 denote the unique singular point of X and $X_{\text{reg}} = X \setminus \{0\}$ the smooth locus. Consider the blow-up of X at 0:

$$f : \widetilde{X} \to X$$

This is a resolution of singularities which is isomorphic to the Springer resolution under the above isomorphism of X with the nilpotent cone. It is an isomorphism over X_{reg} and has fibre $\mathbb{P}^1$ over 0. In particular, the stalks of the direct image of the shifted constant sheaf $f_*\Bbbk_{\widetilde{X}}[2]$ are given by:

	-2	-1	0
X_{reg}	$\Bbbk$	0	0
$\{0\}$	$\Bbbk$	0	$\Bbbk$

One has an isomorphism of $\widetilde{X}$ with the total space of the line bundle $\mathcal{O}(-2)$ on $\mathbb{P}^1$. Under such an isomorphism, the zero section corresponds to $f^{-1}(0)$. In particular, $f^{-1}(0)$ has self-intersection -2 inside $\widetilde{X}$. It follows that:

$$f_*\Bbbk_{\widetilde{X}}[2] \cong \Bbbk_X[2] \oplus \Bbbk_{\{0\}}, \text{ if } \Bbbk \text{ is of characteristic} \neq 2,$$
$$f_*\Bbbk_{\widetilde{X}}[2] \text{ is indecomposable, if } \Bbbk \text{ is of characteristic } 2.$$

If $p = 2$, the complex $f_*\Bbbk_{\widetilde{X}}[2]$ is an archetypal example of a parity sheaf. For further discussion of this example, see Juteau, Mautner, and Williamson [2012, §2.4].

Example 1.11. (The first singular Schubert variety) Let $\mathcal{G}r_2^4$ denote the Grassmannian of 2-planes inside $\mathbb{C}^4$. Fix a two-dimensional subspace $\mathbb{C}^2 \subset \mathbb{C}^4$ and let $X \subset \mathcal{G}r_2^4$ denote the closed subvariety (a Schubert variety)

$$X = \{V \in \mathcal{G}r_2^4 \mid \dim(V \cap \mathbb{C}^2) \geq 1\}.$$

It is of dimension 3, with unique singular point $V = \mathbb{C}^2 \in \mathcal{G}r_2^4$. The space

$$\widetilde{X} = \{V \in \mathcal{G}r_2^4, L \subset \mathbb{C}^2 \mid \dim L = 1, L \subset V \cap \mathbb{C}^2\}$$

is smooth, and the map $f : \widetilde{X} \to X$ forgetting L is a resolution of singularities. This morphism is "small" (i.e. the shifted direct image sheaf $f_*\Bbbk_{\widetilde{X}}[3]$ coincides with the intersection cohomology complex for any field $\Bbbk$) and even. Thus in this example the parity sheaf and intersection cohomology sheaf coincide in all characteristics.

Example 1.12. (Contraction of the zero section) Suppose that Y is smooth of dimension > 0 and that $Y \subset T^*Y$ may be contracted to a point (i.e. there exists a map $f : T^*Y \to X$ such that $f(Y) = \{x\}$ and f is an isomorphism on the complement of Y). In this case x is the unique singular point of X and the intersection form at x is $-\chi(Y)$, where $\chi(Y)$ denotes the Euler characteristic of Y. If Y has vanishing odd cohomology then f is even and $f_*\Bbbk_{T^*Y}$ is parity. The Decomposition Theorem holds if and only if p does not divide $\chi(Y)$.

Example 1.13. (A non-perverse parity sheaf) For $n \geq 1$ consider

$$X = \mathbb{C}^{2n}/(\pm 1) = \operatorname{Spec} \mathbb{C}[x_i x_j \mid 1 \leq i, j \leq 2n].$$

If $\widetilde{X}$ denotes the total space of $\mathcal{O}(-2)$ on $\mathbb{P}^{2n-1}$ then we have a resolution

$$f : \widetilde{X} \to X.$$

It is an isomorphism over $X_{\mathrm{reg}} = X \setminus \{0\}$ with fibre $\mathbb{P}^{2n-1}$ over 0. The intersection form

$$IF_0^j : H_{2n-j}(\mathbb{P}^{2n-1}, \mathbb{Z}) \times H_{2n+j}(\mathbb{P}^{2n-1}, \mathbb{Z}) \to \mathbb{Z}$$

is non-trivial only for $j = -2n+2, -2n+4, \dots, 2n-2$ in which case it is the 1×1 matrix (-2). Thus $f_*\Bbbk_{\widetilde{X}}$ is indecomposable if $p = 2$. Otherwise we have

$$f_*\Bbbk_{\widetilde{X}}[2n] \cong \Bbbk_X[2n] \oplus \Bbbk_0[2n-2] \oplus \Bbbk_0[2n-4] \cdots \oplus \Bbbk_0[-2n+2].$$

Because f is even, $f_*\Bbbk_{\widetilde{X}}[2n]$ is parity. It is indecomposable (and hence is a parity sheaf) if $p = 2$. The interest of this example is that $f_*\Bbbk_{\widetilde{X}}[2n]$ has many non-zero perverse cohomology sheaves. (See Juteau, Mautner, and Williamson [2012, §3.3] for more on this example.)

Example 1.14. (The generalised Kashiwara–Saito singularity) Fix $d \geq 2$ and consider the variety of linear maps

$$\begin{tikzcd} \mathbb{C}^d \arrow[r, "A"] \arrow[d, "D"'] & \mathbb{C}^d \arrow[d, "B"] \\ \mathbb{C}^d \arrow[r, "C"'] & \mathbb{C}^d \end{tikzcd} \quad \text{satisfying} \quad \begin{aligned} BA = CD &= 0, \\ \operatorname{rank}\begin{pmatrix} A \\ D \end{pmatrix} &\leq 1, \\ \operatorname{rank}\begin{pmatrix} B & C \end{pmatrix} &\leq 1. \end{aligned}$$

This is a singular variety of dimension $6d-4$. Let $0=(0,0,0,0)$ denote its most singular point. Consider

$$\widetilde{X}=\left\{(A,B,C,D,H_1,L_2,L_3,L_4)\;\middle|\;\begin{array}{c}H_1\in\mathcal{G}r^d_{d-1},L_i\in\mathcal{G}r^d_1,\\ A\in\mathrm{Hom}(\mathbb{C}^d/H_1,L_2),B\in\mathrm{Hom}(\mathbb{C}^d/L_2,L_4),\\ C\in\mathrm{Hom}(\mathbb{C}^d/L_3,L_4),D\in\mathrm{Hom}(\mathbb{C}^d/H_1,L_3)\end{array}\right\}$$

(where $\mathcal{G}r^d_i$ denotes the Grassmannian if i-planes in $\mathbb{C}^d$). The natural map

$$f:\widetilde{X}\to X$$

is a resolution. We have $F=f^{-1}(0)\cong(\mathbb{P}^{d-1})^4$. The intersection form

$$H_{6d-4}(F)\times H_{6d-4}(F)\to\mathbb{Z}$$

has elementary divisors $(1,\dots,1,d)$. The Decomposition Theorem holds if and only if $p\nmid d$.

The $d=2$ case yields an 8-dimensional singularity which Kashiwara and Saito showed is smoothly equivalent to a singularity of a Schubert variety in the flag variety of SL_8 or a quiver variety of type A_5. It tends to show up as a minimal counterexample to optimistic hopes in representation theory Kashiwara and Saito [1997], Leclerc [2003], and Williamson [2014, 2015]. Polo observed (unpublished) that for any d the above singularities occur in Schubert varieties for SL_{4d}. This shows that the Decomposition Theorem can fail for type A Schubert varieties for arbitrarily large p.

2 The Hecke category

In this section we introduce the Hecke category, a monoidal category whose Grothendieck group is the Hecke algebra. If one thinks of the Hecke algebra as providing Hecke operators which act on representations or function spaces, then the Hecke category consists of an extra layer of "Hecke operators between Hecke operators".

2.1 The Hecke algebra. Let G denote a split reductive group over $\mathbb{F}_q$, and let $T\subset B\subset G$ denote a maximal torus and Borel subgroup. For example, we could take $G=\mathrm{GL}_n$, $B=$ upper triangular matrices and $T=$ diagonal matrices. The set of $\mathbb{F}_q$-points, $G(\mathbb{F}_q)$, is a finite group (e.g. for $G=\mathrm{GL}_n$, $G(\mathbb{F}_q)$ is the group of invertible $n\times n$-matrices with coefficients in $\mathbb{F}_q$). Many important finite groups, including "most" simple groups, are close relatives of groups of this form.

A basic object in the representation theory of the finite group $G(\mathbb{F}_q)$ is the Hecke algebra

$$H_{\mathbb{F}_q}:=\mathrm{Fun}_{B(\mathbb{F}_q)\times B(\mathbb{F}_q)}(G(\mathbb{F}_q),\mathbb{C})$$

of complex valued functions on $G(\mathbb{F}_q)$, invariant under left and right multiplication by $B(\mathbb{F}_q)$. This is an algebra under convolution:

$$(f*f')(g):=\frac{1}{|B(\mathbb{F}_q)|}\sum_{h\in G(\mathbb{F}_q)}f(gh^{-1})f'(h).$$

Remark 2.1. Instead we could replace $G(\mathbb{F}_q)$ by $G(\mathbb{K})$ and $B(\mathbb{F}_q)$ by an Iwahori subgroup of $G(\mathbb{K})$ (for a local field $\mathbb{K}$ with finite residue field), and obtain the affine Hecke algebra (important in the representation theory of p-adic groups).

Let W denote the Weyl group, S its simple reflections, $\ell : W \to \mathbb{Z}_{\geq 0}$ the length function of W with respect to S and $\leq$ the Bruhat order. The Bruhat decomposition

$$G(\mathbb{F}_q) = \bigsqcup_{w \in W} B(\mathbb{F}_q) \cdot wB(\mathbb{F}_q)$$

shows that $H_{\mathbb{F}_q}$ has a basis given by indicator functions t_w of the subsets $B(\mathbb{F}_q) \cdot wB(\mathbb{F}_q)$, for $w \in W$.

Iwahori [1964] showed that $H_{\mathbb{F}_q}$ may also be described as the unital algebra generated by t_s for $s \in S$ subject to the relations

$$\begin{aligned} t_s^2 &= (q-1)t_s + q, \\ \underbrace{t_s t_u \dots}_{m_{su} \text{ factors}} &= \underbrace{t_u t_s \dots}_{m_{su} \text{ factors}} \end{aligned}$$

where $u \neq s$ in the second relation and m_{su} denotes the order of su in W. These relations depend on q in a uniform way and make sense for any Coxeter group. Thus it makes sense to use these generators and relations to define a new algebra H over $\mathbb{Z}[q^{\pm 1}]$ (q is now a formal variable); thus H specialises to the Hecke algebra defined above via $q \mapsto |\mathbb{F}_q|$.

For technical reasons it is useful to adjoin a square root of q and regard H as defined over $\mathbb{Z}[q^{\pm 1/2}]$. We then set $v := q^{-1/2}$ and $\delta_s := vt_s$, so that the defining relations of H become

$$\begin{aligned} \delta_s^2 &= (v^{-1} - v)\delta_s + 1, \\ \underbrace{\delta_s \delta_u \dots}_{m_{su} \text{ factors}} &= \underbrace{\delta_u \delta_s \dots}_{m_{su} \text{ factors}}. \end{aligned}$$

For any reduced expression $w = st \dots u$ (i.e. any expression for w using $\ell(w)$ simple reflections) we set $\delta_w := \delta_s \delta_t \dots \delta_u$. We obtain in this way a well-defined $\mathbb{Z}[v^{\pm 1}]$-basis $\{\delta_x \mid x \in W\}$ for H, the *standard basis*. (This basis specialises via $q \mapsto |\mathbb{F}_q|$ to the indicator functions t_w considered above, up to a power of v.)

There is an involution $d : H \to H$ defined via

$$v \mapsto v^{-1} \qquad \text{and} \qquad \delta_s \mapsto \delta_s^{-1} = \delta_s + (v - v^{-1}).$$

Kazhdan and Lusztig [1979] (see Soergel [1997] for a simple proof) showed that for all $x \in W$ there exists a unique element b_x satisfying

(“self-duality”) $$d(b_x) = b_x,$$

(“degree bound”) $$b_x \in \delta_x + \sum_{y<x} v\mathbb{Z}[v]\delta_y$$

where $\leq$ is Bruhat order. For example $b_s = \delta_s + v$. The set $\{b_x \mid x \in W\}$ is the *Kazhdan–Lusztig basis* of H. The polynomials $h_{y,x} \in \mathbb{Z}[v]$ defined via $b_x = \sum h_{y,x}\delta_y$ are (normalisations of) *Kazhdan–Lusztig polynomials*.

2.2 The Hecke category: geometric incarnation. Grothendieck's function-sheaf correspondence (see e.g. Laumon [1987, §1]) tells us how we should categorify the Hecke algebra $H_{\mathbb{F}_q}$. Namely, we should consider an appropriate category of $B \times B$-equivariant sheaves on G, with the passage to $H_{\mathbb{F}_q}$ being given by the trace of Frobenius at the rational points $G(\mathbb{F}_q)$.[8] Below we will use the fact that the multiplication action of B on G is free, and so instead we can consider B-invariant functions (resp. B-equivariant sheaves) on G/B.

To avoid technical complications, and to ease subsequent discussion, we will change setting slightly. Let us fix a generalised Cartan matrix $C = (c_{st})_{s,t\in S}$ and let $(\mathfrak{h}_{\mathbb{Z}}, \{\alpha_s\}_{s\in S}, \{\alpha_s^\vee\}_{s\in S})$ be a Kac–Moody root datum, so that $\mathfrak{h}_{\mathbb{Z}}$ is a free and finitely generated $\mathbb{Z}$-module, $\alpha_s \in \mathrm{Hom}(\mathfrak{h}_{\mathbb{Z}}, \mathbb{Z})$ are "roots" and $\alpha_s^\vee \in \mathfrak{h}_{\mathbb{Z}}$ are "coroots" such that $\langle \alpha_s^\vee, \alpha_t\rangle = c_{st}$. To this data we may associate a Kac–Moody group $\mathcal{G}$ (a group ind-scheme over $\mathbb{C}$) together with a canonical Borel subgroup $\mathcal{B}$ and maximal torus $\mathcal{T}$. The reader is welcome to take $\mathcal{G}$ to be a complex reductive group, as per the following remark. (For applications to representation theory the case of an affine Kac–Moody group is important.)

Remark 2.2. If $\mathcal{G}$ is a complex reductive group and $\mathcal{T} \subset \mathcal{B} \subset \mathcal{G}$ is a maximal torus and Borel subgroup, then we can consider the corresponding root datum $(\mathcal{X}, R, \mathcal{X}^\vee, R^\vee)$ (where $\mathcal{X}$ denotes the characters of $\mathcal{T}$, R the roots etc.). If $\{\alpha_s\}_{s\in S} \subset R$ denotes the simple roots determined by $\mathcal{B}$ then $(\mathcal{X}^\vee, \{\alpha_s\}, \{\alpha_s^\vee\})$ is a Kac–Moody root datum. The corresponding Kac–Moody group (resp. Borel subgroup and maximal torus) is canonically isomorphic to $\mathcal{G}$ (resp. $\mathcal{B}$, $\mathcal{T}$).

We denote by $\mathcal{G}/\mathcal{B}$ the flag variety (a projective variety in the case of a reductive group, and an ind-projective variety in general). As earlier, we denote by W the Weyl group, ℓ the length function and $\leq$ the Bruhat order. We have the Bruhat decomposition

$$\mathcal{G}/\mathcal{B} = \bigsqcup_{w\in W} X_w \quad \text{where} \quad X_w := \mathcal{B}\cdot w\mathcal{B}/\mathcal{B}.$$

The X_w are isomorphic to affine spaces, and are called *Schubert cells*. Their closures $\overline{X}_w \subset \mathcal{G}/\mathcal{B}$ are projective (and usually singular), and are called *Schubert varieties*.

Fix a field $\Bbbk$ and consider $D^b_{\mathcal{B}}(\mathcal{G}/\mathcal{B};\Bbbk)$, the bounded equivariant derived category with coefficients in $\Bbbk$ (see e.g. Bernstein and Lunts [1994]).[9] This a monoidal category under convolution: given two complexes $\mathcal{F}, \mathcal{G} \in D^b_{\mathcal{B}}(\mathcal{G}/\mathcal{B};\Bbbk)$ their convolution is

$$\mathcal{F} * \mathcal{G} := \mathrm{mult}_*(\mathcal{F} \boxtimes_{\mathcal{B}} \mathcal{G}),$$

where: $\mathcal{G} \times_{\mathcal{B}} \mathcal{G}/\mathcal{B}$ denotes the quotient of $\mathcal{G} \times \mathcal{G}/\mathcal{B}$ by $(gb, g'\mathcal{B}) \sim (g, bg'\mathcal{B})$ for all $g, g' \in \mathcal{G}$ and $b \in \mathcal{B}$; $\mathrm{mult} : \mathcal{G} \times_{\mathcal{B}} \mathcal{G}/\mathcal{B} \to \mathcal{G}/\mathcal{B}$ is induced by the multiplication on $\mathcal{G}$; and $\mathcal{F} \boxtimes_{\mathcal{B}} \mathcal{G} \in D^b_{\mathcal{B}}(\mathcal{G} \times_{\mathcal{B}} \mathcal{G}/\mathcal{B};\Bbbk)$ is obtained via descent from $\mathcal{F} \boxtimes \mathcal{G} \in D^b_{\mathcal{B}^3}(\mathcal{G} \times \mathcal{G}/\mathcal{B};\Bbbk)$.[10] (Note that mult is proper, and so $\mathrm{mult}_* = \mathrm{mult}_!$.)

[8] As is always the case with Grothendieck's function-sheaf correspondence, this actually categorifies the Hecke algebras of $G(\mathbb{F}_{q^m})$ for "all m at once".

[9] By definition, any object of $D^b_{\mathcal{B}}(\mathcal{G}/\mathcal{B};\Bbbk)$ is supported on finitely many Schubert cells, and hence has finite-dimensional support.

[10] The reader is referred to Springer [1982] and Nadler [2005] for more detail on this construction.

Remark 2.3. If $\mathcal{G}$ is a reductive group and we work over $\mathbb{F}_q$ instead of $\mathbb{C}$, then this definition categorifies convolution in the Hecke algebra, via the function-sheaf correspondence.

For any $s \in S$ we can consider the parabolic subgroup

$$\mathcal{P}_s := \overline{\mathcal{B}s\mathcal{B}} = \mathcal{B}s\mathcal{B} \sqcup \mathcal{B} \subset \mathcal{G}.$$

We define the Hecke category (in its geometric incarnation) as follows

$$\mathcal{H}^{\Bbbk}_{\text{geom}} := \langle \Bbbk_{\mathcal{P}_s/\mathcal{B}} \mid s \in S \rangle_{*,\oplus,[1],\text{Kar}}.$$

That is, we consider the full subcategory of $D^b_{\mathcal{B}}(\mathcal{G}/\mathcal{B};\Bbbk)$ generated by $\Bbbk_{\mathcal{P}_s/\mathcal{B}}$ under convolution ($*$), direct sums ($\oplus$), homological shifts ($[1]$) and direct summands (Kar, for "Karoubi").

Remark 2.4. If we were to work over $\mathbb{F}_q$, then $\Bbbk_{\mathcal{P}_s/\mathcal{B}}$ categorifies the indicator function of $\mathcal{P}_s(\mathbb{F}_q) \subset \mathcal{G}(\mathbb{F}_q)$. The definition of the Hecke category is imitating the fact that the Hecke algebra is generated by these indicator functions under convolution (as is clear from Iwahori's presentation).

Let $[\mathcal{H}^{\Bbbk}_{\text{geom}}]_{\oplus}$ denote the split Grothendieck group[11] of $\mathcal{H}^{\Bbbk}_{\text{geom}}$. Because $\mathcal{H}^{\Bbbk}_{\text{geom}}$ is a monoidal category, $[\mathcal{H}^{\Bbbk}_{\text{geom}}]_{\oplus}$ is an algebra via $[\mathcal{F}] \cdot [\mathcal{G}] = [\mathcal{F} * \mathcal{G}]$. We view $[\mathcal{H}^{\Bbbk}_{\text{geom}}]_{\oplus}$ as a $\mathbb{Z}[v^{\pm 1}]$-algebra via $v \cdot [\mathcal{F}] := [\mathcal{F}[1]]$. Recall the Kazhdan–Lusztig basis element $b_s = \delta_s + v$ for all $s \in S$ from earlier. The following theorem explains the name "Hecke category" and is fundamental to all that follows:

Theorem 2.5. *The assignment $b_s \mapsto [\Bbbk_{\mathcal{P}_s/\mathcal{B}}[1]]$ for all $s \in S$ yields an isomorphism of $\mathbb{Z}[v^{\pm 1}]$-algebras:*

$$H \xrightarrow{\sim} [\mathcal{H}^{\Bbbk}_{\text{geom}}]_{\oplus}.$$

(This theorem is easily proved using the theory of parity sheaves, as will be discussed in the next section.) The inverse to the isomorphism in the theorem is given by the *character map*

$$\text{ch} : [\mathcal{H}^{\Bbbk}_{\text{geom}}]_{\oplus} \xrightarrow{\sim} H \qquad \mathcal{F} \mapsto \sum_{x \in W} \dim_{\mathbb{Z}}(H^*(\mathcal{F}_{x\mathcal{B}/\mathcal{B}}))v^{-\ell(x)}\delta_x$$

where: $\mathcal{F}_{x\mathcal{B}/\mathcal{B}}$ denotes the stalk of the constructible sheaf on $\mathcal{G}/\mathcal{B}$ at the point $x\mathcal{B}/\mathcal{B}$ obtained from $\mathcal{F}$ by forgetting $\mathcal{B}$-equivariance; H^* denotes cohomology; and

$$\dim_{\mathbb{Z}} H^* := \sum (\dim H^i) v^{-i} \in \mathbb{Z}[v^{\pm 1}]$$

denotes graded dimension.

[11] The split Grothendieck group $[\mathcal{A}]_{\oplus}$ of an additive category is the abelian group generated by symbols $[A]$ for all $A \in \mathcal{A}$, modulo the relations $[A] = [A'] + [A'']$ whenever $A \cong A' \oplus A''$.

2.3 Role of the Decomposition Theorem. The category $D^b_{\mathcal{B}}(\mathcal{G}/\mathcal{B};\Bbbk)$, and hence also the Hecke category $\mathcal{H}^{\Bbbk}_{\mathrm{geom}}$, is an example of a Krull–Schmidt category: every object admits a decomposition into indecomposable objects; and an object is indecomposable if and only if its endomorphism ring is local.

Recall that the objects of $\mathcal{H}^{\Bbbk}_{\mathrm{geom}}$ are the direct summands of finite direct sums of shifts of objects of the form

$$\mathcal{E}_{(s,t,\dots,u)} := \Bbbk_{\mathcal{P}_s/\mathcal{B}} * \Bbbk_{\mathcal{P}_t/\mathcal{B}} * \cdots * \Bbbk_{\mathcal{P}_u/\mathcal{B}} \in D^b_{\mathcal{B}}(\mathcal{G}/\mathcal{B})$$

for any word $(s,t,\dots,u)$ in S. The Krull–Schmidt property implies that any indecomposable object is isomorphic to a direct summand of a single $\mathcal{E}_{(s,t,\dots,u)}$. Thus in order to understand the objects of $\mathcal{H}^{\Bbbk}_{\mathrm{geom}}$ it is enough to understand the summands of $\mathcal{E}_{(s,t,\dots,u)}$, for any word as above.

For any such word $(s,t,\dots,u)$ we can consider a Bott–Samelson space

$$BS_{(s,t,\dots,u)} := \mathcal{P}_s \times_{\mathcal{B}} \mathcal{P}_t \times_{\mathcal{B}} \cdots \times_{\mathcal{B}} \mathcal{P}_u/\mathcal{B}$$

and the (projective) morphism $m : BS_{(s,t,\dots,u)} \to \mathcal{G}/\mathcal{B}$ induced by multiplication. A straightforward argument (using the proper base change theorem) shows that we have a canonical isomorphism

$$\mathcal{E}_{(s,t,\dots,u)} = m_*\Bbbk_{BS_{(s,t,\dots,u)}}.$$

The upshot: in order to understand the indecomposable objects in $\mathcal{H}^{\Bbbk}_{\mathrm{geom}}$ it is enough to decompose the complexes $m_*\Bbbk_{BS}$, for all expressions $(s,t,\dots,u)$ in S.

Remark 2.6. If $(s,t,\dots,u)$ is a reduced expression for $w \in W$, then the map m provides a resolution of singularities of the Schubert variety $\overline{X}_w$. These resolutions are often called Bott–Samelson resolutions, which explains our notation.

If the characteristic of our field is zero then we can appeal to the Decomposition Theorem to deduce that all indecomposable summands of $m_*\Bbbk_{BS}$ are shifts of the intersection cohomology complexes of Schubert varieties. Thus

$$\mathcal{H}^{\Bbbk}_{\mathrm{geom}} = \langle \mathbf{IC}_x \mid x \in W\rangle_{\oplus,[1]} \quad \text{if } \Bbbk \text{ is of characteristic } 0 \tag{5}$$

where $\mathbf{IC}_x$ denotes the ($\mathcal{B}$-equivariant) intersection cohomology sheaf of the Schubert variety $\overline{X}_x$. It is also not difficult (see e.g. Springer [1982]) to use (5) to deduce that[12]

$$\mathrm{ch}(\mathbf{IC}_x) = b_x \quad \text{if } \Bbbk \text{ is of characteristic } 0. \tag{6}$$

Thus, when the coefficients are of characteristic zero, the intersection cohomology sheaves categorify the Kazhdan–Lusztig basis.

It is known that Bott–Samelson resolutions are even. In particular, $m_*\Bbbk_{BS}$ is a parity complex. Thus, for arbitrary $\Bbbk$ we can appeal to Theorem 1.4 to deduce that all indecomposable summands of $f_*\Bbbk_{BS}$ are shifts of parity sheaves. Thus

$$\mathcal{H}^{\Bbbk}_{\mathrm{geom}} = \langle \mathcal{E}_x \mid x \in W\rangle_{\oplus,[1]} \quad \text{for } \Bbbk \text{ arbitrary}$$

where $\mathcal{E}_x$ denotes the (B-equivariant) parity sheaf of the Schubert variety $\overline{X}_x$.

[12]Roughly speaking, the two conditions ("self-duality" + "degree bound") characterising the Kazhdan–Lusztig basis mirror the two conditions ("self-duality" + "stalk vanishing") characterising the IC sheaf.

Remark 2.7. Recall that for any map there are only finitely many characteristics in which the Decomposition Theorem fails. Thus, for a fixed x there will be only finitely many characteristics in which $\mathcal{E}_x^{\Bbbk} \neq \mathbf{IC}_x^{\Bbbk}$.

2.4 The Hecke category: generators and relations. The above geometric definition of the Hecke category is analogous to the original definition of the Hecke algebra as an algebra of B-biinvariant functions. Now we discuss a description of the Hecke category via generators and relations; this description is analogous to the Iwahori presentation of the Hecke algebra.[13] This description is due to Elias and the author Elias and Williamson [2016], building on work of Elias and Khovanov [2010] and Elias [2016].

Remark 2.8. In this section it will be important to keep in mind that monoidal categories are fundamentally two dimensional. While group presentations (and more generally presentations of categories) occur "on a line", presentations of monoidal categories (and more generally 2-categories) occur "in the plane". For background on these ideas the reader is referred to e.g. Street [1996] or Lauda [2010, §4].

Recall our generalised Cartan matrix C, Coxeter system (W, S) and Kac–Moody root datum from earlier. Given $s, t \in S$ we denote by m_{st} the order (possibly ∞) of $st \in W$. We assume:

$$C \text{ is simply laced, i.e. } m_{st} \in \{2, 3\} \text{ for } s \neq t. \tag{7}$$

(We impose this assumption only to shorten the list of relations below. For the general case the reader is referred to Elias and Williamson [2016].) Recall our "roots" and "coroots"

$$\{\alpha_s\}_{s\in S} \subset \mathfrak{h}_{\mathbb{Z}}^* \quad \text{and} \quad \{\alpha_s^\vee\}_{s\in S} \subset \mathfrak{h}_{\mathbb{Z}}$$

such that $\langle \alpha_s^\vee, \alpha_t \rangle = c_{st}$ for all $s, t \in S$. The formula $s(v) = v - \langle v, \alpha_s^\vee \rangle \alpha_s$ defines an action of W on $\mathfrak{h}_{\mathbb{Z}}^*$. We also assume that our root datum satisfies that $\alpha_s : \mathfrak{h}_{\mathbb{Z}} \to \mathbb{Z}$ and $\alpha_s^\vee : \mathfrak{h}_{\mathbb{Z}}^* \to \mathbb{Z}$ are surjective, for all $s \in S$. (This condition is called "Demazure surjectivity" in Elias and Williamson [ibid.]. We can always find a Kac–Moody root datum satisfying this constraint.)

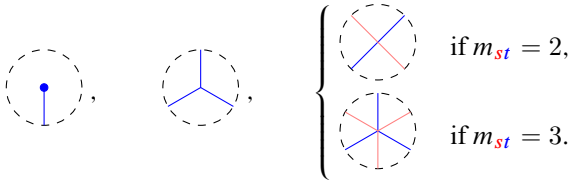
We denote by $R = S(\mathfrak{h}_{\mathbb{Z}}^*)$ the symmetric algebra of $\mathfrak{h}_{\mathbb{Z}}^*$ over $\mathbb{Z}$. We view R as a graded $\mathbb{Z}$-algebra with $\deg \mathfrak{h}_{\mathbb{Z}}^* = 2$; W acts on R via graded automorphisms. For any $s \in S$ we define the *Demazure operator* $\partial_s : R \to R[-2]$ by

$$\partial_s(f) = \frac{f - sf}{\alpha_s}. \tag{8}$$

An S-*graph* is a finite, decorated graph, properly embedded in the planar strip $\mathbb{R} \times [0, 1]$, with edges coloured by S. The vertices of an S-graph are required to be of the

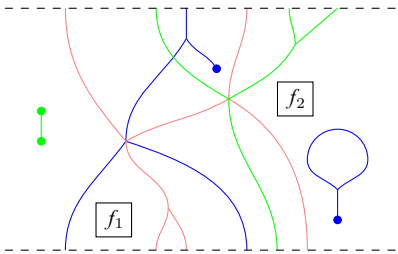
[13] The Iwahori presentation can be given on two lines. Unfortunately all current presentations of the Hecke category need more than two pages!

form:



The regions (i.e. connected components of the complement of our S-graph in $\mathbb{R} \times [0, 1]$) may be decorated by boxes containing homogeneous elements of R.

Example 2.9. An S-graph (with $m_{s,t} = 3$, $m_{s,u} = 2$, $m_{t,u} = 3$, the $f_i \in R$ are homogeneous polynomials):



The *degree* of an S-graph is the sum over the degrees of its vertices and boxes, where each box has degree equal to the degree of the corresponding element of R, and the vertices have degrees given by the following rule: univalent vertices have degree 1, trivalent vertices have degree -1 and $2m_{st}$-valent vertices have degree 0. The boundary points of any S-graph on $\mathbb{R} \times \{0\}$ and on $\mathbb{R} \times \{1\}$ give[14] two words in S, called the *bottom boundary* and *top boundary*.

Example 2.10. The S-graph above has degree $0 + \deg f_1 + \deg f_2$. Its bottom boundary is (s, t, t, s, u, s) and its top boundary is (t, u, s, t, u, u).

We are now ready to define a second incarnation of the Hecke category, which we will denote $\mathcal{H}_{\text{diag}}$. By definition, $\mathcal{H}_{\text{diag}}$ is monoidally generated by objects B_s, for each $s \in S$. Thus the objects of $\mathcal{H}_{\text{diag}}$ are of the form

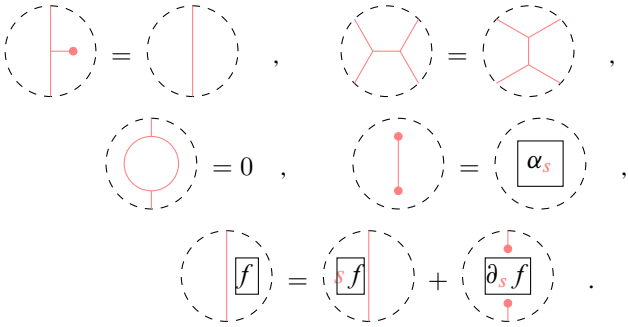
$$B_{(s,t,\ldots,u)} := B_s B_t \ldots B_u$$

for some word $(s, t, \ldots, u)$ in S. (We denote the monoidal structure in $\mathcal{H}_{\text{diag}}$ by concatenation.) Thus $\mathbb{1} := B_\varnothing$ is the monoidal unit. For any two words $(s, t, \ldots, u)$ and $(s', t', \ldots, v')$ in S, $\mathrm{Hom}_{\mathcal{H}_{\text{diag}}}(B_{(s,t,\ldots,u)}, B_{(s',t',\ldots,v')})$ is defined to be the free $\mathbb{Z}$-module generated by isotopy classes[15] of S-graphs with bottom boundary $(s, t, \ldots, u)$ and top boundary $(s', t', \ldots, v')$, modulo the local relations below. Composition (resp. monoidal product) of morphisms is induced by vertical (resp. horizontal) concatenation of diagrams.

[14] we read left to right

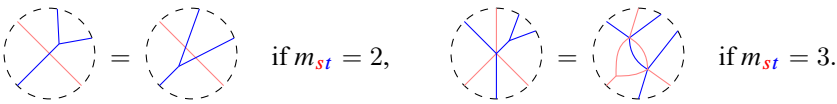
[15] i.e. two S-graphs are regarded as the same if one may be obtained from the other by an isotopy of $\mathbb{R} \times [0, 1]$ which preserves $\mathbb{R} \times \{0\}$ and $\mathbb{R} \times \{1\}$

The one colour relations are as follows (see (8) for the definition of ∂_s):

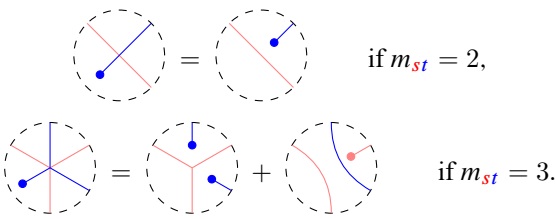


Remark 2.11. The first two relations above imply that B_s is a Frobenius object in $\mathcal{H}_{\mathrm{diag}}$, for all $s \in S$.

There are two relations involving two colours. The first is a kind of "associativity" (see Elias [2016, (6.12)]):



The second is Elias' "Jones–Wenzl relation" (see Elias [ibid.]):

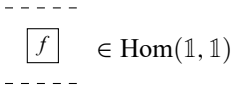


Finally, for each finite standard parabolic subgroup of rank 3 there is a 3-colour "Zamolodchikov relation", which we don't draw here (see Elias and Williamson [2016]). This concludes the definition of $\mathcal{H}_{\mathrm{diag}}$. (We remind the reader that if we drop the assumption that C is simply laced there are more complicated relations, see Elias [2016] and Elias and Williamson [2016].)

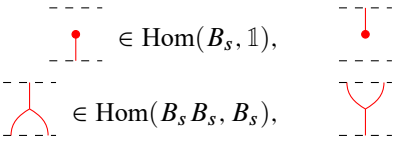
Remark 2.12. Another way of phrasing the above definition is that $\mathcal{H}_{\mathrm{diag}}$ is the monoidal category with:

1. generating objects B_s for all $s \in S$;

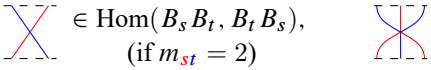
2. generating morphisms



for homogeneous $f \in R$ (recall $\mathbb{1}$ denotes the monoidal unit), as well as


$\in \mathrm{Hom}(B_s, \mathbb{1})$, $\in \mathrm{Hom}(\mathbb{1}, B_s)$,
$\in \mathrm{Hom}(B_s B_s, B_s)$, $\in \mathrm{Hom}(B_s, B_s B_s)$

for all $s \in S$ and


$\in \mathrm{Hom}(B_s B_t, B_t B_s)$, (if $m_{st} = 2$), $\in \mathrm{Hom}(B_s B_t B_s, B_t B_s B_t)$, (if $m_{st} = 3$)

for all pairs $s, t \in S$,

subject to the above relations (and additional relations encoding isotopy invariance).

Remark 2.13. The above relations are complicated, and perhaps a more efficient presentation is possible. The following is perhaps psychologically helpful. Recall that a *standard parabolic subgroup* is a subgroup of W generated by a subset $I \subset S$, and its *rank* is $|I|$. In Iwahori's presentation one has:

$$\text{generators} \leftrightarrow \text{rank } 1,$$
$$\text{relations} \leftrightarrow \text{ranks } 1, 2.$$

In $\mathcal{H}_{\mathrm{diag}}$ one has:

$$\text{generating objects} \leftrightarrow \text{rank } 1,$$
$$\text{generating morphisms} \leftrightarrow \text{ranks } 1, 2,$$
$$\text{relations} \leftrightarrow \text{ranks } 1, 2, 3.$$

(More precisely, it is only the *finite* standard parabolic subgroups which contribute at each step.)

All relations defining $\mathcal{H}_{\mathrm{diag}}$ are homogeneous for the grading on S-graphs defined above. Thus $\mathcal{H}_{\mathrm{diag}}$ is enriched in graded $\mathbb{Z}$-modules. We denote by $\mathcal{H}_{\mathrm{diag}}^{\oplus,[1]}$ the additive, graded envelope[16] of $\mathcal{H}_{\mathrm{diag}}$. Thus $\mathcal{H}_{\mathrm{diag}}^{\oplus,[1]}$ is an additive category equipped with a "shift of grading" equivalence $[1]$, and an isomorphism of graded abelian groups

$$\mathrm{Hom}_{\mathcal{H}_{\mathrm{diag}}}(B, B') = \bigoplus_{m \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{H}_{\mathrm{diag}}^{\oplus,[1]}}(B, B'[m]).$$

For any field $\Bbbk$, we define

$$\mathcal{H}_{\mathrm{diag}}^{\Bbbk,\mathrm{Kar}} := (\mathcal{H}_{\mathrm{diag}}^{\oplus,[1]} \otimes_{\mathbb{Z}} \Bbbk)^{\mathrm{Kar}}$$

where $(-)^{\mathrm{Kar}}$ denotes Karoubi envelope. In other words, $\mathcal{H}_{\mathrm{diag}}^{\Bbbk,\mathrm{Kar}}$ is obtained as the additive Karoubi envelope of the extension of scalars of $\mathcal{H}_{\mathrm{diag}}$ to $\Bbbk$. As for $\mathcal{H}_{\mathrm{geom}}$, let us

[16] Objects are formal sums $F_1[m_1] \oplus F_2[m_2] \oplus \cdots \oplus F_n[m_n]$ where F_i are objects of $\mathcal{H}_{\mathrm{diag}}$ and $m_i \in \mathbb{Z}$; and morphisms are matrices, determined by the rule that $\mathrm{Hom}(F[m], F'[m'])$ is the degree $m' - m$ part of $\mathrm{Hom}_{\mathcal{H}_{\mathrm{diag}}}(F, F')$.

consider the split Grothendieck group $[\mathcal{H}^{\Bbbk,\mathrm{Kar}}_{\mathrm{diag}}]_{\oplus}$ of $\mathcal{H}^{\Bbbk,\mathrm{Kar}}_{\mathrm{diag}}$, which we view as a $\mathbb{Z}[v^{\pm 1}]$-algebra in the same was as for $\mathcal{H}^{\Bbbk}_{\mathrm{geom}}$ earlier. The following is the analogue of Theorem 2.5 in this setting:

Theorem 2.14 (Elias and Williamson [2016]). *The map $b_s \mapsto [B_s]$ for all $s \in S$ induces an isomorphism of $\mathbb{Z}[v^{\pm 1}]$-algebras:*

$$H \xrightarrow{\sim} [\mathcal{H}^{\Bbbk,\mathrm{Kar}}_{\mathrm{diag}}]_{\oplus}.$$

The proof is rather complicated diagrammatic algebra, and involves first producing a basis of morphisms between the objects of $\mathcal{H}_{\mathrm{diag}}$, in terms of light leaf morphisms Libedinsky [2008]. The following theorem shows that $\mathcal{H}_{\mathrm{diag}}$ does indeed give a "generators and relations description" of the Hecke category:

Theorem 2.15 (S. Riche and Williamson [2015, Theorem 10.3.1]). *We have an equivalence of graded monoidal categories:*

$$\mathcal{H}^{\Bbbk,\mathrm{Kar}}_{\mathrm{diag}} \xrightarrow{\sim} \mathcal{H}^{\Bbbk}_{\mathrm{geom}}.$$

Remark 2.16. Knowing a presentation of a group or algebra by generators and relations opens the possibility of defining representations by specifying the action of generators and verifying relations. Similarly, studying actions of monoidal categories is sometimes easier when one has a presentation. In principle, the above presentation should allow a detailed study of categories acted on by the Hecke category. For interesting recent classification results, see Mazorchuk and Miemietz [2016] and Mackaay and Tubbenhauer [2016]. One drawback of the theory in its current state is that the above relations (though explicit) can be difficult to check in examples. The parallel theory of representations of categorified quantum groups is much better developed (see e.g. Chuang and Rouquier [2008] and Brundan [2016]).

Remark 2.17. An important historical antecedent to Theorems 2.14 and 2.15 is the theory of Soergel bimodules. We have chosen not to discuss this topic, as there is already a substantial literature on this subject. The above generators and relations were first written down in the context of Soergel bimodules, and Soergel bimodules are used in the proof of Theorem 2.15. We refer the interested reader to the surveys Riche [2017] and Libedinsky [2017] or the papers Soergel [1990a, 1992, 2007].

2.5 The spherical and anti-spherical module. In this section we introduce the spherical and anti-spherical modules for the Hecke algebra, as well as their categorifications. They are useful for (at least) two reasons: they are ubiquitous in applications to representation theory; and they often provide smaller worlds in which interesting phenomena become more tractable.

Throughout this section we fix a subset $I \subset S$ and assume for simplicity that the standard parabolic subgroup W_I generated by I is finite. We denote by w_I its longest element. Let H_I denote the parabolic subalgebra of H generated by δ_s for $s \in I$; it is canonically isomorphic to the Hecke algebra of W_I. Consider the induced modules

$$M_I := H \otimes_{H_I} \mathrm{triv}_v \quad \text{and} \quad N_I := H \otimes_{H_I} \mathrm{sgn}_v$$

where triv_v (resp. sgn_v) is the rank one H_I-module with action given by $\delta_s \mapsto v^{-1}$ (resp. $\delta_s \mapsto -v$). These modules are the *spherical* and *anti-spherical* modules respectively. If W^I denotes the set of minimal length representatives for the cosets W/W_I then $\{\delta_x \otimes 1 \mid x \in W^I\}$ gives a (*standard*) basis for M_I (resp. N_I), which we denote by $\{\mu_x \mid x \in W^I\}$ (resp. $\{\nu_x \mid x \in W^I\}$). We denote the canonical bases in M_I (resp. N_I) by $\{c_x \mid x \in W^I\}$ (resp. $\{d_x \mid x \in W^I\}$) (see e.g. Soergel [1997]).

We now describe a categorification of M_I. To I is associated a standard parabolic subgroup $\mathcal{P}_I \subset \mathcal{G}$, and we may consider the partial flag variety $\mathcal{G}/\mathcal{P}_I$ (an ind-variety) and its Bruhat decomposition

$$\mathcal{G}/\mathcal{P}_I = \bigsqcup_{x \in W^I} Y_x \quad \text{where} \quad Y_x := \mathcal{B} \cdot x\mathcal{P}_I/\mathcal{P}_I.$$

The closures $\overline{Y}_x$ are Schubert varieties, and we denote by $\mathbf{IC}_{x,I}$ (resp. $\mathcal{E}_{x,I}$) the intersection cohomology complex (resp. parity sheaf) supported on $\overline{Y}_x$.

Given any $\mathcal{G}$-variety or ind-variety Z the monoidal category $D^b_{\mathcal{B}}(\mathcal{G}/\mathcal{B};\Bbbk)$ acts on $D^b_{\mathcal{B}}(Z;\Bbbk)$. (The definition is analogous to the formula for convolution given earlier.) In particular, $\mathcal{H}^{\Bbbk}_{\mathrm{geom}}$ acts on $D^b_{\mathcal{B}}(\mathcal{G}/\mathcal{P}_I;\Bbbk)$. One can check that this action preserves

$$\mathfrak{M}^{\Bbbk}_I := \langle \mathcal{E}_{x,I} \mid x \in W^I \rangle_{\oplus,[1]}$$

and thus $\mathfrak{M}^{\Bbbk}_I$ is a module over $\mathcal{H}^{\Bbbk}_{\mathrm{geom}}$. We have:

Theorem 2.18. *There is a unique isomorphism of $H = [\mathcal{H}^{\Bbbk}_{\mathrm{geom}}]_{\oplus}$-modules*

$$M_I \xrightarrow{\sim} [\mathfrak{M}^{\Bbbk}_I]_{\oplus}$$

sending $\mu_{\mathrm{id}} \mapsto [\Bbbk_{\mathcal{P}_I/\mathcal{P}_I}]$ (we use the indentification $H = [\mathcal{H}^{\Bbbk}_{\mathrm{geom}}]_{\oplus}$ of Theorem 2.5).

The inverse to the isomorphism in the theorem is given by the *character map*

$$\mathrm{ch} : [\mathfrak{M}^{\Bbbk}_I]_{\oplus} \xrightarrow{\sim} M_I \qquad \mathcal{F} \mapsto \sum_{x \in W^I} \dim_{\mathbb{Z}}(H^*(\mathcal{F}_{x\mathcal{P}_I/\mathcal{P}_I}))v^{-\ell(x)}\mu_x \in M_I.$$

(The notation is entirely analogous to the previous definition of ch in §2.2.)

We now turn to categorifying N_I. The full additive subcategory

$$\langle \mathcal{E}_x \mid x \notin W^I \rangle \subset \mathcal{H}^{\Bbbk}_{\mathrm{geom}}$$

is a left ideal. In particular, if we consider the quotient of additive categories

$$\mathfrak{N}^{\Bbbk}_I := \mathcal{H}^{\Bbbk}_{\mathrm{geom}}/\langle \mathcal{E}_x \mid x \notin W^I \rangle$$

this is a left $\mathcal{H}^{\Bbbk}$-module. We denote the image of $\mathcal{F} \in \mathcal{H}^{\Bbbk}_{\mathrm{geom}}$ by $\overline{\mathcal{F}}_I$. The objects $\overline{\mathcal{E}}_{x,I}$ for $x \in W^I$ are precisely the indecomposable objects of $\mathfrak{N}^{\Bbbk}_I$ up to shift and isomorphism. We have:

Theorem 2.19. *There is a unique isomorphism of right $H = [\mathcal{H}^{\Bbbk}_{\mathrm{geom}}]_{\oplus}$-modules*

$$N_I \xrightarrow{\sim} [\mathfrak{N}^{\Bbbk}_I]_{\oplus}$$

sending $\nu_{\mathrm{id}} \mapsto [\overline{\mathcal{E}}_{\mathrm{id},I}]$ (we use the identification $H = [\mathcal{H}^{\Bbbk}_{\mathrm{geom}}]_{\oplus}$ of Theorem 2.5).

The inverse $\mathrm{ch} : [\mathfrak{N}_I^{\Bbbk}]_{\oplus} \xrightarrow{\sim} N^I$ is more complicated to describe.

Remark 2.20. It is also possible to give a geometric description of $\mathfrak{N}_I^{\Bbbk}$ via Iwahori–Whittaker sheaves S. Riche and Williamson [2015, Chapter 11].

2.6 The p-canonical basis. Suppose that $\Bbbk$ is a field of characteristic $p \geq 0$. Consider the Hecke category $\mathcal{H}_{\mathrm{geom}}^{\Bbbk}$ with coefficients in $\Bbbk$. Let us define

$$ {}^p b_x := \mathrm{ch}(\mathcal{E}_x) \in H. $$

Because $\mathcal{E}_x$ is supported on $\overline{X}_x$ and its restriction to X_x is $\Bbbk_{X_x}[\ell(x)]$, it follows from the definition of the character map that

$$ {}^p b_x = \delta_x + \sum_{y<x} {}^p h_{y,x} \delta_y \tag{9} $$

for certain ${}^p h_{y,x} \in \mathbb{Z}_{\geq 0}[v^{\pm 1}]$. Thus the set $\{{}^p b_x \mid x \in W\}$ is a basis for H, the *p-canonical basis*. The base change coefficients ${}^p h_{y,x}$ are called *p-Kazhdan–Lusztig polynomials*, although they are Laurent polynomials in general.

The p-canonical basis has the following properties (see Jensen and Williamson [2017, Proposition 4.2]):

$$ d({}^p b_x) = b_x \quad \text{for all } x \in W; \tag{10} $$

$$ \text{if } \ {}^p b_x = \sum_{y \leq x} {}^p a_{y,x} b_y \ \text{ then } {}^p a_{y,x} \in \mathbb{Z}_{\geq 0}[v^{\pm 1}] \text{ and } d({}^p a_{y,x}) = {}^p a_{y,x}; \tag{11} $$

$$ \text{if } \ {}^p b_x {}^p b_y = \sum_{z \in W} {}^p \mu_{x,y}^z {}^p b_z \ \text{ then } {}^p \mu_{x,y}^z \in \mathbb{Z}_{\geq 0}[v^{\pm 1}] \text{ and } d({}^p \mu_{x,y}^z) = {}^p \mu_{x,y}^z; \tag{12} $$

$$ \text{for fixed } x \in W \text{ we have } {}^p b_x = {}^0 b_x = b_x \text{ for large } p. \tag{13} $$

Remark 2.21. Recall that the Kazhdan–Lusztig basis is uniquely determined by the "self-duality" and "degree bound" conditions (see §2.1). The p-canonical basis satisfies self-duality (10), but there appears to be no analogue of the degree bound condition in general (see Example 2.26 below).

Remark 2.22. There is an algorithm to calculate the p-canonical basis, involving the generators and relations presentation of the Hecke category discussed earlier. This algorithm is described in detail in Jensen and Williamson [ibid., §3].

Remark 2.23. The Kazhdan–Lusztig basis only depends on the Weyl group (a fact which is rather surprising from a geometric point of view). The p-canonical basis depends on the root system. For example, the 2-canonical bases in types B_3 and C_3 are quite different (see Jensen and Williamson [ibid., §5.4]).

Remark 2.24. The above properties certainly do not characterise the p-canonical basis. (For example, for affine Weyl groups the p-canonical bases are distinct for every prime.) However in certain situations they do appear to constrain the situation quite rigidly. For example, the above conditions are enough to deduce that ${}^p b_x = b_x$ for all primes p, if $\mathcal{G}$ is of types A_n for $n < 7$ (see Williamson and Braden [2012]). See Jensen [2017] for further combinatorial constraints on the p-canonical basis.

We can also define p-canonical bases in the spherical and anti-spherical module. Let $I \subset S$ be as in the previous section, and $\Bbbk$ and p be as above. For $x \in W^I$, set

$$ {}^p c_x := \mathrm{ch}(\mathcal{E}_{x,I}) = \sum_{y \in W^I} {}^p m_{y,x} \mu_y \in M_I, $$
$$ {}^p d_x := \mathrm{ch}(\overline{\mathcal{E}}_{x,I}) = \sum_{y \in W^I} {}^p n_{y,x} \nu_y \in N_I. $$

We have ${}^p m_{y,x} = {}^p n_{y,x} = 0$ unless $y \leq x$ and ${}^p m_{x,x} = {}^p n_{x,x} = 1$. Thus $\{{}^p c_x\}$ (resp. $\{{}^p d_x\}$) give *p-canonical* bases for M_I (resp. N_I). We leave it to the reader to write down the analogues of (10), (11), (12) and (13) that they satisfy.

We define spherical and anti-spherical analogues of the "adjustment" polynomials ${}^p a_{y,x}$ via:

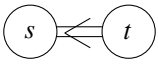
$$ {}^p c_x = \sum_{y \in W^I} {}^p a^{\mathrm{sph}}_{y,x} c_y \quad \text{and} \quad {}^p d_x = \sum_{y \in W^I} {}^p a^{\mathrm{asph}}_{y,x} d_y. $$

These polynomials give partial information on the p-canonical basis. For all $x, y \in W^I$ we have:

$$ {}^p a_{y w_I, x w_I} = {}^p a^{\mathrm{sph}}_{y,x} \quad \text{and} \quad {}^p a_{y,x} = {}^p a^{\mathrm{asph}}_{y,x}. \tag{14} $$

We finish this section with a few examples of the p-canonical basis. These are intended to complement the calculations in §1.5.

Example 2.25. Let $\mathcal{G}$ be of type B_2 with Dynkin diagram:



The Schubert variety $\overline{Y}_{st} \subset \mathcal{G}/\mathcal{P}_s$ has an isolated singularity at $\mathcal{P}_s/\mathcal{P}_s$, and a neighbourhood of this singularity is isomorphic to X from Example 1.10. From this one may deduce that

$$ {}^2 c_{st} = c_{st} + c_{\mathrm{id}}. $$

For a version of this calculation using diagrams see Jensen and Williamson [2017, §5.1].

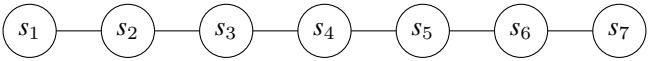
Example 2.26. Here we explain the implications of Example 1.13 for the p-canonical basis. The singularity $\mathbb{C}^{2n}/(\pm 1)$ occurs in the affine Grassmannian for Sp_{2n}, which is isomorphic to $\mathcal{G}/\mathcal{P}_I$, where $\mathcal{G}$ is the affine Kac–Moody group of affine type C_n with Dynkin diagram

and $I = \{s_1, \ldots, s_n\}$ denotes the subset of finite reflections. After some work matching parameters, one may deduce that

$$ {}^2 c_{w_n w_{n-1} s_0} = c_{w_n w_{n-1} s_0} + (v^{2n-2} + v^{2n-4} + \cdots + v^{-2n+2}) \cdot c_{\mathrm{id}}. $$

where w_n (resp. w_{n-1}) denotes the longest element in the standard parabolic subgroup generated by $\{s_1, \ldots, s_n\}$ (resp. $\{s_2, \ldots, s_n\}$).

Example 2.27. Let $\mathcal{G} = \mathrm{SL}_8(\mathbb{C})$ with simple reflections:



Let

$$w = s_1 s_3 s_2 s_4 s_3 s_5 s_4 s_3 s_2 s_1 s_6 s_7 s_6 s_5 s_4 s_3$$

and consider w_I where $I = \{s_1, s_3, s_4, s_5, s_7\}$. The singularity of the Schubert variety $\overline{X}_w$ at w_I is isomorphic to the Kashiwara–Saito singularity from Example 1.14 (with $d = 2$). It follows that

$${}^2 b_w = b_w + b_{w_I}.$$

This is one of the first examples for SL_n with ${}^p b_x \neq b_x$.

2.7 Torsion explosion. In this section we assume that $\mathcal{G} \cong \mathrm{SL}_n(\mathbb{C})$ and so $W = S_n$, the symmetric group. Here the p-canonical basis is completely known for $n = 2, 3, \ldots, 9$ and difficult to calculate beyond that. The following theorem makes clear some of the difficulties that await us in high rank:

Theorem 2.28 (Williamson [2017c]). *Let $\underline{\gamma}$ be a word of length l in the generators*

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

with product

$$\gamma = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix}.$$

For non-zero $m \in \{\gamma_{11}, \gamma_{12}, \gamma_{21}, \gamma_{22}\}$ and any prime p dividing m there exists $y \in S_{3l+5}$ such that ${}^p b_y \neq b_y$.

The moral seems to be that arithmetical issues ("which primes divide entries of this product of elementary matrices?") are hidden in the question of determining the p-canonical basis.[17]

We can get some qualitative information out of Theorem 2.28 as follows. Define

$$\Pi_n := \{p \text{ prime} \mid {}^p b_x \neq b_x \text{ for some } x \in S_n\}.$$

Because any Schubert variety in $\mathrm{SL}_n(\mathbb{C})/\mathcal{B}$ is also a Schubert variety in $\mathrm{SL}_{n+1}(\mathbb{C})/\mathcal{B}$ we have inclusions $\Pi_n \subset \Pi_{n+1}$ for all n. By long calculations by Braden, Polo, Saito and the author, we know the following about Π_n for small n:

$$\Pi_n = \varnothing \quad \text{for } n \leq 7,$$
$$\Pi_n = \{2\} \quad \text{for } n = 8, 9,$$
$$\{2, 3\} \subset \Pi_{12}.$$

[17] Another example of this phenomenon from Williamson [2017c]: for any prime number p dividing the lth Fibonacci number there exists $y \in S_{3l+5}$ with ${}^p b_y \neq b_y$. Understanding the behaviour of primes dividing Fibonacci numbers is a challenging open problem in number theory. It is conjectured, but not known, that infinitely many Fibonacci numbers are prime.

The most interesting values here are $2 \in \Pi_8$ (discovered by Braden in 2002, see Williamson and Braden [2012, Appendix]) and $3 \in \Pi_{12}$ (discovered by Polo in 2012). More generally, Polo shows that $p \in \Pi_{4p}$ for any prime p, and hence Π_n exhausts all prime numbers as $n \to \infty$ (see Example 1.14).

For applications to representation theory, it is important to know how large the entries of Π_n grow with n.[18] Some number theory, combined with Theorem 2.28, implies the following:

Corollary 2.29 (Williamson [2017c, Theorem A.1])**.** *For n large, $n \mapsto \max \Pi_n$ grows at least exponentially in n. More generally, Π_n contains many exponentially large prime numbers.*

Remark 2.30. Let us try to outline how Theorem 2.28 is proved. To $\underline{\gamma}$ and m we associate a reduced expression $\underline{x} = (s_1, \dots, s_n)$ for some particular $x \in S_{3l+5}$. (There is a precise but complicated combinatorial recipe as to how to do this, which we won't go into here. Let us mention however that the length of x grows quadratically in l.) Associated to this reduced expression we have a Bott–Samelson resolution

$$f : BS_{\underline{x}} \to \overline{X}_x.$$

We calculate the intersection form at a point $w_I\mathcal{B}/\mathcal{B}$ (corresponding to the maximal element of a standard parabolic subgroup) and discover the 1×1-matrix (m). Thus for any p dividing m the Decomposition Theorem fails for f at the point $w_I\mathcal{B}/\mathcal{B}$, which is enough to deduce the theorem. The hard part in all of this is finding the appropriate expression $\underline{x}$ and calculating the intersection form. The intersection form calculation was first done in Williamson [ibid.] using a formula in the nil Hecke ring discovered by He and Williamson [2015]. Later a purely geometric argument was found Williamson [2017b].

Remark 2.31. Let us keep the notation of the previous remark. In general we do not know whether ${}^p a_{w_A,x} \neq 0$ for any p dividing m, only that there is some y with $w_A \leq y \leq x$ and ${}^p a_{w_A,y} \neq 0$. Thus, in the statement of Theorem 2.28 we don't know that ${}^p b_x \neq b_x$, although this seems likely.

Remark 2.32. By a classical theorem of Zelevinskiĭ [1983], Schubert varieties in Grassmannians admit small resolutions, and hence the p-canonical basis is equal to the canonical basis in the spherical modules for one step flag varieties (we saw a hint of this in Example 1.11). It is an interesting question (suggested by Joe Chuang) as to how the p-canonical basis behaves in flag varieties with small numbers of steps and at what point (i.e. at how many steps) the behaviour indicated in Theorem 2.28 begins.

Remark 2.33. Any Schubert variety in $\mathrm{SL}_n(\mathbb{C})/\mathcal{B}$ is isomorphic to a Schubert variety in the flag varieties of types B_n, C_n and D_n. In particular, the above complexity is present in the p-canonical bases for all classical finite types.

[18] For example, the Lusztig conjecture would have implied that the entries of Π_n are bounded linearly in n, and the James conjecture would have implied a quadratic bound in n, see Williamson [ibid.].

2.8 Open questions about the p-canonical basis. In this section we discuss some interesting open problems about the p-canonical basis. We also try to outline what is known and point out connections to problems in modular representation theory.

In the following a Kac–Moody root datum is assumed to be fixed throughout. Thus, when we write ${}^p b_x$, its dependence on the root datum is implicit. Throughout, p denotes the characteristic of $\Bbbk$, our field of coefficients.

Question 2.34. *For $x \in W$ and p a prime, when is ${}^p b_x = b_x$?*

Remark 2.35. This question is equivalent to asking whether $\mathbf{IC}_x^{\Bbbk} \cong \mathcal{E}_x^{\Bbbk}$.

A finer-grained version of this question is:

Question 2.36. *For $x, y \in W$ and p a prime, when is ${}^p h_{y,x} = h_{y,x}$?*

Remark 2.37. If $h_{y,x} = v^{\ell(x)-\ell(y)}$ then Question 2.36 has a satisfactory answer. In this case $y\mathcal{B}/\mathcal{B}$ is a rationally smooth point of the Schubert variety $\overline{X}_x$ and ${}^p h_{y,x} = h_{y,x}$ if and only if $\overline{X}_x$ is also p-smooth at $y\mathcal{B}/\mathcal{B}$; moreover, this holds if and only if a certain combinatorially defined integer (the numerator of the "equivariant multiplicity") is not divisible by p, see Juteau and Williamson [2014] and Dyer [2001]. (See Fiebig [2010] and Fiebig and Williamson [2014] for related ideas.) It would be very interesting if one could extend such a criterion beyond the rationally smooth case.

In applications the following variants of Question 2.34 and 2.36 (for particular choices of Z) are more relevant:

Question 2.38. *Fix $Z \subset W$. For which p does there exist $x \in Z$ with ${}^p b_x \neq b_x$?*

Question 2.39. *Fix $Z \subset W^I$.*

1. *For which p is ${}^p c_x = c_x$ for all $x \in Z$?*
2. *For which p is ${}^p d_x = d_x$ for all $x \in Z$?*

Remark 2.40. If $\mathcal{G}$ is finite-dimensional then ${}^p b_x = b_x$ for all $x \in W$ if and only if a part of Lusztig's conjecture holds (see Soergel [2000]). The results of §2.7 give exponentially large counter-examples fo the expected bounds in Lusztig's character formula Lusztig [1980] and Williamson [2017c].

Remark 2.41. With $\mathcal{G}$ as in the previous remark, Xuhua He has suggested that we might have ${}^p b_x = b_x$ for all x, if $p > |W|$. This seems like a reasonable hope, and it would be wonderful to have a proof.

Remark 2.42. Suppose $\mathcal{G}$ is an affine Kac–Moody group and $I \subset S$ denotes the "finite" reflections (so that $W_I = \langle I \rangle$ is the finite Weyl group). Then there exists a finite subset $Z_1 \subset W^I$ for which Question 2.38(1) is equivalent to determining in which characteristics Lusztig's character formula holds, see Achar and Riche [2016b, §11.6] and Williamson [2017a, §2.6]. Because Z_1 is finite, ${}^p c_x = c_x$ for all $x \in Z_1$ for p large, which translates into the known fact that Lusztig's conjecture holds in large characteristic.

Remark 2.43. Suppose that $\mathcal{G}$ and I are as in the previous remark. There exists a subset $Z(p) \subset W^I$ (depending on p) such that if ${}^p d_x = d_x$ for all $x \in Z(p)$ then Andersen's conjecture on characters of tilting modules (see Andersen [2000, Proposition 4.6]) holds in characteristic p (this is a consequence of the character formula proved in Achar, Makisumi, Riche, and Williamson [2017b]). Note that Andersen's conjecture does not give a character formula for the characters of all tilting modules and is not known even for large p.

The following questions are also interesting:

Question 2.44. *For which $x \in W$ and p is ${}^p a_{y,x} \in \mathbb{Z}$ for all $y \in W$?*

Remark 2.45. This is equivalent to asking when $\mathcal{E}_x^{\Bbbk}$ is perverse.

Question 2.46. *Fix $? \in \{\mathrm{sph}, \mathrm{asph}\}$. For which $x \in W^I$ and p is ${}^p a^?_{y,x} \in \mathbb{Z}$ for all $y \in W^I$?*

Remark 2.47. Example 2.26 shows that in the affine case ${}^p a_{y,x}$ can be a polynomial in v of arbitrarily high degree. An example of Libedinsky and the author Libedinsky and Williamson [2017] shows that there exists x, y in the symmetric group S_{15}, for which ${}^2 a_{y,x} = (v + v^{-1})$. Recently P. McNamara has proposed new candidate examples, which appear to show that for any p, the degree of ${}^p a_{y,x}$ is unbounded in symmetric groups.

Remark 2.48. Suppose $\mathcal{G}$ and I are as in Remark 2.42. It follows from the the results of Juteau, Mautner, and Williamson [2016] and Mautner and Riche [2018] that ${}^p a_{y,x} \in \mathbb{Z}$ if x is maximal in $W_I x W_I$. More generally, it seems likely that ${}^p a^{\mathrm{sph}}_{y,x} \in \mathbb{Z}$ for all $x, y \in W^I$ and large p (depending only on the Dynkin diagram of $\mathcal{G}$). This is true for trivial reasons in affine types A_1 and A_2.

Remark 2.49. In contrast, recent conjectures of Lusztig and the author Lusztig and Williamson [2018] imply that, if $\mathcal{G}$ is of affine type $\widetilde{A_2}$ then it is never the case (for any $p \neq 2$) that ${}^p a^{\mathrm{asph}}_{y,x} \in \mathbb{Z}$ for all $y, x \in W^I$. In fact, our conjecture implies that

$$\max \{\deg({}^p a^{\mathrm{asph}}_{y,x}) \mid y, w \in W^I\} = \infty.$$

This contrast in behaviour between the p-canonical bases in the spherical and anti-spherical modules is rather striking.

3 Koszul duality

In this section we discuss Koszul duality for the Hecke category. This is a remarkable derived equivalence relating the Hecke categories of Langlands dual groups. It resembles a Fourier transform. Its modular version involves parity sheaves, and is closely related to certain formality questions. In this section we assume that the reader has some background with perverse sheaves and highest weight categories.

3.1 Classical Koszul duality. Let $C, \mathcal{G}, \mathcal{B}, \mathcal{T}, W, \Bbbk$ be as previously. We denote by $\mathcal{G}^\vee, \mathcal{B}^\vee, \mathcal{T}^\vee$ the Kac–Moody group (resp. Borel subgroup, resp. maximal torus) associated to the dual Kac–Moody root datum. We have a canonical identification of W with the Weyl group of $\mathcal{G}^\vee$.

In this section we assume that $\mathcal{G}$ is a (finite-dimensional) complex reductive group, i.e. that C is a Cartan matrix. We denote by $w_0 \in W$ the longest element. For any $x \in W$ let $i_x : X_x = \mathcal{B} \cdot x\mathcal{B}/\mathcal{B} \hookrightarrow \mathcal{G}/\mathcal{B}$ denote the inclusion of the Schubert cell and set

$$\Delta_x := i_{x!}\Bbbk_{X_x}[\ell(x)] \quad \text{and} \quad \nabla_x := i_{x*}\Bbbk_{X_x}[\ell(x)].$$

Let $D^b_{(\mathcal{B})}(\mathcal{G}/\mathcal{B};\Bbbk)$ denote the derived category, constructible with respect to $\mathcal{B}$-orbits and let $\mathbf{P}_{(\mathcal{B})}(\mathcal{G}/\mathcal{B};\Bbbk) \subset D^b_{(\mathcal{B})}(\mathcal{G}/\mathcal{B};\Bbbk)$ denote the subcategory of perverse sheaves. The abelian category $\mathbf{P}_{(\mathcal{B})}(\mathcal{G}/\mathcal{B};\Bbbk)$ is highest weight Beilinson, Ginzburg, and Soergel [1996] and Beilinson, Bezrukavnikov, and Mirković [2004] with standard (resp. costandard) objects $\{\Delta_x\}_{x\in W}$ (resp. $\{\nabla_x\}_{x\in W}$). For $x \in W$, we denote by $\mathcal{P}_x, \mathcal{I}_x$ and $\mathcal{T}_x$ the corresponding indecomposable projective, injective, and tilting object. The corresponding objects in $\mathbf{P}_{(\mathcal{B}^\vee)}(\mathcal{G}^\vee/\mathcal{B}^\vee;\Bbbk)$ are denoted with a check, e.g. $\mathbf{IC}^\vee_x, \Delta^\vee_x$ etc.

Let us assume that $\Bbbk = \mathbb{Q}$. Motivic considerations, together with the Kazhdan–Lusztig inversion formula (see Kazhdan and Lusztig [1979])

$$\sum_{z\in W}(-1)^{\ell(x)+\ell(y)}h_{y,x}h_{yw_0,zw_0} = \delta_{x,z}, \tag{15}$$

led Beilinson and Ginsburg [1986] to the following conjecture[19]:

1. There exists a triangulated category $D^{mix}_{(\mathcal{B})}(\mathcal{G}/\mathcal{B};\mathbb{Q})$ equipped with an action of the integers $\mathcal{F} \mapsto \mathcal{F}\langle m\rangle$ for $m \in \mathbb{Z}$ ("Tate twist") and a "forgetting the mixed structure" functor

$$\phi : D^{mix}_{(\mathcal{B})}(\mathcal{G}/\mathcal{B};\mathbb{Q}) \to D^b_{(\mathcal{B})}(\mathcal{G}/\mathcal{B};\mathbb{Q}),$$

such that

$$\mathrm{Hom}(\phi(\mathcal{F}),\phi(\mathcal{G})) = \bigoplus_{n\in\mathbb{Z}}\mathrm{Hom}(\mathcal{F},\mathcal{G}\langle n\rangle)$$

for all $\mathcal{F},\mathcal{G} \in D^{mix}_{(\mathcal{B})}(\mathcal{G}/\mathcal{B};\mathbb{Q})$. Furthermore, "canonical" objects (e.g. simple, standard, projective etc. objects) admit lifts[20] to $D^{mix}_{(\mathcal{B})}(\mathcal{G}/\mathcal{B};\mathbb{Q})$.

2. There is an equivalence of triangulated categories

$$\kappa : D^{mix}_{(\mathcal{B})}(\mathcal{G}/\mathcal{B};\mathbb{Q}) \xrightarrow{\sim} D^{mix}_{(\mathcal{B}^\vee)}(\mathcal{G}^\vee/\mathcal{B}^\vee;\mathbb{Q}) \tag{16}$$

such that $\kappa \circ \langle -1\rangle[1] \cong \langle 1\rangle \circ \kappa$, and such that κ acts on standard, simple and projective objects (for an appropriate choice of lift) as follows:

$$\Delta_x \mapsto \nabla^\vee_{x^{-1}w_0}, \qquad \mathbf{IC}_x \mapsto \mathcal{I}^\vee_{x^{-1}w_0}, \qquad \mathcal{P}_x \mapsto \mathbf{IC}^\vee_{x^{-1}w_0}.$$

[19] to simplify the exposition we have modified the statement of their original conjecture slightly (they worked with Lie algebra representations and sought a contravariant equivalence).

[20] $\mathcal{F} \in D^b_{(\mathcal{B})}(\mathcal{G}/\mathcal{B};\mathbb{Q})$ admits a lift, if there exists $\widetilde{\mathcal{F}} \in D^{mix}_{(\mathcal{B})}(\mathcal{G}/\mathcal{B};\mathbb{Q})$ such that $\mathcal{F} \cong \phi(\widetilde{\mathcal{F}})$.

Remark 3.1. To understand why the extra grading (provided by the mixed structure) as well as the relation $\kappa \circ \langle -1 \rangle [1] \cong \langle 1 \rangle \circ \kappa$ is necessary, one only needs to ask oneself where the grading on extensions between simple modules should go under this equivalence.

Remark 3.2. One can deduce from (15) and the Kazhdan–Lusztig conjecture that the assignment $\Delta_x \mapsto \nabla_{x^{-1}w_0}$ on mixed categories forces $\mathbf{IC}_x \mapsto \mathcal{J}_{x^{-1}w_0}$ and $\mathcal{P}_x \mapsto \mathbf{IC}_{x^{-1}w_0}$ on the level of Grothendieck groups.

This conjecture was proved by Beilinson, Ginzburg and Soergel in the seminal paper Beilinson, Ginzburg, and Soergel [1996], where they interpreted κ in the framework of Koszul duality for graded algebras. The authors give two constructions of the mixed derived category: one involving mixed étale sheaves (here it is necessary to consider the flag variety for the split group defined over a finite field), and one involving mixed Hodge modules.

Remark 3.3. Both constructions of the mixed derived category in Beilinson, Ginzburg, and Soergel [ibid.] involve some non-geometric "cooking" to get the right result. Recently Soergel and Wendt have used various flavours of mixed Tate motives to give a purely geometric construction of these mixed derived categories Soergel [1990b].

After the fact, it is not difficult to see that the mixed derived category admits a simple definition. Indeed, the results of Beilinson, Ginzburg, and Soergel [1996] imply that one has an equivalence

$$\sigma : K^b(\mathrm{Semis}(\mathcal{G}/\mathcal{B};\mathbb{Q})) \xrightarrow{\sim} D^{mix}_{(\mathcal{B})}(\mathcal{G}/\mathcal{B};\mathbb{Q}). \tag{17}$$

Here $\mathrm{Semis}(\mathcal{G}/\mathcal{B};\mathbb{Q})$ denotes the full additive subcategory of $D^{mix}_{(\mathcal{B})}(\mathcal{G}/\mathcal{B};\mathbb{Q})$ consisting of direct sums of shifts of intersection cohomology complexes ("semi-simple complexes"), and $K^b(\mathrm{Semis}(\mathcal{G}/\mathcal{B};\mathbb{Q}))$ denotes its homotopy category. Note that there are two shift functors on $\mathrm{Semis}(\mathcal{G}/\mathcal{B};\mathbb{Q})$: one coming from its structure as a homotopy category (which we denote $[1]$); and one induced from the shift functor on $\mathrm{Semis}(\mathcal{G}/\mathcal{B};\mathbb{Q})$ (which we rename (1)). Under the equivalence σ, Tate twist $\langle 1 \rangle$ corresponds to $[1](-1)$.

Now, if $\mathcal{H}^{\mathbb{Q}}$ denotes the Hecke category we have an equivalence

$$\mathbb{Q} \otimes_R \mathcal{H}^{\mathbb{Q}}_{\mathrm{diag}} \xrightarrow{\sim} \mathrm{Semis}(\mathcal{G}/\mathcal{B};\mathbb{Q}).$$

Moreover, the left hand side can be described by generators and relations. In particular, Koszul duality can be formulated entirely algebraically as an equivalence

$$\kappa : K^b(\mathbb{Q} \otimes_R \mathcal{H}^{\mathbb{Q}}_{\mathrm{diag}}) \xrightarrow{\sim} K^b(\mathbb{Q} \otimes_R \mathcal{H}^{\vee,\mathbb{Q}}_{\mathrm{diag}}).$$

The existence of such an equivalence (valid more generally for any finite real reflection group, with $\mathbb{Q}$ replaced by $\mathbb{R}$) has recently been established by Makisumi [2017]. (The case of a dihedral group was worked out by Sauerwein [2018].)

3.2 Monoidal Koszul duality. The above results raise the following questions:

1. How does Koszul duality interact with the monoidal structure?
2. Can Koszul duality be generalised to the setting of Kac–Moody groups?

The first question was addressed by Beilinson and Ginzburg [1999]. They noticed that if one composes Koszul duality κ with the Radon transform and inversion, one obtains a derived equivalence

$$\widetilde{\kappa} : D^{mix}_{(\mathcal{B})}(\mathcal{G}/\mathcal{B};\mathbb{Q}) \xrightarrow{\sim} D^{mix}_{(\mathcal{B}^\vee)}(\mathcal{B}^\vee \setminus \mathcal{G}^\vee;\mathbb{Q}) \tag{18}$$

with $\widetilde{\kappa} \circ \langle -1\rangle[1] \cong \langle 1\rangle \circ \widetilde{\kappa}$ as previously, however now

$$\mathbf{IC}_x \mapsto \mathcal{T}_x^\vee, \quad \Delta_x \mapsto \Delta_x^\vee, \qquad \nabla_x \mapsto \nabla_x^\vee, \qquad \mathcal{T}_x \mapsto \mathbf{IC}_x^\vee. \tag{19}$$

The new equivalence $\widetilde{\kappa}$ is visibly more symmetric than κ. It also has the advantage that it does not involve the longest element w_0, and hence makes sense for Kac–Moody groups. Moreover, Beilinson and Ginzburg conjectured that $\widetilde{\kappa}$ can be promoted to a monoidal equivalence (suitably interpreted).

Remark 3.4. It has been a stumbling block for some time that (18) cannot be upgraded to a monoidal equivalence in a straightforward way. This is already evident for SL_2: the "big" tilting sheaf $\mathcal{T}_s \in D^b_{(\mathcal{B})}(\mathbb{P}^1;\mathbb{Q})$ does not admit a $\mathcal{B}$-equivariant structure.

Subsequently, Bezrukavnikov and Yun [2013] established a monoidal equivalence

$$\widetilde{\kappa} : (D^{mix}_{\mathcal{B}}(\mathcal{G}/\mathcal{B};\mathbb{Q}), *) \xrightarrow{\sim} (\widehat{D}^{mix}(\mathcal{B}^\vee \backslash\!\backslash \mathcal{G}^\vee /\!\!/ \mathcal{B}^\vee;\mathbb{Q}), \star) \tag{20}$$

which induces the Koszul duality equivalence above after killing the deformations, and is valid for any Kac–Moody group.[21] Here $\widehat{D}^{mix}(\mathcal{B}^\vee \backslash\!\backslash \mathcal{G}^\vee /\!\!/ \mathcal{B}^\vee;\mathbb{Q})$ denotes a suitable ("free monodromic") completion of the full subcategory of mixed $U^\vee$-constructible complexes on $\mathcal{G}^\vee/U^\vee$ which have unipotent monodromy along the fibres of the map $\mathcal{G}^\vee/U^\vee \to \mathcal{G}^\vee/\mathcal{B}^\vee$. The construction of this completion involves considerable technical difficulties. The proof involves relating both sides to a suitable category of Soergel bimodules (and thus is by "generators and relations").

3.3 Modular Koszul duality. We now discuss the question of how to generalise (18) to coefficients $\Bbbk$ of positive characteristic. A first difficulty is how to make sense of the mixed derived category. A naïve attempt (carried out in Riche, Soergel, and Williamson [2014]) is to consider a flag variety over a finite field together with the Frobenius endomorphism and its weights, however here one runs into problems because one obtains gradings by a finite cyclic group rather than $\mathbb{Z}$. Achar and Riche took the surprising step of simply defining

$$D^{mix}_{(\mathcal{B})}(\mathcal{G}/\mathcal{B};\Bbbk) := K^b(\mathrm{Par}(\mathcal{G}/\mathcal{B};\Bbbk))$$

where $\mathrm{Par}(\mathcal{G}/\mathcal{B};\Bbbk)$ denotes the additive category of $\mathcal{B}$-constructible parity complexes on $\mathcal{G}/\mathcal{B}$, and the shift $[1]$ and twist (1) functors are defined as in the paragraph following (17). (The discussion there shows that this definition is consistent when $\Bbbk = \mathbb{Q}$.) In doing so one obtains a triangulated category with most of the favourable properties one expects from the mixed derived category. In this setting Koszul duality takes the form:

[21] Actually, Bezrukavnikov and Yun use mixed ℓ-adic sheaves, and no non-geometric "cooking" is necessary.

Theorem 3.5. *There is an equivalence of triangulated categories*

$$\kappa : D^{mix}_{(\mathcal{B})}(\mathcal{G}/\mathcal{B};\Bbbk) \xrightarrow{\sim} D^{mix}_{(\mathcal{B}^\vee)}(\mathcal{B}^\vee\backslash\mathcal{G}^\vee;\Bbbk)$$

which satisfies $\kappa \circ \langle -1\rangle[1] \cong \langle 1\rangle \circ \kappa$ *and*

$$\kappa(\Delta_w) \cong \Delta_w^\vee, \quad \kappa(\nabla_w) \cong \nabla_w^\vee, \quad \kappa(\mathcal{E}_w) \cong \mathcal{T}_w^\vee, \quad \kappa(\mathcal{T}_w) \cong \mathcal{E}_w^\vee.$$

Remark 3.6. The important difference in the modular case is that tilting sheaves correspond to parity sheaves (rather than IC sheaves).

Remark 3.7. For finite-dimensional $\mathcal{G}$ this theorem was proved in Achar and Riche [2016a] (in good characteristic). For general $\mathcal{G}$ this theorem is proved in Achar, Makisumi, Riche, and Williamson [2017a,b], as a corollary of a monoidal modular Koszul duality equivalence, inspired by Bezrukavnikov and Yun [2013].

Remark 3.8. The appearance of the Langlands dual group was missing from the original conjectures of Beilinson and Ginsburg [1986] and only appeared in Beilinson, Ginzburg, and Soergel [1996]. However in the settings considered there ($\Bbbk = \mathbb{Q}$), the Hecke categories associated to dual groups are equivalent. This is no longer the case with modular coefficients, and examples (e.g. B_3 and C_3 in characteristic 2) show that the analogue of Theorem 3.5 is false if one ignores the dual group.

Remark 3.9. A major motivation for Achar, Makisumi, Riche, and Williamson [2017a,b] was a conjecture of Riche and the author S. Riche and Williamson [2015, §1.4] giving characters for tilting modules for reductive algebraic groups in terms of p-Kazhdan–Lusztig polynomials. In fact, a recent theorem of Achar and Riche [2016b] (generalising a theorem of Arkhipov, Bezrukavnikov, and Ginzburg [2004]) combined with (a variant of) the above Koszul duality theorem leads to a solution of this conjecture. We expect that modular Koszul duality will have other applications in modular representation theory.

Remark 3.10. One issue with the above definition of the mixed derived category is the absence of a "forget the mixed structure" functor $\phi : D^{mix}_{(\mathcal{B})}(\mathcal{G}/\mathcal{B};\Bbbk) \to D^b_{(\mathcal{B})}(\mathcal{G}/\mathcal{B};\Bbbk)$ in general. For finite-dimensional $\mathcal{G}$ its existence is established in Achar and Riche [2016a]. Its existence for affine Weyl groups would imply an important conjecture of Finkelberg and Mirković [1999], see also Achar and Riche [2016b].

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DYNAMICAL SYSTEMS EVOLVING

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Abstract

This is an expanded version of a presentation given at ICM2018. It discusses a number of results taken from a cross-section of the author's work in Dynamical Systems. The topics include relation between entropy, Lyapunov exponents and fractal dimension, statistical properties of chaotic dynamical systems, physically relevant invariant measures, strange attractors arising from shear-induced chaos, random maps and random attractors. The last section contains two applications of Dynamical Systems ideas to Biology, one to epidemics control and the other to computational neuroscience.

I am grateful for the opportunity to share my ideas with the mathematics community. My field is *Dynamical Systems*, an area of mathematics concerned with time evolutions of processes natural and engineered, of iterative schemes and algorithms. It is the study of change, of moving objects and evolving situations, and the goal is to describe, analyze, explain, and ultimately predict.

I will be discussing mostly my own work, but to provide context – mostly for the benefit of readers not working in Dynamical Systems – I would like to start with a brief introduction to the field, where it has been, and where we are today. Dynamical Systems is now a little over 100 years old. As with any $2-3$ page overview of such a vast body of ideas, my account will necessarily be a gross simplification, a biased and intensely personal one at that, but still I hope it will help put the field in perspective.

Dynamical systems: a very brief overview

To facilitate referencing, I will divide the growth of the subject into four distinct phases, with the understanding that these are, for the most part, artificial boundaries.

I. A field was born. It is generally accepted that the origin of Dynamical Systems goes back to two sets of truly great ideas before and around the turn of the 20th century. One is the introduction of qualitative analysis into the study of differential equations by Poincaré in his work on celestial mechanics; the other consists of ideas surrounding the Ergodic Hypothesis.

Prior to the time of Poincaré, understanding a system defined by a set of (ordinary) differential equations meant solving analytically those equations, or approximating them with infinite series expansions. As we know now, most differential equations

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are not analytically solvable, and series expansions do not always shed light. Poincaré showed that a wealth of information about the qualitative behavior of a system, local and global, can be deduced without solving the equations explicitly, thereby introducing the world to the idea of geometric or qualitative analysis.

The Ergodic Hypothesis came from statistical mechanics. It asserts, roughly speaking, that because trajectories explore the phase space, time averages of an observable along individual trajectories should reflect the spatial average. In part because the needed language did not exist at the time, there is little consensus on how exactly this Hypothesis was first formulated or to whom to attribute it, though Boltzmann clearly had some of the ideas. However it came about, the impact that this profound idea has had on Dynamical Systems is undisputed.

II. Laying of foundation. Though Dynamical Systems was not recognized as a branch of mathematics until later, most of its foundation was laid in the first part of the twentieth century through the 1960s.

The framework for Ergodic Theory as the study of measure-preserving transformations was put on firm footing by the Ergodic Theorems of Birkhoff and von Neumann in the 1930s, and through the introduction of important invariants such as entropy later on. Ergodic theory can be seen as a probabilistic approach to Dynamical Systems, an approach in which one is not concerned with every initial condition but focuses instead on averages and almost-sure behaviors.

On the geometric-analytic side, the subject saw an explosion of creative activity in the middle of the 20th century for two distinct types of dynamical systems. I think of them as occupying the two ends of an "ordered-disordered" spectrum:

At the "ordered" end are quasi-periodic systems. In integrable Hamiltonian systems, orbits move about in an orderly fashion on highly constrained surfaces. The theory of Kolmogorov, Arnold and Moser (KAM) guarantees the persistence of this order on large parts of the phase space when such systems are perturbed.

At the "disordered" end are chaotic dynamical systems (though the word "chaos" was not yet in use at that time). Smale led a bold attempt to axiomatize chaotic behavior. Globalizing the idea of dynamics near a fixed point of saddle type, he invented as models of chaotic behavior the idea of hyperbolic invariant sets, characterized by exponential separation of nearby orbits on the set. His seminal work led to what eventually became the hyperbolic theory of dynamical systems.

III. Maturation and diversification. In the next few decades, Dynamical Systems gained formal recognition as an area of mathematics. The ideas outlined above continued to blossom, both conceptually and technically. New topics of research opened up, and the subject became more diverse. I mention below a sample of these developments:

Smooth ergodic theory, the use of ergodic theory techniques to study differentiable dynamical systems, was pioneered in the then Soviet Union in the early 1970s. The result was a nonuniform hyperbolic theory, which generalizes the uniformly hyperbolic invariant sets of Smale to a version where hyperbolicity occurs only *almost everywhere* and its onset is nonuniform in time.

KAM theory gave rise to two major topics: one studies structures that are remnants of KAM tori as one moves farther from integrable systems, and the other, pioneered by

Arnold, studies the diffusion of orbits in near-integrable systems having more than two degrees of freedom, where motion is not blocked by invariant tori.

In the 1980s and 90s, a large number of researchers flocked to the study of one dimensional maps, real and complex, attracted by the fact that (i) the low dimensionality of the phase space made these systems more tractable, yet (ii) such "simple" systems already exhibited rich dynamical behaviors. The subject flourished; a number of deep and very striking theorems were proved. One dimension, however, is very special, and implications of 1D results for higher dimensions remain to be explored.

There was also growing interest in the analysis of concrete systems. Well known examples include the Lorenz system (3-mode truncation of convection in 2D), the periodically forced van der Pol Equation (a slow-fast system the forcing of which produces complex dynamics), and a class of dynamical systems known as "billiards" and "hard balls". Billiard systems are models of uniform motion of point particles in bounded domains, making elastic collisions with the walls of the domain.

Finally, smooth actions of groups other than $\mathbb{Z}$ and $\mathbb{R}$ on manifolds extending the theory of diffeomorphisms and flows became an active area. Many results concern "rigidity", as relations on group elements can severely limit the actions that can exist.

IV. Making connections. Starting from its inception, Dynamical Systems has always had contact with other fields. These contacts have intensified in recent years for multiple reasons: While it makes sense to have a general theory, what can be said about general systems without more specific context is limited. At the same time, the fact that many parts of mathematics and science can benefit from Dynamical Systems thinking became clearer than ever before.

Though I don't know enough to write about them individually, I know that Dynamical Systems has interactions with many branches of pure mathematics, including Number Theory, Geometric Group Theory, Differential Geometry, Analysis and Probability, some through problems of mutual interest, or analogous phenomena, and also because many problems are naturally studied through iterative procedures.

Opportunities of collaborative interaction with applied mathematics, engineering, the physical and biological sciences abound, as almost all systems evolve with time. Many such collaborations are already in place or ongoing, but here I think we have some distance to go to achieve the full potential of Dynamical Systems' usefulness. I also think that to do that, some retooling may be necessary on our part.

This concludes the brief overview.

Turning now to my own research, I entered the field during Phase III in the Overview. In the sections to follow, I want to share with the reader some highlights of my work, the bulk of which lies in the ergodic theory of chaotic dynamical systems, the topic that descended from what I described as the "disordered" end of the spectrum in Part II — though my interests diversified with time. I have chosen to present several snapshots taken from a cross-section of my work, as opposed to an in-depth discussion of one or two results, because I believe that will give you a more balanced view of what I do, and of what my field is about.

This article is written with the aim of communicating with the broader mathematics community. I hope you will find it readable except for a few things here and there that I have to ask you to accept on faith.

1 Entropy, Lyapunov exponents, and fractal dimension

The middle third Cantor set can be seen as the invariant set

$$\Lambda := \left\{ x \in [0,1] : f^n x \in \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right] \forall n \in \mathbb{Z}^+ \right\}$$

of the map $f(x) = 3x \bmod 1,\ x \in [0,1]$. When iterated, the complexity of f restricted to Λ can be thought of as growing like 2^n, as in each iterate there is the possibility of getting sent to two distinct intervals. Clearly, $(f^n)' = 3^n$, and an elementary exercise shows that the Hausdorff dimension of Λ, $\mathrm{HD}(\Lambda) = \log 2/\log 3$.

The main result of this section is a relation among three invariants for general dynamical systems: entropy, which measures the average growth in complexity, Lyapunov exponents, which measures average derivative growth, and the dimension of an invariant measure. The results presented will hold in all dimensions. In one dimension, it generalizes the Cantor set computation above, illustrating a concept at the heart of Ergodic Theory, namely that quantities that vary from point to point in the phase space can be represented by time averages along orbits.

For the benefit of readers not familiar with the subject, I have included in the first half of Sect. 1.1 some background material in smooth ergodic theory. The idea of Lyapunov exponents, in particular, will appear many times in Sections 1–4.

1.1 Setting and background information. Let M be a C^∞ compact Riemannian manifold without boundary. We consider a pair (f, μ), where $f : M \circlearrowleft$ is a diffeomorphism of M onto itself, of differentiability class $C^{1+\alpha}$ for some $\alpha > 0$, and μ is a Borel probability measure preserved by f, i..e., for every Borel subset $E \subset M$, $\mu(f^{-1}(E)) = \mu(E)$.

Given (f, μ), there are two important, and conceptually different, ways to quantify the dynamical complexity of the system. One is *entropy*, and the other is *Lyapunov exponents*. The entropy of f with respect to μ, denoted $h_\mu(f)$, is defined to be

$$h_\mu(f) = \sup_\eta h_\mu(f;\eta)$$

where the supremum is taken of all finite measurable partitions η of M and

$$h_\mu(f;\eta) = \lim_{n\to\infty} \frac{1}{n} H\Big(\bigvee_{i=0}^{n-1} f^{-i}\eta\Big) .$$

Here $H(\eta) = \int I(\eta)d\mu$ where $I(\eta)(x) = -\log \mu(\eta(x))$, so $h_\mu(f;\eta)$ has the interpretation of average information gain, or average uncertainty removed, per iteration of the

map. (See Walters [1982] for an introductory text.) The Lyapunov exponent of f at x in the direction of a tangent vector v is the growth rate of $\|Df_x^n(v)\|$, i.e.,

$$\lambda(x,v) = \lim_{n\to\infty} \frac{1}{n} \log \|Df_x^n(v)\|$$

if this limit exists. By the well known Multiplicative Ergodic Theorem Oseledec [1968], given (f,μ), at μ-a.e. x, there is a set of numbers

$$\lambda_1(x) > \lambda_2(x) > \cdots \lambda_{r(x)}(x)$$

and a splitting of the tangent space T_xM at x into a direct sum of subspaces

$$T_xM = E_1(x) \oplus E_2(x) \oplus \cdots \oplus E_{r(x)}(x)$$

such that for all $v \in E_i(x)$, $v \neq 0$, we have $\lambda(x,v) = \lambda_i(x)$. The functions $x \mapsto r(x), \lambda_i(x), E_i(x)$ are Borel measurable, and since $\lambda_i(x) = \lambda_i(fx)$, we have that if (f,μ) is ergodic, then the Lyapunov exponents are given by a finite set of numbers $\lambda_1 > \cdots > \lambda_r$ with multiplicities $m_1, \cdots, m_r$ where $m_i = \dim(E_i)$.

I started to learn about these ideas when a number of exciting results had just been proved. Among them are the following relations between entropy and Lyapunov exponents: Most basic are Ruelle's inequality Ruelle [1978a], which asserts that

$$h_\mu(f) \le \int \sum_i \lambda_i^+(x) m_i(x) d\mu\,, \qquad a^+ = \max\{a,0\}\,, \tag{1}$$

and Pesin's entropy formula J. B. Pesin [1977], which asserts that if μ is volume on M, then the inequality in (1) is an equality. These results can be interpreted as saying that all uncertainty is created by expansion, and that in conservative systems, all expansion is used to create entropy. The entropy formula was subsequently generalized to hold for SRB measures Ledrappier and Strelcyn [1982], i.e., μ is required only to have conditional densities on unstable manifolds; it does not have to be volume.

My contribution to this topic is to explain the gap in (1) in terms of the dimension of μ. For an arbitrary finite Borel measure ν on a metric space, we define the *dimension* of ν at x, $\dim(\nu,x)$, to be δ if $\nu(B(x,\rho)) \sim \rho^\delta$ for small ρ where $B(x,\rho)$ is the ball of radius ρ centered at x. More precisely,

$$\dim(\nu,x) := \lim_{\rho\to 0} \frac{\log \nu(B(x,\rho))}{\log \rho} = \delta$$

assuming this limit exists, and write $\dim(\nu)$ if $\dim(\nu,x)$ is constant ν-a.e. So Lebesgue measure on $\mathbb{R}^d$ has dimension d, but δ in general does not have to be an integer; it is a notion of fractal dimension.

1.2 Results. My first result in this topic was for surface diffeomorphisms. The notation is as above.

Theorem 1. L. S. Young [1982] *Let $dim(M) = 2$, and let f be an arbitrary $C^{1+\alpha}$ diffeomorphism of M. We assume (f, μ) is ergodic with $\lambda_1 > 0 > \lambda_2$. Then $dim(\mu)$ is well defined and is given by*

$$dim(\mu) = h_\mu(f)\left[\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right].$$

The idea, roughly speaking, is that the dimension of μ "in the unstable direction" is equal to $h_\mu(f)/\lambda_1$, it is equal to $h_\mu(f)/(-\lambda_2)$ "in the stable direction", and $\dim(\mu)$ is the sum of these two contributions. The number $h_\mu(f)/\lambda_1$ is the direct analog of $\log 2/\log 3$ in the middle third Cantor set example at the beginning of this section.

When there are multiple positive Lyapunov exponents, the nonconformality makes the situation considerably more complicated. The next result, proved in a joint work with Ledrappier, deals with that.

Theorem 2. Ledrappier and L.-S. Young [1985b] *Let f be a $C^{1+\alpha}$ diffeomorphism of a compact Riemannian manifold of arbitrary dimension, and let (f, μ) be ergodic, with positive Lyapunov exponents $\lambda_1 > \cdots > \lambda_r$ and multiplicities $m_1, \ldots, m_r$ respectively. Then $dim(\mu|W^u)$, the dimension of the conditional measures of μ on unstable manifolds, is well defined, and there are numbers $\delta_i \in [0, m_i], i = 1, \cdots, r$, such that*

$$h_\mu(f) = \sum_{i=1}^{r} \lambda_i \delta_i \qquad \textit{and} \qquad \dim(\mu|W^u) = \sum_{i=1}^{r} \delta_i\ .$$

The numbers δ_i have the interpretation of being the "partial dimension" of μ in the direction of E_i, the subspace corresponding to the Lyapunov exponent λ_i. Interpreting $\lambda_i\delta_i$ as the entropy of (f, μ) in the direction of E_i, the first equality in Theorem 2 asserts that $h_\mu(f)$ is the sum of the "partial entropies" in the different expanding directions, while the second equality asserts that the dimension of $\mu|W^u$ is the sum of the partial dimensions. To be technically correct, I should say that the system (f, μ) has a hierarchy of unstable manifolds $W^1 \subset W^2 \subset \cdots \subset W^r = W^u$ defined μ-a.e., with W^k tangent to $E_1 \oplus \cdots \oplus E_k$, but there need not be invariant manifolds tangent to each E_i; and that in the actual proof, the δ_i are the dimensions of quotient measures on W^i/W^{i-1}, and $\lambda_i\delta_i = h_i - h_{i-1}$ where h_i can be made precise as a notion of entropy along the invariant foliation W^i.

Clearly, one can apply Theorem 2 to f^{-1}, obtaining an analogous result for $\dim(\mu|W^s)$. The result of Barreira, Y. Pesin, and Schmeling [1999], which states that in the absence of zero Lyapunov exponents,

$$\dim(\mu) = \dim(\mu|W^u) + \dim(\mu|W^s)\ ,$$

completes this circle of ideas.

I want to stress again the generality of the results in this section: they hold for *all diffeomorphisms and all invariant Borel probability measures*, with no restriction whatsoever on the geometry or the defining equations of the maps, or on the invariant measure.

2 Statistical properties of chaotic dynamical systems via Markov extensions

In Paragraph II of the Overview in the Introduction, I portrayed, impressionistically, all dynamical systems as lying on an "ordered-disordered" spectrum. Let (f, μ) be as in Section 1, and think of it as being on the "disordered" side of this spectrum. Given an observable $\varphi : M \to \mathbb{R}$, how random can sequences of the form

$$\varphi(x),\ \varphi(fx),\ \varphi(f^2x),\ \ldots,\ \varphi(f^nx), \ldots$$

be for randomly chosen initial condition $x \in M$? Can these sequences be, for example, as random as functions of coin flips? This is a tricky question, for dynamical systems generated by maps or flows are *deterministic* in the sense that given an initial condition, the entire future trajectory is fully determined and nothing is left to chance. At the same time, if f has chaotic dynamics, then it may be hard to predict $\varphi(f^nx)$ from approximate knowledge of x. Indeed uniformly hyperbolic systems have been shown to produce statistics that obey some the same probabilistic limit laws as genuinely random stochastic processes.

A fairly complete theory of the statistical properties of uniformly hyperbolic or Axiom A systems (à la Smale) was developed in the 1970s; see Ruelle [1978b] and Bowen [1975]. After that, the community began to move beyond Axiom A, to confront the challenges of examples such as the Lorenz attractor, Hénon maps and various billiard systems to which previously developed analytical techniques did not apply. It was against this backdrop that the work presented in this section was carried out.

2.1 A unified view for predominantly hyperbolic systems. First, a basic definition: Let $\Lambda \subset M$ be a compact invariant set of a diffeomorphism $f : M \circlearrowleft$. We say $f|_\Lambda$ ("f restricted to Λ") is *uniformly hyperbolic* if the tangent bundle $T_\Lambda M$ over Λ can be split into the direct sum of two Df-invariant subbundles

$$T_\Lambda M = E^u \oplus E^s$$

with the property that $Df^n|_{E^u}$ is uniformly expanding and $Df^n|_{E^s}$ is uniformly contracting. As mentioned in the Overview, this idea was introduced by Smale in his ground-breaking work in Smale [1967]. It is slightly inaccurate, but I will use the term "uniformly hyperbolic" and "Axiom A" interchangeably in this article.

The existence of hyperbolic invariant sets in a dynamical system is often seen as the presence of chaotic behavior. Many of the examples that challenged the community in the 1980s are not uniformly hyperbolic, but their dynamics are dominated by large hyperbolic invariant sets. This prompted me to propose the following construction as an attempt to provide a unified view for a class of dynamical systems with weaker forms of hyperbolicity than Axiom A. The material in the rest of this subsection is taken from L.-S. Young [1998].

Given $f : M \circlearrowleft$, my proposal was to construct, where possible, a *countable Markov extension*, i.e., a map $F : \Delta \circlearrowleft$ with

$$\begin{array}{ccc} \Delta & \xrightarrow{F} & \Delta \\ \pi \downarrow & & \downarrow \pi \\ M & \xrightarrow{f} & M \end{array} \tag{2}$$

where $F : \Delta \circlearrowleft$ has the structure of a countable state Markov chain. To be clear, F is not a Markov chain, or isomorphic to one in any sense, only that there is a countable partition on Δ with respect to which the action of F has the flavor of a Markov chain.

Finite Markov partitions were constructed for Axiom A systems and were instrumental in the study of their statistical properties, see e.g. J. G. Sinaĭ [1972], Bowen [1975], and Ruelle [1978b]. What I proposed was a generalization of these ideas to systems with weaker hyperbolicity. The usefulness of this proposal will, of course, depend on what one can do with it, and I will discuss that in Sects. 2.2 and 2.3.

It's a little technical, but I will give more detail on $F : \Delta \circlearrowleft$. We look for a set Λ_0 with a nice hyperbolic structure, and study "hyperbolic returns" to this set. More precisely, we seek Λ_0 of the form

$$\Lambda_0 = \left(\bigcup_{\gamma^u \in \Gamma^u} \gamma^u \right) \bigcap \left(\bigcup_{\gamma^s \in \Gamma^s} \gamma^s \right)$$

where Γ^s and Γ^u are two families of local stable and unstable manifolds, each element of $\gamma^s \in \Gamma^s$ intersecting each $\gamma^u \in \Gamma^u$ transversally in a unique point. We call a subset $\Theta \subset \Lambda_0$ a *u-subset* if $\Theta = \Lambda_0 \cap (\cup_{\gamma^u \in \tilde{\Gamma}^u} \gamma^u)$ for some subset $\tilde{\Gamma}^u \subset \Gamma^u$; *s-subsets* of Λ_0 are defined analogously. Pictorially, if one thinks of Γ^u as a stack of roughly "horizontal" disks and Γ^s a stack of roughly "vertical" disks, then Λ_0 is the lattice of intersection points of the disks in the two families, a u-subset of Λ_0 is a sublattice that runs from left to right, and an s-subset is a sublattice that runs from top to bottom.

Now suppose there is a decomposition of Λ_0 into

$$\Lambda_0 = \bigcup_{i=1}^{\infty} \Lambda^s_{0,i} \qquad \text{(disjoint union)}$$

with the property that each $\Lambda^s_{0,i}$ is a s-subset, and for each i, there exists $r_i \in \mathbb{Z}^+$ such that $f^{r_i}(\Lambda^s_{0,i}) = \Lambda^u_{0,i}$ where $\Lambda^u_{0,i}$ is an u-subset of Λ_0. This is what I meant by a "hyperbolic return". The function $R : \Lambda_0 \to \mathbb{Z}^+$ given by $R(x) = r_i$ for $x \in \Lambda^s_{0,i}$ is called the *return time* function. (We do not require R to be the first return time, and the $\Lambda^u_{0,i}$ do not have to be pairwise disjoint.)

For $f : M \circlearrowleft$ admitting the construction above, we let $\Delta = \bigcup_{i=0}^{\infty} \Delta_i$, where each Δ_i is a copy of $f^i(\{x \in \Lambda_0 : R(x) > i\})$, let π is the identification map, and define $F : \Delta \circlearrowleft$ so the diagram in (2) commutes. This in our Markov extension.

2.2 Statistical properties of systems with Markov extensions. To address the question raised at the beginning of this section, we begin with the following definitions: Let

(f, μ) be as in Section 1, and let $\mathfrak{F}$ be a class of functions on M. We say (f, μ) has *exponential decay of time correlations* for test functions in $\mathfrak{F}$ if there exists $\tau < 1$ such that for all $\varphi, \psi \in \mathfrak{F}$, there exists $C = C(\varphi, \psi)$ such that

$$\left| \int (\varphi \circ f^n)\psi d\mu - \int \varphi d\mu \int \psi d\mu \right| \leq C\tau^n \qquad \forall n \geq 1 .$$

Thus if $\varphi \circ f^n$ represents the observation on "Day n", then (f, μ) having exponential correlation decay means in particular that $\varphi \circ f^n$ and φ decorrelate exponentially fast in n. Informally, on Day n, there is still some memory of Day 0, but memory is fading exponentially fast. *Polynomial decay of correlations* is defined similarly.

In ergodic theory, a measure-preserving transformation (f, μ) is said to be *mixing* if for all measurable sets A, B, $\mu(f^{-n}A \cap B) \to \mu(A)\mu(B)$ as $n \to \infty$. The decay of time correlations to 0 is just a functional form of the same idea, i.e., exponential correlation decay means exponential mixing of sets in the phase space.

We say φ satisfies the *Central Limit Theorem* (CLT) with respect to (f, μ) if

$$\frac{1}{\sqrt{n}} \left(\sum_{i=0}^{n-1} \varphi \circ f^i - n \int \varphi d\mu \right) \to \mathfrak{N}(0, \sigma) ,$$

the normal distribution with variance σ^2 for some $\sigma \geq 0$.

Consider now a map $f : M \circlearrowleft$ admitting a Markov extension. To this construction we now add a *reference measure*: Recall that Γ^s and Γ^u are the families of stable and unstable disks defining the hyperbolic product set Λ_0. We let $m = m_{\gamma^u}$ be the Riemannian measure on γ^u, and assume that

$$m(\gamma^u \cap \Lambda_0) > 0 \qquad \text{for some} \qquad \gamma^u \in \Gamma^u . \tag{3}$$

Furthermore, we require only that the return time function R be defined on m-a.e. $x \in \Lambda_0 \cap \gamma^u$. The reason for this focus on m is that we are primarily interested in SRB measures, the conditional measures on unstable manifolds of which are equivalent to Lebesgue. The importance of SRB measures is discussed in Sect. 3.1.

For systems satisfying the conditions above, I proved the following results:

Theorem 3. L.-S. Young [1998, 1999] *Let $f : M \circlearrowleft$ be as above, i.e., there is a set $\Lambda_0 \subset M$ satisfying (3) with return time function $R : \Lambda_0 \to \mathbb{Z}^+$ and reference measure m on γ^u, and let $F : \Delta \circlearrowleft$ be its Markov extension. Then the following hold:*

(a) If $\int R dm < \infty$, then f has an SRB measure μ.

(b) If $\int R dm < \infty$ and $\gcd\{R\} = 1$, then (f, μ) is mixing.

Here gcd = *greatest common divisor. Assume below that the conditions in (b) hold. All results pertain to (f, μ), and all test functions are Hölder continuous.*

(c) If $m\{R > n\} \leq C\theta^n$ for some $\theta < 1$, then correlation decay is exponential.

(d) If $m\{R > n\} = \mathcal{O}(n^{-\alpha}), \alpha > 1$, then correlation decays at $\mathcal{O}(n^{-\alpha+1})$.

(e) If $m\{R > n\} = \mathcal{O}(n^{-\alpha}), \alpha > 2$, then the CLT holds.

The results in Theorem 3 were first proved for $F : \Delta \circlearrowleft$ and then passed to f, the idea being that proofs for F are simpler because $F : \Delta \circlearrowleft$ has the topological structure of a countable state Markov chain. Parts (a) and (b) have obvious analogs with Markov chain theory: finite expectation of return times is equivalent to positivity of recurrence, $\gcd\{R\} = 1$ means periodicity. Parts (c)–(e) establish the connection between statistical properties of the system and the tails of "renewal" times. Though I was not aware of it until later, results along similar lines for (real) Markov chains were obtained around that same general time frame Meyn and Tweedie [2009]. My "Markov extensions" are not Markov chains as I have indicated earlier, but the ideas are similar.

2.3 Periodic Lorentz gas and other potential applications. The method discussed in Sects. 2.1 and 2.2 was used to study the statistical properties for a number of examples of dynamical systems that have hyperbolic properties but are not necessarily uniformly hyperbolic (see e.g. L.-S. Young [1992], L.-S. Young [1998, 1999], N. Chernov [2006], and Rey-Bellet and L.-S. Young [2008]). I will discuss in some detail one example, the 2-dimensional periodic Lorentz gas, and finish with some remarks on other applications.

The 2D periodic Lorentz gas. The Lorentz gas is a model for electron gases in metals. Mathematically, the 2D periodic case is represented by the motion of a point mass in $\mathbb{R}^2$ bouncing elastically off a fixed periodic configuration of convex scatterers. It was first studied by Sinai around 1970 J. G. Sinaĭ [1970], and is sometimes called the Sinai billiard. Putting the dynamics on the 2-torus $\mathbb{T}^2$, we obtain the billiard flow φ_t on $\Omega \times \mathbb{S}^1$ where $\Omega = \mathbb{T}^2 \setminus \Omega_i$ and the Ω_i's are disjoint convex regions with C^3 boundaries. Points in $\Omega \times \mathbb{S}^1$ are denoted (x, θ) where $x \in \Omega$ is the footpoint of the arrow pointing in direction θ, the direction of the motion. A section to the billiard flow is the collision manifold $M = \partial\Omega \times [-\frac{\pi}{2}, \frac{\pi}{2}]$. We consider the collision map f of the billiard flow, or the first return map from M to itself. It is straightforward to check that f leaves invariant the probability measure $\mu = c \cos\theta dx d\theta$ where θ is the angle the arrow makes with the normal pointing into Ω and c is the normalizing constant. For simplicity, we will assume the finite horizon condition, which requires that the time between collisions be uniformly bounded.

Following the strategy outlined in Sects. 2.1 and 2.2, i.e., by constructing for f a countable Markov extension and investigating the tail properties of the return time function R, I proved the following result. Let C^α denote the class of Hölder functions on M with Hölder exponent α.

Theorem 4. L.-S. Young [1998] *Let (f, μ) be as above. Then correlation decays exponentially fast for observables in C^α. More precisely, there exists $\beta = \beta(\alpha) > 0$ such that for every $\varphi, \psi \in C^\alpha$,*

$$\left| \int (\varphi \circ f^n)\psi d\mu - \int \varphi d\mu \int \psi d\mu \right| \le Ce^{-\beta n} \qquad \forall n \ge 1$$

for some $C = C(\varphi, \psi)$.

For more information on statistical properties of billiards and hard balls, see N. Chernov and L. S. Young [2000]. A weaker version of this result showing "stretched exponential decay" was first proved for this class of billiards in Bunimovich, Y. G. Sinaĭ, and N. I. Chernov [1991] along with other statistical properties like the CLT. Markov extensions are, needless to say, not the only way to study correlation decay of dynamical systems. Stronger results on larger classes of billiard maps were obtained by N. Chernov [2006], and exponential decay of the billiard *flow*, a much harder problem, was solved only recently by Baladi, Demers, and Liverani [2018].

Summary and other applications. The ideas of this section were intended for systems possessing a good amount of hyperbolicity but are not necessarily uniformly hyperbolic. The approach I proposed was to set aside individual characteristics of the dynamical system, focus on return times to a good reference set, and to deduce the statistical properties of the system from tail properties of these return times.

This method has been particularly effective for systems with a localized source of nonhyperbolicity, an identifiable set that spoils the hyperbolicity of orbits passing near it. The simplest examples are 1D maps of the form $f(x) = 1 - ax^2, x \in [-1, 1]$, $a \in [0, 2]$. Here the "bad set" is $\{x = 0\}$, where $f' = 0$. I showed in L.-S. Young [1992] that if the orbit of 0 does not approach the point 0 too fast, then the map has exponential correlation decay. In the Lorentz gas example above, the "bad set" is where particle trajectories graze the scatterers. In other examples, it can take the form of "traps" or "eddies", where orbits linger for long periods of times, thereby slowing down the speed with which different regions of the phase space are mixed. Two examples in this category are neutral fixed points of 1D maps L.-S. Young [1999] and billiard maps with parabolic regions such as the stadium Markarian [2004], both of which have polynomial decay.

Yet another kind of "bad set" is where directions of expansions and contractions are switched, i.e., as the orbit passes near the bad set, tangent vectors that have been growing in length get rotated, causing them to shrink in subsequent iterates. Bad sets of this type are very challenging to deal with, as we will see. I want to mention that the results of this section have also been applied successfully to prove exponential correlation decay for the "good maps" in Theorem 7 Wang and L.-S. Young [2013].

3 Strange attractors from shear-induced chaos

This was one of my first attempts to connect the abstract theory of chaotic dynamical systems to concrete settings. An immediate question is: *Which invariant measure should one consider?* Except in the case of Hamiltonian or volume preserving flows, dissipative systems such as those with attractors do not come equipped with a natural invariant measure. In general, the number of invariant measures is very large, and not all of them are relevant for purposes of describing what one sees.

Sect. 3.1 discusses general mathematical issues associated with observable chaos. Sect. 3.2 describes some examples and Sect. 3.3 the analysis behind these examples.

3.1 Observable chaos and SRB measures. It is one thing for a dynamical system to have orbits that behave chaotically, another for this chaotic behavior to be observable. In finite-dimensional dynamics, one often equates positive Lebesgue measure sets with observable events. Adopting such a view, we say $f : M \circlearrowleft$ has *observable chaos* if $\lambda_{\max} > 0$ on at least a positive Lebesgue measure set, where

$$\lambda_{\max}(x) := \liminf_{n\to\infty} \frac{1}{n} \log \|Df_x^n\| ,$$

i.e. $\lambda_{\max}$ is the largest Lyapunov exponent at x when that is defined. In the rest of this section, I will write "$\lambda_{\max} > 0$" as abbreviation for "observable chaos".

Hyperbolic invariant sets such as Smale's horseshoes contain orbits with chaotic dynamics, but the presence of a horseshoe does not imply $\lambda_{\max} > 0$, for the horseshoe itself occupies a zero Lebesgue measure set, and its presence does not preclude the possibility that orbits starting from Leb-a.e. $x \in M$ may tend eventually to a stable equilibrium, called a "sink". By contrast, $\lambda_{\max} > 0$ implies that instability is not only observable (in the sense of Lebesgue measure), but it persists for all future times. It is a much stronger form of chaos than the presence of horseshoes alone.

The question, then, is: which systems have $\lambda_{\max} > 0$? Following Eckmann and Ruelle [1985], we call an invariant probability measure *physically relevant* if it reflects the properties of initial conditions on positive Lebesgue measure sets. For Hamiltonian or volume-preserving systems, Liouville measure or Riemannian measure are clearly physically relevant. For systems that are not conservative (no technical meaning intended), the situation is more subtle: In such systems, trajectories of most orbits tend toward *attractors*. For a set to attract, there is often volume contraction, and when there is volume contraction, all invariant probability measures are necessarily singular with respect to Lebesgue measure. *A priori*, it is not clear if the concept of a physically relevant invariant measure makes sense for attractors.

An important discovery in the 1970s by Sinai, Ruelle and Bowen was that every uniformly hyperbolic or Axiom A attractor Λ admits a special invariant probability measure μ which plays the role of Liouville measure for Hamiltonian systems, see J. G. Sinaĭ [1972], Ruelle [1976], and Bowen [1975]. This measure – when Λ is not a sink – is called an *SRB measure*. SRB measures are generally singular with respect to Lebesgue. Their physical relevance is derived from the fact that they have smooth conditional measures on unstable manifolds. A precise definition of physical relevance here is that for every continuous observable $\varphi : M \to \mathbb{R}$, Leb-a.e. x in the basin of the attractor has the property that

$$\frac{1}{n}\sum_{i=0}^{n-1} \varphi(f^i x) \to \int \varphi d\mu . \tag{4}$$

I stress that (4) is not a consequence of Birkhoff's Ergodic Theorem: In general, μ is supported on the attractor Λ, a zero Lebesgue measure set Λ; yet it governs the large-time dynamics of trajectories starting from Leb-a.e. point in the basin of attraction, a much larger open set containing Λ.

The idea of SRB measures was generalized in the 1980s to a significantly broader context by Ledrappier and myself, meaning we identified a special class of invariant measures for general diffeomorphisms and flows that play a similar role as SRB measures for Axiom A attractors Ledrappier and Strelcyn [1982], Ledrappier and L.-S. Young [1985a], and L.-S. Young [1985]. These measures continue to be characterized by their smooth conditional measures on unstable manifolds, though the picture is more complicated and less tidy than in the uniform hyperbolic case. Importantly, if a system admits an ergodic SRB measure μ with no zero Lyapunov exponents, then (4) holds for initial conditions from a positive Lebesgue measure set Pugh and Shub [1989], implying $\lambda_{\max} > 0$.

Generalizing the idea of SRB measures to arbitrary dynamical systems does not, however, guarantee their *existence* in this larger context. Uniformly hyperbolic systems are special in that they have well aligned subspaces E^u consisting of vectors that are uniformly expanded from one iterate to the next. They satisfy what is called an "invariant cones" condition. For systems without continuous families of invariant cones – and that is the case for most dynamical systems – $\|Df_x^n(v)\|$ is likely to sometimes grow and sometimes shrink as n increases for most tangent vectors v, making it very challenging to identify aligned directions of exponential growth, a prerequisite for SRB measures.

Though SRB measures are thought to be prevalent among systems with chaotic attractors, current state of the art is that few instances of (genuinely) nonuniformly hyperbolic attractors have been shown rigorously to possess SRB measures. I will give some concrete examples in Sect. 3.2. All currently known examples in fact belong in the class of "rank one" attractors, which I will discuss in Sect. 3.3.

3.2 Shear-induced chaos in periodically kicked oscillators. A mechanism for producing $\lambda_{\max} > 0$ is shear-induced chaos, by which I mean the following: Start with a system with tame, nonchaotic dynamics, but some amount of shearing. The idea, in a nutshell, is that external forcing that magnifies the underlying shear can produce stretching and folding of the phase space, leading to $\lambda_{\max} > 0$.

A perfect setting for this mechanism is the periodic forcing of oscillators. The idea goes back nearly 100 years, to van der Pol and van der Mark, who observed in the course of their work on vacuum tube triode circuits that periodic forcing of relaxation oscillators could lead to "irregular noise". The problem was studied analytically by many authors: Cartwright, Littlewood, Levinson in the 1940s, Levi and others much later. The existence of horseshoes was proved analytically for a linearized system by Levi and for the original van der Pol equation by Haiduc with a computer assisted proof. See Haiduc [2009] for references on the above.

As explained in Sect. 3.1, the existence of horseshoes does not imply observable chaos. Noting that it is technically simpler to decouple the effects of the periodic drive from the dynamics of the unforced oscillator, my co-author Wang and I used pulsatile forcing, or *kicks*, with relatively long periods of relaxation in between. The phenomenology already manifests itself in the following very simple example:

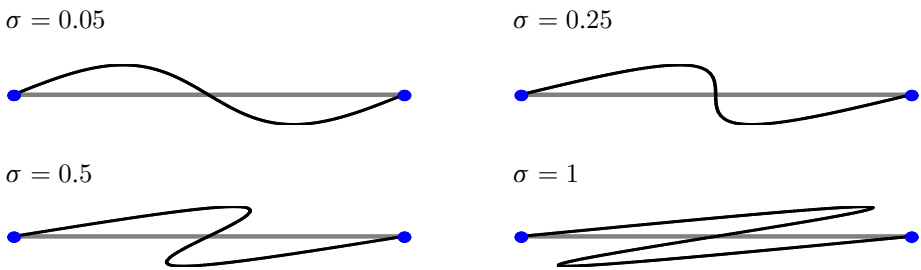

Figure 1: **Effect of increasing shear.** Images of $\Psi_\tau(\gamma)$ for $T = 10$ and $\lambda = A = 0.1$.

Linear shear flow.[1] This model is given by

$$\begin{aligned} \dot{\theta} &= 1 + \sigma y \,, \\ \dot{y} &= -\lambda y \;+\; A \cdot \sin(2\pi\theta) \cdot \textstyle\sum_{n=-\infty}^{\infty} \delta(t - nT) \,, \end{aligned} \tag{5}$$

where $(\theta, y) \in \mathbb{S}^1 \times \mathbb{R}$, $\mathbb{S}^1 \equiv \mathbb{R}/\mathbb{Z}$, and σ, λ, A and T are constants with $\sigma, \lambda > 0$ and $T \gg 1$. Here, Eq. (5) with $A = 0$ is the unforced equation. Letting Φ_t denote the unforced flow, one sees that for all $z \in \mathbb{S}^1 \times \mathbb{R}$, $\Phi_t(z)$ tends to the limit cycle $\gamma = \{y = 0\}$ as $t \to \infty$. With T being the period of the forcing, it is natural to consider the time-T map $F_T = \Phi_T \circ \kappa$ of the forced system, where the effect of the instantaneous kick is given by $\kappa(\theta, y) = (\theta, y + A\sin(2\pi\theta))$.

We observed that for λT large, i.e., if the contraction between kicks is strong, the ratio

$$\frac{\sigma}{\lambda} A \;=\; \frac{\text{shear}}{\text{contraction}} \cdot \text{kick amplitude}$$

is key to determining whether the system is chaotic. See Fig 1. Our results can be summarized informally as follows:

Theorem 5. Wang and L.-S. Young [2002] *For each choice of σ, λ, A and T, the time-T map F_T has a (maximal) attracting set $\Lambda = \Lambda(\sigma, \lambda, A, T)$ with the following properties:*

(i) For $\frac{\sigma}{\lambda} A$ small, Λ is a closed invariant curve.

(ii) As $\frac{\sigma}{\lambda} A$ increases, the invariant curve breaks; the dynamics of F_T is initially of gradient type (with sinks and sources); then horseshoes start to develop.

(iii) In the case of large $\frac{\sigma}{\lambda} A$, regarding T as a parameter:

(a) F_T has horseshoes and sinks on an open set of T, and

(b) provided that $e^{-\lambda T}$ is sufficiently small, F_T has an SRB measure, and $\lambda_{\max} > 0$ Leb-a.e. for a positive measure set of T.

[1] After Wang and I had completed the study reported here, we learned that G. Zaslavsky, a physicist colleague of mine, had studied numerically a similar example 30 years earlier Zaslavsky [1978/79].

The ideas in this linear oscillator example are valid for general limit cycles in arbitrary dimensions, except that the effects of σ, λ and A cannot be separated. Instead, one has to interpret "shear" as the degree to which the kick "scrambles" the limit cycle, its variation measured with respect to the strong stable foliation of the unforced system. See the review article Lin and L.-S. Young [2010] for more detail.

The following is another illustration of the same mechanism at work.

Periodically forced Hopf bifurcations in ODEs and PDEs. Consider first the usual picture of a Hopf bifurcation in $2D$, for an equation $\dot{x} = F_\mu(x)$ where μ here is the bifurcation parameter. We assume $F_\mu(0) = 0$ for all μ, and that $x = 0$ undergoes a generic supercritical Hopf bifurcation at $\mu = 0$, so that a limit cycle of radius $\sim \sqrt{\mu}$ emerges as 0 destabilizes. Writing all this in normal form, we have

$$\dot{z} = k_0(\mu)z + k_1(\mu)z^2\bar{z} + k_2(\mu)z^3\bar{z}^2 + \text{h. o. t.}$$

We define the *twist* of the system to be

$$\tau = \frac{\operatorname{Im} k_1(0)}{-\operatorname{Re} k_1(0)}.$$

As we will see, this quantity is the analog of the shear in the previous example. The result below applies to dynamical systems defined by ODEs on phase spaces of any dimension $d \geq 2$ Wang and L.-S. Young [2003] as well as to systems defined by PDEs Lu, Wang, and L.-S. Young [2013]. As illustration of the breadth of its applicability, I will state it for dissipative parabolic PDEs undergoing a Hopf bifurcation.

Consider, for example, a 1-parameter family of semilinear parabolic equations on a bounded domain $\Omega \subset \mathbb{R}^d$ with smooth boundary:

$$\begin{aligned} u_t &= D\Delta u + f_\mu(u), \qquad x \in \Omega,\ u \in \mathbb{R}^m, \\ u(x,t) &= 0, \qquad x \in \partial\Omega,\ t \geq 0. \end{aligned}$$

Here D is a diagonal matrix with positive entries, μ is a parameter, and $f_\mu : \mathbb{R}^d \to \mathbb{R}^m$ is a polynomial with $f_\mu(0) = 0$ for all μ. We assume, in a sense to be made precise, that the solution $u(x,t) \equiv 0$ is stable for $\mu < 0$, and that it loses its stability at $\mu = 0$. To this equation, we add a periodic forcing that is close to impulsive, i.e.,

$$u_t = D\Delta u + f_\mu(u) + \rho\varphi(x)p_T(t) \tag{6}$$

where $\rho \in \mathbb{R}$ is a constant, $\varphi : \Omega \to \mathbb{R}^m$ is a smooth function satisfying mild conditions, $p_T = \sum_{n=-\infty}^{\infty} \varepsilon^{-1} I_{[nT, nT+\varepsilon]}$ and I_A is the indicator function on A. As a *dynamical system* on $\left(H_0^1(\Omega)\right)^m$, (6) is a special case of an abstract system generated by an equation of the form

$$\dot{\mathbf{u}} = \mathcal{A}\mathbf{u} + \mathbf{f}_\mu(\mathbf{u}) + \rho\Box(\mathbf{u})p_T(t). \tag{7}$$

The result below applies to evolutionary equations of the form (7); we have motivated with a concrete PDE but the exact PDE that gives rise to it is immaterial.

Theorem 6. Lu, Wang, and L.-S. Young [2013] *Let $\mathbb{H}$ be a Hilbert space. Our unforced equation has the form*

$$\dot{u} = \mathbf{A}u + \mathbf{F}(u, \mu), \qquad u \in \mathbb{H}, \quad \mu \in (-\mu_1, \mu_1) \subset \mathbb{R}, \tag{8}$$

where $-\mathbf{A}$ is a sectorial operator and $\mathbf{F} : \mathbb{H}^\sigma \times (-\mu_1, \mu_1) \to \mathbb{H}$ is C^5 for some $\sigma \in [0, 1)$.[2] *We assume $\mathbf{F}(0, \mu) = 0$ for all μ, and rewrite (8) to obtain*

$$\dot{u} = \mathcal{A}_\mu u + f_\mu(u), \quad f_\mu(0) = 0, \quad \partial_u f_\mu(0) = 0. \tag{9}$$

This system is assumed to undergo a generic supercritical Hopf bifurcation at $\mu = 0$, with a limit cycle γ_μ emerging from $u = 0$ for $\mu > 0$.

To (9) we add a forcing term, resulting in

$$\dot{u} = \mathcal{A}_\mu u + f_\mu(u) + \rho\Phi(u)p_T(t). \tag{10}$$

Here $\rho > 0$ is a constant, $\Phi : \mathbb{H}^\sigma \to \mathbb{H}^\sigma$ is C^5 with uniformly bounded C^5-norms, and p_T is as above. Let $\mathbb{H}^\sigma = E^c_\mu \oplus E^s_\mu$ be the decomposition into $\mathcal{A}_\mu$-invariant center and stable subspaces, E^c_μ corresponding to the two leading eigenvalues in the Hopf bifurcation. We assume $\Phi(0) \notin E^s_0$, and normalize to give $|P^c_0(\Phi(0))| = 1$ where P^c_0 is the projection of $\mathbb{H}^\sigma$ onto E^c_0.

Let F_T be the time-T map of the semiflow defined by (10). For $T > \text{const} \cdot \mu^{-1}$, F_T has an attractor Λ_μ near the limit cycle γ_μ of (9), attracting all points in an open neighborhood $\mathcal{U}$ of Λ_μ in $\mathbb{H}^\sigma$. Assume further that

$$|\tau|\rho\mu^{-\frac{1}{2}} > L_0 \qquad \text{for a certain } L_0 .$$

Then we have for each small enough $\mu > 0$ a roughly T-periodic positive measure set $\Delta_\mu \subset (M_\mu, \infty)$ for some $M_\mu \gg \mu^{-1}$ with the property that for every $T \in \Delta$,

(a) F_T has an ergodic SRB measure and

(b) $\lambda_{\max} > 0$ "almost everywhere" in $\mathcal{U}$.

The term "almost everywhere" above certainly requires justification, as $\mathcal{U}$ is an open set in an infinite dimensional function space. We discussed in Sect. 3.1 the idea of equating observable events with positive Lebesgue measure sets. There is, of course, no Lebesgue measure on Banach spaces, but one can use finite parameter families of initial conditions, and use Lebesgue as a reference measure on parameter space. In Theorem 6, "almost everywhere" refers to Lebesgue-a.e. initial condition in every 2-parameter family of initial conditions transversal to the strong stable foliation in $\mathcal{U}$. See Lian, L.-S. Young, and Zeng [2013], Lu, Wang, and L.-S. Young [2013], and Blumenthal and L.-S. Young [2017] for more information.

[2] $\{\mathbb{H}^\sigma\}$ are a family of interpolating subspaces called fractional power spaces; see Henry [1981]

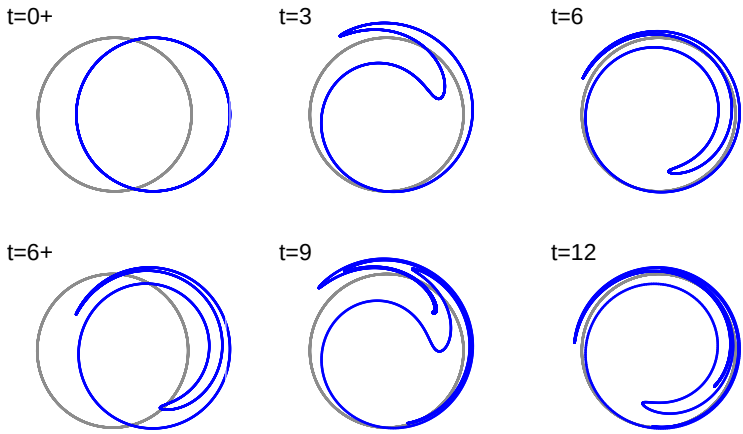

Figure 2: **Hopf attractor.** Grey circle is the limit cycle γ_μ of the unforced Hopf bifurcation. At time $t = 0$, the system receives a "kick", sending γ_μ to the blue circle. Between $t = 0$ and $t = 6$, the unforced flow brings the blue circle black to the grey, rotating counterclockwise with points farther from the center of the grey circle rotating at a higher speed due to a nonzero twist as explained in the text. At $t = 6$, the image of γ_μ has the shape shown, due to the differential in rotational speeds. The kick is repeated once every 6 units of time, i.e., $T = 6$. Here $\rho\mu^{-\frac{1}{2}} = \frac{1}{2}$ and $\tau = 10$.

3.3 A theory of rank-one attractors. The $\Lambda_{\max} > 0$ results in Sect. 3.2 were obtained by appealing to a general theorem on "rank-one attractors", the name my co-author Qiudong Wang and I coined for a class of attractors that we introduced and studied Wang and L.-S. Young [2001, 2008, 2013]. These attractors can live in phase spaces of any dimension. They are so called because they have only one direction of instability, with (strong) contraction in all other directions. Rank-one attractors are, in many ways, the simplest and least chaotic among chaotic attractors. They occur naturally, often following the loss of stability.

Of interest to us here are those rank-one attractors that possess SRB measures and exhibit observable chaos. The existence and abundance of such attractors is guaranteed by the theorem below. The precise statement of this result is technical, and since this article is not the right forum for technical details, I will suppress some of them, referring the reader to Wang and L.-S. Young [2008]. (A 2D version of this result was first proved in Wang and L.-S. Young [2001], but Wang and L.-S. Young [2008] is both more general and more readable.)

Theorem 7. Wang and L.-S. Young [2001, 2008] (Informal version) *Let $M = I \times D_m$ where I is either an interval or a circle and D_m is an m-dimensional disk, $m \geq 1$. For each $\varepsilon > 0$, let $F_{a,\varepsilon} : M \circlearrowleft$ be a C^3 family of embeddings with $|\det(DF_{a,\varepsilon})| \sim \varepsilon^m$. Assume*

(a) as $\varepsilon \to 0$, $F_{a,\varepsilon} \to F_{a,0}$ in C^3 where $F_{a,0}$ is a family of maps from $M \to I \times \{0\}$;

(b) letting $f_a = F_{a,0}|_{I\times\{0\}}$, we obtain a family of 1D maps with

(i) nondegenerate critical points and

(ii) sufficiently strong expansion away from critical sets;

(c) the mappings $F_{a,0}$ satisfy certain nondegeneracy and transversality conditions.

Then for each sufficiently small $\varepsilon > 0$, there exists a positive measure set Δ_ε such that for all $a \in \Delta_\varepsilon$:

(i) $F_{a,\varepsilon}$ has an ergodic SRB measure;

(ii) $\lambda_{\max} > 0$ Leb-a.e. on M.

This is a perturbative result. The idea is to embed the systems of interest in a family that can be passed, in a meaningful way, to a *singular limit*, which is an object of a lower dimension (in this case 1D), the idea being that lower-dimensional objects are more tractable. The tricky part is to "unfold" the 1D results at $\varepsilon = 0$ to recover the dynamical picture for small $\varepsilon > 0$.

Theorem 7 is a generalization of the work of Benedicks and Carleson [1991] and Benedicks and myself Benedicks and L.-S. Young [1993] on the Hénon maps; Benedicks and Carleson [1991] and Benedicks and L.-S. Young [1993] can be seen as an extension of Jakobson's theorem Jakobson [1981] in 1D. Theorem 7 extends the core ideas in Benedicks and Carleson [1991] and Benedicks and L.-S. Young [1993], permitting the attractor to be embedded in a phase space of arbitrary dimension and replacing the

formula-based arguments in Benedicks and Carleson [1991] by geometric conditions that imply the existence of chaotic rank-one attractors with SRB measures.

The proof of Theorem 7 is technically quite involved, as it required a delicate parameter selection. This proof together with the study of dynamical properties in Wang and L.-S. Young [2013] for the "good maps" $F_{a,\varepsilon}$ in Theorem 7 occupy well over 150 pages. To avoid having to repeat a parallel analysis every time a similar situation is encountered, we have formulated Theorem 7 in such a way that the conclusion of SRB measure and $\lambda_{\max} > 0$ holds once certain conditions are met. The results in Sect. 3.2 were proved by checking these conditions. Another application was to slow-fast systems Guckenheimer, Wechselberger, and L.-S. Young [2006].

To summarize, Theorem 7 provides a general framework for producing rank-one attractors with SRB measures and $\lambda_{\max} > 0$, and all currently known examples of nonuniformly hyperbolic attractors with observable chaos belong in this class.

4 Random dynamical systems

Most realistic systems are governed by laws that are neither purely deterministic nor purely stochastic but a combination of the two. Noise terms are routinely added to differential equations to model uncontrolled fluctuations or forces not accounted for. Now it is known that solutions of stochastic differential equations have representations as *stochastic flows of diffeomorphisms*, i.e., for each ω corresponding to a realization of Brownian path, there is a 1-parameter family of diffeomorphisms $x \mapsto \varphi_t(x;\omega)$ satisfying $\varphi_{s+t}(x;\omega) = \varphi_t(\varphi_s(x;\omega);\sigma_s(\omega))$ where σ_s is time-shift along the path. See e.g. Kunita [1990]. Thus systems modeled by SDEs can be seen as *i.i.d.* sequences of *random maps*, and as such, they have been studied a fair amount.

In Sect. 4.1, I will discuss two sets of results, both illustrating the fact that the averaging effect of randomness makes deterministic systems nicer and more tractable. An application of random dynamical systems is discussed in Sect. 4.2.

4.1 Extensions of deterministic theory to random maps. Consider first a *random maps system* $\mathcal{X}$ defined as follows. Fix a probability ν on Ω, the space of diffeomorphisms of a compact manifold M. We consider the composition

$$\cdots \circ f_n \circ \cdots \circ f_2 \circ f_1 \,, \quad n = 1, 2, \ldots \,,$$

where $f_1, f_2, \ldots$ are chosen independently with law ν. Such a sequence defines a Markov chain on M, its transition probabilities being given by $P(A|x) = \nu\{f \in \Omega : f(x) \in A\}$ for Borel subsets $A \subset M$. A probability measure μ on M is called *stationary* for $\mathcal{X}$ if $\mu(A) = \int P(A|x)d\mu(x)$. Given a stationary measure μ, under the usual integrability conditions, Lyapunov exponents $\{\lambda_i\}$ are defined μ-a.e. for $\nu^{\mathbb{N}}$-almost every sequence $\underline{\mathbf{f}}^+ = (f_i)_{i=1}^{\infty}$ and are nonrandom. The (pathwise) entropy h_μ for $\mathcal{X}$ is as defined in the case of nonrandom maps, replacing the iteration of a single map by compositions of $(f_i)_{i=1}^{\infty}$; it is also nonrandom. See Kifer [1986] for more information.

While μ is invariant under the Markov chain, i.e., when averaged over all random maps, there is the following pathwise notion of invariant measure: First we extend the

sequence of maps $\underline{\mathbf{f}}^+ = (f_i)_{i=1}^\infty$ to a bi-infinite sequence $\underline{\mathbf{f}} = (f_i)_{i=-\infty}^\infty$, chosen *i.i.d.* Viewing the system as having started from time $-\infty$, one obtains *sample measures* $\{\mu_{\underline{\mathbf{f}}}^-\}$ defined for a.e. $\underline{\mathbf{f}} = (f_i)_{i=-\infty}^\infty$ by conditioning μ on "the past". That is to say, $\mu_{\underline{\mathbf{f}}}^-$ describes the distribution at time 0 given that the maps $(f_i)_{i<0}$ have been applied. Equivalently,

$$\mu_{\underline{\mathbf{f}}}^- = \lim_{n\to\infty} (f_{-1} \circ f_{-2} \circ \cdots \circ f_{-n})_* \mu \ ;$$

the limit exists by martingale convergence. It is easy to see that $\mu_{\underline{\mathbf{f}}}^-$ depends only on $(f_i)_{i<0}$, that $\int \mu_{\underline{\mathbf{f}}}^- d\nu^{\mathbb{Z}}(\underline{\mathbf{f}}) = \mu$, and that $\mu_{\underline{\mathbf{f}}}^-$ is invariant in the sense that $(f_0)_* \mu_{\underline{\mathbf{f}}}^- = \mu_{\sigma\underline{\mathbf{f}}}^-$ where $\sigma\underline{\mathbf{f}}$ is the shifted sequence. See e.g. Kifer [1986] and Ledrappier and L.-S. Young [1988b].

Now for a bi-infinite sequence $\underline{\mathbf{f}} = (f_i)_{i=-\infty}^\infty$, unstable manifolds at time 0 depend also only on the past (while stable manifolds depend on the future). It therefore makes sense, given a random maps system $\mathcal{X}$, to define $\mu_{\underline{\mathbf{f}}}^-$ to be a *random SRB measure* if $\lambda_{\max} > 0$ $\mu_{\underline{\mathbf{f}}}^-$-a.e. and the conditional measures of $\mu_{\underline{\mathbf{f}}}^-$ on unstable manifolds have densities, following the definition in the deterministic case. The next result is roughly parallel to Theorem 2, to which we refer the reader for notation.

Theorem 8. *Let $\mathcal{X}$ be as above with an ergodic stationary measure μ. We assume that*

$$\int \log^+ \|f\|_{C^2} \nu(df), \quad \int \log^+ \|f^{-1}\|_{C^2} \nu(df) < \infty .$$

(1a) Le Jan [1987] *If $\lambda_{\max} < 0$, then $\mu_{\underline{\mathbf{f}}}^-$ is supported on a finite set of points for $\nu^{\mathbb{Z}}$-a.e. $\underline{\mathbf{f}}$.*

(1b) Ledrappier and L.-S. Young [1988b] *If μ has a density and $\lambda_{\max} > 0$, then*

$$h_\mu = \sum \lambda_i^+ m_i$$

and $\mu_{\underline{\mathbf{f}}}^-$ is a random SRB measure for $\nu^{\mathbb{Z}}$-a.e. $\underline{\mathbf{f}}$.

(2) Ledrappier and L.-S. Young [1988a] *Assume in addition to $\lambda_{\max} > 0$ that the backward derivative process associated with the Markov chain on the Grassmannian bundle of M has absolutely continuous transition probability kernels (see* Ledrappier and L.-S. Young [ibid.]*). Then there is an i_0 such that the partial dimensions δ_i (see Sect. 1.2) satisfy*

$$\delta_i = 1 \ \ \textit{for all } i < i_0, \qquad \delta_i = 0 \ \ \textit{for all } i > i_0 .$$

Moreover, $\dim(\mu_{\underline{\mathbf{f}}}^-) = \sum_i \delta_i m_i$.

The condition that μ has a density is very natural for random maps; a sufficient (but not necessary) condition is that the transition probabilities $P(\cdot|x)$ have densities. Item (1) in Theorem 8 says that except when $\lambda_{\max} = 0$, $\mathcal{X}$ either has *random sinks*, i.e., almost all solutions coalesce in time into at most a finite number of (evolving) trajectories, or they have *random attractors* with *random SRB measures*, i.e., attracting sets having the geometric characteristics of attractors with SRB measures in the deterministic case

– except that these attractors too evolve with time. Item (2) says that when randomly perturbed, sample measures align with the most expanding directions. With the configuration of δ_i in Theorem 8, the quantity $\sum_i \delta_i m_i$ is in fact another way to write the Kaplan–Yorke dimension Frederickson, Kaplan, E. D. Yorke, and J. A. Yorke [1983]. Thus Theorem 8(2) proves that the Kaplan–Yorke conjecture holds for random maps.

Positivity of Lyapunov exponents via random perturbations. For deterministic maps, we discussed at the end of Sect. 3.1 the challenges in proving the existence of SRB measures. Similar challenges exist for proving $\lambda_{\max} > 0$ for volume-preserving maps. The *standard maps* family, a 1-parameter family of area-preserving maps $f_L : \mathbb{T}^2 \circlearrowleft$ with the property that $\|Df_L\| \sim L$ for $L > 1$, symbolizes the enormity of the challenge: In spite of considerable effort by leading researchers, no one has been able to prove – or disprove – the positivity of Lyapunov exponents for f_L for any L, however large.

The following result illustrates what the addition of random noise can do. Two points are of note: One is that the amount of noise needed is tiny; the other is that unlike Theorem 7, no parameter selection is needed.

Theorem 9. Blumenthal, Xue, and L.-S. Young [2017] *Let $f_{L,a} : \mathbb{T}^2 \circlearrowleft$ be given by*

$$f_{L,a}(x, y) = (L \sin(2\pi x) + a - y,\ x)\,.$$

To $f_{L,a}$, we add a random perturbation of the form $(x, y) \mapsto (x + \xi, y)$, $\xi \in [-\varepsilon, \varepsilon]$ uniformly distributed. Then given any $\alpha, \beta > 0$, there exists $C > 0$ such that for all L sufficiently large, for all a, and for all $\varepsilon \geq L^{-CL^{1-\beta}}$, we have

$$\lambda_{\max} > (1 - \alpha) \log L \qquad \textit{Lebesgue-a.e. on } \mathbb{T}^2\,.$$

4.2 Interpretation as reliability of driven systems. Consider a continuous-time dynamical system defined on a manifold M. A signal $I(t) \in \mathbb{R}^n, t \in [0, \infty)$, is presented to the system at time 0. Think of it as an external input being switched on in an engineered system, or the onset of a stimulus in a biological system. The response of the system at time $t > 0$ is given by $F(t) = F(x_0, \{I(s)\}_{0 \leq s < t}; t)$, where $x_0 \in M$ is the internal state of the system at time 0. A system is called *reliable* with respect to a class of signals $\mathcal{I}$ if for almost all $I \in \mathcal{I}$, the dependence of $F(t)$ on x_0 vanishes with time. Initially, some dependence of $F(t)$ on x_0 is unavoidable. The idea is that a reliable system will, after a transient, entrain to the signal $I(t)$ and lose memory of its own initial state; see e.g. Lin, Shea-Brown, and L.-S. Young [2009].

In the simple setting where $I(t)$ are (frozen) realizations of white noise, this setup can be described by a stochastic differential equation. For a typical sample Brownian path ω and $t_1 < t_2$, flow-maps $F_{t_1,t_2,\omega}$ from time t_1 to t_2 are well defined under mild regularity assumptions on the coefficients of the SDE Kunita [1990]. This puts us in the setting of random maps. Let μ be the invariant probability measure, which we assume to be unique. As the sample measures μ_ω^- are given by $\mu_\omega^- = \lim_{t\to\infty}(F_{-t,0,\omega})_*\mu$ (see Sect. 4.1), we may assume that for large enough t, the observed distribution of x_t starting from μ at time 0 is approximately $\mu_{\sigma_t(\omega)}^-$ where σ_t is the time shift.

The interpretation therefore is as follows: In the case $\lambda_{\max} < 0$, μ_ω^- is a finite set for a.e. ω by Theorem 8(1)(a). Under suitable conditions, it consists of a single point.

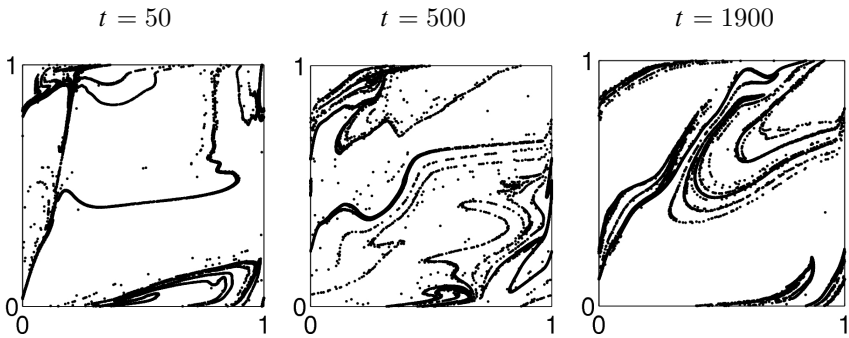

Figure 3: **Random attractors.** Snapshopts of distributions of x_t starting from μ for two coupled phase oscillators driven by a white-noise stimulus. These distributions approximate the sample measures $\mu^-_{\sigma_t(\omega)}$. The phase space is the 2-torus, and the system is unreliable. The curves seen are unstable manifolds of random strange attractors on which SRB measures are supported. The attractors evolve perpetually with time, retaining certain basic characteristics throughout.

That is to say, the approximate location of x_t is largely independent of x_0 for large t, the definition of a reliable system. If $\lambda_{\max} > 0$, then by Theorem 8(1)(b), almost surely μ^-_ω has the characteristic geometry of a random SRB measure. That means x_t may be very different depending on x_0, and this dependence on x_0 will persist for all $t > 0$. The system is unreliable, and the effective dimension of the random SRB measure given by Theorem 8(2) describes the extent of its unreliability.

5 Applications to biology

In the last 5-10 years, I have taken an interest in the application of Dynamical Systems ideas to the biological sciences. I would like to report here on work in two different directions: Sect. 5.1 discusses a study on an idealized model of epidemics control, while Sect. 5.2 contains a glimpse into some work in computational neuroscience.

5.1 Control of epidemics via isolation of infected hosts. This work is on a simple model of infectious diseases. When an outbreak is unforeseen, the only available means of containment is the isolation of infected individuals. Isolation is very effective when implemented immediately and in full; one simply cuts off all contact between infected hosts and the rest of the population. But such perfect implementation is not feasible in reality: facilities to house the infected, and medical personnel to care for them have to be available at a moment's notice, and infected hosts have to be identified as soon as they become infectious.

Below I describe the results of a theoretical study L.-S. Young, Ruschel, Yanchuk, and Pereira [2017] the goals of which are to quantify minimum response capabilities needed to squash incipient outbreaks, and to predict the consequences when containment

fails. It is an idealized model built on the well known susceptible - infected - susceptible (SIS) model of epidemics.

Consider, to begin with, a random network of N nodes; each node represents a host, and nodes that are linked by edges are neighbors. The mean degree at a node is given by $\langle k \rangle$, so the density of connections is denoted by $m = \langle k \rangle / N$. In the absence of any control mechanism, the situation is as follows. Each host is in one of three discrete states: healthy and susceptible (S), infectious (I) and incubating (E). Infectious hosts infect their neighbors at rate β; infected nodes incubate the disease for a period of σ units of time during which they are assumed to be neither symptomatic nor infectious. At the end of the incubation period they become infectious. Infectious nodes remain in that state until they recover and rejoin the susceptible group. The rate of recovery is γ. This, roughly speaking, is the SIS model.

To the setting above, we introduce the following isolation protocol: If a host remains infectious for τ units of time without having recovered, it enters a new state, Q (for *quarantine*), with probability p. The hosts that do not enter state Q at time τ remain infectious until they recover on their own. A host that enters state Q remains in this state for κ units of time, at the end of which it is discharged and rejoins the healthy and susceptible pool.

The quantities of interest in this model are $S(t)$, $E(t)$, $I(t)$, and $Q(t)$, representing the fractions of the population in the *susceptible*, *incubating*, *infectious* and *quarantine* states respectively at time t starting from some initial condition, and the new parameters of our model are $\tau, \kappa > 0$ and $p \in [0, 1]$: p is the probability that an infected individual will be isolated, τ is the time between becoming infectious and entering isolation, and κ is the duration of isolation.[3]

Assuming that links between the infectious and susceptible nodes are uncorrelated, we obtain, by moment closure, the following system of Delay Differential Equations in the continuum limit as $N \to \infty$:

$$
\begin{aligned}
\dot{S}(t) &= -\beta m S(t) I(t) + \gamma I(t) + \beta m \varepsilon S(t-\sigma-\tau-\kappa) I(t-\sigma-\tau-\kappa), \\
\dot{E}(t) &= \beta m [S(t) I(t) - S(t-\sigma) I(t-\sigma)] \\
\dot{I}(t) &= \beta m S(t-\sigma)\, I(t-\sigma) - \gamma I(t) - \beta m \varepsilon S(t-\sigma-\tau)\, I(t-\sigma-\tau)\,, \\
\dot{Q}(t) &= \beta m \varepsilon [S(t-\sigma-\tau) I(t-\sigma-\tau) - S(t-\sigma-\tau-\kappa) I(t-\sigma-\tau-\kappa)]\,,
\end{aligned}
$$

where β and m are as above and $\varepsilon = p e^{-\gamma\tau}$.

Let $C := C\left([-\sigma-\tau-\kappa, 0], \mathbb{R}^4\right)$, the Banach space of continuous functions. Given an initial function $\phi \in C$, the solution $x(t, \phi) \in \mathbb{R}^4$, $t \geq 0$, to the initial value problem exists and is unique. Standard results imply, in fact, that the system above defines a C^1 semi-flow on C with the sup norm. Observe that by the conservation of mass property, if $\phi = (\phi_S, \phi_E, \phi_I, \phi_Q)$ and $x(t; \phi) = (S(t), E(t), I(t), Q(t))$, then $S(t) + E(t) + I(t) + Q(t) = \phi_S(0) + \phi_E(0) + \phi_I(0) + \phi_Q(0)$ for all $t \geq 0$. Therefore, the 3-D hyperplane $H^3 := \{S(t) + E(t) + I(t) + Q(t) = 1\} \subset \mathbb{R}^4$ is left invariant by

[3] We have not built into the model the idea of *immunity*, which is of course very relevant in the long run, but less so for shorter time scales, such as the time evolution following a single outbreak.

the semi-flow. To obtain biologically relevant solutions we further restrict to the subset of H^3 on which $S(t), E(t), I(t), Q(t) \geq 0$.

In the SIS model, what determines whether the disease will propagate is the *disease reproductive number* $r := \beta m/\gamma$: that is, without an isolation protocol, a small outbreak is contained if $r < 1$, and it spreads if $r > 1$.

Theorem 10. L.-S. Young, Ruschel, Yanchuk, and Pereira [2017] *(1) Given p, τ and κ, the* effective disease reproductive number

$$r_e = r(1-\varepsilon) = r(1 - pe^{-\gamma\tau}) \,.$$

Starting from an initial condition $\phi = (\phi_S, \phi_E, \phi_I, \phi_Q)$ with $\phi_I > 0$ near the disease-free equilibrium $(1, 0, 0, 0)$, the following response is needed to ensure $r_e < 1$:

(a) there is a minimum isolation probability $p_c = 1 - \frac{1}{r}$ *so that to contain the outbreak, one must have $p > p_c$; and*

(b) for $p > p_c$, there is a critical identification time

$$\tau_c(p) = \frac{1}{\gamma} \log \frac{p}{p_c} \,,$$

so that for $\tau < \tau_c(p)$ the disease dies out and for $\tau > \tau_c(p)$ it spreads.

(2) Let p, τ, κ and ϕ be as above. If $r_e > 1$, then there is at most one possible endemic equilibrium, given by constant functions $(S_{\text{eq}}, E_{\text{eq}}, I_{\text{eq}}, Q_{\text{eq}})$ with

$$S_{\text{eq}} = \frac{1}{r_e} = \frac{1}{r(1-\varepsilon)} \qquad \textit{and} \qquad I_{\text{eq}} = \frac{(1-\varepsilon)}{\sigma\gamma + \kappa\gamma\varepsilon + (1-\varepsilon)} \cdot (1 - S_{\text{eq}}).$$

We conjecture that when $r_e > 1$, all solutions starting from ϕ in the theorem converge to the predicted endemic equilibrium. A proof is out of reach for now, but numerical evidence is strongly in favor of the conjecture for reasonable parameters.

5.2 Dynamics of the brain. First, why the brain? *The brain is a dynamical system*, a large and complex dynamical system, one of the most fascinating naturally occurring systems I have ever seen. It is made up of a large number of smaller subsystems, namely neurons, coupled together in a hierarchical network. Single neurons are themselves complicated biological entities. While one must eventually combine information across scales, it is simpler, to begin with, to first focus on either network level activity or cellular/subcellular properties of individual neurons. This section is about the former, that is to say, I will suppress detailed properties of individual neurons and focus on their dynamical interactions.

A crash course on neuronal dynamics. Neurons communicate with one another by electrical signals which they produce when they spike. When an Excitatory (E) neuron spikes, it brings the recipients of its signal, called its postsynaptic neurons, a little

closer to their own spiking thresholds. Inhibitory (I) neurons do the opposite to their postsynaptic neurons.

Neglecting biochemical processes and keeping track of electrical properties only, the dynamics of a neuron are governed by its membrane potential and are described by the standard Leaky Integrate-and-Fire (LIF) equation

$$\frac{dv}{dt} = -g_R v - g_E(t)(v - V_E) - g_I(t)(v - V_I)\,. \tag{11}$$

Here v is membrane potential in normalized units: Without external input, v tends to 0 at rate g_R (a constant). The Excitatory conductance $g_E(t)$ of a neuron is elevated for a brief period of time ($10 - 20$ milliseconds) when the neuron receives excitatory input, such as a spike from a presynaptic E-neuron. The term $g_E(t)(v - V_E)$ is called the Excitatory current; it drives the membrane potential towards the value V_E ($= 14/3$ in the present normalization), called the excitatory reversal potential. But $v(t)$ never gets that high: by the time it reaches 1, the neuron spikes and v is reset to 0. The Inhibitory conductance $g_I(t)$ is elevated similarly when the neuron receives inhibitory input as a result of the spiking of a presynaptic I-neuron; this current drives the membrane potential towards $V_I = -2/3$.

To summarize, the membrane potential of a neuron swings up and down depending on the input it receives. When it reaches its spiking threshold 1, the neuron spikes, sending signals to other neurons thereby affecting their time evolutions, and its own membrane potential is reset to 0.

Network architecture, or wiring, is of course important. The primate cerebral cortex (which is what I know best) is organized into regions and subregions and layers, with neurons in local populations having similar preferences and functional roles. Within local circuitries, E- and I-neurons are relatively homogeneously though sparsely connected, whereas connections between distinct regions or layers are specific and tend to be Excitatory only.

Computational modeling of the visual cortex. Turning now to my own modeling work in Neuroscience, I have worked mostly with the visual cortex, the part of the brain responsible for the processing of visual information. The modeling I do is heavily data-driven, benchmarked by dozens if not hundreds of sets of experimental data from monkey, whose visual cortex is quite similar to our own. From point-to-point representation of visual images in the retina, cortex extracts, through multiple stages of processing, information such as edges, shapes, color, movement, and texture — features that enable the brain to make sense of visual scenes. This complex task is accomplished through the dynamical interaction of neurons. My immediate goal in this research is to unravel the dynamical processes responsible for feature extraction. My larger goals are to connect dynamical events on the neuronal level to cortical functions and ultimately to human perception and behavior.

Working with a small team of neuroscientists and postdocs, I have been involved in building a comprehensive model of the primary visual cortex, or V1. This is the area of cortex closest to sensory input. It is also the largest and most complex of all the visual cortical areas. We have worked mostly on the input layer of the magnocellular pathway,

one of the two main pathways in V1. *Our challenge is to deduce underlying dynamical mechanisms from experimental data documenting how V1 responded to various stimuli,* i.e., to reverse-engineer the dynamical system from its outputs in response to stimuli.

This is less straightforward than one might think, because the brain does not simply reproduce a copy, pixel by pixel, of the stimulus presented. It extracts data, dissects and recombines, suppresses and enhances; it *processes* the information. Below I will illustrate this point with an example; the material is taken in part from Chariker, Shapley, and L.-S. Young [2016].

To get a sense of size, think of the moon as projecting to a region of diameter $\sim 1/2$ degree in our retina, near the center of one's visual field. A somewhat surprising fact is that to cover a region that projects to $1/4$ deg $\times 1/4$ deg of the retina, there are, in the magno-pathway, only about 10 cells in the Lateral Geniculate Nucleus (LGN), the body that relays information from the retina to V1. Of these 10 cells in the LGN, about 5 are ON, and 5 are OFF. The ON-cells spike vigorously when luminance in its visual field goes from dark to light, and the OFF cells do the same when it goes from light to dark. All that is simple enough, an array of cells reporting the changes in luminance in their visual fields.

Now LGN neurons project directly to V1, and one of the most salient features of V1 is that its cells are *orientation selective*: most cells have a preference for edges of a particular orientation, and when such an edge passes through its receptive field, the cell gets excited and spikes. Orientation selectivity (OS) is a very important property; it helps us detect contours in the visual scene. The question is, how do V1 neurons acquire their OS, since LGN neurons have no such selectivity?

Half a century ago, Hubel and Wiesel, who eventually won the Nobel Prize, proposed an explanation for OS in terms of the alignment of LGN cells that converge on V1 cells Hubel and Wiesel [1962]. Based on currently available data, our model confirmed that this idea is very likely the source of OS, but contrary to conventional wisdom, we found that the signal from LGN is weak — an idea that is only slowly gaining traction in the neuroscience community. In our model, the feedforward signal is substantially amplified (and mollified) through dynamical interactions among cortical neurons, which after all provide the bulk of the current received by neurons in V1. See Fig 4.

The modeling work above is computation based. Realistic models as complex as ours are not amenable to rigorous analysis at the present time, yet much of our model building has been guided by Dynamical Systems ideas, the only difference being that simulations played the role of proofs in the validation of conjectures.

6 Looking forward

Dynamical Systems was born a little over 100 years ago, inspired by curiosity about natural phenomena. The field has blossomed and matured; we have developed a rich collection of ideas and techniques that we can proudly call our own. Advances within the field have suggested new avenues of research and new questions, but it is also important to reconnect with original goals, and to build new connections.

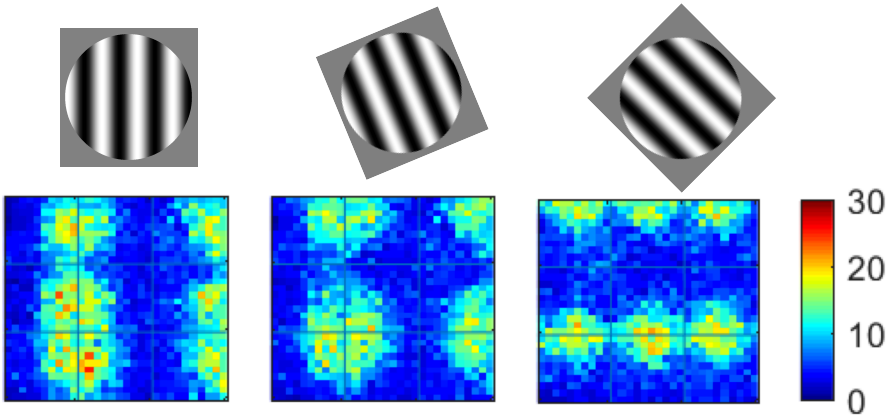

Figure 4: **Model response to drifting gratings.** In the top row are three drifting gratings oriented at $0°$, $22.5°$ and $45°$ from the vertical; these stimuli are used a great deal in experiments in neuroscience, as they elicit a strong response from V1 neurons having a preference for orientations aligned with the gratings. The bottom three panels show a small piece of our model V1 cortex (layer 4Cα), about 1.5×1.5 mm^2 in actual size. Each tiny box corresponds to ~ 30 neurons, its color indicating the number of spikes fired per neuron over a 1 sec interval; see color bar. Visible in each panel is the group of neurons preferring the orientation of the grating above the panel. **Important:** The only input to the model are light intensity maps, i.e., functions $g(x,t)$ where x is location in visual space, t is time, and $g(x,t)$ is the luminance of the grating at location x at time t. LGN relays this information to cortex via the spiking of ON and OFF cells as explained in the text, and our model cortical neurons, the evolutions of which are governed by (11), interact dynamically to produce the response shown.

Building connections will require developing new techniques and adopting new viewpoints. Let me give just one example, drawing from my own limited experience: Most naturally occurring dynamical systems are *large* – large in the sense of extended phase spaces, many degrees of freedom, a more complex dynamical landscape. The concerns are very different when studying "small" and "large" dynamical systems. For small systems, complexity means chaotic behavior, positive Lyapunov exponents, positive entropy. In "large" systems, it is not clear what chaotic behavior means. The focus is more naturally on *emergent* phenomena, behaviors that cannot be predicted from local rules and that emerge as a result of interaction among components. To gain insight, one looks for ways to reduce system complexity by identifying parameters, or modes of behavior, that are more important than others.

Since Dynamical Systems is naturally connected to many parts of mathematics and many scientific disciplines, I see a future full of possibilities, and I encourage my fellow dynamicists to embrace the challenge. Not only is Dynamical Systems a study of moving objects and evolving situations, the subject itself is also evolving. It *must* evolve, if it is to remain interesting, vibrant and relevant.

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STRENGTHENING MATHEMATICS IN THE DEVELOPING WORLD

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Abstract

The panel and poster session entitled "Strengthening Mathematics in the Developing World" at the 2018 International Congress of Mathematicians (ICM) in Rio de Janeiro, Brazil was organized by the Commission for Developing Countries (CDC) of the International Mathematical Union (IMU). The objective was to share information about mathematical development activities with mathematicians at the ICM and to serve as a catalyst for interactions between mathematicians, organizations and funding agencies. The panel had representatives from seven organizations including international mathematical unions, commissions supporting the developing world, supporting women, a funding agency and the mathematical society of the host nation to the ICM (Brazil). There were ten other organizations supporting the developing world represented in the poster session, several of which work closely with the organizations in the panel. The panel and poster session had an attendance of approximately 400 and led to a large number of interactions between the representatives of the organizations and the audience.

1 Introduction

The panel and poster session on strengthening mathematics in the developing world addressed the need for more understanding of the situation of mathematics in developing countries and of the activities currently being carried out to support development. The developing world has a huge amount of mathematical talent that is currently not being supported. This results in both unsolved problems for mathematics as a whole and unfulfilled potential for individuals. The organizations represented in the panel and poster session are addressing the challenges from different perspectives and within varied cultural and socioeconomic realities. The common thread in the efforts is a love of mathematics and creativity in addressing the relevant issues which range from encouraging elementary school students to participate in mathematical olympiads to supporting high level research.

The panel and poster session was organized by Mama Foupouagnigni, Angel R. Pineda, and Polly W. Sy who comprised a committee of the CDC created for that task

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in collaboration with Lena Koch, the IMU administrator for the CDC. The philosophy of the committee was to identify organizations that are conducting work which would be of interest to the participants of the ICM and which would be able to give a global perspective on the issues arising in mathematical development. During the panel, there was an overview of activities from most regions around the world which was made in 10 minute presentations by each of the panelists with the discussion occurring in the poster session following the panel.

Figure 1: (A) Panelists and moderator (from left to right): Alejandro Jofré, Nouzha El Yacoubi, Jose Maria P. Balmaceda, Paolo Piccione,Yuri Tschinkel, Marie-Françoise Roy, Wandera Ogana, and Angel R. Pineda, (B) Presenters and participants during the poster session.

The panelists (Figure 1A) were representing the following organizations: IMU Commission for Developing Countries President (Wandera Ogana), IMU Committee for Women in Mathematics Chair (Marie-Françoise Roy), Simons Foundation, Mathematical and Physical Sciences Director (Yuri Tschinkel), Brazilian Mathematical Society President (Paolo Piccione), Southeast Asian Mathematical Society President (Jose Maria Balmaceda), African Mathematical Society President (Nouzha El Yacoubi) and Mathematical Union of Latina American and the Caribbean Secretary (Alejandro Jofré).

The organizations participating in the poster session (Figure 1B) along with those in the panel were:

- African Institute for Mathematical Sciences (AIMS)
- African Mathematics Millennium Science Initiative (AMMSI)
- Centre International de Mathématiques Pures et Appliquées (CIMPA)
- European Mathematical Society Committee for Developing Countries (EMS-CDC)
- Heidelberg Laureate Forum
- International Centre for Theoretical Physics (ICTP)
- International Commission on Mathematical Instruction (ICMI)
- International Science Programme (ISP), Uppsala University

- London Mathematical Society (LMS)
- Research in Germany

More information about the panel and poster session including links to all of the organizations which participated, the slides used in the panel and photos can be found in the IMU CDC website.

In the following sections, we summarize the presentations of the panelists.

2 IMU Commission for Developing Countries: Wandera Ogana

The mandate of the Commission for Developing Countries (CDC) is "To manage all initiatives of the IMU in support of mathematics in developing and economically disadvantaged countries." In that role, the CDC manages a wide range of programs supporting mathematicians around the world. Those programs can be classified into several categories:

- **Conference Support Program:** Gives partial support to conferences organized in developing and economically disadvantaged countries and supports a few major international conferences occurring in developed countries. The maximum amount is 3,500 euros.

- **Project Support Program:** Supports capacity building projects and activities in mathematics and mathematics education, be they international, regional or local initiatives in developing countries.

- **Lecturing and Mentoring:**

  The **Volunteer Lecturer Program (VLP)** offers universities in the developing world lecturers for intensive 3-4 week courses in mathematics at the advanced undergraduate or master's level. Funding for the volunteer is provided by CDC. The program regularly receives support from the American Mathematical Society (AMS) and the Niels Henrik Abel Board (NHAB). During 2014-2017 a total of 34 courses in 16 countries were supported.

  The **African Diaspora Mathematicians Program (ADMP)** offers partnerships between African diaspora mathematicians and mathematics academic units in Africa. Initially, the program is run as a pilot program in Africa for two years (May 2017 - June 2019) and involves a total of three partnerships (Ethiopia, Cameroon, Zimbabwe). The plan is to extend it to other regions of developing countries, depending on success of the pilot program.

- **Individual Research Travel Grants:**

  The **Abel Visiting Scholar Program** is designed to offer the opportunity for a research sabbatical. All travel and living expenses of the grantees will be covered by the fellowship up to 5,000 USD. It supports applicants under 40 years of age at the day of the application deadline.

The **IMU – Simons African Fellowship Program** supports research sabbaticals for mathematicians from African developing countries employed in Africa to travel to an internationally known mathematical centre of excellence (worldwide) for collaborative research. All travel and living expenses of the grantees will be covered by the fellowship up to 5,000 USD.

The **Individual Research Travel Support Program** supports travel costs for research visits (minimum stay is four weeks) by mathematicians based in developing and economically disadvantaged countries. The host institution must cover local living expenses like accommodation and boarding.

- **Graduate Support Programs:**

The **IMU Breakout Graduate Fellowship Program** is funded by the donations of the winners of the Breakthrough Prizes in Mathematics and is given to excellent students from developing countries. In 2016, three students (from Botswana, Colombia and Vietnam) were awarded fellowships.

The **Graduate Research Assistantships in Developing Countries Program (GRAID)** is for research assistantships to graduate (MSc or PhD) students of emerging research groups working in a developing country listed in Priority 1 or 2 of the IMU CDC definition of developing countries. The GRAID Program is funded by voluntary donations from mathematicians or mathematical institutions worldwide. In 2017, two research groups were selected for support, one in Cameroon and the other in Morocco.

- **Other Activities and Programs:**

The **Library Assistance Scheme** offers limited financial support for shipment costs to individual scientists or institutions wishing to donate books in the mathematical sciences to libraries in developing countries.

Special Activities during ICMs include MENAO (2014) and this CDC Panel and Poster Session in 2018. The IMU CDC organized a day-long symposium prior to the opening of the ICM 2014 in Seoul, Korea entitled Mathematics in Emerging Nations: Achievements and Opportunities (MENAO). At the ICM 2014, the CDC also organized three networking sessions for the recipients of travel grants from developing countries (NANUM grants).

Reports on mathematics research and graduate education in emerging nations: CDC members and regional partners prepared in 2013 and 2014 three reports about the current state of mathematics in Africa, Asia and Latin America and on opportunities for new initiatives to support mathematical development.

CDC's principal source of income is an annual grant from the International Mathematical Union (IMU). IMU in turn receives its financial support from IMU member countries. Additionally CDC has received financial support from a number of organizations and individual mathematicians. Furthermore, CDC has many volunteers who dedicate their time to support CDC activities

The Commission for Developing Countries would like to express its profound thanks to all institutions and individuals supporting CDC and its activities. CDC is looking forward to future collaboration with all its supporters, cooperation partners and volunteers.

More information about CDC and its activities can be found on its website.

As part of his presentation, Wandera Ogana was asked to summarize some of the activities from the International Commission on Mathematical Instruction (ICMI) and the Committee on Electronic Information and Communication (CEIC).

ICMI was founded in 1908 and has been a commission of IMU since 1952. The purposes are:

- To promote international programs of activities and publications that improve the collaboration, exchange and dissemination of ideas and information on all aspects of the theory and practice of contemporary mathematical education.
- To foster efforts to improve the quality of mathematics teaching and learning worldwide.
- To support and assist the International Congress on Mathematical Education (ICME) and meetings or conferences of ICMI affiliated organizations.

Some activities of ICMI are:

- ICME: The major responsibility of ICMI is to plan and organize the quadrennial International Congress on Mathematical Education (ICME). This congress is attended by many participants from developing countries.
- CANP: ICMI organizes the Capacity and Network Project (CANP) whose aim is to provide teacher educators in developing countries with enhanced mathematical expertise.
- The Klein Project: This is an IMU/ICMI project that began in 2008 with the aim of producing mathematics resources for secondary school teachers on contemporary mathematics.
- ICMI Regional Conferences: ICMI lends its name to regional meetings on mathematics education, particularly in developing countries.

More information about ICMI and its activities can be found on its website.

CEIC is a standing committee of the IMU Executive Committee (EC) formed in 1998 to advise the EC on trends on the effects of electronic information and communication on society in general and mathematicians in particular. The following is the mandate of CEIC:

- To advise the IMU on aspects of its operations related to information and communication, including technical, legal and financial implications, and keep it informed of new developments.
- To review the development of electronic information, communication, publication, instruction, and archiving so as to keep the IMU abreast of current and emerging issues.

- To advise the IMU about potential opportunities to endorse standards and articulate best practice recommendations on issues related to the field, as well as potential opportunities to foster the growth and development of electronic infrastructure.

Some activities of CEIC are:

- Publications: Over the years, CEIC has produced a number of publications which address the mandate given by EC.
- Some activities at the ICM:

  ICM2010: Round table on "The Use of Metrics in Evaluating Research"

  ICM2014: Organized 3 panels: Massive Open Online Courses (or MOOCs), Electronic Publishing and Open Access, World Digital Mathematical Library

  ICM2018: Organized 2 panels: Machine-Verified Proof: State of the Art, International Mathematical Knowledge Trust (IMKT): an Update on the Global Digital Mathematics Library

More information about ICMI and its activities can be found on its website.

3 IMU Committee for Women in Mathematics: Marie-Françoise Roy

The IMU Committee for Women in Mathematics (CWM) is the only worldwide structure for women in mathematics.

Its mandate is to:

- To promote international contacts between national and regional organizations for women in mathematical sciences
- To maintain the Women in Mathematics pages on the IMU website
- To organize an electronic community of women mathematicians
- To work with groups, committees and commissions of IMU on topics pertaining to women mathematicians and their representation
- To publicize and suggest, working practices that ensure equal opportunities for women mathematician
- To report annually to the EC and propose actions

Some activities of CWM are:

- **CWM website**: Unique and important function as the only platform coordinating such diverse worldwide activity. Initiated by Ingrid Daubechies and launched before CWM at the Seoul ICM. Includes items (events, new women in maths organizations, newsworthy items, resources, etc.) added every week. Thirty six countries are listed with some form of organization.

- **CWM ambassadors**: A network of 120 women (and supporters) in 76 countries disseminate information from CWM and circulate information about initiatives.

- **Gender gap in science project**: Major project with a budget of 100K euros per year for 2017-19 and 11 partners[1]. IMU is the lead union. The project is funded by the International Council of Science (ICS). This project aims to produce sound data to support the choices of interventions that ICS and its member unions can feasibly undertake to reduce the gender gap. The project has a focus on developing countries. The survey of scientists is open until October 31, 2018.

- **World Meeting for Women in Mathematics, (WM)^2**: Satellite event of ICM 2018 on July 31 2018 with over 350 participants and over 1/3 supported by Open Arms. The meeting had a Latin American focus, included a memorial to Maryam Mirzakhani and the world premiere of the first part of the film Journeys of Women in Mathematics.

Some activities funded by CWM are:

- **Networks**: Annual call for supporting networks of female mathematicians on a regional basis in developing or emerging countries. In 2016-2018 : over 150 applications, 29 funded, amounts up to 3 K euro each. In 2017, networks in Brazil, Canada (Mathematical Congress of the Americas), Morocco, Chile, India, Nepal, Tunisia, South Africa, Iran, Vietnam, Mexico, and Japan were funded for 1500 participants in total in 2016-2017.

- Short film **Faces of Women in Mathematics** (Eugenie Hunsicker, Chair of the LMS Women in Maths Committee, Irina Linke, filmmaker): Sequence of film clips of women saying: "I am (name) from (country), and I am a mathematician!" in the language of their choice. One hundred and forty six clips featured 243 women mathematicians from 36 different countries speaking in 31 different languages.

- **Journeys of Women in Mathematics** is a film created by CWM, filmed and edited by Micro-Documentaries, made possible by a grant from the Simons Foundation. Neela Nataraj (India) , Aminatou Pecha (Cameroon) and Carolina Araujo (Brazil) featured in their home countries and a group of six women from various Latin American countries interviewed at (WM)^2. Documents both the successes and barriers for women in mathematics, in the words of the women themselves. First version premiered at (WM)^2:

- **Maryam Mirzakhani Memorial**: Created at the initiative of CWM by Thaís Jordão and Rafael Meireles Barroso. The memorial at ICM 2018 contains an exhibition of 15 original posters, with two books containing all the papers of Maryam Mirzakhani as well as a book with papers about her. Book of condolences at the disposal of ICM attendees to write their thoughts and feelings about Maryam Mirzakhani.

More information about IMU CWM and its activities can be found on its website.

[1] IMU, IUPAC,IUPAP, ICIAM, IAU, IUBS, ACM, IUHPST, UNESCO, OWSD, GenderInSite

4 Simons Foundation: Yuri Tschinkel

The Simons Foundation was founded by Jim and Marilyn Simons in 1984. The mission is to advance the frontiers of research in mathematics and the basic sciences. The mission has grown in scope over the years, starting with the Simons Foundation Autism Research Initiative (SFARI) in 2007, Mathematical and Physical Sciences (MPS) in 2009, Life Sciences in 2013, the Simons Center for Data Analysis (SCDA) in 2013, Outreach and Education in 2014 which includes Math for America (since 2004) and Quanta Magazine. In 2015, the Center for Computational Astronomy (CCA) and the Center for Computational Biology (CCB) were added, followed by the Flatiron Institute in 2016 and the Center for Computational Quantum Materials (CCQ) in 2017.

In Mathematical and Physical Sciences, the Simons Foundation has the following funding categories:

- Grants to Individuals
- Simons Investigators (US, Canada, UK, and Ireland)
- Simons Fellows (US and Canada)
- Targeted Grants in MPS (e.g., Simons Observatory)
- Simons Collaborations
- Grants to Institutes
- Conferences
- Infrastructure (Magma, ArXiv)

The Simons Collaborations in MPS is one of the funding areas of particular interest. The goal of this program is to stimulate progress on fundamental scientific questions of major importance in mathematics, theoretical physics and theoretical computer science. There are 12 collaborations for 4 years with a possible 3 year extension. There are about 10 PIs per collaboration at a funding level of 2-2.5 million USD per year per collaboration in the areas of algorithms and geometry, homological mirror symmetry, special holonomy, arithmetic geometry, number theory, and computation and nonlinear waves.

The Grants to Institutes is another funding area of interest to the international community. MPS currently supports 29 institutes in 18 countries, e.g., Simons Institute for the Theory of Computing (Berkeley), MSRI, IHES, KITP Aspen Center for Physics, Oberwolfach, Vietnam Academy of Science and Technology, Mandelstam Institute for Theoretical Physics at the University of the Witwatersrand, TATA (Bangalore), Banach Center of the Polish Academy, ICTP, IMU-CDC and the EMS-CDC.

The Simons Foundation also has a program designed to support mathematics research in Africa, the Africa Mathematics Project. The three current awardees are the Université des Sciences et Techniques de Masuku, Gabon, Botswana International University of Science and Technology and Universite Cheikh Anta Diop de Dakar, Senegal.

As a final note, the Simons Foundation is interested in hearing ideas of innovative projects to support mathematics around the world. Of special interest are projects not currently supported by governments or traditional funding sources.

More information about the Simons Foundation and its activities can be found on its website.

5 Brazilian Mathematical Society: Paolo Piccione

The Brazilian Mathematical Society (SBM) is a non-profit professional association founded in 1969 to promote mathematics in Brazil. It serves the national and international community of mathematicians through publications, meetings, competitions and other programs and activities.

SBM publishes and distributes journals and books in Portuguese and English. It also provides a distribution channel for other organizations such as IMPA (Instituto de Matemática Pura e Aplicada) at a national level. SBM also has an agreement with Springer-Verlag for the publication of the selected research papers of renowned Brazilian mathematicians.

The Brazilian Mathematical Society also runs several initiatives to support and improve mathematical education at all levels. The most important of these is Profmat, a master's degree program for school teachers. In the nationwide network PROFMAT takes 1,800 new students every year, in about 100 campuses, in all Brazilian states. It has 75 associated institutions in 96 cities and over 3,900 graduates.

Meetings are also an important activity of SBM since they serve to increase knowledge through the exchange of ideas. There are periodic national meetings and international meetings including bi-national meetings between Brazil and Italy, France, Spain and Portugal.

Through prizes and awards, SBM encourages research and investigation in the mathematical field recognizing the most brilliant works and efforts. It awards:

- The **SBM Prize** which rewards the best original research paper in mathematics published in the last three years by a young mathematician working in Brazil.
- The **IMPA-SBM Journalism Prize** which stimulates print media publications on mathematics, science and technology.
- The **SBMAC Prize** awards young students' papers on undergraduate research projects.
- The **Gutierrez Prize** which selects the best doctoral thesis in mathematics for the current year.

Outreach activities by SBM promote awareness and appreciation of mathematics as a broad phenomenon and the community interest in mathematics. These include:

- The **Biennium of Mathematics 2017-2018** with included a series of activities aiming at disseminating and popularizing mathematics throughout the whole society: children and their families, students and teachers.

- The **Math's Festival 2017** was a fair-type event offered a wide range of activities, from lectures and exhibits to workshops and games, designed for all audiences. It attracted more than 18,000 visitors in four days.

- The **Brazilian Mathematical Olympiad (OBM)** was founded by the SBM in 1978 and involves about 500,000 students every year, from 6th grade to the undergraduate level.

- The **Brazilian Math Olympiad for Schools (OBMEP)** is a national project targeting Brazilian public and private schools, organized by IMPA, with the support of the Brazilian Society of Mathematics and promoted with resources from the Brazilian Ministry of Education and the Ministry of Science, Technology, Innovation and Communications - MCTIC. It enrolls 18.2 million students in 5,545 Brazilian municipalities. Five hundred and seventy six gold medals, 1,727 silver medals, 5,178 bronze medals and 44,386 honorary mentions are awarded. OBMEP reaches across the entire of Brazil including remote areas.

More information about the Brazilian Mathematical Society and its activities can be found on its website.

6 Southeast Asian Mathematical Society: Jose Maria P. Balmaceda

Southeast Asia's socio-economic, cultural and geographic diversity poses many challenges. The region includes economic powerhouses like Singapore and Hong Kong, but also Myanmar, Laos and Cambodia. The mathematical needs span the full range from most basic to the most advanced. Any developmental activity should begin with a clear understanding of context (individual histories, culture, tradition, resources, priorities, etc).

Even in mid-tier countries like Indonesia, Philippines and Vietnam, poverty, lack of basic infrastructure and competent teachers are huge obstacles. A big goal is not to transform cramped classrooms into high-tech ones but to empower countries to develop mathematics that is relevant to the socio-cultural needs of their people. Developed countries also contend with several issues, such as low satisfaction and negative attitudes of students towards math, including widespread anxiety especially with national exams.

The Southeast Asian Mathematical Society (SEAMS) was founded in 1972. The member societies and their foundation dates are:

Singapore Mathematical Society	1952
Vietnam Mathematical Society	1966
Malaysian Mathematical Sciences Society	1970
Mathematical Society of the Philippines	1973
Indonesia Mathematical Society	1976
Mathematical Association of Thailand	1978
Hong Kong Mathematical Society	1979
Cambodian Mathematical Society	2005
Mathematical Society of Myanmar	2013
Nepal Mathematical Society	1979

Nepal is an affiliate member of SEAMS. Singapore, Vietnam, Malaysian, Philippines, Indonesia, Thailand and Hong Kong are members of the IMU; Cambodia and Nepal are associate members of IMU.

Regional societies like SEAMS facilitate collaboration and exchanges between member societies. Their members can learn from their different experiences. Regional members face many common challenges, often with common root causes. Solutions and best practices are shared in "South–South" cooperation. An example is for graduate studies in universities within the region, like a Cambodian student getting a master's degree in the Philippines. Costs are often lower and culture shock is lessened than graduate study far away.

Some activities of SEAMS are:

- Organization of the Asian Mathematical Conference every 4 years (AMC 2020 will be in Vietnam)
- Establishment of regional/ bilateral partnerships and reciprocity arrangements with other math societies
- Establishment of and support to national math societies (e.g. Cambodia, Myanmar; next: Laos, Brunei)
- Publication of the journal Southeast Asian Bulletin of Mathematics

Common activities of individual member-societies include:

- Organization and sponsorship of conferences, workshops on math and math education
- Organization of local/national math olympiads and training of IMO teams
- Popularization of math, teacher-training (basic and higher education)
- Publication of journals (research and popular)
- Involvement in national education initiatives and reform (e.g. new K-12 basic education in the Philippines)

The SEAMS Schools, launched in 2010, are 7 to 10 day intensive study programs for advanced undergraduates and master's students in the region. These schools were inspired by the EMALCA program in Latin America and the Caribbean. The schools

are partially supported by CIMPA. Lecturers come mainly from the host country and region which makes them less costly. These schools encourage women and students from less developed countries to be mathematicians.

As part of his presentation Jose Maria Balmaceda was asked to share some information about India and Central Asia. There was a description of the number of scientific publications for countries in the region which identifies a common feature with Southeast Asia: the unevenness in the mathematical growth and levels from country to country, and within the large countries like India.

Balmaceda concluded his presentation by sharing some thoughts on how to promote mathematics in the developing world in general and Southeast Asia in particular:

- Good partnerships are based on mutual respect and appreciation of differences, leading to self reliance for the less-developed partner
- Information and communications technology can be a game-changer and MOOCs can be effective, but not in places where electricity and connectivity are still major problems
- Offer more opportunities for women and the poor, including indigenous peoples (local culture is also a rich source of math)
- Encourage "PhD sandwich programs" (cheaper, often a deterrent to brain-drain)
- Provide opportunities for research visits and post-doc experience, possibly within the region
- Raise more awareness and campaign against insidious practices like predatory journals or conferences that often victimize those from less developed countries

More information about the SEAMS and its activities can be found on its website.

7 African Mathematical Union: Nouzha El Yacoubi

The African Mathematical Union (AMU) was founded during the first Pan African Congress of Mathematicians, in Rabat, Morocco in July 1976, with the mission:

> "To coordinate and promote the quality of teaching, research and outreach activities in all areas of mathematical sciences throughout Africa. Advancing mathematical research and education includes efforts and contributions towards the economic, social and cultural development of the continent."

For the achievement of its goals, in 1978, the AMU Executive Committee created the AMU Journal: Afrika Matematika, initially with the aim to make mathematical research originating within Africa more widely known. Since 2011, Afrika Matematika has been published by Springer. The submission rate has been growing since that date and it has truly increased in the last couple of years. Since 2013, the improved status of the journal was evidenced by the fact that it has been listed by Scopus.

In 1986, four AMU Commissions were established: Mathematics Education in Africa, Pan African Mathematics Olympiads, History of Mathematics in Africa and African

Women in Mathematics. In 2009, the AMU Commission for Research and Innovation in Mathematical Sciences was added.

The most important AMU scientific event is the Pan African Congress of Mathematicians (PACOM). Pan African Mathematics Olympiads (PAMO) are organized annually, and every four years, they are organized in the framework of PACOM. Also, AMU medals are awarded to outstanding African mathematicians during each PACOM. Until now 9 editions have been organized in various African countries. The last PACOM was hosted by the Mohammed V University in Rabat in July 2017 (PACOM 2017), with the theme: "Mathematics at the heart of technological innovation and economic development of Africa". A new Executive Committee has been elected for the period 2017-2021. In December 2018, for the first time PACOM 2017 Proceedings will be published in a special issue of Afrika Matematika.

Now, more than forty years after the creation of the African Mathematical Union with a mission, statutes and objectives designed according to the various impediments, challenges, opportunities, etc. of the moment, our responsibility for this new mandate (2017-2021) is to define a new vision consisting on redesigned goals with an improved and better adapted strategy for a relevant strengthening and a fruitful development of Mathematics in Africa, with aims to meet the requirements and expectations of this new millennium. To this end, AMU has plans to develop a database of African mathematical association, societies, and non-governmental organizations (NGOs), in order to invite them to adhere to AMU and to encourage them and support them to adhere to the International Mathematical Union (IMU). Also the AMU is determined to be innovative by developing mathematics that can impact critical sectors of the economy collaborating with business and industry and adhering to ICIAM, Math-in, etc. and by creating Africa Math-In.

A specific example of this effort is the first International African Mathematical Union Commission for African Women in Mathematics workshop on Science , Technology, Innovation (STI) and Mathematics: (STIMATH 2018), to be held at Botswana International University of Science and Technology (BIUST), November 22-23, 2018. This meeting will support women in mathematics addressing the challenges of STI in Africa.

There is also a current fruitful collaboration between AMU and the International Centre for Pure and Applied Mathematics (CIMPA) consisting in successfully organizing the African Mathematical Schools (EMA).

More information about the AMU and its activities can be found on its website.

8 Mathematical Union of Latin America and the Caribbean: Alejandro Jofré

UMALCA is the Mathematical Union of Latin America and the Caribbean. The goal of UMALCA is to improve the communication between the mathematical efforts in the member countries by promoting scientific exchanges by researchers, doctoral and postdoctoral students as well as carrying out programs to support mathematics in countries with less mathematical development.

UMALCA's programs include:

- **Mathematical School of Latin America and the Caribbean (EMALCA)** The goal is to put in contact college students in locations with limited access to main stream mathematics and mathematicians with experienced research mathematicians. They were started in 1998 and now there are several every year. The EMALCA schools are supported by CIMPA (Regular sponsor), IMU CDC, Mathematical Congress of the Americas (MCA) and ICTP- Trieste (in process). In recent years, this is where the EMALCA schools have taken place:

  2018: Guatemala, El Salvador, Argentina, Venezuela, Cuba, Mexico.

  2017: Colombia, Argentina, Ecuador, Venezuela, Bolivia y Mexico

  2016: Chile, Mexico, Dominican Republic, Venezuela

  2015: Argentina, Bolivia, Mexico, Nicaragua, Peru, Venezuela

- **Travel Grants:** Support for research visits or for attending conferences within the Latin America region. Travel grants are given mostly to young researchers and post-doctoral students.

- **Latin American Congress of Mathematicians (CLAM):** region-wide meeting modeled on the ICM, organized every 4 years. At each CLAM, the UMALCA prizes are awarded to mathematicians working in Latin American who have done extraordinary research and are under the age of 40. These have been the locations of CLAM and the UMALCA Prize winners:

  CLAM 2000 Rio de Janeiro Brazil; UMALCA Prize winner: Marcelo Viana (Brazil)

  CLAM 2004 Cancun, Mexico; UMALCA Prize winner: Enrique R. Pujals (Brazil)

  CLAM 2009 Santiago de Chile, Chile; UMALCA Prize winners:Alejandro Maass (Chile), Universidad de Chile, Carlos Gustavo Moreira (Brazil) Federico Rodriguez Hertz (Uruguay)

  CLAM 2012 Cordoba, Argentina; UMALCA Prize winner: Fernando Codá Marques (Brazil)

  CLAM 2016 Barranquilla, Colombia; UMALCA Prize winners: Henrique Bursztyn (Brazil), Robert, Morris (Brazil); Andres Navas (Chile); Pablo Shmerkin (Argentina)

  Next meeting: CLAM 2020 Montevideo, Uruguay

UMALCA works to increase the quality of mathematics produced in UMALCA's countries by increasing the sense of community of mathematicians, by supporting young researchers and promoting mathematics in isolated communities (EMALCA). The level of mathematics within UMALCA has improved in quality and volume the last decade. Furthermore, we have today a network of Centers of Excellence in Mathematics in Argentina, Chile, Brazil and Mexico. All of this leads to a strong international network, which cooperates in addressing fundamental problems in mathematics as well as applications that have country and regional impact such as energy, mining, forestry, environment, natural catastrophes, etc. Examples of other programs in Latin America are

two programs supported by CNRS (France) MathAmsud and SticAmsud, which are research projects organized by UMALCA 's members. Another example is the Mathematical Congress of the Americas (MCA).

All these efforts are leading to stronger PhD and Postdoctoral programs as well as international prizes including a Field medal. UMALCA is connected with the main associations in other latitudes. The development of the region can be seen in that Brazil, a member of UMALCA, is the host of ICM 2018.

More information about the AMU and its activities can be found on its website.

9 Conclusions

The panel and poster session provided a mechanism for an exchange of ideas possible only at the ICM because of it's worldwide participation. Through the presentation and discussion of the mechanisms being used to support mathematics around the world, it was possible to see the efforts that were shared. An example of a common effort are the schools for students at the advanced undergraduate or master's level in Latin America, Southeast Asia and Africa. It was also helpful in terms of seeing the potential future in countries with strong mathematical development such as Brazil which has both a strong olympiads program at the elementary school level and research communities able to host the ICM. Understanding the perspectives from a funding agency helps us see the different viewpoints which are all working toward solving the same problem. As we know, there is a large fraction of the world where mathematical talent is not being fully developed. This event helped us identify ways in which we can be partners in improving this situation.

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THE GENDER GAP IN MATHEMATICAL AND NATURAL SCIENCES FROM A HISTORICAL PERSPECTIVE

June Barrow-Green, Silvina Ponce Dawson and Marie-Françoise Roy

Abstract

The panel organised by the Committee for Women in Mathematics (CWM) of the International Mathematical Union (IMU) took place at the International Congress of Mathematicians (ICM) on August 2nd, 2018. It was attended by about 190 people, with a reasonable gender balance (1/4 men, 3/4 women). The panel was moderated by Caroline Series, President of the London Mathematical Society and Vice-Chair of CWM. Presentations were made by Marie-Françoise Roy, Chair of CWM, June Barrow-Green, Chair of the International Commission on the History of Mathematics, and Silvina Ponce Dawson, Vice-President at Large and Gender Champion of the International Union of Pure and Applied Physics (IUPAP). The presentations were followed by general discussion. Marie-Françoise briefly outlined the history and activities of CWM and described the ongoing "Gender Gap in Science" project which is being carried out under the leadership of IMU and the International Union of Pure and Applied Chemistry (IUPAC), with the participation of IUPAP and many other scientific unions. June gave some insights into the historical context of the gender gap in mathematics, while Silvina gave an overview of activities undertaken by the IUPAP Working Group on Women in Physics to evaluate and improve the situation of female physicists.

What follows are the authors' accounts of their presentations together with some notes on the subsequent discussion.

1 The International Mathematical Union (IMU) Committee for Women in Mathematics (CWM)

1.1 Creation and organization of CWM. CWM was created by the IMU Executive Committee (EC) in March 2015 with the following terms of reference:

(1) To promote international contacts between national and regional organizations for women in mathematical sciences;

(2) To maintain up-to-date content on the Women in Mathematics part of the IMU website and, with appropriate assistance from the IMU, to ensure its technical development;

(3) To consider how best to facilitate electronic communications among the community of women mathematicians internationally;

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(4) To work with groups, committees and commissions of IMU on topics pertaining to women mathematicians and their representation;

(5) To publicize, and where needed to suggest, working practices that ensure equal opportunities for women mathematicians in universities and research institutions, for example appropriate funding arrangements, family friendly policies and facilities;

(6) To report annually to the IMU Executive Committee and to propose actions that would foster equal treatment of women in the mathematical community and lead to an increase in the representation of women in mathematics at all levels.

The CWM has a chair and a vice-chair, and 6 to 8 members at large, who are appointed for four years in accordance with the EC terms, and whose country of residence should be distributed internationally, reflecting the global character of CWM.

The list of committee members in the period 2015–2018 was as follows: Marie-Françoise Roy, France (Chair, in charge of electronic communication); Caroline Series, UK, (Vice Chair, in charge of the CWM website); Carolina Araujo, Brazil; Bill Barton, New Zealand; Ari Laptev, UK and Sweden; Kristin Lauter, USA; Sunsook Noh, S. Korea; Marie-Françoise Ouedraogo, Burkina Faso; Sujatha Ramdorai, Canada; Betül Tanbay, Turkey. CWM also had two associate members, Neela Nataraj (India - Coopted for coordinating grant reports) and Petra Bonfert-Taylor (USA - Coopted for website). The liaison with EC was made by John Toland, UK.

The impetus for creating CWM followed the formation of a group of female mathematicians led by Ingrid Daubechies whose aim was to collect and organise information for a new Women in Maths section of the IMU website. The CWM website was launched at the ICM in Seoul 2014. When CWM started, the website was updated accordingly; publicity for the new committee was done at the same time. Items (events, new women in maths organisations, newsworthy items, resources etc.) are added every week. The site lists an impressive number of events related to women mathematicians in 2015–18 in all parts of the world. In addition, 36 countries are listed with some form of organisation, activities or contacts. The site has a unique and important function as the only platform for coordinating so much diverse activity worldwide.

CWM has in addition established a network of (currently) 120 *CWM Ambassadors*, each of whom is responsible for disseminating relevant information such as CWM funding calls within her geographical or mathematical area and of keeping CWM informed about relevant activities or initiatives.

1.2 CWM call for proposals. At its first meeting in Cortona, Italy in September 2015, CWM decided to use most of its budget to support the formation of networks of female mathematicians on a regional basis in developing or emerging countries. Annual calls were organized in 2016, 2017 and 2018. A total of 155 applications were received and 31 were supported, most of them in developing or emerging countries from Africa (Ethiopia, Morocco, Kenya, Nigeria, Senegal, South Africa, Tunisia), Asia (India, Indonesia, Iran, Japan, Kazakhstan, Nepal, Uzbekistan, Vietnam), Latin America (Brazil, Chile, Colombia, El Salvador, Mexico, Uruguay) as well as Austria, Canada, Italy and Macedonia. Also, almost all grants in developed countries support the participation of

Figure 1: World Meeting for Women in Mathematics $(WM)^2$

women from developing countries. Around 2500 people, most of them women, participated in the various events.

CWM initiated the World Meeting for Women in Mathematics $(WM)^2$ project. The first $(WM)^2$ in Rio, with main organizer Carolina Araujo, had a strong Latin American focus and took place immediately prior to the ICM, on Tuesday July 31st, 2018. A professional designer created a logo and poster, the domain name worldwomeninmaths.org was reserved and a website company designed the website. $(WM)^2$ was approved as a satellite meeting of the ICM on 15 February 2016. The scientific committee chaired by Georgia Benkart (University of Wisconsin-Madison) has selected a key-note lecturer (Monique Laurent), three invited lecturers (Alicia Dickenstein, Salome Martinez, Maria Eulalia Vares) and a public lecturer (Maria Esteban). The program also included group discussion, presentation of 100 posters (both mathematical and on women in mathematics), and a tribute to Maryam Mirzakhani. The 293 Female OpenArms grantees were offered free registration for $(WM)^2$. Finally more than 350 people attended, more of one third of them from the OpenArms programme.

1.3 Women mathematicians in film. A short film for International Women's Day 2018, *Faces of Women in Mathematics*, was suggested by Eugenie Hunsicker, Chair of the London Mathematical Society Women in Maths Committee, and Irina Linke, a filmmaker. It consists of an edited sequence of film clips of women saying into the camera "I am (name) from (country), and I am a mathematician!" in the language of their choice. The 146 clips, sent in as a result of a message circulated to CWM ambassadors, featured 243 women mathematicians from 36 different countries speaking in 31 different languages. CWM used a small part of its budget for editing the clips and the film has been greatly appreciated by all who have seen it.

With a recommendation from Ingrid Daubechies, CWM proposed creating a short film *Journeys of Women in Mathematics* to the Simons Foundation. Micro-Documentaries has been selected to film and edit the film. The focus is on the diversity of women mathematicians worldwide. The first version, shown at $(WM)^2$, presents three women active in organizing regional networks: Carolina Araujo (Rio, Brazil) the $(WM)^2$ main

organizer and also an ICM lecturer; Neela Nataraj (Mumbai India), an active member of Indian Women in Mathematics and Aminatou Pecha (Yaounde, Cameroon), the founding chair of the Cameroon Women in Mathematics Association. Micro-Documentaries met them in their home countries. The second augmented version will highlight six women from Latin America, including the three invited lecturers at $(\mathrm{WM})^2$.

1.4 Maryam Mirzakhani Memorial exhibition. Made for $(\mathrm{WM})^2$ and remaining posted throughout the entire ICM2018, the exhibition consisted of 18 posters inspired by Maryam Mirzakhani's achievements and her premature death, two specially printed books containing all her mathematical papers, one book with articles about her, and a book of condolences. The Maryam Mirzakhani Memorial exhibition will be permanently hosted by Stanford University and will also be made available at scientific events by agreement of the organizers with CWM.

1.5 CWM flier and posters. Using a professional designer, CWM created a flyer and poster, which was distributed through CWM Ambassadors and organizers of CWM funded events. A CWM roll-up for use in displays has been designed in the same style. Three posters (one for Africa, one for Asia, one for Latin-America) reporting on CWM funded activities since ICM Seoul have being designed for display at $(\mathrm{WM})^2$ and ICM Rio.

1.6 Interdisciplinary project Gender Gap in Science. CWM joined forces with the International Union of Pure and Applied Physics (IUPAP) and the International Union of Pure and Applied Chemistry (IUPAC) and submitted a project entitled *A Global Approach to the Gender Gap in Mathematical, Computing and Natural Sciences: How to Measure It, How to Reduce It?* to the International Council for Science (ICSU) in 2016. Lead by the IMU with IUPAC as co-lead applicant, the application was approved on

7 February 2017 (with a budget from ICSU of €100 000 per year in 2017, 2018 and 2019). There are 9 other partners in the project: International Union of Pure and Applied Physics (IUPAP), International Astronomical Union (IAU), International Council for Industrial and Applied Mathematics (ICIAM), International Union of Biological Sciences (IUBS), International Union of History and Philosophy of Science and Technology (IUHPST), United Nations Educational, Scientific and Cultural Organization (UNESCO), Gender in Science, Innovation, Technology and Engineering (GenderInSITE), Organization of Women in Science for the Developing World (OWSD), Association for Computing Machinery (ACM).

Barriers to achievement by women in mathematical, computing and natural sciences persist, especially in developing countries. The aim of the project is to produce sound data to support the choices of interventions that ICSU and member unions can feasibly undertake. It will provide evidence for informed decisions, including trends – since the situation for women continues to change around the world, with some negative developments – and will provide easy access to materials proven to be useful in encouraging girls and young women to study and work in these fields. It will do this through three tasks: (Task 1) a global survey planned to reach 45,000 respondents in more than 130 countries using at least 8 languages; (Task 2) a study on publication patterns which will analyze the effects of gender and location on scientists publication behavior using data on more than 500,000 scientists since 1970. Finally, it is impossible to ignore that there are many initiatives around the world that aim to enhance the participation of girls and women in science and mathematics. Which ones work? What is the evidence for effectiveness? Can effective practices developed in one place be used in other contexts? These are some of the questions that will structure an online database of good practices (Task 3).

The project's part focussing on publication patterns is based on methods and research questions from a recent study in Mathematics Mihaljević-Brandt, Santamaría, and Tullney [2016]. Based on four decades of data from Zentralblatt Math, the authors showed

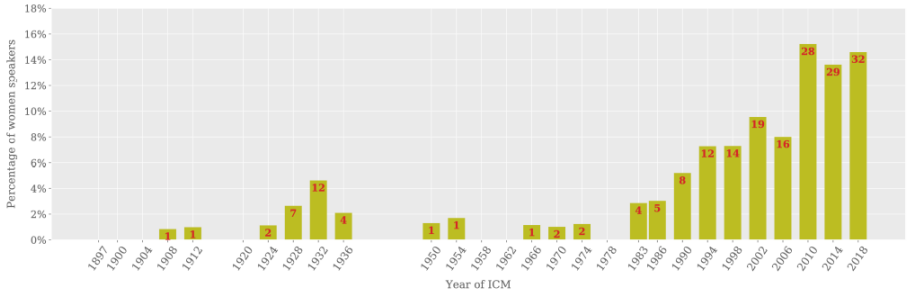

Figure 2: Percentage of women speakers at ICM (as well as numbers)

a systemic gender imbalance in the publication distribution of mathematicians: women mathematicians tripled their number since 1970, but they publish less than men at the beginning of their careers and leave academia at a higher rate; high-ranked journals publish fewer articles by women, some showing less than 5% authorships by women with no change over time; women publish fewer single authored papers, although their coauthor networks are similar in size to those of men.

As part of the project, the Gender Gap inside among ICM's invited speakers has also been studied. As presented at $(WM)^2$ in a poster by Helena Mihaljević and Marie-Françoise Roy the number of women sums up to 201, less than 5% in total. Of all lectures delivered by women in the history of ICM, 80% took place after 1990. In the recent three ICMs the share of women lecturers has grown to approximately 14%. The poster provided glimpses into individual career paths of some of the invited women speakers, and documents the interesting and nonlinear development of women's presence within this community.

2 The Historical Context of the Gender Gap in Mathematics

2.1 Introduction. In 1971 the *Association for Women in Mathematics* (AWM), the first organisation for supporting women in mathematics, was established in the United States Blum [1991]. There are now many organisations worldwide supporting women in mathematics, and the number continues to grow, with the IMU's *Committee for Women in Mathematics* (CWM) providing a focus point, for more details see Math Union web site. Nevertheless, despite the extensive work that has been done since 1971 to address the particular challenges which confront women in mathematics, women still face particular difficulties within their professional careers. Many of these difficulties have a long history stemming from deeply embedded cultural attitudes. What follows is a glimpse at some of the challenges that women mathematicians have had to face during the last two hundred and fifty years.

2.2 The 18th and 19th centuries. The first woman in the modern period to make a substantial contribution to mathematics was the Italian Maria Agnesi (1718–1799) who in 1748 published one of the earliest textbooks on the calculus. Two years later she was

appointed to the chair of mathematics in Bologna on the recommendation of the Pope, Benedict XIV, but she never took up the position, choosing to devote her life to works of charity. In fact Agnesi never even went to Bologna although her name remained on the rolls of the university. With reference to her work and the more general question of how women mathematicians were perceived in the 18th century, an interesting remark was made by the French historian of mathematics, Jean-Étienne Montucla (1725–1799), who said that he wished that the *Instituzioni Analitiche* had been translated into French by a French female mathematician, thus implying he believed there was something intrinsically feminine about the text.

Agnesi, along with other women in the 18th and early 19th century, such as Émilie du Châtelet (1706–1749), Ada Lovelace (1815–1852) and Mary Somerville (1780–1872), all of whom made lasting and significant contributions to mathematics, were not prevented from doing mathematics, in fact sometimes rather the opposite. For example, Ada Lovelace was encouraged by her mother to study mathematics with Augustus De Morgan Hollings, Martin, and Rice [2017]. One thing these women all had in common was that they came from a social class which allowed them to attend social functions where they could discuss mathematics and natural philosophy with men on equal terms. Both Mary Somerville and Ada Lovelace attended the scientific soirées of Charles Babbage (1791–1871) and together they frequently called on him in order to see and to discuss his analytical engine. That Élisabeth Ferrand (1700–1752), an important influence on Abbé de Condillac and a friend of Alexis Claude Clairaut, chose a page from Voltaire's influential *Eleménts de la philosophie de Newton* (1738)—the book which introduced Newtonian physics to France—as the backdrop to her portrait is indicative of the acceptability of such learning among women in Enlightenment circles (Figure 3). For biographical information about Ferrand and a discussion of Maurice-Quentin de La Tour's pastel portrait, see Jeffares [2016].

However, although women could mix socially in mathematical and scientific circles, they could not hold an official position. Mary Somerville could make money from the sales of her books—her *Mechanism of the Heavens*, Somerville [1831], an acclaimed translation and commentary on the celestial mechanics of Pierre-Simon Laplace (1749–1827), Laplace [1798], became a recommended text for men studying for the Mathematical Tripos at Cambridge—and she could have a paper published by the Royal Society but there was no question of her being admitted as a Fellow of the Royal Society, Mason [1992]. The first women mathematician to be admitted was Mary Cartwright (1900–1998) in 1947.

The first woman to be a professional academic mathematician was the Russian Sofia Kovalevskaya (1850–1891). Championed by the Swedish mathematician Gösta Mittag-Leffler, who overcame strong opposition to secure her appointment at the Stockholm Högskola, she became a full professor in 1889. But despite Kovalevskaya's internationally recognised mathematical talent—she was awarded the Prix Bordin of the French Académie des Sciences for her work on the spinning top—there was no chance for her to gain a position in one of the mathematical centres of Europe, such as Paris or Berlin.Koblitz [1983, pp. 215–217], Kovalevskaya herself reported an example of the prejudice that existed against women mathematicians during the period. In 1869, early in her career, when she was making one of her visits to the London salon of the novelist

Figure 3: "Mlle Ferrand méditant sur Newton" by Maurice-Quentin de La Tour

George Eliot (Mary Anne Evans) she found Eliot, who had an interest in mathematics, very keen to introduce her to the philosopher Herbert Spencer because, as Eliot said, Spencer denied "the very existence of a woman mathematician." Kovalevsakya [1978, p. 359].

As a gifted female mathematician, Kovalevskaya inevitably attracted attention, but not only because of her mathematics. James Joseph Sylvester's assistant declared that she was "the first handsome mathematical lady" he had ever seen. (Of course one can wonder how many mathematical ladies he had ever seen!) Beauty it seems was not expected in a female mathematician. After Kovalevskaya's death – she died unexpectedly aged only 41 – her fame escalated and interest in her appearance intensified. But no longer was there a consensus – for some she was beautiful for others she was not and there was no general agreement. Opinions about her looks still abound and, as has been observed, the differing nature of these opinions, provide an insight into the changing views about female mathematicians Kaufholz-Soldat [2017].

2.3 Cambridge University. During the 19th century, Cambridge was the beating heart of British mathematics and the Mathematical Tripos the most prestigious examination in Britain. It is hard to over-estimate the kudos attached to being senior wrangler, the top student of the year. Kudos that went far beyond the bounds of Cambridge. Although from 1869/1872 women could study mathematics at Girton/Newnham, women did not have the right to sit the examination—permission had to be granted— and they

could not obtain a degree. Indeed the colleges were not officially part of the University (that had to wait until 1948).

In 1880 Charlotte Scott (1858–1931) created a sensation by being judged equal to the 8th wrangler.[1] The newspapers and periodicals were full of her success—she had done better than 93 of the 102 men taking the examination—and the reports provide an interesting insight into prevailing attitudes. *The Spectator* was typical:

> "Miss Scott has answered papers set for the mathematical tripos in a manner which would have brought her high on the list of Wranglers, an achievement of no common kind. … We hope that the ability which the new system brings out and fosters in women, will not be of a kind to give to those who possess it a character for deficiency in feminine gentleness. We do not believe that it will be so. But even in the rare cases where it is so, the world should remember that there have always been women of the masculine type—only that they have hitherto lacked the means of proving what they could do, though possessing amply the means of proving what they could not be." *The Spectator* [1880, p.163].

Scott's achievement generated a growth in support for women students with the result that from 1881 they were given the right to take the examinations and their results were published, albeit separately from the men. But still they could not be awarded degrees.

An even greater sensation was created when, in 1890, Philippa Fawcett (1868–1948) was judged to be above the senior wrangler. She had achieved what many had believed impossible. Nevertheless, when the Tripos list was published, her name still appeared below that of all the men. After Fawcett's success, the clamor for women to be awarded degrees grew louder but still not loud enough. Cambridge did not fully open its doors to women until 1947. Those who wanted degrees had to go to London or, from 1920, Oxford. Those who wanted to study for higher degrees had to go abroad – the PhD did not come to Britain until after the First World War.

Grace Chisholm (1868–1944), who sat the Tripos in 1892, completed her studies with Felix Klein in Göttingen (see below) and in 1895 became the first British women to gain a PhD in mathematics (and the first woman in Germany to gain a conventional PhD). Shortly afterwards she married the mathematician W. H. Young. Young was content for her to continue with mathematical research but, as he told her rather directly in 1902, publishing mathematical papers was a man's game:

> "The fact is that our papers ought to be published under our joint names, but if this were done neither of us get the benefit of it. No. Mine the laurels now and the knowledge. Yours the knowledge only. Everything under my name now, and later when the loaves and fishes are no more procurable in that way, everything or much under your name."Grattan-Guinness [1972, p. 141].

[1] 1880 was a strong year with Joseph Larmor, future Lucasian professor, being senior wrangler, and J. J. Thomson, future Nobel laureate, being second wrangler.

Although such a shocking situation no longer pertains, recent analysis has shown that "a systemic gender imbalance" in the publication distribution of mathematicians still exists Mihaljević-Brandt, Santamaría, and Tullney [2016].

Young was not alone as a man supporting women mathematicians. Charlotte Scott studied algebraic geometry with Arthur Cayley, the Sadleirian Professor, and it was Cayley who recommended her for the position of head of mathematics at the newly founded Bryn Mawr College in the United States, a position which she took up in 1895, no equivalent opening being available to her in Britain. But for a long time men like Young and Cayley were in the minority—the belief that women were not capable of doing serious mathematics proved extremely hard to shift in Cambridge.

After 1947, women could be awarded degrees at Cambridge but further progress towards gender equality in mathematics was and continues to be glacially slow. Mary Cartwright (1900–1998), who in 1947 was the first female mathematician to be elected a Fellow of the Royal Society, and who in the following year became Mistress of Girton, was not deemed worthy of a professorship. The first woman professor of mathematics in Cambridge, the applied mathematician Anne Davis, was appointed only in 2002. As at 2018 no female professor in pure mathematics has ever been appointed at Cambridge. Furthermore, the gender imbalance in mathematics students remains considerably greater than elsewhere.[2] In 2014 the Faculty of Mathematics at Cambridge achieved an Athena SWAN bronze award.

2.4 Germany. In 1764 Immanuel Kant had pronounced that women who succeeded in mathematics "might as well have a beard."Kant and Goldthwait [2004, p. 79]. His point being that if women did succeed in mathematics then they would not be women they would be men! The first concrete sign of progress was in 1874 when Kovalevskaya, having studied privately with Karl Weierstrass in Berlin, was awarded a PhD in Göttingen in absentia. But it remained an isolated incident until the 1890s when Felix Klein, and subsequently David Hilbert, in Göttingen began to allow women to audit lectures.[3] But as Klein observed in 1895, the general opinion in Germany was that the study of mathematics should be as good as inaccessible to women. At that time he himself had had six women successfully participating in his higher mathematics lectures but all were foreigners (American, English and Russian) which prompted him to remark:

> "No one would wish to assert, however, that these foreign nations possess some inherent and specific talent that evades us, and thus that, with suitable preparation, our German women should not be able to accomplish the same thing." Tobies [n.d.].

Klein also encouraged his women students to publish in *Mathematische Annalen*, the journal of which he was the chief editor. The American Mary Winston (1869–1859), whom Klein had originally met in 1893 when she attended both the Mathematics

[2]For a discussion about the current situation with respect to mathematics students in Cambridge, see the *Varsity* interview of 2 November 2017 with Julia Gog.

[3]In 1891 the American Ruth Gentry was permitted to audit the lectures of Lazarus Fuchs and Ludwig Schlesinger in Berlin for one term before permission was revoked.

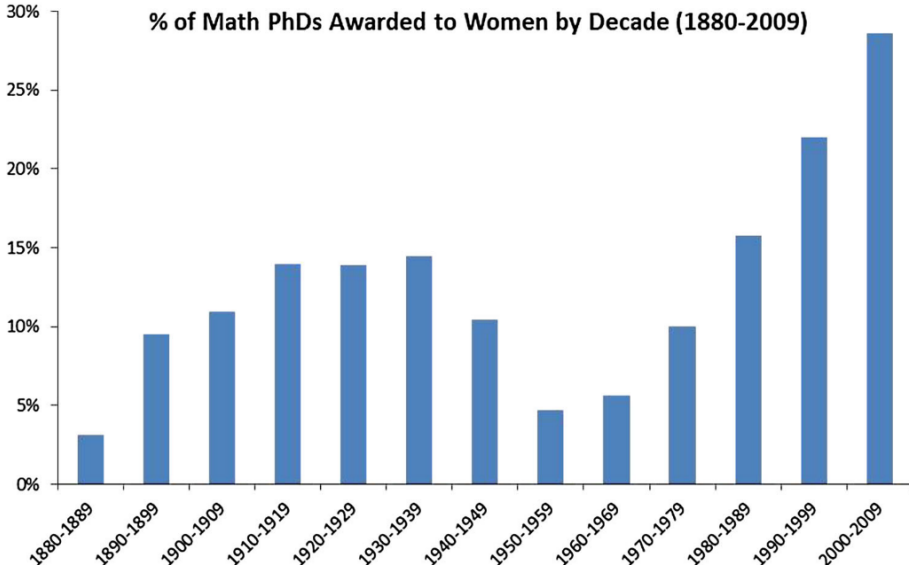

Figure 4

Congress in Chicago and his Evanston Lectures that followed it, was the first, in 1895, with a short note on the hypergeometric function.

The most prolific female author in *Mathematische Annalen* under Klein's editorship was Emmy Noether (1882–1935). Noether's life and extraordinary talent for mathematics have been well documented but recently more information has come to light with regard to her application in 1928 for a professorship at Kiel, information which underlines the tremendous difficulty she faced in trying to get a position in Germany Siegmund-Schultze [2018].

2.5 United States of America. Thanks to the detailed work of Judy Green and Jeanne LaDuke, there is now a wealth of information available about the 228 American women mathematicians who earned PhDs before 1940 Green and LaDuke [2009] including supplement material by the same authors. Added to that is the very informative article by Sarah Greenwald, Anne Leggett and Jill Thomley on the AWM which brings the picture in the United States up to date Greenwald, Leggett, and Thomley [2015]. What is striking about their findings is how the percentage of women mathematics PhDs rose fairly steadily decade on decade from the end of the 19th century up to the beginning of the Second World War only then to drop off significantly. As can be seen from the graph in the article by Greenwald et al. (Figure 4), the 1930s percentage was only really surpassed in the 1990s Greenwald, Leggett, and Thomley [ibid., p. 13]

In the pre-WW2 period, certain institutions stand out. Bryn Mawr, the women's college founded in 1895, benefited from having Charlotte Scott at its mathematical helm. Scott supervised seven PhD students and her colleague, and successor as head of mathematics, Anna Johnson Pell Wheeler (1883–1966) supervised six. Both of them, together with Olive Hazlett (1890–1974) who spent a short time as a lecturer at Bryn Mawr,

are distinguished for being the only starred women mathematicians in (the inaccurately named) *American Men of Science* from 1903 to 1943. In Chicago Leonard E. Dickson supervised 18 women PhDs (27% of his output), and Gilbert A. Bliss supervised 12 women PhDs (23% of his output). Meanwhile in Cornell Virgil Snyder supervised 14 women PhDs (37%) of his output. In addition, as already noted, Klein in Göttingen also supported American women mathematicians.

In the aftermath of WW2, the social conditions conspired against women mathematicians as it did against women in other fields. It was not until the 1970s, with the advent of organisations supporting women mathematicians, that significant improvements could be seen.

2.6 The growth of institutional support for women in mathematics. In general, national mathematical societies have been welcoming to women members. However, the same is not true of their governing bodies. The American Mathematical Society was exceptional in appointing Charlotte Scott as a Vice-President in 1906, although it took the Society until 1983 before it appointed its first woman president, Julia Robinson. The first Society to appoint a woman president was the *Société Mathématique de France* when they elected Marie-Louise Dubreil-Jacotin in 1952. Even in recent times, the number of women in senior roles within societies has not accurately reflected the contribution of women to mathematics as a whole.

After the formation of the AWM in 1971, a number of other organisations supporting women in mathematics were established in North America and Europe: *The Joint Committee on Women in the Mathematical Sciences* (1971), *European Women in Mathematics* (1986), *Femmes et Mathématiques* (1987), *The Women in Mathematics Committee of the European Mathematical Society* (1991), the *Canadian Society Committee for Women in Mathematics* (1992) and the *London Mathematical Society Women in Mathematics Committee* (1999).

At the First European Congress of Mathematics in 1992, there was a Round Table on Women in Mathematics organised by the Women in Mathematics committee (WiM) of the *European Mathematical Society* (EMS). The aim of the Round Table was to look at the proportion of women involved in mathematics in various countries. Its report contains a wealth of information and data providing a detailed picture of the situation, Bayer-Fluckiger [1994].

The WiM committee, with the help of the EMC, gathered data about women mathematicians across Europe in 1993 and in 2005 (Figure 5). Although the data shows a substantial increase in the percentage of women mathematicians during the intervening period, it also reveals a significant difference between north and south, highlighting where the greatest efforts need to be made.

Since 2000 the number of organisations set up to support women in mathematics has grown worldwide. There are now organisations in Australia, Cameroon, Chile, China, India, Israel, Iran, Kenya, Korea, Pakistan, Peru, Poland, Russia, Senegal, Spain,

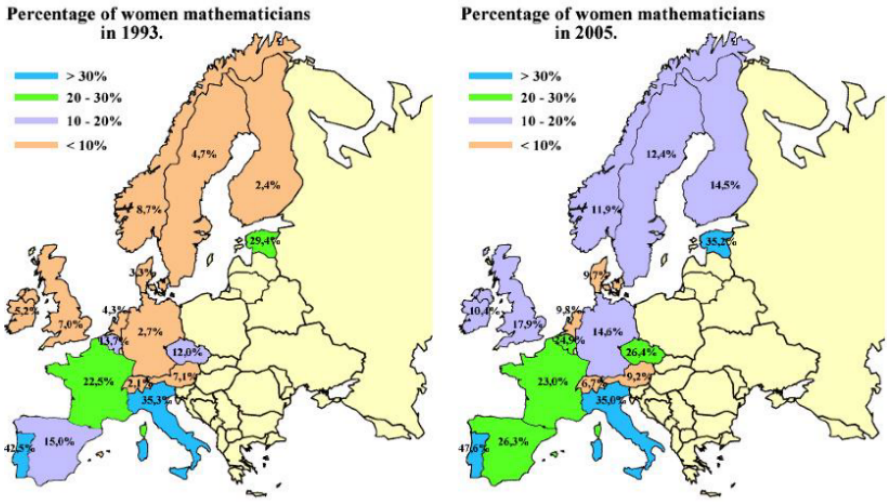

Figure 5

Tunisia, and Turkey, as well as umbrella organisations for African Women in Mathematics and Central Asian Women in Mathematics. Information about all of these organisations, and much more, can be found on the website of the IMU Committee for Women in Mathematics.

3 Gender-related policies of the International Union of Pure and Applied Physics

3.1 The International Union of Pure and Applied Physics. The International Union of Pure and Applied Physics (IUPAP) was created in 1922 to assist in the worldwide development of physics, to foster international cooperation in physics, and to help in the application of physics toward solving problems of concern to humanity. Membership in the IUPAP is through country representation. Currently 68 countries are represented in the Union. The IUPAP is governed by the General Assembly that meets once very three years. Its main executive body is the Executive Council which, among other things, oversees the activities of 19 specialized Commissions. There are also Working Groups that are created with a finite time duration to address specific problems.

3.2 The Working Group on Women in Physics. In 1999 the IUPAP General Assembly decided to create the Working Group on Women in Physics with the mandate of evaluating the situation of women physicists worldwide and suggesting ways to improve it. Since then, the Working Group has engaged in very intense activity that helped bring the issue upfront in many countries and made the physics community aware that there

was a problem that called for specific actions. It also prompted the formation of Working Groups in all continents leading to the creation of a network of women physicists that spans more countries than IUPAP members.

The first activities of the Working Group included: subcontracting the Statistical Research Group of the American Institute of Physics (AIP) to perform a survey to analyze the situation of women physicists, encouraging the creation of Working Groups in all IUPAP country members and organizing the International Conference on Women in Physics (ICWIP).

3.3 International Conference on Women in Physics (ICWIP). The first ICWIP, which took place in Paris in 2002, was attended by about 300 people from more than 65 countries. This conference established the main guidelines that were applied to all following ICWIPs. Participation is by country. This led to the formation of country Working Groups that are in charge of collecting local data on the situation of women physicists and of taking the necessary steps to induce change in their own place. To ensure a fair distribution of ICWIP participants from more or less developed countries, a limit has been set on the number of country team members that can attend a conference. Countries wanting to surpass this limit have to fund the attendance of as many participants from countries in financial need as the number of members they want to include beyond the limit. ICWIP participation is not limited to women. On the contrary, the Working Group recommends that a minimal country representation should include one senior female physicist, one female graduate student and one man. Since 2002, ICWIPs have been organized with the same frequency as the IUPAP General Assembly (once every three years) rotating throughout the world (Rio de Janeiro, Brazil; Seoul, South Korea; Stellenbosch, South Africa; Waterloo, Canada; Birmingham, UK). ICWIPs have five main types of activities: plenary talks given by recognized female physicists about their research with some recollection of their personal lives; parallel break-out sessions devoted to discuss issues directly related to gender equity in physics; poster sessions where country teams report on the situation of women physicists in their own country, a scientific poster session whose aim is to facilitate the establishment of research networks between participants and a final conference assembly. The outcome of the break-out sessions is to draft a set of recommendations that are presented, modified and voted on in the final assembly. All ICWIPs also organize outreach activities for school children and the general public destined to remove stereotypes and change the perception about women physicists.

3.4 Travel program. Another important activity of the Working Group is to award Travel Grants for young female physicists from developing countries to attend conferences or schools outside their home institutions. Between 20 and 40 awards of this type are given out in "non-ICWIP" years. Otherwise, the grants are used to fund attendance to the ICWIP. As a way to create awareness of the situation of women physicists, the working group has also decided to celebrate the International Women in Physics Day on February 11th, the same date as the International Day of Women and Girls in Science as

established by the United Nations. Finally, the Group is currently finalizing the elaboration of the "Waterloo Charter", a declaration of principles on inclusivity in physics with a set of guidelines that will be presented for approval of the Executive Council. This Charter is based on the rubrics of the Baltimore Charter and the Pasadena Recommendations that were formulated by the American Astronomical Society and is shaped and guided by the principles dictated by the Project Juno of the Institute of Physics in the UK.

3.5 Survey of Physicists. After the first two surveys of physicists, each of which were responded to by about 2000 women, the Working Group subcontracted the AIP Research Center to carry on a global survey of physicists that was available in 8 languages, remained open for one year in 2009–2010 and was responded by 15,000 people of all genders. This Global Survey showed the relevance of early educational experiences for choosing physics, that the personal lives of women physicists were more affected by their careers than those of men, that male physicists were more likely than women to have spouses who did not work and took care of home, and that women had their children earlier than men during their careers, something that was directly implicated in their slower progression. But the difficulties in making personal lives and careers compatible were not the only reason behind this (on average) slower progression. The survey also showed that women had a harder time than men finding certain professional opportunities such as international visits, invitations to speak, supervisory experiences, serving on influential committees, serving as journal editors and advising graduate students. The IUPAP is now collaborating with several other scientific unions in the Gender Gap in Science project which, with the leadership of IMU, has among its primary tasks the realization of a Global Survey of Scientists that will provide updated information on this situation not only about physics but also about mathematical, computing and natural sciences.

3.6 New guidelines for the IUPAP. Based on the general discussions at ICWIP, the Working Group elaborates recommendations and resolutions to be presented at the IUPAP General Assembly to be upheld by the union. Among the several recommendations that had a direct impact on IUPAP policies were: to include women in its commissions; to enforce that women be among invited speakers and serve on conference committees; to require that conferences had an associated outreach activity; to consider women for prizes and awards. Recommendations and resolutions can be found on the group's website, and also in the conference proceedings that are published after each ICWIP Hartline and Li [2002], Hartline and Michelman-Ribeiro [2005], Hartline, Horton, and Kaicher [2009], Cunningham [2013], and Cunningham, O'Riordan, and Ghose [2016].

More recently, the IUPAP has also decided to appoint one of its Vice-Presidents at large as Gender Champion. The role of this VP is to liaise with the Working Group to suggest gender-related policies to the Executive Committee and to oversee that they are observed. The existence of this position proved to be key in establishing new guidelines for IUPAP sponsored or endorsed conferences, among them, the requirement that at least 10% of the invited speakers and of conference committee members should be female,

that all conferences should have someone appointed to handle problems of harassment and that a special session on inclusion and diversity in physics should be included.

4 Discussion

Several questions were raised and answers were variously given by the chair, the panelists and people in the audience. We include some of them in what follows.

It seems that there are more women in mathematics in less developed countries than in more developed countries, is it really so ? It is certainly the case in many less developed countries, but not all. In Iran for example there are many women in mathematics, and it is also the case in several Asian countries, but it is much less so in Africa. It maybe related to the prestige of mathematics in a given country. Most of the time, the more prestigious the position, the less women! On the other hand, to have childcare or household help in developed countries is much more expensive than in less developed ones. There are also cultural differences to take into account. Both in the more developed and less developed countries, the percentage of women drops as we advance in the academic career. This phenomenon is similar in mathematics and in physics.

Is there a difference in the proportion of women in different areas of mathematics? If it is so, why ? The number of women in a specific area seems to have a lot to do with a major figure in the subject in the past, woman or man, encouraging women's participation. Some subfields are more congenial to young people, to women. Rather than asking women to be able to compete like men do, we need a more friendly atmosphere in the community that allows and values other types of behaviour. Maybe the values and the sociology of the community need to be revised so that it becomes more women-friendly and more inclusive in general.

Everbody agrees that we need women role models, but when we look at the prizes awarded by the ICM in 2018, we do not see any woman. Are there initiatives to push IMU for giving prizes to women ? There are efforts to have more women speaking at ICM. We have seen that there has been a change, the proportion of women speakers is about 15% now. Prizes are another issue since each committee works independently. In the prize committees, the percentage of women is reasonable. But, as we have seen, women in committees does not imply women receiving prizes. It is important to nominate women. If you believe in an outstanding woman, you need to make sure she is nominated. Very often the committees do not receive enough female nominations.

On the stage there were no woman prize winner, but five women were present: one banker representing a sponsor and four women helping from the staff. What image does that give to the young Olympiad medalists in the audience? Indeed, it was a real caricature. In ICM Seoul, Ingrid Daubechies, IMU president, Park Geun-hye, President of Korea, and Maryam Mirzakhani, first woman Fields medalist on stage at

the same time made a great picture of three strong women together! Unfortunately this great event remains a singular isolated point.

Should not we force a gender balance for the composition of committees? The committee percentages were not so bad. IMU, LMS and other organisations are making a lot of effort to have women in their committees. A complete balance would become impossible for the women researchers, as there are currently many fewer women than men.

Often, even women do not think of women, they think of men to nominate. How do we organize ourselves for prizes and committees? We need guidelines. It is true that often both men and women prefer men. This phenomenon is well documented. Identical job applications for academic positions submitted with one female name /one male name had different answers to the advantage of the male name, even though the evaluators included women! We need to modify such unconscious biases.

Acknowledgments. We would like to thank Caroline Series for her chairing of the panel and help in the editing of the report, Carolina Araujo for the local organization of the venue and the reception and Betül Tanbay for her notes on the discussion.

Many thanks to Ingrid Daubechies for her contributions to the discussion, and all the attendees of the panel for their active participation.

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MACHINE-ASSISTED PROOFS

James Davenport, Bjorn Poonen, James Maynard,
Harald Helfgott, Pham Huu Tiep and Luís Cruz-Filipe

Abstract

The panel took place on the 7th August 2018. After the moderator had introduced the topic, the panelists presented their experiences and points of view, and then took questions from the floor.

1 Introduction – James Davenport

1.1 A (very brief, partial) history.

1963 "Solvability of Groups of Odd Order": 254 pages,[1] Feit and Thompson [1963]. Also Birch and Swinnerton-Dyer [1963] published the algorithms underpinning their conjectures.

1976 "Every Planar Map is Four-Colorable": 256 pages + computation: Appel and Haken [1976].

1989 Revised Four-Color Theorem proof published by Appel and Haken [1989].

1998 Hales announced proof of Kepler Conjecture.

2005 Hales' proof published in an abridged form "uncertified",[2] Hales [2005b].

2008 Gonthier [2008] stated formal proof of Four-Color Theorem.

2012 Gonthier and Théry [2012] stated[3] formal proof of Odd Order Theorem , see also Gonthier, Asperti, et al. [2013].

2013 Helfgott [2013] published (arXiv) proof of ternary Goldbach Conjecture.

2014 Flyspeck project announced formal proof of Kepler Conjecture, Hales [2014].

MSC2010: primary 65Z05; secondary 52C23.
Keywords: Machine proof.

[1] "one of the longest proof to have appeared in the mathematical literature to that point." Gonthier, Asperti, et al. [2013].

[2] *Mathematical Reviews* states "Nobody has managed to check all the details of the proof so far, but the theoretical part seems to be correct. The whole proof is considered and assumed to be correct by most of the mathematical community." Tóth [2006].

[3] "Both the size of this proof and the range of mathematics involved make formalization a formidable task" Gonthier, Asperti, et al. [2013].

2015 Maynard [2015] published "Small gaps between primes".

2017 Flyspeck paper published: Hales, Adams, Bauer, Dang, Harrison, Hoang, et al. [2017].

The Odd Order Theorem is important, but chiefly because it leads to the classification of finite simple groups. One might ask when the classification will be formally proved, and indeed I did ask Georges Gonthier this question. He answered that for the Odd Order Theorem, he worked, not so much from Feit and Thompson [1963] itself as from Bender, Glauberman, and Carlip [1994] and Peterfalvi [2000], two substantial books which between them described much work simplifying and clarifying the argument, and that such work had yet to be done for the full classification.

1.2 Questions for Consideration. What are the implications for

- authors
- journals and their publishers
- the refereeing process (we note that, although Hales [2005b] took seven years not to be completely refereed, Hales, Adams, Bauer, Dang, Harrison, Hoang, et al. [2017] still took three years to be refereed: "formal" is not the same as "simple".)
- readers
- the storage and curation of such proofs, and, if necessary, the software necessary to run such proofs?

These questions are not independent: the refereeing process is run by journals, and one can ask whether the journal should keep the machine-readable proof, as with Hales [2005b], or whether Helfgott is right with "available on request", or maybe Maynard's "at `www.arxiv.org`".

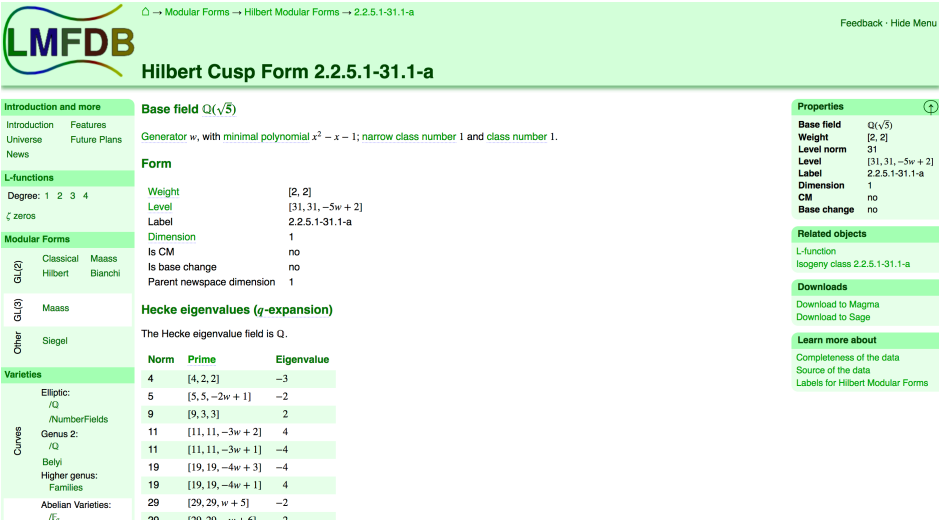
Acknowledgements. I am extremely grateful to Ingrid Daubechies for her hard work convening this panel.

2 Bjorn Poonen

2.1 Kinds of machine assistance. There are various forms of machine-assisted proof, and they need different approaches.

- **Experimental mathematics**: Humans design experiments for the computers to carry out, in order to discover or test new conjectures.
- **Human/machine collaborative proofs**: Humans reduce a proof to a large number of cases, or to a detailed computation, which the computer carries out.

Figure 1: Modular Forms Database



- **Formal proof verification**: Humans supply the steps of a proof in a language such that the computer can verify that each step follows logically from previous steps.

- **Formal proof discovery**: The computer searches for a chain of deduction leading from provided axioms to a theorem.

Cloud computing services make it possible to do computations much larger than most people would be able to do with their own physical computers.

Example 1. *My colleague Andrew Sutherland at MIT did a 300 core-year computation in 8 hours using (a record) 580,000 cores of the Google Compute Engine for about $20,000.* As he says,

> "Having a computer that is 10 times as fast doesn't really change the way you do your research, but having a computer that is 10,000 times as fast does."

2.2 Large databases. There are now many databases of mathematical facts, such as the Atlas of Finite Simple Groups (Conway, Curtis, Norton, Parker, and Wilson [1985]) and the Online Encyclopedia of Integer Sequences (Sloane [2003, 2007]). In number theory, there is the LMFDB — The L-functions and modular forms database (`www.lmfdb.org`), which I personally have found useful.

2.3 Refereeing computations. Almost all math journals now publish papers depending on machine computations.

- Should papers involving machine-assisted proofs be viewed as more or less suspect than purely human ones?

- What if *the referee is not qualified* to check the computations, e.g., by redoing them?
- What if the computations are *too expensive* to do more than once?
- What if the computation involves *proprietary software* (e.g., MAGMA) for which there is no direct way to check that the algorithms do what they claim to do?
- Should one insist on **open-source software**, or even software that has gone through a **peer-review process** (as in Sage, for instance)?
- In what form should computational proofs be published?

2.4 Some opinions.

- The burden should be on authors to provide understandable and verifiable code, just as the burden is on authors to make their human-generated proofs understandable and verifiable.
- Ideally, code should be written in a high-level computer algebra package whose language is close to mathematics, to minimize the computer proficiency demands on a potential reader.

Acknowledgements. Many thanks are due to my colleague Andrew V. Sutherland.

3 James Maynard

3.1 Opportunities and challenges of use of machines. The use of computers in mathematics is widespread and likely to increase.

This presents several **opportunities**:

- Personal assistant: Guiding intuition, checking hypotheses
- Theorem proving: Large computations, checking many cases
- Theorem checking: Formal verification

But equally it presents several **challenges**:

- Are computations rigorous?
- Are proofs with computation understandable to humans?
- How do we find errors?

3.2 My use of computation. I use computation daily to guide my proofs and intuition. Two particular kinds of computations occur in my proofs.

The first is when I want to understand the spectrum of infinite dimensional operator, but of course computers can't handle these as such. Then I consider well-chosen finite-dimensional subspace and do explicit computations there.

- Do non-rigorous computation to guess good answers, then compute answers using exact arithmetic as in Maynard [2015].
- Happy trade-off between quality of numerical result and computation time.

The second is when, after doing theoretical manipulations, I need to show that some messy explicit integral is less than 1, as in Maynard [2013].

- My computations are non-rigorous!
- Very difficult to referee — many potential sources for error.

4 Harald Helfgott

4.1 The varieties of machine assistance: Main issues. When we say that our work on a proof is "machine-assisted", we may mean any of several related things. First, we may simply refer to exploratory work – computations and plots that give us an idea of which statements are true. We may also speak of computations that are truly part of a proof – dealing with case-work, say, or numerical computations, which of course ought to be rigorous. Lastly, we may speak of *formal proofs*: their production and verification could in principle be done by hand, but part of their point is that they can be verified mechanically; moreover, as far as the working mathematician is concerned, they were essentially a useful fiction before computer assistance in their creation became possible.

My present concerns center on the second kind of machine assistance just listed, namely, the use of computations – numerical or symbolic – as part of a proof. The first issue to discuss is what is meant by rigorous computation. (That would be less of an issue for exploratory work, where all one wants is results that are sufficiently precise and almost certainly correct.) Another issue is that of error. Are computer errors at all likely? How do we reduce the possibility of programming errors, or, more generally, errors in machine-human interaction? For that matter, can computers help us check for errors in proofs altogether, whether the process of proof was computer-assisted or not?

This last question does lead naturally to the matter of formal proofs. That matter gives rise to another sort of concern: what is the meaning and purpose of a proof, for us, as human beings practicing mathematics? I will not do more than touch upon that domain. What can be discussed more factually is the extent to which formal proofs are already a reality, particularly in number theory, and *how* they are a reality; such a discussion has to precede and inform more philosophical considerations.

4.2 Rigorous computation. We will be first of all concerned with the production of numerical results that are guaranteed to be correct. (The same matter goes by the

names of "reliable computing", or "validated numerics".) In exploratory work, we may ask, say, for the value of $\sin(0.1)$; however, a moment's thought shows that the true numerical value of $\sin(0.1)$ cannot even be stored in a computer – a decimal or binary expansion has to be cut off at some point. What can we do?

4.2.1 Basics on interval arithmetic. Rigorous computations with real numbers must take into account rounding errors and other sources of lack of precision. As we were saying, a generic element of $\mathbb{R}$ cannot even be represented in a computer. A transcendental number is, in general, given by an infinite sequence of digits, and computers have only finite space. Furthermore, computers are built so that they deal with integers, and, by extension, with rational numbers; they are fastest when working with rationals whose denominators are powers of 2.

Interval arithmetic provides a way to keep track of rounding errors automatically, while providing data types for work in $\mathbb{R}$ and $\mathbb{C}$. The basic data type is an *interval* $[a, b]$, where $a, b \in \mathbb{Q}$. For obvious reasons of efficiency, we may restrict a and b to be elements of the localization Q_2 of $\mathbb{Z}$ by the powers of 2, i.e., the ring Q_2 of rationals whose denominators are powers of 2. (We also may decide to work with $Q_2 \cup \{-\infty, \infty\}$ instead of Q_2.)

A procedure is said to implement a function $f : \mathbb{R}^k \to \mathbb{R}$ if, given $B = ([a_i, b_i])_{1 \le i \le k}$ where $a_i, b_i \in Q_2$, the procedure returns an interval $[a, b]$ containing $f(B)$, with $a, b \in Q_2$. Of course, one would expect a good implementation not to return an interval (a, b) much larger than it needs to be.

Interval arithmetic in this sense was first proposed in the 1950s and 1960s (Moore [1966]; see also Young [1931], Warmus [1956], Sunaga [2009], among others). There are several commonly used open-source implementations. The package ARB (Johansson [2017]) implements a slight variant, *ball arithmetic*. A good interval-arithmetic package should implement not just the four basic operations and some basic functions such as exp, log or sin, but also as many commonly used transcendental functions as possible.

The main drawback of interval arithmetic is its time consumption. Very roughly speaking, a procedure coded using a good interval-arithmetic package will typically take about 8 times as long as the same procedure coded without interval arithmetic.

There is also the fact that the underlying floating-point routines must give validated results, that is, results with a precise bound on the error term; it is best if results are correct up to the last bit, with whatever rounding type is specified, as all further bits after the last guaranteed bit are of course useless. Correctly rounded floating-point results may feel like a basic right, but the fact is that most processors do not guarantee this arguable right for anything other than the four basic operations, and in fact often violate it. Thus, transcendental functions have to be implemented in software. See, e.g., Daramy-Loirat, Defour, de Dinechin, Gallet, Gast, Lauter, and Muller [2006]. (Of course, it may not be fair to count the need for a software implementation as part of the overhead of interval arithmetic; we should not use wrong routines implemented in hardware to begin with.)

The speed of computers in our days makes large-scale computations in interval arithmetic possible; see, for instance, D. Platt's verification of the Riemann Hypothesis for zeroes with imaginary part $\leq 1.1 \cdot 10^{11}$ (Platt [2011]).

Alternatives. It is possible to avoid interval arithmetic (or rather its usage during the computation, with the attendant overhead) by keeping track of rounding errors *a priori*, that is, analyzing carefully to what extent an error of a given size in the input to a given procedure can affect the output. This is one of the classic subjects of numerical analysis (Wilkinson [1994], Higham [2002]). One clear drawback is that proceeding in this fashion for anything outside a small set of well-studied tasks involves saving computer time at the expense of human time, and creates one more occasion for human error.

It would make sense for it to be possible to be able to analyze rounding errors a priori *with the help of a computer*. (The same goes for errors that result from, say, truncating a series.) Systems for doing as much exist, and in fact produce formal proofs (q.v.): FPTaylor (which produces certificates for the formal system HOL Light), Gappa (which gives certificates for the formal system Coq), Real2Float (Coq again)…

However, none of these systems can treat loops, at least not if the floating-point computation carried out in an iteration of the loop depends on the floating-point computations carried out in previous iterations. This limitation is severe: it makes these systems unusable for one the kinds of computations most common in number theory, namely, a computation or verification involving sums or some other quantities defined by an iterative or recursive process. (For an example, see the discussion of the sum $m(x)$ in section 4.3.)

On the other hand, systems for analyzing rounding errors can be and have been used to verify that at least some of the routines used by interval-arithmetic packages are correct. Moreover, there are presently tools that, if developed further, should become able to give formal certificates for at least some kinds of computations with loops in a fairly near future.

4.2.2 Comparisons. Maxima and minima. In exploratory work, it may be fine to plot two functions $f, g : I \to \mathbb{R}$, where $I = [0, 1]$ (say) and decide that $f(x) < g(x)$ for all $x \in I$ because the graph shows that it is so. Of course, that would not do as a proof. Fortunately, just as we can let a computer plot f and g, we can let a computer prove that $f(x) < g(x)$ for all $x \in I$.

In fact, it is particularly simple to do so by means of interval arithmetic. Let f and g be implemented in interval arithmetic. If $f(I)$ and $g(I)$ do not overlap, we obtain either that $f(x) < g(x)$ for all $x \in I$ (if $f(I)$ is to the left of $g(I)$) or that $f(x) > g(x)$ for all $x \in I$ (otherwise). If $f(I)$ and $g(I)$ do overlap, we divide I in half, and recur: we do what we have just described to each half. We stop when we obtain that the statement we are trying to prove is false on some subinterval, or when we have divided I into subintervals on each of which the statement is true. While this approach runs into difficulties for x close to a point x_0 such that $f(x_0) = g(x_0)$, we can usually resolve such a situation by comparing the derivatives or higher derivatives of f and g at or near x_0.

A very similar version of the bisection method in interval arithmetic (combined, if necessary, with automatic differentiation) can be used to locate maxima and minima, as well as roots.

4.2.3 Numerical integration (quadrature). There are several well-known ways to estimate an integral and bound the error in the estimate, starting with, say, the trapezoid rule, or Simpson's rule, and including Euler-Maclaurin or Gaussian quadrature, for instance. It is generally possible to implement these methods in interval arithmetic.

Matters can become more complicated if the integrand is not everywhere differentiable, or even if it not differentiable on the whole *closure* of the interval we are working on. A common case is that of an integrand of the form $|f(x)|$, where f is everywhere differentiable, but $|f|$ is not differentiable at all zeroes of $f(x) = 0$. That case and several other ones ought to be done automatically; they still require ad-hoc work at the time of writing.

Even more ad-hoc work is required (for now) if we must compute a complex integral. For instance, it is possible to show that, for $P_r(s) = \prod_{p \le r}(1 - p^{-s})$ and R the straight path from $200i$ to $40000i$,

$$\frac{1}{\pi i}\int_R |P(s+1/2)P(s+1/4)|\frac{|\zeta(s+1/2)||\zeta(s+1/4)|}{|s|^2}|ds| = 0.009269 + \text{error},$$

where $|\text{error}| \le 3 \cdot 10^{-6}$. However, "it is possible" means for now "one can obtain this result by e-mailing ARB's author (F. Johansson), editing the code he kindly sends you, and then running it overnight". It is clear that the situation should (and will) improve.

4.3 Combining asymptotics and computations. Let us now discuss the way in which the need for computations typically arises in number theory. (It is not the only way; one can also need, say, verifications of finite chunks of the Riemann or Generalized Riemann Hypotheses of the kind we have already mentioned.) Often, methods from analysis yield estimates of the following form:

$$\textbf{expression}(n) = f(n) + \text{error}, \tag{1}$$

where $|\text{error}| \le g(n)$ and $g(n)$ is much smaller than $|f(n)|$ for n sufficiently large. The question is what to do when n is not sufficiently large.

One answer is, of course, that one may compute **expression**(n) for n not sufficiently large, that is, n smaller than a constant N. If we reduce our bound $g(n)$, then we are reducing the constant N such that $g(n)$ is sufficiently smaller than $|f(n)|$ for $n \ge N$; reducing N to a reasonable level is thus a challenge for us as analysts. Computing **expression**(n) reasonably efficiently for $n < N$ is a challenge for us as programmers, or, what is almost always more interesting, as algorithm designers. Whether the algorithm for computing **expression**(n) for $n \le N$ runs in time $O(N)$, $O(N^2)$ or $O(N^3)$ is obviously more important than the extent to which the code has been optimized.

An interesting example is given by the proof of the ternary Goldbach conjecture, taken as a whole. (The first version of the proof is available online, both as a series of preprints and as a book draft (Helfgott [2015]); the version to be published is in

preparation.) As is well-known, the ternary Goldbach conjecture states that every odd integer $n \geq 7$ can be written as the sum of three prime numbers.

Almost all of the effort involved in the proof went into proving an estimate of the form (1) for **expression**(n) defined to be the number of ways one can write n as the sum of three primes p_1, p_2, p_3 (counted with certain weights). It was then very easy to show that $g(n) < |f(n)|$ for $n \geq 10^{27}$ odd. It followed that every odd number $n \geq 10^{27}$ can be written as the sum of three primes, i.e., ternary Goldbach holds for $n \geq 10^{27}$. It remained to show that every odd number $7 \leq n \leq 10^{27}$ can also be written as the sum of three primes; then the ternary Goldbach conjecture would follow.

Checking each odd number $\leq 10^{27}$ is out of current computational reach. However, there is the following well-known trick. We can easily construct an increasing succession of primes, all of them $< N$ but for the last one, such that the difference between any two consecutive primes in the list is at least 4 and at most $M - 2$, say; the time taken is not much larger than $O(N/M)$. (Platt and I checked that for $N = 8.875 \cdot 10^{30}$ and $M = 4 \cdot 10^{18}$; by now, there is even a formal proof of the same for $N = 10^{28}$ and $M = 4 \cdot 10^{18}$ (see MARELLE [2014]).) Then we can use a verification by brute force that the strong Goldbach conjecture holds for all even numbers $4 \leq n \leq M$, i.e., every even number $4 \leq n \leq M$ can be written as the sum of two primes. It then follows that every odd number $7 \leq n \leq N$ can be written as the sum of three primes: just let p be the largest prime in the sequence such that $n - p \geq 4$, and apply strong Goldbach to $n - p$, which has to be at most M. We obtain that $n - p = p_1 + p_2$. Thus $n = p + p_1 + p_2$, and so we are done. The total time taken is not much larger than $O(M + N/M)$, so we can simply set $M = \sqrt{N}$, and obtain an algorithm running in time not much more than $O(\sqrt{N})$, which is not too much for $N = 10^{27}$.

As it happens, it had already been checked that strong Goldbach holds for all even $n \leq M = 4 \cdot 10^{18} + 2$ (Oliveira e Silva, Herzog, and Pardi [2014], plus a trivial check for $4 \cdot 10^{18} + 2$). Hence, we can simply set $M = 4 \cdot 10^{18} + 2$, and the proof of the ternary Goldbach conjecture is done.

Let us see a different, much smaller example. We would like to bound the sum $m(x) = \sum_{n \leq x} \mu(n)/n$ for general $x > 0$. (That sum and its variants play an important role in the proof of ternary Goldbach.) The strongest explicit bound is due to Ramaré [2015], who proved that

$$|m(x)| \leq \frac{0.0144}{\log x} \tag{2}$$

for $x \geq 96955$. (We would have a bound of $O_\epsilon(1/x^{1/2-\epsilon})$ if we assumed the Riemann hypothesis, but that is out of reach; there are also unconditional non-explicit bounds that are qualitatively much stronger than (2) as $x \to \infty$, but there are serious obstacles to making them explicit with constants reasonable enough for the bounds to be usable.)

Clearly, we can compute $m(x)$ for all small integer values of x. I wrote a C program using interval arithmetic to do so, and obtained, after a couple of weeks on a rather ordinary PC, that

$$|m(x)| \leq \sqrt{2/x} \tag{3}$$

for all real $0 < x \leq N = 10^{14}$, with 0.569449 instead of $\sqrt{2}$ if $x \geq 3$ is assumed. Note that, if a standard conjecture holds, the bound (3) is actually too strong to hold for all x.

Is it overshooting to test (3) for all $x \leq 10^{14}$? Perhaps, but, since the obvious algorithm for checking (3) for all $x \leq N$ takes time essentially linear on N, we might as well allow ourselves a large N. Also to the point – it is possible to combine (2) and (3) to prove a result valid for all x, small and large: given any $y > 1$ and any $0 < x \leq y$,

$$|m(x)| \leq \sqrt{\frac{2}{x}} + 0.0144 \cdot \frac{y^c}{\log y} \frac{1}{x^c},$$

where $c = 1/\log N = 1/\log 10^{14}$. The bound here is a sum of powers of x, something highly convenient for several purposes. The larger N is, the better the bound.

Of course, in this example, it was easy to give a fast algorithm. For some other sums I had to deal with, the obvious algorithm would have run in time $O(N^2)$ or $O(N^3)$, and finding an algorithm that would run in, say, time $O(N \log N)$ and space roughly $O(\sqrt{N})$ was a challenge.

4.4 Error avoidance. In practice, computer errors in the strict sense are barely an issue, except perhaps for very large computations. (By "computer error" we mean a malfunction in a correctly designed circuit. There is also the issue of incorrectly designed circuits, such as those in the original Pentium chip, whose flaw, as it happens, was found through a computation in number theory (Nicely [n.d.], Cipra [1995]). Such situations are nowadays largely avoided through *formal verification* – another interesting area.) Computers have long had various inbuilt tools (e.g., checksums) for error detection and correction. Moreover, there is the obvious fact that the same computation can be run time and again, possibly on different hardware. Such a thing would be cumbersome or expensive only if the computation were very large indeed. (Admittedly, that is also the case when the probability of computer error might not be overwhelmingly negligible.)

Almost all of the time, the actual issue is human error: errors in programming, and errors in input/output. Of course, human beings also make mistakes when they don't use computers. There are conceptual mistakes, and then there are silly errors, especially in computations or tedious case-work. Presumably, whatever discipline we adopt to avoid errors in programming and input/output can also be adapted to weed out silly mistakes in general.

Before we discuss formal proofs, we should touch upon the more humble matter of everyday discipline.[4] Here I can simply describe my current practice, particularly in the context of my proof of the ternary Goldbach conjecture.

When facing computational tasks, I tend to program a great deal at first. However, I try to minimize the number of programs used in the submitted version of a book or paper, and keep them short. Nowadays I classify tasks in two categories. For time- and space-intensive computations, I am whittling down my code to a few programs in C,

[4]Specialists in formal proofs remind me that "everyday discipline" in the sense I am about to discuss is also an issue for them.

submitted or to be submitted to referees. The programs are or will be available upon request.

For small computations, I now use Sage/Python code, almost all of it included in the TEX source of the book. (The code for small computations that take more than a couple of seconds has been submitted together with the programs in C just mentioned.) It is an easy matter to keep Python code brief and readable. Thanks to SageTEX, input and output are automated, in that the output of Sage/Python code is displayed automatically in the file that TEX outputs. For instance, given two variables called `result1` and `result2`, with values 5 and 7, we can type the TEX code

```
Hence,  $f(x)\leq \sage{result1 + result2}$.
```

and obtain

Hence, $f(x) \leq 12$.

In this way, the possibility of human error in copying input/output and updating material is reduced. All code can be run afresh between two compilations of the TEX file; indeed, I run it afresh (and have to run it afresh) after any change to any code in the file. Someone downloading the source code may decide to run the code or not; it is very easy to do so for anyone with Sage installed. It is also very easy to display all code; all that is involved is switching a single flag at the beginning of the source code.

There is a question that cannot be avoided, namely, whether it is right to use a large computer-algebra system in a proof. It is in general unsatisfactory to say "By M.,…"[5] before a claim, as some papers do. Consequently, in the first version of my proof of ternary Goldbach, I used Sage only in an exploratory role, and did all coding in C. At the same time, Sage is open-source, peer-reviewed and highly modular (though some of its larger, older components may not themselves be highly modular or peer-reviewed in quite the same way). Because Sage is modular, we depend only on the correctness of the relatively small parts of it that we use in the proof; in my case, that amounts to Python, basic symbolic algebra and ARB. In part for this reason, and in part for the reasons outlined above (code readability, code organization, reduction of human error in input/output), I decided that the advantages of using Sage/Python for small computations that can be embedded in the TEX code overwhelmed any reasonable misgivings one might have about using Sage in a proof in this particular way.

4.5 Automated proofs. Formal proofs.

4.5.1 Automated proofs in practice. Informally speaking, Gödel's incompleteness theorem states that, in a general axiomatic system – in particular, in any finite axiomatic system that includes basic arithmetic, including exponentiation – there are truths that cannot be proved starting from the axioms. In particular, we cannot ask a machine to prove all true statements.

Moreover, even in axiomatic systems that *are* complete, the problem of finding a proof may be (and in fact is) computationally hard. Indeed, the first problem to be

[5] Here M. stands for Mammoth, or anything else that is closed-source and very large.

proven to be NP-complete was that of Boolean satisfiability (SAT); in other words, if we could determine rapidly whether the value of the variables in a Boolean formula can be set so that the formula is true, we could solve rapidly *any* problem in a very broad class (essentially, any problem whose solution can be verified rapidly).

Does that mean that fully automated theorem proving is hopeless? Not necessarily. First, a machine could prove *some* true statements, even if it cannot prove every true statement; after all, human beings are in that same situation. Second, a computer can proceed by heuristics to solve quickly some cases of a problem that is computationally hard in general. Third, a problem may be small enough (in terms of its number of variables, say) that an inefficient algorithm running on a computer can still solve the problem in a reasonable time, even when its solution would be non-obvious or cumbersome to a human.

I have had little experience with automated theorem proving – and in fact was surprised to see that it could be used in practice at all. At some point, I wanted to prove the following lemma: for all $0 < x \le y_1, y_2 < 1$ with $y_1^2 \le x$, $y_2^2 \le x$,

$$1 + \frac{y_1 y_2}{(1 - y_1 + x)(1 - y_2 + x)} \le \frac{(1 - x^3)^2(1 - x^4)}{(1 - y_1 y_2)(1 - y_1 y_2^2)(1 - y_1^2 y_2)}. \tag{4}$$

Now, this is a statement in the theory of real closed fields, which is in fact complete; a proof has to exist. It turns out that there is a freely available program QEPCAD (Hong and Brown [2011]) implementing an algorithm that proves statements in the theory of real closed fields (by *CAD*, that is, cylindrical algebraic decomposition, Collins [1975]). Can it deal with (4)? Not on my desktop, and that is not surprising: the computational complexity of CAD is doubly exponential in n, with base proportional to the maximal degree of the polynomials involved. However, thinking a little, it is not hard to eliminate a variable in (4), in that one can show that the maximum of the left side minus the right side has to be attained when either $y_1 = y_2$, $y_i = \sqrt{x}$ or $y_i = x$ holds for at least one of $i = 1, 2$. QEPCAD proves the resulting inequality in two variables with ease.

Since (4) has quantifiers of only one kind ($\forall x \forall y_1 \forall y_2$), there are alternative algorithms that solve the problem in time exponential on the number of variables (see Basu, Pollack, and Roy [2006, Ch. 11] and references therein). However, there does not seem to be a practical, reasonably efficient implementation of the exponential-time algorithm commonly available just yet.

In the end, I found an alternative way to prove what I truly wanted, and so I no longer needed (4) at all. It was still interesting to learn that automatic provers could sometimes establish useful auxiliary lemmas. (Of course, in some sense, the simple combinations of the bisection method and interval arithmetic we have already discussed also fall into this category.) It was also interesting to learn of the extent to which practice still fell behind theory – and of how "computationally hard" does not always mean "hopeless" in practice.

Again, my personal comments should be understood as coming from the possibly naïve perspective of a number theorist. I have recently learned that Boolean-satisfiability (SAT) solvers are being used intensively in combinatorics. There, one of the main issues

is error in the human/computer interface: the interface of an automated solver is not necessarily intuitive, and some claimed proofs in the literature are the result of inputting the problem into a SAT solver incorrectly.

At any rate, it seems clear that the role of automated provers in the foreseeable future will be one of *assistance* to the human prover. Thus we are led to our next, broader subject.

4.5.2 Formal proofs and proof assistants. Contrary to what we may sometimes tell ourselves, what we call a proof in our everyday practice is not quite the same as what we would call a proof in a logic course. The latter – called a *formal proof* for clarity – is a sequence of symbols whose correctness is a purely syntactic property that can be checked by a monkey grinding an organ. In contrast, a proof, for the working mathematician, is a convincing argument that can in principle be turned into a formal proof.

Until relatively recently, "in principle" came with an enormous caveat: none but the simplest sort of proofs was turned into formal proofs. Our imaginary organ and monkey were first replaced by a (real) computer system in 1967 (De Bruijn [1970] AUTOMATH), but the task of producing a formal proof remained solely in the hands of humans, and was very cumbersome.

The situation has gradually changed thanks to *proof assistants*, programs whose role it is to help a human being write a formal proof. There is now a number of such assistants: Mizar, Coq, Isabelle, HOL Light,…

One notable recent success was the second proof of Hales' sphere packing theorem. As is well-known, the first proof (Hales [2005b]; see also Lagarias [2011]) was computer-assisted in the more traditional ways discussed in previous sections. That first proof met with some misgivings: it involved a great deal of case-work (thus increasing the possibility of human error, in part because the refereeing process became very onerous), and the computer code ran up to about 40000 lines (making the code practically impossible to referee). The second-generation proof in Hales, Adams, Bauer, Dang, Harrison, Hoang, et al. [2017] is a formal proof, written and verified by means of the Isabelle and HOL Light proof assistants.

In which fields of mathematics is such an effort now feasible? Large areas, including much of basic real analysis, remained uncovered until recently. As far as analytic number theory is concerned: the first formal proof of the Prime Number Theorem by Avigad, Donnelly, Gray, and Raff [2007] was constructed in 2005, by means of Isabelle; it is based on Selberg's elementary proof, which is often seen as more difficult, or less natural, than more traditional proofs by complex analysis. The first analytic proof, based on Newman's simplified version of the traditional approach (Newman [1980]), was given in Harrison [2009]; it relied on the fact that some of real and complex analysis had become available in HOL Light (Harrison [2005, 2007]), and also necessitated giving formal proofs of some basic facts about the Riemann zeta function.

In analytic number theory, we are, then, perhaps just past the beginning. The next natural step would be to formalize much of a first-year graduate textbook – say, the results in H. Davenport [1967] and Montgomery and Vaughan [2007], possibly with

different proofs, together with some sieve theory, and also Titchmarsh [1986]; then we would need large parts of Iwaniec and Kowalski [2004]. (Replace chapters of the older books here by more modern sources when needed.) Then we would be able to start working on newer or new material. Of course, one can also proceed backwards, setting oneself a challenge (several of the major results in Iwaniec and Kowalski [ibid.] would do nicely) and working backwards from it, proving whatever basic results one needs, much as Gonthier, Asperti, et al. [2013]. did with the Feit–Thompson Theorem.

This last strategy is, in a sense, similar to what I did in the end for ternary Goldbach; I had to ask about, learn about, and prove explicit analogues of many basic results in analytic number theory. Sometimes, to my naïve surprise, I had to do without a standard technique or result, since no practical explicit analogue existed or could realistically be proved. We will presumably face the same kind of challenge when we try to give formal proofs of the main results of twentieth-century number theory, including, why not, Vinogradov's three-prime theorem.

What about formal proofs of explicit results, or of statements whose proofs make crucial use of explicit intermediate results? Why not a formal proof of ternary Goldbach? Why not indeed, in the long run, but we have to be realistic about the fact that both making a proof explicit and making it formal take plenty of work, and of course can also lengthen the proof greatly. It may be that the de Bruijn factor – that is, the informal quantity defined as the ratio of the length of the formal version of the proof to the original, conventional ("informal"?) version, however stored – is lower for some explicit results, as their proofs tend to include a level of detail not always needed for non-explicit results. Time will tell. Of course one can also say that formal proofs are more sorely needed for explicit results; while, in general, they do not elicit the same suspicion as results with plenty of case-work, they may be more likely to be in the "fixable, but incorrect as stated" category than their non-explicit counterparts. (Asymptotic notation is a carpet under which both known and unnoticed dirt can be conveniently swept.) Thus it would be a very worthwhile task – for the fairly near future – to give formal proofs for basic explicit estimates that are used again and again.

4.6 Final considerations. A point often made in connection with computer-assisted proofs is that the purpose of proof is not just to establish truth, but to demonstrate and advance understanding. The worth of a proof understandable to no one would thus be limited.

This viewpoint is valid, but may not really be specially relevant to computer-assisted proofs as they are currently developing. It could certainly be a point against automated provers, particularly if their output is not intelligible. However, in most fields, automated provers seem likely to continue playing at most an auxiliary role, proving small lemmas that would be cumbersome and not particularly enlightening for a human to prove. (Does (4) have any "meaning"?) Computer-assisted formal proofs are a different matter: there, we start with a proof in the everyday sense, that is, a proof that is produced by human beings and understandable to a (hopefully proper) superset of the same; we then formalize it, with the help of a computer, having the more solid establishment of truth as the primary or sole aim.

As for case-work: while it is often locally easy to understand (so to speak), it is true that it can feel meaningless, unaesthetic, and an invitation to error by means of tedium. However, that is an issue with case-work in general, and not with computer-based case-work in particular. Computers can, at least, play a role in reducing or eliminating error from case-work.

Moreover, the kind of finite verification typical of number theory cannot really be said to be case-work in this sense. Instead of having many slightly different cases, we have typically a single equality or inequality that must be verified for very many integers, or for all values of a variable under a certain threshold. Moreover, what we verify is often far from meaningless: if a computation verifies that the first 10^9 non-trivial zeroes of the Riemann zeta function lie on the critical line, or that $\sum_{n \leq x} \mu(n)$ is bounded by not much more than $\sqrt{x}$ for all $x \leq N$, we are verifying finite parts or finite consequences of standard conjectures that we have very good reasons to believe in – and good reasons to believe out of reach.

The assistance of computers can lead us both to results that were previously unattainable and to a higher standard in certainty and rigor. It may be some time before the standard of rigor set by formal proofs becomes widespread, but we have come to the point where computers, correctly used, can quell misgivings rather than give rise to them.

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5 Pham Huu Tiep

Machine-Assisted Proofs in Group Theory and Representation Theory. Typically, many proofs in mathematics rely on *mathematical induction*. In group theory and representation theory, this inductive approach often follows a modified strategy, which can be described as follows. Suppose the goal is to prove a certain statement $(\mathcal{P})$ concerning a (finite or algebraic) group G.

(i) Then the first step is to prove a *reduction theorem* to reduce to the case where G is (very close to being) *simple*, using perhaps the *Classification of Finite Simple Groups*. (One should note that these reduction theorems usually require one to establish a much stronger condition $(\mathcal{P}^*)$ for simple groups than the original condition $(\mathcal{P})$. See Isaacs, Malle, and Navarro [2007] and Navarro and Tiep [2011] for some recent reduction theorems.)

(ii) Next, one works out a uniform proof, which handles the simple groups G of *large enough* order.

(iii) The *induction base* is then to treat all "small" simple groups G.

In either strategy, the induction base usually needs a completely different treatment, rather than the uniform case of large groups, which often involves the sometimes massive use of computer calculations.

Let us illustrate this on the example of the proof of the *Ore conjecture*:

Conjecture 2 (Ore [1951]). *Every element g in any finite non-abelian simple group G is a commutator, i.e., can be written as $g = xyx^{-1}y^{-1}$ for some $x, y \in G$.*

Many important but partial results on the conjecture were established by Ore himself, Miller, R. C. Thompson, Neubüser–Pahlings–Cleuvers, and most notably, Ellers–Gordeev. The conjecture was finally proved in 2010:

Theorem 3 (Liebeck, O'Brien, Shalev, and Tiep [2010]). *Conjecture 2 holds for all finite non-abelian simple groups.*

Even building on all previous results, the proof of this "LOST-theorem" is still 70 pages long. So how does this proof go? A detailed account of it was given in the 2013 Bourbaki seminar Malle [2013]. A key ingredient of the proof is the following formula, where $\mathrm{Irr}(G)$ denotes the set of all complex irreducible characters of the finite group G:

Lemma 4 (Frobenius character sum formula). *Given a finite group G and an element $g \in G$, the number of pairs $(x, y) \in G \times G$ such that $g = xyx^{-1}y^{-1}$ is*

$$|G| \cdot \sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(g)}{\chi(1)}.$$

So in order to show that a given element $g \in G$ is a commutator, one just needs to show that $\sum_{\chi \in \mathrm{Irr}(G)} \chi(g)/\chi(1) \neq 0$. Now, let G be one of the groups in the induction base for the proof by Liebeck, O'Brien, Shalev, and Tiep [2010] of the Ore conjecture.

(a) In many cases, the character table of G is well known, either published and/or publicly available, in which case we can just use Lemma 4.

(b) In the remaining cases, where the character table of G is not available, but $|G|$ is not too large, then we construct the character table of G and proceed as before. To construct the character table of such a group G, one starts with a "nice" presentation or representation of G. Then one can try to use various operations with group characters to produce enough characters of G to generate the full group $\mathbb{Z}\mathrm{Irr}(G)$ of complex characters of G. With respect to the usual inner product

$$[\alpha, \beta] = \frac{1}{|G|} \sum_{x \in G} \alpha(x)\overline{\beta(x)},$$

$\mathbb{Z}\mathrm{Irr}(G)$ is a Euclidean lattice, whose minimal vectors are (up to sign) precisely the *irreducible* characters of G that we are after. So one can try to follow, say, the *LLL-algorithm* to find these irreducible characters. In practice (and in Liebeck, O'Brien, Shalev, and Tiep [ibid.]), one can use Unger's algorithm (Unger [2006]), implemented in MAGMA (Bosma, Cannon, and Playoust [1997]).

(c) But some of the groups G in the induction base of Liebeck, O'Brien, Shalev, and Tiep [2010], like $\mathrm{Sp}_{10}(\mathbb{F}_3)$, $\Omega_{11}(\mathbb{F}_3)$, or $\mathrm{U}_6(\mathbb{F}_7)$, are still too big for the computation in (b). For these too-big groups G, we implement another strategy. Namely, for any

given $g \in G$, we run a randomized search for $y \in G$ such that y and gy are conjugate. (In fact, one needs to do it only for one representative of each conjugacy class of G. So, for the largest sporadic simple group, the *Monster*, of order about $8 \cdot 10^{53}$, one needs to work with only 194 such representatives.) Once such a y can be found, then we have $gy = xyx^{-1}$ for some $x \in G$, i.e., $g = [x, y]$, which is what we wanted to show!

The first obvious question that arises is: How *long* was this computation? All in all, it took us about 150 weeks of CPU time of a 2.3GHz computer with 250GB of RAM to complete (by April 2008) all the computations needed for the proof in Liebeck, O'Brien, Shalev, and Tiep [ibid.] of the Ore conjecture. (Certainly, this amount of CPU time could be significantly reduced with the better computational algorithms available now, ten years later.)

The second, and more important, question is: How *reliable* is this computation? In the cases of (a) and (b) where we used or computed the character table of G, the relevant character tables have been subjected to various checks which attest to their accuracy. The tables are also publicly available in character table libraries, so they can be checked by others and used for independent verification of the conjecture. Next, in the larger cases of (c), the randomized computation was used to find y that gy and y *should* be G-conjugate for a given g, and then one checks directly (using the given presentation or representation of G) that gy and y are *indeed* G-conjugate. One should notice that this is quite different in nature to other machine-assisted proofs which reduce an elaborate proof to many cases – each is then decided by machine, often reporting "yes" or "no" to the existence of some object.

In group theory and representation theory, as in many other areas of mathematics, perhaps even more important than the machine-assisted proofs are

- *machine-assisted discovered theorems*, and
- *machine-assisted discovered counterexamples*.

Let us mention a couple of examples of these two kinds.

- The *Galois–McKay conjecture* was formulated by Navarro [2004] after many, many days of computing in GAP [2004].
- Also after some long experimental computations with the symmetric groups S_n, $n \leq 50$, Isaacs, Navarro, and I found a natural *McKay correspondence* (for the prime 2), which should hold for all symmetric groups, and which was subsequently proved in the joint paper with Isaacs, Navarro, Olsson, and Tiep [2017].
- An old conjecture in character theory states that if a finite group G is rational (that is, all $\chi \in \mathrm{Irr}(G)$ are rational-valued), then so are the Sylow 2-subgroups of G. However, after some long (but directed!) search using the "SmallGroups" databases contained in the computer packages GAP [2004] and Magma (Bosma, Cannon, and Playoust [1997]), Isaacs and Navarro have been able to find two counterexamples of order $2^9 \cdot 3$ to this conjecture, see Isaacs and Navarro [2012].

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6 Luís Cruz-Filipe

The awareness of the mathematical community towards the use of computers in proofs of mathematical results started in 1976 with the announcement of the proof of the Four-Color Theorem in Appel and Haken [1976].

However, there are earlier examples of proofs that were partially or completely done by a machine. An interesting example is Floyd and Knuth's proof of optimality of the 16-comparator sorting network on 7 inputs (Theorem 5 in Floyd and Knuth [1973]), which starts:

> "This theorem was proved by exhaustive enumeration on a CDC G-21 computer at Carnegie Institute of Technology in 1966."

Optimality of sorting networks is an interesting problem in combinatorics. The question we want to answer is: what is the length $S(n)$ of the shortest sequence of compare-and-swap operations (i.e., atomic operations consisting of sorting a pair of values) that will sort all inputs of a given length n? Floyd and Knuth's work cited above addresses this problem for $2 \leq n \leq 7$. Except for the values of $S(4)$ and $S(6)$, which are derived from those of $S(3)$ and $S(5)$ by application of a more general theorem, all values are derived by exhaustively enumerating the possible sequences of length $S(n) - 1$ and showing that, for each of them, there is an input of length n that they do not sort. As the authors note in their conclusion, this method is however "quite unsatisfactory for higher values of n": the number of cases that need to be analyzed for establishing $S(7)$ is unmanageable for a human being, and lay at the limits of what could be computed in 1966.

Two styles of proving. Floyd and Knuth's proof, as well as the proof of the Four-Color Theorem, are examples of one family of machine proofs: they establish a property by means of an *ad-hoc* computer program that is written for that specific purpose. Nowadays, this approach is not very common: verifying such proofs also demands that the program be verified, a task that does not fall under the usual scope of the peer-reviewing process. Instead, it is generally considered preferable to use a *theorem prover*: a general-purpose program that can construct and/or verify proofs in a particular logic.

Trusting a proof produced with the help of a theorem prover also has some implications. First, one still needs to trust a computer program – the theorem prover itself. The difference with respect to using *ad-hoc* programs is that we are now considering a program that has been subject to much wider scrutiny, and it can be reasonably argued that the ensuing proof is as trustworthy as a published mathematical proof that has been subject to peer-reviewing. The other point that needs to be checked is that the encoding of the actual mathematical problem in the theorem prover's logic is correct: this is generally accepted as part of the peer-reviewing process, as these encodings are typically

included and discussed in submitted articles. (This aspect is also an issue when using *ad-hoc* computer programs in proofs, but in that scenario the encodings used tend to be much more direct.)

Theorem provers come in a wide variety of styles and flavors, ranging from very general purpose to more specifically tailored for a particular family of problems. They use different logics, and are often uncomparable in their usefulness. A very encompassing overview of the world of theorem provers can be found in Wiedijk [2006].

Machine-assisted proofs today. Forty years after the announced proof of the Four-Color Theorem, machine-assisted proofs are everywhere. Arguably, their widest area of application is hardware and software verification, of which Floyd and Knuth's problem is an example. In an area where we are increasingly dependent on computers to perform critical tasks – from controlling air traffic to administering medicine to patients – it is more and more important that there be no errors in the execution of those tasks, whether due to programming errors or to hardware flaws.

As regards hardware, many useful properties can be verified in a fully automated way, see Kropf [1999]. As such, formal verification is becoming more commonplace in industry. Software verification tends to be more complex, but the number of success stories is increasing. In recent years there have been large projects addressing formalization of large fragments of widely-used programming languages, making it possible for non-experts to experiment with proving properties of the programs they write.

Much more challenging is the formalization of mathematical proofs in a computer. These proofs tend to be much less mechanic and systematic, requiring constant interaction between the computer and an expert – where the expert "guides" the computer through the main steps of the proof. Mathematics also tends to build upon itself, and researchers often find that the biggest challenge in formally verifying an "interesting" result lies on the unavailability of libraries formalizing the relevant underlying theories. Nevertheless, recent formalizations of e.g. the proof of the Four-Color Theorem in Gonthier [2008] and the proof of Kepler's conjecture in Hales [2014] show that this goal is not hopeless. (A recent contribution to this list is a formal proof of Floyd and Knuth's results on sorting networks by Cruz-Filipe, Larsen, and Schneider-Kamp [2017], along with the recently established value of $S(9)$ in Codish, Cruz-Filipe, Frank, and Schneider-Kamp [2016].)

A different approach to machine-verified proofs of mathematical results, which some may argue is less elegant, capitalizes on the success of SAT solvers. A SAT solver is a program that tackles the Boolean satisfiability problem: given a propositional formula, decide whether it has a satisfying assignment. Although this problem is NP-complete, years of intensive investment in researching efficient techniques for solving it has resulted in extremely powerful tools that can solve formulas with millions of variables and billions of clauses. To test the limits of these programs, several researchers have experimented with encoding open mathematical problems (typically from combinatorics) as propositional formulas, which were then successfully proven to be unsatisfiable, see e.g. Heule and Kullmann [2017] for an overview.

Until recently, SAT solvers were viewed by some with skepticism: their complexity made them impossible to analyze in practice, and in the case a formula was claimed to be unsatisfiable no independently verifiable guarantee of this fact was provided. The situation changed recently, and the majority of today's SAT solvers also output traces that allow independent, formally verified systems to check proofs of unsatisfiability of propositional formulas, see Cruz-Filipe, Marques-Silva, and Schneider-Kamp [2017] and Cruz-Filipe, Heule, Hunt Jr., Kaufmann, and Schneider-Kamp [2017].

7 Questions

Q Isn't there a problem with proprietary systems (e.g. Magma, Maple or Mathematics)? You might not be able to find referees who have them.

A There is a problem here, but it is more about rarity than about the proprietary nature, and in fact you are more likely to find a referee who knows (and has) one of these than you are some rare open-source package. The real problem is the proprietary nature of the algorithms. "Yes, this large piece of code, which may be proprietary or open-source, gives me this result, but do I trust it?" For computer algebra, the question is discussed in J. H. Davenport [2017].

On the other hand, it is also the case that Sage has made installing open-source mathematics programs easier, in so far as it seems to incorporate most of them.

Q What about a system that produces verified code, which is then compiled and run?

A That's a good question, and there is some of this in Flyspeck (Hales, Adams, Bauer, Dang, Harrison, Hoang, et al. [2017]). See also Paulson's MetiTarski project: Akbarpour and Paulson [2010]. Of course, one would need a verified compiler, but such things exist these days.

Q Is there much consistency between journals on how these proofs are treated?

A There isn't even much consistency within a given journal. Hales [2005b] and Maynard [2015] were both in *Annals of Mathematics*, yet seem to have been treated differently. There is no caution on Maynard [ibid.], and it was published much more rapidly than Hales [2005b]. Conversely the computer programs underpinning Hales [ibid.] are on the *Annals of Mathematics* website, whereas Maynard [2015] simply says "An ancillary Mathematica(R) file detailing these computations is available alongside this paper at `www.arxiv.org`". The computations in Maynard [ibid.] were relatively simple, and carried out exactly: what would *Annals of Mathematics* have made of Maynard [2013]?

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MATHEMATICIANS AND PLANETARY CHALLENGES

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Maria J. Esteban, Edward Lungu and Claudia Sagastizábal

Abstract

The discussion panel with the topic "How can mathematicians contribute to planetary challenges" met on Tuesday, August 7, 2018.

1 Introduction

The panelists were prepared to discuss the following six questions:

1. What are the roles that the mathematical sciences can play in addressing planetary challenges?
2. Where are there new opportunities for mathematical research and interdisciplinary work arising from planetary challenges?
3. How is mathematical modeling done in work on planetary challenges?
4. How can mathematical work lead to recommendations and decisions that benefit all?
5. How to engage students? How to train the next generation of mathematical scientists for work on planetary challenges?
6. How use work on planetary challenges to reach out to the wider public?

The panel session began with a short overview of planetary challenges. This overview is summarized in section 2. The panelists then took turns addressing questions 1–3 and 5. Their comments are reported in section 3. There were also comments and questions from the audience,, summarized in section 4. Questions 4 and 6 were not discussed during the panel, but some panelists submitted written comments, which are summarized in section 5. The concluding section 6 of this report places the panel topic into the broader context of the United Nations Sustainable Development Goals of 2015 and of the mathematical activities related to Mathematics of Planet Earth 2013+.

Sections 3 and 4 are based on an audio recording of the panel. Panelists are identified with the two letter acronyms given in the list above. The corresponding author has summarized and edited responses by the panelists and from the audience in order to increase overall clarity. Any omissions or inaccuracies are solely his responsibility.

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2 What are Planetary Challenges?

While it is tempting for a mathematician to define the term "planetary challenges", it is perhaps more useful to list some criteria:

Planetary challenges are serious. They affect many people (millions or billions, perhaps future generations). They go beyond national boundaries, due to mechanisms and causes that affect the entire planet, such as climate change, or due to mechanisms that spread and amplify them beyond national borders, such as international travel or trade. Meeting such challenges may go beyond the capabilities of nation states, making international efforts necessary. Such challenges then also require efforts of many scientific disciplines and more broadly a wide range of human skills. Social, political, and economic aspects all have to be taken into account, together with physical and environmental limitations.

These goals and constraints may be represented in a "doughnut" analogy, proposed first in Raworth [2017] and shown in the figure. The inner ring of the doughnut represents the resources and circumstances we need to lead a good life, such as food, clean water, housing, sanitation, energy, access to education and healthcare, democracy. The outer ring depicts the Earth's environmental limits. The doughnut analogy also allows one to identify sectors where we are currently exceeding the planet's ecological ceiling or where we are falling short of social goals.

Mathematics is a universal human activity that knows no borders and that is fundamental to many sciences. The mathematical community can therefore play a basic and unifying role in meeting many of these challenges. And indeed mathematicians and mathematical research already are very important for many current efforts related to tasks that affect large portions of humanity. Examples are weather forecasting and climate projections, logistics and planning for global travel and trade, risk assessment and (re)insurance for disasters small and large, and tracking, forecasting, and helping control epidemics for humans and animals.

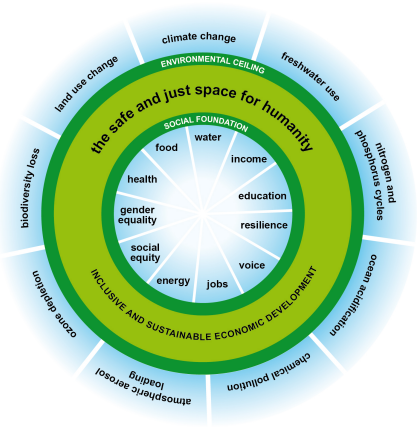

Figure 1: The "doughnut": Finding a just and safe space between social needs and environmental constraints.

General background material on challenges that affect the entire planet may be found on websites of international organizations, such as the Food and Agriculture Organization, the Intergovernmental Panel on Climate Change, and the World Health Organization. The website of the Gapminder Foundation is full of detailed visualizations of data for human wellbeing and development.

The slides that were used by Hans Engler to add detail about the panel topics are summarized next. The goal is to identify some themes which are amenable to mathematical work, perhaps because mathematical models already exist, quantitative studies have been done, or mathematical researchers have become interested in them, without claiming to be complete or comprehensive.

Food and Water. Out of more than 7 billion people who are currently alive, about 3 billion are malnourished (without sufficient access to calories or to micronutrients), while about 2 billion adults are obese. The intersection of these two sets is nontrivial. During the next 12 years or so an extra billion people will be added to the planet's population, increasing the demands on the food system. In addition, people as a whole are becoming wealthier, leading to expanding dietary demands. On the other hand, currently about 30% of all food is wasted, some of it already at the production stage (the leading source of food waste in developing countries) and much at the consumption stage (the most important factor in food waste in developed countries), and of course also at processing and distribution stages in between. The worldwide demand for fresh water is increasing, with needs for drinking, cooking, sanitation, agriculture and aquaculture, and many industrial uses all competing.

Energy and Climate. Planet-wide greenhouse gas emissions continue to increase at a steady rate. Atmospheric CO_2 content is currently at around 410 parts per billion, which is about 50% above preindustrial levels. There is universal scientific consensus that climate change is real, harmful to humanity, and largely caused by human activity. At the same time, economic and technological change together with economic incentives is leading to substantial developments of renewable energy sources, which are disrupting traditional energy markets. For example, in the US electrical energy produced from wind already tends to be cheaper than coal energy, but it is available only intermittently. Global climate projections are now possible for many different greenhouse gas emissions scenarios, with a resolution that is sufficiently high to resolve local geographic features and the diurnal cycle.

Health. Malnutrition causes 1 million child deaths per year, and about 200 million children under five are too small or underweight for their age, leading to disadvantages for their entire lives. Chronic diseases such as cancer and cardiovascular diseases, diarrhea with its many causes, and epidemics such as HIV/AIDS, tuberculosis and malaria are on the rise or continue to threaten the lives of many people. At the same time, new communicable diseases such as ebola and zika are making headlines, while the emerging antibiotic resistance of harmful bacteria is a growing threat for the future..

Communities. The planet's population is expected to grow to more than 10 billion people by 2050. Population growth will then level off in the second half of the 21st century. More than half of the worlds population is already living in urban areas, and about one of eight people living in conurbations with more than 10 million people. Four of the last five International Congresses (Beijing 2002, Hyderabad 2010, Seoul 2014, Rio de Janeiro 2018) have been held in such megacities. Local and regional traffic, regional and international migration, and the resulting changes and increases in transportation networks have transformed the lives of everybody. Real or perceived economic opportunities have accompanied these changes. At the same time, economic inequality

appears to be on the rise worldwide. Worldwide virtual communities and social networks with planetary reach have been emerging for the last two decades, with profound impact on society and on almost everybody's day-to-day lives.

Life on Land and in Oceans. There is broad scientific consensus that we are currently experiencing a phase of mass extinctions - an accelerated loss of biodiversity, with many biological species disappearing before they even have been discovered. There is increased understanding of ecological dependencies and networks. Our understanding of the world's oceans is rapidly improving due to advances in remote sensing and automatic ocean drifters. There is also a worldwide increase in ocean acidification and pollution which is of great concern.

3 Panel Questions

Roles of the Mathematical Sciences. The first question posed to the panel was "What are the roles that the mathematical sciences can play in addressing planetary challenges?". The panelists responded by reporting on success stories from their own experience and by outlining mathematical tasks for the future.

Pedro Dias began the discussion by outlining a number of topics in his response, all broadly related to weather and climate. Mathematics plays a central role through the development and validation of numerical models in meteorology and climate prediction. The numerical models here are discretizations of partial differential equations, leading to initial value problems with $O(10^{10})$ state variables that describe the behavior of the atmosphere including trace gas components, oceans and cryosphere, and vegetation and biosphere, as depicted in the figure.

These models use multiple temporal and spatial scales and are based on multiple physical processes (multiphysics). New algorithms are required to solve the model equations accurately and with sufficient speed to be useful. The sheer size of these models together with the large scale of observations, currently typically $O(10^7)$ from satellites, sensors, etc. also lead to new challenges in the analysis of large datasets. Here methods from machine learning are becoming useful. The connection between observations and numerical models is made with data assimilation methods, first developed in the 1980s, which are based on optimization techniques and/or Bayesian approaches and are capable of using the large amounts of data that have become available through satellite observations since around 1980. These methods became operational world wide around the turn of the century, resulting in substantial improvements in the quality of weather forecasts. A current goal in this field is to create models that are capable of seamless prediction from weeks to seasons and beyond. This has led to interesting questions for dynamical systems about energy interactions between different time scales, e.g. from the diurnal cycle to synoptic (5-7 day) and interseasonal (30 - 60 day) scales, interactions with the annual climate cycle and on to interannual, decadal or multidecadal phenomena, such as El Niño/La Niña (ENSO) and the North Atlantic Oscillation (NAO). These models are known to have noisy inputs and are also expected to have modeling errors and gaps, requiring new approaches to quantify uncertainties. The emerging area of attribution

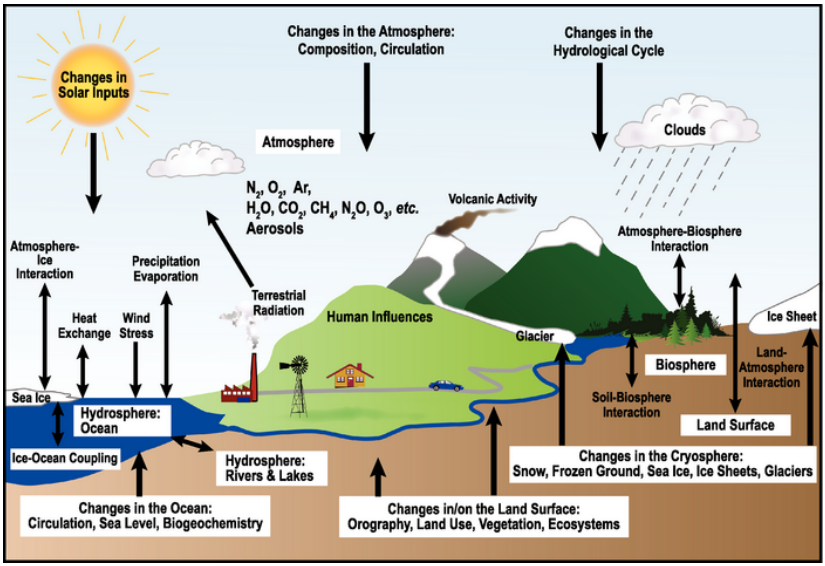

Figure 2: Schematic sketch of the Earth's climate system, from Intergovernmental Panel on Climate Change.

science requires new mathematical and statistical tools to assess to what extent climate change and weather phenomena are natural or induced by humans.

The response of **Claudia Sagastizábal** focused on models for the energy system. Mathematical and computational models here help with planning and allocation. Models are credible and reproducible and thus add continuity to decision and planning. As a concrete example from Brazil, where renewable hydroelectric energy plays a large role, the management of this energy source needs to address fluctuations in precipitation and water flow that are sometimes hard to predict, due to phenomena such as El Niño. Some years ago, the models then in use predicted unusually high energy prices in certain regions of the country. This led to investment decisions that turned out to be poor when these forecasts did not materialize. An analysis of the models led to the conclusion that some of the algorithms terminated incorrectly in certain situations that may occur with small positive probability. Essentially this was a situation where the assumptions of the Borel–Cantelli Lemma from probability theory were very nearly satisfied. Uncovering this was a striking example of using abstract mathematical results in a concrete situation.

Amit Apte returned to the topic of weather and climate prediction, giving the example of the Indian monsoon. This important annual phenomenon is still poorly understood, despite a century of meteorological study that involved mathematical and statistical methods from the very beginning. There is lack of understanding especially of the role of clouds. There are no simple mathematical models of monsoon phenomena that have significant predictive power (e.g predict strength and timing better than standard climatology). This deficit also exists for interdecadal phenomena such as ENSO

and multidecadal phenomena such as the NAO. There are however simplified conceptual models (typically, low-dimensional systems of differential equations) that provide good insight and reproduce qualitative features well. He also commented on advances in data assimilation, specifically combining Bayesian methods with techniques for partial differential equations. Recent work here has successfully applied work on optimal transport problems. There are resulting theoretical challenges for the understanding of dynamical systems, also related to data assimilation. There is emerging understanding of the good performance of such methods for short-term prediction, but for longer term problems, completely new ideas will be needed.

Maria J. Esteban did not talk about any particular topic and instead commented on the way in which mathematical sciences can contribute to planetary challenges. She emphasized the importance of collaboration between mathematicians and other scientists in the study of planetary phenomena. Collaboration is required at all stages - when the evolution or the impact of natural or man-made processes are quantified, models are developed, predictions are made, risks are assessed, and results are communicated to decision makers. Constructing well-founded models, analyzing and using these models rigorously, and communicating the results so they can be used for decisions is an essential mathematical contribution.

Edward Lungu began his response by reminding the audience of his origin – Botswana, located in a semi-arid zone of the planet that is now turning arid. The economy of Botswana depends both on its wildlife through ecotourism and also on its livestock, which is the mainstay of the agricultural sector. Limited land and water resources lead to competition between these sectors, and every one to five years decisions must be made to handle this competition, managing resources and maintaining the sustainability the nation's ecosystem. Mathematicians have helped with this task, using stochastic differential equation modeling. Mathematical challenges also arise in the modeling of diseases of humans and animals. Cross-border movements of wild animals such as buffaloes and their mingling with domestic cattle leads to questions beyond standard models for epidemics. A third area of mathematical applications comes from the energy sector. Especially in Africa, the need for fuel leads to deforestation. Adjusting price structures for energy can change these developments, and mathematics has a role to play here.

New Opportunities for Mathematical Research. The panel then addressed the question "Where are there new opportunities for mathematical research and interdisciplinary work arising from planetary challenges?". In its answers, it identified scientific opportunities for research and collaboration, but also opportunities coming from societal expectations and new possibilities for research funding.

Edward Lungu started off by explaining that at his academic home, a new university of science and technology, the government expects the mathematical sciences to play an enabling, integrating, and leading role. In response, mathematicians are involved in various applications related to the life sciences, including ecology, epidemics, and resource management, especially in arid regions. More broadly there are opportunities related to biotechnology, including the mathematical training of biologists, in order to address

food insecurity. This has led to many interdisciplinary collaborations, e.g. between biologists, mathematicians, and computer scientists. More unexpected opportunities for mathematical work will very likely emerge as a result.

Maria J. Esteban responded by pointing to new opportunities related to clean energy production. Besides its connections to economic modeling, this area has connections to electrical engineering and material science (for solar energy). It continues to generate interesting mathematical problems.

Amit Apte noted the necessity of developing mathematical approaches for the interaction of physical processes and social processes, such as models that couple climate change and agriculture or migration. This will require interdisciplinary efforts and possible quite different types of mathematical work, since the outcomes must be usable for policy makers.

Claudia Sagastizábal followed up on Maria J. Esteban's comments regarding mathematical work on problems related to energy. She noted that the possibility of very detailed energy metering has implications for privacy and the protection of data. This has implications for the design of mathematical decision algorithms. Regarding renewable energy, there is the big technical challenge of storing intermittently produced energy, mostly wind and solar energy. Current solutions such as seasonal thermal energy storage require broader mathematical models than currently in use and lead to interesting new problems, e.g. for optimal control theory.

Pedro Dias saw opportunities of support of mathematics resulting from the threat of climate change. This has brought new stakeholders on the scene, such as the insurance business, which needs to assess the associated risks with mathematical techniques. By funding such work, insurance companies are creating opportunities for support of mathematical research. A similar situation exists with respect to renewable energy in Brazil, where the law requires energy companies to invest certain amounts in research. The mathematical community could also use this possibility for support, jointly with other scientific disciplines. Applications of climate and weather research are also important in agriculture, which is a significant component of Brazil's national economy. Here, the private sector has become more interested in new approaches that involve cooperative models with the mathematical sciences. Since academic institutions such as universities have become much more open to interdisciplinary scientific work in Brazil in recent decades, such collaborative efforts are becoming more acceptable.

Mathematical Modeling. "How is mathematical modeling done in work on planetary challenges?" This was the next question taken up by the panel.

Maria J. Esteban began with general remarks on modeling, addressed at mathematicians in the audience who might be less familiar with this activity. A mathematical model serves to describe phenomena or processes from reality in a quantitative way. Problems from many areas are amenable to mathematical models. The physical, biological, economic and medical sciences have long histories of successful model construction and model use. Before a quantitative model can be developed, it must first be possible to measure the phenomena of interest. The development of a model is usually a collaborative effort of mathematical scientists and domain experts, and it is important that the mathematical partners understand the phenomena under study, just like the domain expert must be familiar with the mathematical language. This always takes time.

Even so, such models often are built by experts in other fields, sometimes jointly with mathematicians. This is frequent practice in engineering and the physical sciences and is becoming more common also in the life sciences. To build such a model, an appropriate mathematical language must be found or – if necessary – developed. The language is often based on existing mathematical techniques, such as systems of linear equations, theories for ordinary or partial differential equations, probability and statistics, or optimization theory. The result then has at its core a set of equations or inequalities, or an energy function that is to be minimized, or statements about the distributions of certain random quantities, or combinations of all these. In addition to a rigorous formulation, an essential component of a good model is always a well-founded explanation of its range of validity and its explanatory or predictive quality. In the words of the statistician George Box, "all models are wrong, but some are useful". Finally, mathematical models tend to be studied, used, and further improved for a long time. As noted earlier, they tend to lend continuity to the study of a phenomenon, generate new questions and hypotheses, and remain an object of mathematical thought.

Pedro Dias continued with comments on modeling for planetary challenges, specifically for use in meteorology and climate science. In this case, operational weather and climate prediction needs fast numerical codes to provide users with information in real time. After suitable discretizations, the dimension of state vectors is very high: currently $O(10^{10})$, soon to surpass $O(10^{11})$. This is addressed numerically with massive parallelization. Up to a certain point, this is a computational problem, but progress often requires new algorithms. Models also need to deal with the complexity that comes from multiscaling and multiphysics in climate models, leading to challenges for data assimilation and uncertainty evaluation. Finally (and echoing comments by other panelists), it is necessary to introduce human dimensions in climate modeling, e.g. by coupling global or regional economy models to climate models in order to incorporate the interaction e.g. between economic change and emissions of greenhouse gases. Existing mathematical tools to achieve this may come from game theory and from multiagent modeling and will require further development.

In additional brief comments, **Claudia Sagastizábal** likened the range of validity of a mathematical model to the mental map of a commuter fore her trips between home and work. A relatively coarse model describing only highways is usually sufficient for the trip, but it may be necessary to augment and refine a model, for example by a mental map of side streets, in case of a traffic jam. Put differently, models are in practice always evolving. She added that mathematics furnishes a common language, which must however sometimes first be established between mathematicians and practitioners.

Edward Lungu also added a comment on the need to combine models with data, He reported that at the beginning of the recent Ebola outbreak in West Africa there were no data available at all with which model parameters could be estimated, leading to unrealistic expectations that the epidemic would be cleared within eight weeks. In fact, this process took more than a year. In this case, as in many others, the complex dynamics of this disease had to be understood, but then it was also essential to estimate the model parameters as data became available. The need to obtain good data to tune an appropriate model creates an additional layer of difficulty.

Training the Next Generation. At the end of the session, panelists responded to the question "How to engage students? How to train the next generations of mathematical scientists for work on planetary challenges?" Since time was limited, these responses were quite succinct, but nevertheless covered a lot of ground, ranging from secondary to graduate education. The comments also reflect specific situations in the panelists' countries of origin.

Edward Lungu voiced the opinion that the mathematical community as a whole has a done a good job in this regard so far. He identified several specific challenges for Africa, where students are already eagerly working on these themes. To broaden the perspectives and skill sets of these students, African mathematicians have invited colleagues from abroad (the United States and the United Kingdom) to participate in this training. These efforts have been supported by the Simons Foundation. Maria J. Esteban saw the need to design curricula for mathematical work on planetary challenges and to design and offer suitable courses for interested students. Amit Apte added that lowering the existing or perceived barriers between pure and applied mathematics and emphasizing the commonalities of all mathematical work would generally be very helpful. He noted that in India pure mathematics tends to enjoy higher recognition than applied mathematics. Reducing these perceived distinctions would go a long way towards attracting more students to work on issues related to Planet Earth. Claudia Sagastizábal added that in her experience, good funding is required to attract good students. Senior mathematicians may therefore have to include areas in their research agendas in which it easier to attract funding. One such area nowadays is perhaps machine learning. While this is a reality we have to live with, it should be added that there are also good scientific reasons for including these methods in one's research, as some of the previous comments indicated. Pedro Dias reported on an experience at his home institution, where the astronomy program started training high school teachers about a decade ago with the long term goal to attract more and better students into the program. These efforts are now paying off, and the program is beginning to get very good students who already received a good preparation in high school. Well-trained high school teachers act as multipliers, with positive effects for the health of the entire field.

4 Audience Contributions

After the introductory slides and before the panel discussion proper started, the audience was asked to use hand signals to respond to a few questions. The question "Have you done mathematical work related to one of these [planetary challenges] topics?" was answered affirmatively by about half the audience. The audience was then asked to indicate which of the general areas from the slides were most important to them. Most audience members raised their hands for several areas, with no particular emphasis on any of them. For the third question, "Do you know how to find partners or opportunities to work on one of these [planetary challenges] areas?" far fewer audience members responded affirmatively.

During the second half of the panel, members of the audience had the opportunity to pose questions to the panel or report on their own experiences and views, followed each

time by responses from panel members. On the whole, these questions took the panel discussion into more general directions.

The first comment from the audience followed up on the question of communication between mathematicians and other scientists, taking the specific case of research on solar energy. The audience member proposed an electronic forum, e.g. a website, as a way to connect interested stakeholders, non-mathematical scientists, and mathematicians with the right expertise. Two panelists responded, agreeing broadly that such an electronic forum may not be sufficient. Pedro Dias reported on his experience as director of a national laboratory for computational science in Brazil, where getting scientists with different backgrounds together to develop mathematical models was a rewarding activity. He recalled that it often took a sustained effort to get such collaborations started, usually by first talking to the prospective partners separately and only then putting the groups together. Starting with such an interaction right away tended to be less successful. Maria J. Esteban reported on her observations of a government funded research initiative that tried to initiate collaborations between mathematicians and other scientists on problems related to solar energy. Effective responses to the request for proposals often used already existing connections between scientific partners with the appropriate background, while ad hoc collaborations that were formed in response to the initiative were not common and less successful.

Several audience comments pointed out that a substantial amount of applied mathematical work currently is directed towards maintaining and increasing unsustainable developments, with the goal of ever-increasing consumption and production. It was also noted that mathematics does not only provide strong explanatory tools, it also has the potential of inventing, creating, and shaping new and sustainable ways of living. This comment received strong agreement from the panelists who pointed out that mathematical modeling always has this forward-looking aspect. A related comment anticipated that mathematical research that is merely interesting and fascinating in its own right might become a broader human activity in a future society, where fewer jobs will be available and more human activity could happen within an ethical framework of curiosity, discovery, sharing, and rationality. The panel was not quite ready to follow this line of thought, but emphasized the common ground of core and applied mathematics, which both generate and require passion and are driven by curiosity. One panelist observed rather whimsically that over the course of her career, the division between pure and applied mathematics diminished for her, while the difference between pure and applied mathematicians became more apparent.

5 Additional Panel Responses

Due to time limitations, two of the six questions were not discussed during the panel. Instead, some panelists prepared written responses, which are summarized here.

Communication with Decision Makers. The panel question is "How can mathematical work lead to recommendations and decisions that benefit all? How to communicate with decision makers?"

The panelists who responded agreed that successful communication with decision makers is always a long term project that may take years. It requires finding a common language and agreeing on the importance of a project, resulting in a joint sense of purpose. It is more likely to succeed if some of the partners already have a technical background, and less likely if attempted in a "top down" fashion. Long term preparation require more than just scientific work: It involves building connections, opening and maintaining channels of communication, participating in committees and meetings, and so on. It may also require substantial involvement in training future decision makers, in order to convince them of the usefulness of mathematics and to increase the likelihood that they will seek mathematical advice later. Every opportunity to communicate with decision makers should be used. It is essential to communicate clearly and concisely, preferably with good graphics if done in writing. One should not expect these partners to have the time to digest technical detail. On the other hand, the chances for success increase if the results are easy to quote or use otherwise by these partners. Such users typically expect predictions and risk assessment, not so much explanations after the fact.

Outreach to the Wider Public. The detailed question is "How to use mathematical work on planetary challenges to reach out to the wider public? Can we use this to show that mathematical work is relevant and accessible?"

The panelist responses emphasized that there must be good stories, told by good communicators. For example, success stories such as the impact of Compressed Sensing on medical imaging techniques (see the talk by David Donoho in these Proceedings) are perfect in this respect, because they help relating the abstract approach of mathematics to very practical concerns that affect us all, like early detection of tumors and other preventive health issues. At the same time, mathematicians should be ready to speak and write at every possible opportunity, accept the challenge to write articles in generalist publications, speak to young students, high school students, etc., precisely because they have such compelling stories to tell. As an aside, physicists seem to be much more ready than we are to try to pass on what they know, and it works for their field. Planetary challenges is an area where mathematicians can communicate. It is important, however, that their comments and contributions are based on rigorous and scientific work when they speak as mathematicians. Since planetary challenges tend to have political and societal aspects and may have controversies and conflict at their core, mathematicians who are publicly active in this area should carefully avoid mixing scientific views and personal opinions.

6 Conclusion

In 2015, the United Nations formulated a set of 17 "sustainable development goals", see SDG. The themes formulated for this panel were in part taken from that set of political goals, which will doubtlessly shape future development initiatives, specify goals and milestones for economic and technological development, provide themes for funding initiatives, conferences, and scientific committees, and even influence the way in which information is organized and data are being collected. Mathematicians who want to work on planetary challenges should consider becoming familiar with these goals.

Starting in 2013, an international effort of national mathematical societies, mathematical research institutes, teacher associations, science museums, and many other partners called "Mathematics of Planet Earth" has worked towards bringing mathematicians together to work on pressing global challenges. Initially called MPE2013 and now morphed into MPE2013+, this effort has led to activities and events around the world that are continuing. New doctoral training programs have been established, research communities have been formed, scientific communities such as the Society for Industrial and Applied Mathematics and the European Geosciences Union have started or reoriented interest groups, there are book series, and conferences, lecture series, and even art competitions have been held, all with the MPE theme. Planetary challenges will stay with us for the future, and mathematicians will continue to respond.

Figure 3: Mathematics of Planet Earth logo.

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LESSON STUDY IN MATHEMATICS: CURRENT STATUS AND FURTHER DIRECTIONS

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1 Introduction

Lesson study[1] (LS), job-embedded, teacher-oriented, student-focused teacher professional development approach, has been in place in Japan and China for over a century Chen and F. Yang [2013] and Lewis and Tsuchida [1997]. The approach has played crucial roles in transforming mathematics teaching, promoting students' learning, developing teachers, and implementing mathematics curriculum reforms in both countries system wide L. Gu and Y. Yang [2003], Huang, Gong, and Han [2016], Lewis and Takahashi [2013], and Lewis [2016]. It was the seminal work on introducing Japanese lesson study to the mathematics education community in the West in the late 1990s e.g. Lewis and Tsuchida [1997] and Stigler and Hiebert [1999] that has attracted educators around the world to adapt Japanese LS in their own countries, and consequently LS has spread the globe Lewis and Lee [2017]. LS has been widely incorporated in both pre-service teacher preparation and in-service teacher professional development programs. Although many positive effects of LS on improving teaching, improving students' learning and implementing new curriculum have been documented around the world, the challenges and obstacles of adapting LS in other education systems were revealed Huang and Shimizu [2016] and H. Xu and Pedder [2015]. The goals of this chapter are to provide an overview of LS with examples in several selected aspects and discuss the directions of further development of LS. Thus, the chapter is organized into six sections. The first section is about overview and conceptualization of Japanese LS. The second section focuses on introduction into Chinese LS. The third section shows a case of adaptation of LS with pre-service teacher in Malawi. The fourth section presents a case of adaptation of LS with in-service teachers in Thailand. The fifth section discusses theoretical and methodological issues of studies on LS with an example in Switzerland. The chapter ends with discussions about further directions of LS.

MSC2010: primary 18D50; secondary 55P62, 57Q45, 81T18, 57R56.

[1] Lesson Study is the English Translation of the Japanese term, *Jyugyou Kenkyuu*

2 Overview of Japanese LS

A Japanese LS typically includes four steps Lewis [2016] as follows: (1) Goal Setting: Consider students' current characteristics, their long-term goals and development. Identify gaps between these long-term goals and current reality. Formulate the research theme. (2) Planning: Based on studies of teaching materials (textbooks, curriculum and teacher guides, relevant research), collaboratively plan a "research lesson" to address the identified goals. (3) Research Lesson: One team member teaches the research lesson while other members of the team observe. The observers collect data of students' learning. (4) Reflection: Share and discuss the data collected from research lesson, draw implications for redesign, teaching and student learning broadly, and write a report. The revision and re-teaching of the research lesson is optional Lewis, Perry, and Hurd [2009] and Murata [2011]. Murata [2011] identified major features of LS: Centering on teachers' interest, focusing on students' learning surrounding a research lesson, and collaborative processes, and reflective processes. Furthermore, Lewis and Takahashi [2013] illustrated four types of LS in Japan: local schools, districts, university-attached laboratory schools, and professional associations. Although different types of LS focus on different aspects of teaching or implementation of curriculum, they all provide opportunities to observe teaching and learning, to analyze and discuss data collected during the research lesson, and to network with other educators and build professional learning communities. Thus, LS in Japan has resulted in "teaching for understanding in both mathematics and science, successfully spreading some major instructional innovations" Lewis [2015, p. 50] and is a type of improvement science Langley, Moen, K. M. Nolan, T. W. Nolan, Norman, and Provost [2009].

Since adaptation of Japanese LS globally, a great deal of studies have documented the major benefits of implementing LS including: teacher collaboration and development of a professional learning community, development of professional knowledge, practice and professionalism, more explicit focus on pupil learning, and improved quality of classroom teaching and learning H. Xu and Pedder [2015]. Regarding LS with in-service mathematics teachers, Huang and Shimizu [2016] highlight the following effects of LS: promoting teacher learning, improving teaching and student learning, implementing curriculum, sharing instructional products, and building connections between theory and practice.

However, researchers also identified obstacles and challenges when adapting LS in other countries Fujii [2014], Huang and Shimizu [2016], Larssen et al. [2018], and Ponte [2017]. With in-service teachers, Fujii [2014] revealed six misconceptions of LS such as regarding LS as a workshop. Huang and Shimizu [2016] classified factors influencing the success of LS into two broad categories. At macro-level, these factors include cultural value in general, teaching and teacher learning culture in particular, teacher professional development system, professional learning community, and leadership of district leaders and school leaders. At micro-level, these factors include appropriate content and pedagogical knowledge of teachers, developing inquiry stances such as critical lens as researcher, critical lens as curriculum developer, and critical lens as

student; classroom observation with a focus on student learning, teachers' commitment and so on.

With regarding LS with pre-service teachers, Ponte [2017] identified the challenges such as defining the aims of LS, establishing the relationships among participants, scaling up of LS, and adapting or simplifying lesson studies for the particular purpose of educating future teachers. Larssen et al. [2018] put forward the challenges in adapting LS with initial teacher education (ITE) programs including how to prepare student-teachers to observe (professional noticing being a promising option), the wide variation in the focus of classroom observation in ITE lesson studies, and discussion of what is understood by learning needs to stand at the heart of preparation for lesson studies in ITE. To maximum the benefits of LS and to address challenges facing implementing LS, it is critical to conceptualizing LS by consideration of the variation and adaptation of LS Huang and Shimizu [2016].

While there are many projects related to LS outside of Japan, they have been met with varying degrees of success, often because they diverge from authentic Japanese LS. In order to maximize the impact of LS, Japanese mathematics education researchers and researchers of mathematics LS in the US worked collaboratively to examine the process of LS in Japanese schools e.g. Fujii [2016], Takahashi [2014b,a], and Watanabe [2014]. As a result of the collaboration Collaborative Lesson Research (CLR) was proposed as a program to adapt LS to schools outside of Japan, Takahashi and McDougal [2016]. The term is drawn from Catherine Lewis's original translation of *jyugyou kenkyuu* as "Lesson Research," which we reverse in order to emphasize the research aspect Lewis and Hurd [2011] and Takahashi, Watanabe, Yoshida, and Wang-Iverson [2005]. CLR is defined as having the following components:

1. A clear research purpose

2. *Kyouzai kenkyuu*, the "study of teaching materials"

3. A written research proposal

4. A live research lesson and discussion

5. Knowledgeable others

6. Sharing of results

CLR is an investigation undertaken by a group of educators, usually teachers, using live lessons to answer shared questions about teaching and learning. If any component is missing, then the activity cannot be called CLR. CLR has been piloting in several US urban school districts and schools in UK, Qatar, and other countries, Takahashi and McDougal [2018].

3 Introduction into Chinese LS

3.1 Introduction. Chinese students' strong performance in mathematics and science in international assessments Fan and Zhu [2004] and OECD [2014] have attracted scholars to explore mathematics education in China from various perspectives Leung [2005], Y. Li and Huang [2013, 2018], Ma [1999], and Stevenson and Stigler [1992]. In particular, Ma's seminal study Ma [1999] reveals a paradox of Chinese mathematics teachers: the elementary mathematics teachers possessed profound understanding of fundamental mathematics (PUFM) while they received very limited academic training. Researchers Fang and Paine [2008] and Ma [1999] speculated that Chinese mathematics teachers extensively studied curriculum and other instructional materials, collaboratively planned lessons, and frequently observed public lessons within school-based teaching research groups may have promoted their subject matter knowledge and pedagogical knowledge. Yet how and why they have developed that knowledge still remains a significant research problem to explore. Recently, researchers Chen [2017], Fang [2017], and Huang, Fang, and Chen [2017] have started to investigate the core component of teaching research activities, known as Chinese LS (CLS hereafter), from different lens. The major findings of relevant studies on CLS are reviewed and presented in three aspects: historical and cultural perspectives of CLS, underlying infrastructures of CLS, and studies on CLS.

3.2 Chinese LS from historical and cultural perspectives. Chinese Lesson Study (CLS), an approach known for teacher professional development in China, includes cycles of collaborative activities, such as lesson planning, delivering planned lessons along with team observation of classroom teaching, post-lesson debriefing (including talk lesson) and reflection, followed by continued revisions for improvement Huang and Bao [2006] and Y. Yang and Ricks [2013]. It can be generally seen as different from Japanese LS Lewis [2016] in four major aspects Huang and Han [2015] and Huang, Fang, and Chen [2017]. First, for a long time, CLS is content-focused, oriented to developing best teaching strategies for specific subject contents for student learning, while Japanese lesson study was regarded as focusing on general and long-term education goals. Second, there are various kinds of lesson studies for teachers at different stages of their professional development in China with a focus on demonstrating and/or developing exemplary lessons to demonstrate effective teaching. Yet, Japanese model focuses more on the process of teacher learning than the product of a "perfect lesson". Third, rehearsal teaching is repeated multiple times until teachers involved feel satisfied with the goals they set out to achieve, while Japanese teachers might or might not repeat teaching the improved lesson. Fourth, knowledgeable others could be involved throughout the entire process of LS, while Japanese LS, knowledgeable others may also be involved, but not necessary, or not throughout the entire process.

From a historical perspective, Li (in press) illustrated how Chinese LS has evolved through 3 major stages and finally developed its own characteristics. At the first phase

(1896–1949), the initial forms of LS activities were first introduced and practiced. During that period, the affiliated schools to normal university provided a venue for pre-service teachers' teaching practicum experience. The goals of student teachers' teaching practices were: (1) experienced teachers observe and provide feedback on lessons taught by student teachers; (2) experienced teachers demonstrate how to teach to student teachers. These two types and their variations have been and are currently practiced in schools in China. At the second stage (1949–1999), adapted from former Soviet Union, Chinese government officially established a nationwide teaching research group (TRG) system. Teaching Research Groups focused on three main themes: (1) in-depth analysis of textbooks, other instructional materials, and pedagogy; (2) collective lesson planning by teachers in the same group; and (3) observing an exemplary lesson taught by expert teachers or an experimental lesson involving new teaching strategies; teaching or observing a public lesson, followed by the audience commenting on the lesson; mutually observing lessons taught by other teachers in the same group, and providing feedback S. Li [2014]. At the third stage (2000–), to address the challenges of implementing new curriculum, teacher educators have developed their own style of LS, by adopting ideas of community of practices Lave [1988] and Japanese LS. The most recognized work was done by the Action Research Project team at Qingpu County in Shanghai L. Gu and Y. Yang [2003]. Based on over 20 years of teaching experiment and empirical research in mathematics classrooms, the project team formulated an Action Education model for teacher professional development that incorporates LS as the main platform for teacher learning, planning, teaching, reflections, and new behaviors F. Gu and L. Gu [2016] and Huang and Bao [2006].

From a cultural perspective, Chen [2017] explained the major features of Chinese LS. Based on a careful analysis of classic texts on Chinese cultural thoughts and related literature, ground approaches to analyzing the data collected from Chinese teachers' LS activities and interviews with teachers, the characteristics of CLS could be interpreted from three aspects. First, in terms of their actions, the Chinese teachers enact their understanding of teaching in public lessons through unity of knowing and doing (知行合一) more than conceptual explication. Second, with regard to their thinking, the Chinese teachers use practical reasoning (实践推理) in deliberate practice of repeated teaching through group inquiry and reflection. Third, a tendency of emulating those better than oneself (见贤思齐) is evident in novice teachers' learning from "good" examples by expert teachers. These cultural interpretations can contribute to a deeper understanding about the persistence and importance of LS in the Chinese education history. These cultural roots help to explain the legitimacy of why the repeated teaching is stressed, why learning from experts or peers is valued.

3.3 Two infrastructures laying a foundation for conducting LS in China system wide. A teaching research system consisting of hierarchical Teaching Research Offices (TROs, hereafter) from national to district levels is one of the most important infrastructures for supporting teacher professional development and implementing curriculum reform in China Huang, Ye, and Prince [2016] and Wang [2013]. "Teaching

research activities" refer to various types of professional development organized by the TROs. Mathematics *Teaching Research Specialists* (TRSs, hereafter) employed in TROs are officially responsible for organizing teaching research activities at different levels. Based on official documents and requirements of TRSs in China, Huang, S. Xu, and Su [2012] found that Chinese TRSs should have expertise in conducting effective teaching, doing teaching research, effectively organizing school-based teaching activities, and evaluating teachers' teaching and students' learning. Huang, Zhang, Mok, Zhou, Wu, and Zhao [2017] further identified that TRSs valued the following knowledge and skills including subject matter and interdisciplinary knowledge, student learning, mathematics instruction, evaluating student learning, evaluating teacher instruction and mentoring teaching research activities, and leadership in cultivating master teachers and being educational policy consultants.

Correspondingly, a professional promotion system for primary and secondary teachers has motivated teachers to participate in various teaching research activities Huang, Ye, and Prince [2016]. Similar to university promotion system, there are different professional ranks for secondary and primary teachers in China. The senior-rank level includes full-senior and senior teachers. The intermediate-rank level is called level 1 teacher, and the primary-rank level consists of level 2 and level 3 teachers. There are specific criteria for each level. For example, full-senior teachers should meet the following criteria: (1) have high, professional aspirations and firm professional beliefs; have experience working as a teacher for a long time and serve as a guide and steering role in prompting students' growth, and have been an excellent class supervisor and student counselor, and have made a great accomplishment in educating students; (2) have a profound understanding and mastery of curriculum standards and subject knowledge; achieved excellent performance in education and teaching, demonstrated an adept in teaching arts, and developed a unique teaching style; (3) have an ability to organize and guide education and teaching research; achieved creative results in educational ideas, curriculum reform, teaching methods, and applied them in teaching practices, and exerted a demonstration and steering role; (4) Make exceptional contributions to mentoring and cultivating teachers at level 1, 2 and 3; maintain a high reputation in subject teaching, and have been well-recognized as an education and teaching expert; and (5) normally hold a bachelor or above degree, and have served as an advanced teacher at least five years. For another example, a level 3 teacher (entry level) should meet the following criteria: (1) basic mastery of the principles and methods of educating students, and should be able to appropriately educate and guide students; (2) have educational, psychological and pedagogical knowledge, and basic mastery of subject matter knowledge and pedagogical knowledge in the subject being taught, and be able to teach a subject; and (3) holds an associate degree or above, and one year successful teaching probation. Thus, the system defines what professional knowledge and skills are required in order to get promotion to a level. Moreover, senior teachers are required to mentor and supervise novice teachers.

3.4 Studies on the effect of Chinese lesson study (CLS) in mathematics education. Studies have shown the effects of CLS on many educational areas such as curriculum reform Chen and F. Yang [2013], L. Gu and Y. Yang [2003], Fang [2017], and Cravens and Wang [2017], improving teaching Huang and Y. Li [2009] and Y. Yang [2009], promoting teacher learning Huang, Su, and S. Xu [2014], Huang and Y. Li [2009], and Huang and Han [2015], and student learning Huang, Gong, and Han [2016] and L. Y. Gu and Wang [2003]. In Huang and Han [2015]'s study, the authors evidenced how teachers developed their instructional expertise through cross-district LS. Huang, Gong, and Han [2016] reported how teachers improved their instruction that promoted student learning through CLS. Fang [2017] illustrated that some school-based teaching research projects and activities have translated the official curriculum into classroom instruction based on CLS. Moreover, Cravens and Wang [2017] further revealed that how expert teachers played crucial roles in making transaction of curriculum from intended to enacted, and promoting junior/novice teachers' growth district-wide while conducting teaching research activities. Moreover, Huang et al. (in press) demonstrated the mechanism of transfer of a curriculum idea into classroom practice through CLS. Combined, Huang, Fang, and Chen [2017] argued that CLS is a deliberate practice from a deliberate practice for developing instructional expertise; a research methodology for linking research and practice, and an improvement science for instruction and school improvement system wide.

3.5 Discussion and conclusion. Researchers have interpreted the major features of CLS from historical and cultural perspectives. The persistence and predominance of repeated teaching, learning from knowledgeable others and peers, perfecting research lessons through deliberate practice are legitimate in CLS due to a cultural value of teachers' learning. Meanwhile, research has documented the effect of CLS in various aspects. However, there are several issues that need to be addressed in order to further develop and maximize the effects of LS in China. First, due to the emphasis of perfecting research lesson and learning from others, Chinese LS often focuses on improving teacher's instructional techniques, with less attention to student learning. It is still a challenge how to elicit and use student thinking in larger classrooms in China. Second, due to the popularity of using online systems, it is quite common to watch online videos developed by expert teachers in China. However, it is largely unknown how the nature of teacher's learning is impacted through online systems or how to use online systems for enhancing LS. Finally, although a few studies explored the adaptation of CLS in Italy and the US Bartolini Bussi, Bertolini, Ramploud, and Sun [2017] and Huang, Haupt, and Barlow [2017], it is a new area to explore the value and possibility of adaptation of CLS in other countries.

4 Adaptation of LS with pre-service teachers in Malawi

4.1 Background. LS has been introduced in many contexts outside of Japan with the aim of improving teaching e.g. Dudley [2014] and Lewis and Hurd [2011]. However,

research has shown that there are some challenges and misconceptions in adapting and implementing LS outside of Japan Fujii [2014]. Furthermore, it has been observed that some fundamental aspects of LS are not easily adapted into new contexts e.g. Cheung and Wong [2014], Ponte [2017], and H. Xu and Pedder [2015].

In more recent decades, LS has also been introduced and researched in teacher education. For example in Norway, a study on LS in initial teacher education found that in planning their research lessons, student teachers did not always start with a research question. In addition, student teachers' mathematics research lessons were not structured in a way that would make students' learning visible. In many cases students were working on tasks individually. Consequently, there was lack of observation of students learning Bjuland and Mosvold [2015].

In Canada, Chassels and Melville [2009] researched LS to teacher education focusing on affordances and challenges. They found that LS raised student teachers' consciousness about the needs of students, the importance of teaching strategies that address students' needs, and value of working collaboratively to improve teaching. The challenges that were observed were related to constraints in time and administrative structures to support the LS. These challenges seem to be common to new contexts as they were also observed in the Norwegian context Bjuland and Mosvold [2015].

In their review of literature of LS in teacher education, Larssen et al. [2018] show that there are differences in adaptations of LS in different contexts in terms of how the LS cycle is conducted and the discussion among researchers. For instance, they found that there was no common understanding of the process of observation. Ponte [2017] reviewed research on LS in secondary teacher education and observed that adaptation of LS in new contexts is not yet well understood, and thus called for further research with a critical view in exploring affordances and challenges in LS.

LS has been adapted in some schools in a few African countries such as Malawi, Uganda and Zambia e.g. Fujii [2014, 2016] and Ono and Ferreira [2009], However, H. Xu and Pedder [2015] observed that between 2001 and 2013 there were only two studies published from the African context. Furthermore, Larssen et al. [2018] observed that there were no studies from African context of LS in initial teacher education. Therefore, researching LS in teacher education in African contexts is important to contribute to worldwide understanding of adaptation of LS in different contexts.

4.2 The study. In the remainder of this section, we discuss findings of a study that explored how mathematics teacher educators in Malawi understand LS and how they intended to implement the LS in their teaching. The study is part of a wider project whose aim is to improve quality of mathematics teacher education in Malawi. One objective is to enhance capacity of mathematics teacher educators through professional development. The professional development is in the form of LS where mathematics teacher educators at each teacher college work together to plan and implement one LS cycle. The entire process takes 7 months; it starts with a three-day workshop in May where teacher educators are introduced to LS and ends with another three-day workshop in November where they report about their LS. After the first workshop, mathematics

teacher educators from each college work together to draft their research lesson plan and send it to the authors who act as knowledgeable others Takahashi [2014b], and comment on the lesson plans. The teacher educators then revise the lesson plans following the comments. The process of drafting and refining the lesson plan takes at least 10 weeks. The research lessons are then taught and video recorded and post lesson discussions are also videotaped. At the second workshop, teacher educators report on their LS cycle and describe what they learned from their research lesson.

For the purpose of this paper, data was collected from five teacher education colleges and the unit of analysis is the teacher educators' written lesson plans; the draft lesson plan and the revised lesson plan. We analyzed the lesson plans using qualitative content analysis both deductively and inductively, and we focused on three aspects, namely, research question, prediction and observation.

4.3 Findings.

	Draft lesson plan	**Revised lesson plan**
Research question	2 had research question though not explicit. 2 had lesson titles which implied question but not clearly 1 had no research question	2 more explicit research question 2 still lesson titles but more explicit 1 still no research question
Prediction	1 had prediction, 2 had assumptions of students' knowledge, 2 had no prediction nor assumptions	1 had more specific prediction, 2 had same assumptions of students' knowledge 2 still had neither predictions or assumptions
Observation	4 had few points of observation (average 3) 1 had no explicit observation	3 of the 4 had more and well-focused points of observations (average 8) 1 not revised 1 still no explicit observation

4.3.1 Research Question, Prediction and Observations in Teacher Educators' lesson plans. Across the research lesson plans from the five teacher colleges, we observe that it is difficult for the teacher educators who are newcomers to LS to understand the importance of a research question as a starting point for planning their research lessons.

This supports findings in Norway where Bjuland and Mosvold [2015] found that student teacher did not always see a research question as an important starting point for their research lesson. For prediction, we observe that it was not easy for the teacher educators to predict how students would respond to their mathematical tasks. It appears that prediction is difficult for teacher educators just as it is difficult for student teachers as observed by Munthe, Bjuland, and Helgevold [2016]. For observation, we observe that this was better and more commonly understood by the teacher educators than research question and prediction. As we can see from the table all draft lesson plans except one had some points of observations and the revised plans improved in both the number of observation points and the focus. However, two of the five teacher colleges still had challenges in developing explicit points of observations. This seems to be related to the lack of a clear research question which is important in focusing research lesson on students learning and consequently what and how to observe the students learning.

4.4 Conclusion. Mathematics teacher educators understanding of LS varied across the five teacher colleges, especially in terms of research question and prediction. The variation highlights the complexity of LS in new contexts and by inexperienced participants. Nevertheless, it is important to note that in general the teacher educators' lesson plans improved from the draft to the revised version in all three aspects. The improvement varied across the colleges and in some cases was very minimal. The little and varied improvement further highlights the complexity of LS in new contexts. We view the improvement as emphasizing the importance of knowledgeable others in LS especially in new contexts and by inexperienced participants. The lack of much improvement challenges us of the crucial role of knowledgeable others that would make a difference to the quality of LS in new contexts.

5 Adaptation of LS with in-service teachers in Thailand

5.1 Introduction. After the release of *"The Teaching Gap"* by Stigler and Hiebert [1999], Japanese LS has become well-known around the world, Inprasitha [2015]. Many countries have been trying to adapt this approach. Thailand has also been adapting this Japanese LS since 2002, Inprasitha [2003]. At the early state, the research project had focused on introducing "new school mathematics" to teaching mathematics. In 2002, a LS team of student teachers pioneered in using "open-ended problems" as new mathematical contents for developing rich mathematical activity to challenge their students. Later during 2003–2005, a LS team of in-service teachers at a number of schools had been challenged to use 5 open-ended problems in their traditional classrooms, see Inprasitha, Loipha, and Silanoi [2006]. Since 2006, Japanese mathematics textbooks have been used in the first two LS project schools. At present, Thai version of Japanese mathematics textbooks are being used in 120 LS project schools across the country. Through an analysis of the project mathematics textbook, it concludes that focusing on students' ideas can provide a new approach for teachers to teach new school mathematics, rather than just follow mathematical topics as they appear in the

traditional textbooks. In this way, it is found that it is useful in helping to bridging the gap, as Klein (1902–1908); 2004 cited in Biehler and Peter-Koop [2008] put it more than one hundred years ago: *"there is a gap between university mathematics and school mathematics"*.

For years, school mathematics has not had its own unique identity Kilpatrick [1992] and Sierpinska [1994]. It has been taught on the demand of university mathematics. For example, in order to prepare school students to learn calculus at the university level, they have to learn pre-calculus at the school level. In other words, school mathematics is what teachers simplify university mathematics to teach at the school level. Pòlya [1945] mentioned that the opportunity is lost if the student regards mathematics as a subject in which he has to earn so-and-so much credit and which he should forget after the final examination as quickly as possible. Furthermore, mathematics has been viewed as a static arena Dossey [1992] and Ernest [1988] and school mathematics as content is viewed separately from teaching mathematics. Teachers just transmit the content to the students. Treating school mathematics and pedagogy this way hinders the development of both school mathematics and pedagogy to teach mathematics for many decades in many countries. This is especially true in the case of Thailand and some developing economies, or even in the United States Stigler and Hiebert [1999]. Thus, this paper provides new ideas on how we treat school mathematics, how it relates to teaching mathematics, and how we can support teachers with teaching new school mathematics through an analysis of the project mathematics textbook.

5.2 A new idea for school mathematics. Many mathematicians, philosophers, and mathematics educators have long been proposing that we could have an alternative approach of school mathematics. Pioneered with Felix Klein (1902–1908); 2004 cited in Biehler and Peter-Koop [2008], as a mathematician marks the history of mathematics education by highlighting the gap between university and school mathematics. He proposed analytical geometry as school mathematics, which is more accessible to school students, rather than traditional calculus. Followed by another great mathematician, George Pòlya [1945], who proposed a new aspect of mathematics that is school mathematics as doing problem solving. This idea later became an agenda for action of school mathematics highlighted by NCTM (1980) as *"the central of school mathematics is mathematical problem solving."* To name just a few, ideas of school mathematics as "a quasi-mathematics, experimental mathematics etc., Lakatos [1976] are in concerted with those of Klein, Pólya and Freudenthal.

A mathematics educator and philosopher like Paul Ernest [1989, 1991, 1992] proposes a philosophical view on school mathematics seen by different groups of teachers as follows: 1) mathematics as an instrumentalist, 2) mathematics as Platonist, and 3) mathematics as problem solving. The third view is a new view for school mathematics emerged from students as his/her own experiences in solving mathematics by themselves. In the late the 20th century, it became clear that *"problem solving approach is the central issue for school mathematics around the world."*

5.3 Mathematical problem solving as school mathematics. Traditionally, when we think about mathematical problem solving, mathematics as a content is on one side and solving that mathematical problem is on the other side. Goldin and McClintock [1979] and Lester [1994] put it this way: there was a general agreement that problem difficulty is not so much a function of various task variables such as content and context variables, structure variables, syntax variables and heuristic behavior variables as it is of characteristics of the problem solver. Similarly, Nohda [1982, 1991, 2000] put it, "Opening up the hearts of students toward mathematics. In the open-approach method, it is intended to provide students with rich situations by using open problems that have possibility to serve for individual differences among students both in their abilities and interests and in the development of mathematical ways of thinking, and to support the investigative process of solving and generating problems". And the most well-known citations are from those ideas of *"metacognition as a driving forces of mathematical problem solving*" Lester [1985], articulated by Schoenfeld and Herrmann [1982] and Silver [1985]. This quote informs us that an idea of mathematics, especially school mathematics is not a separate entity from students' problem solving. Steffe [2002] has confirmed that there is such a thing as 'students' mathematics, or mathematics of students'. From these points of views, school mathematics is intertwined between mathematical contents and students' problem solving and if we should put these ideas in school contexts, especially, contexts for teaching and learning mathematics. School mathematics could be seen as components of these following: 1) students' ideas emerge while they engage in mathematical tasks or problem situations, 2) students' individual differences become rich resources of mathematical aspects, 3) students' cognitive development become constraints of school mathematics at each grade level. These ideas contrast to the way typical mathematics textbooks provide school mathematics according to mathematical topics from the authors or the teachers' points of views, which is a rather definitional approach.

5.4 Distinguishing between mathematical tasks and students' authentic problem or real problem. Usually, mathematics teachers take for granted the distinction between students' authentic or real mathematical problems and others' sources of mathematical problems such as teachers' or mathematics textbooks. A typical approach to dealing with students' individual differences is for teachers to provide each student with "a number of mathematical problems" expecting that each student should have the opportunity to solve those problems according to their capability (See Fig. 1). However, students are busy with solving those problems superficially and teachers do not have much time to observe students' solving processes. Accordingly, students do not engage deeply with the problems while the teachers do not have time to learn from students' ideas. Thus, students' individual differences become a hindrance for teaching and learning mathematics. On the contrary, more than 100 years ago Japanese teachers developed innovations for dealing with this issue (See Fig. 2).

Figure 2 shows an innovation for engaging a number of students in a class with a particular task or problem. In this situation, the teachers have much more time in the

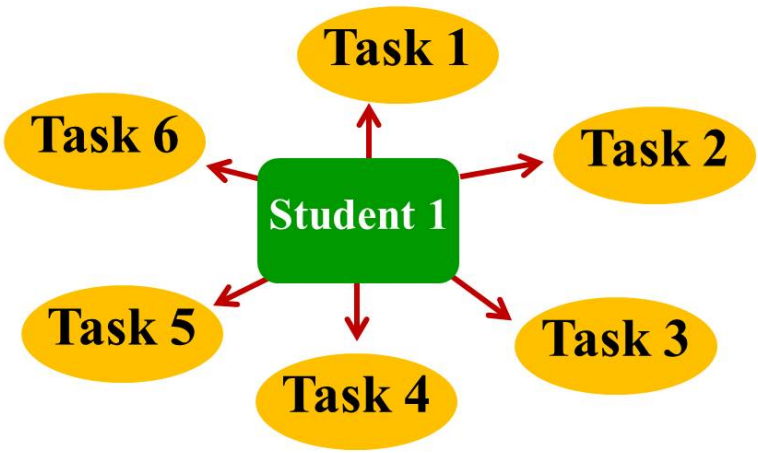

Figure 1: One student engaging in multiple tasks

class to engage students to collaboratively solve a particular problem and provide time for teachers to invest in students' individual differences.

5.5 Students' ideas as an origin of mathematical problem solving. Typically, mathematical problem solving focuses more on the problem-solving phase but ignores the problem posing phase Silver [1985]. Worse than that, in the problem-solving phase, teachers emphasize on getting right or wrong answers instead of the process of students' problem-solving. To complement these two phases, Brown and Walter [1990] highlight the importance of problem posing phase. They put it this way: there are two phase of problem posing: problem accepting and problem challenging. To engage students in mathematical tasks or problem situations in order that they come to accept the problem or they have their "own problem" is very crucial for future problem-solving phase. According to figure 2, if the teachers carefully 'designed' mathematical task or problem situations based on an anticipation of students' ideas, it will guarantee that students have been provided multiple chances to develop their own problems. Mathematical tasks or problem situations which are embedded in the students' real world will be easily accessible for most of the students in these classes. Each student's ideas emerge in collaborative problem solving and become rich resources for productive discussions in various aspects of a particular task or problem situation. Step 3 and 4 of innovation for teaching mathematics like Open Approach Inprasitha [2011, 2017] provide chances for the students in the class to compare and discuss among many ideas, later each student becomes 'aware of' each idea as a 'how to' for future solving an 'unknown' task or problem situation. Thus, students' ideas are an origin of mathematical problem solving.

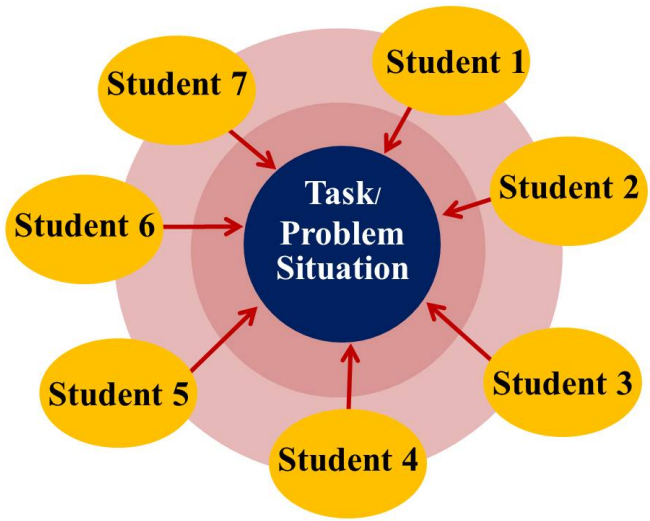

Figure 2: One problem engaged by multiple students

5.6 Open approach LS: Focusing on Japanese mathematical textbooks. The unique adaptation of Japanese LS in Thailand since 2002 is the focus of introducing new school mathematics as an innovation for teaching mathematics through problem-solving. Between 2002–2005, the selected open-ended mathematical problems had been used in a number of schools in the pioneered phase. Since 2006, 4 steps of Open Approach as an innovation for teaching mathematics have been incorporated into 3 steps of LS Inprasitha [2011]. Reading mathematics textbooks Plianram and Inprasitha [2012] is the main activity of LS team in the LS process implemented in a number of the project schools. For the first 6 years (2006–2010), most in-service teachers found it very difficult to read mathematics textbooks (translation version) due to many constraints such as: 1) the rigidity of indicators of curriculum standards, and 2) the teachers' familiarity with reading content through topics. However, Japanese mathematics textbooks are structured based on students' ideas beginning with students' real-world problems and progressing towards typical mathematics textbook problems. If the teachers cannot anticipate the students' ideas when they engage in the given problem task or situation, it is very difficult for the teachers to follow students' problem-solving process in the classroom. However, through working in LS team and teaching through 4 steps of Open Approach, the teachers have improved their deep pedagogical content knowledge on teaching new school mathematics. The weekly cycle of Open Approach LS Inprasitha [2017] is a promising innovation for teaching and learning mathematics in Thailand.

6 Methodological and theoretical issues of studying LS

6.1 Introduction. With the increased popularity of LS around the world there have been calls to deepen our understandings of how it contributes to teacher learning Lewis, Perry, and Hurd [2009] and Widjaja, Vale, Groves, and Doig [2017]. While studies have demonstrated the potential of LS to develop teacher community and enhance teacher knowledge e.g. Lewis and Perry [2017], Ni Shuilleabhain [2016], and Warwick, Vrikki, Vermunt, Mercer, and van Halem [2016], research has also highlighted the need to explore the theoretical underpinnings of mathematics teacher learning in LS Clivaz [2015, 2018], Miyakawa and Winsløw [2009], and H. Xu and Pedder [2015]. Many discussions about theoretical and methodological issues can be found in two recent books: *Mathematics lesson study around the world: Theoretical and methodological issues* Quaresma, Winsløw, Clivaz, Ponte, Ni Shuilleabhain, and Takahashi [2018] and *Theory and practices of lesson study in mathematics: An international perspective* Huang, Takahashi, and da Ponte [2019]. The latter comprises the main content of discussion of this panel. In this book, several theoretical frameworks have been called to analyze in detail what is happening in LS in terms of teacher learning: knowledge integration environment, self-determination theory, self-efficacy theory and pedagogies of practice Lewis, Friedkin, Emerson, Henn, and L. Goldsmith [2019], cultural-historical activity theory Wei [2019], deliberate practice Han and Huang [2019], interconnected model of professional growth Widjaja, Vale, Groves, and Doig [2019], Teaching for Robust Understanding Schoenfeld, Dosalmas, Fink, Sayavedra, Tran, Weltman, and Zuniga-Ruiz [2019] or, Theory of Didactical Situation Bahn and Winsløw [2019] and Clivaz and Ni Shuilleabhain [2019].

As an illustration of "finding an appropriate theoretical perspective to approach LS and glean its advantages Wei [2019]" this panel contribution highlights the types of knowledge employed by teachers in LS (for the full paper, please see Clivaz and Ni Shuilleabhain [2019]. Based on a case study conducted with eight primary school teachers in Switzerland Ni Shuilleabhain and Clivaz [2017], we detail their participation across one cycle of LS utilizing a combination of the theoretical frameworks of and Ball, Thames, and Phelps [2008] and Margolinas, Coulange, and Bessot [2005]. This fine-grained analysis demonstrates the constituents of both subject and pedagogical content knowledge employed by teachers, at varying levels of pedagogical activity, for each phase of the LS cycle. In this case study teachers' pedagogical content knowledge, particularly related to their consideration of content, was the most utilized form of knowledge incorporated in their LS work. Teachers' values and considerations about teaching and learning was also apparent throughout their planning and reflection of the research lesson. These findings provide a detailed representation of the types of knowledge employed by teachers across the phases of LS and contribute to our understanding of how and what teachers may learn in their participation in LS. In addition, our analysis demonstrates that each phase of LS need not necessarily take place in succession, but rather occur in a confluence of teachers' conversations over one full cycle.

6.2 Theoretical framework. Ball, Thames, and Phelps [2008] developed a practice based theory of the knowledge required "to carry out the work of teaching mathematics", presented as a framework of Mathematical Knowledge for Teaching (MKT) (p. 395). This model built on Shulman's theoretical suggestion of PCK as a specific type of knowledge unique to teachers and distinguished it from subject matter or content knowledge (1986, 1987). In their model, Ball and her colleagues highlighted particular categories of knowledge within the subject matter delineations and PCK delineations:

Subject Matter Knowledge

- Common Content Knowledge (CCK)
- Horizon Content Knowledge (HCK)
- Specialized Content Knowledge (SCK)

Pedagogical Content Knowledge

- Knowledge of Content and Teaching (KCT)
- Knowledge of Content and Students (KCS)
- Knowledge of Content and Curriculum (KCC).

In their review of the conceptualization and evidence of PCK in mathematics education research, Depaepe, Verschaffel, and Kelchtermans [2013] noted the MKT model as "probably the most influential re-conceptualizations of teacher PCK within mathematics education" (p. 13). However, Steinbring [1998] and Margolinas [2004] suggest that in Shulman's proposed framework of teacher knowledge Shulman [1986], on which the MKT framework is modelled, fixed categories of teacher knowledge are "not a good model for teacher's activity, which is more complicated" Margolinas, Coulange, and Bessot [2005, p. 207].

In the 1970s, Brousseau's theory of didactical situations Brousseau [1997] first modelled a learning situation by focusing on student learning without explicitly incorporating the role of the teacher Bloch [2005]. However, in analyzing student learning from the 1990s, the importance of the teacher's role became increasingly evident in the theorization of classroom situations Bloch [1999], Dorier [2012], and Roditi [2011]. This new lens provided opportunity to introduce a situated theory embedded in the context of the classroom, through which the various levels of practices, skills and knowledge required of mathematics teachers could be analyzed. Based on this theory of didactical situations, Margolinas [2002] developed a model of a mathematics teacher's milieu to describe the teacher's activities, both in and outside of the classroom. This model was designed to acknowledge the complexity of teachers' actions and while also capturing the broad range of activities contained in teaching and learning Margolinas, Coulange, and Bessot [2005]. Centering on the action of the classroom, the model depicts the various levels at which a teacher must situate themselves within their pedagogical practices. In this model (see Fig. 3), level +3 refers to teachers' values and conceptions about learning and teaching, level +2 concerns teachers' actions and discourses about

the global didactic project, level +1 pertains to the local didactic project, level 0 is the didactic action, and level –1 deals with the observation of pupils' activity. The teacher's point of view can be related to his or her considerations and reflections at different levels of generality. Observing students' work (including noticing student talk) relates to a more deliberate focus of the teacher on individual students and, hence, relates to level -1. Planning the local didactic project (about the lesson) refers to the preparation and sequencing of content within the lesson and, hence, is placed at level +1. At level +2, the teacher considers the didactic project in a more holistic or global sense (e.g. teaching a particular element of a topic as one lesson in a series of lessons or throughout a term). While at level +3, the teacher draws upon their beliefs about the teaching and learning of mathematics, which can be related to how the didactic projects may be constructed and to how the teacher will engage with individual students. The model is not intended as a linear interpretation of teachers' work, but rather identifies the multidimensional tensions involved in teaching Margolinas, Coulange, and Bessot [ibid.]. At every level the teacher not only has to deal with the current, most prescient, level of activity, but also with the levels directly before and after and, in some instances, with levels extending beyond.

In our research Ni Shuilleabhain and Clivaz [2017], we proposed a combination Prediger, Bikner-Ahsbahs, and Arzarello [2008] of these two existing frameworks of Mathematical Knowledge for Teaching Ball, Thames, and Phelps [2008] and Levels of Teacher Activity Margolinas, Coulange, and Bessot [2005] to analyze the knowledge incorporated by teachers in two case studies. The graphical representation of this framework (Fig. 3) shows that the categorization of knowledge lies in one plane (the *egg*), while the levels of activity are characterized in a contrasting cross-sectional plane (the *cake*). In this chapter, we develop this work and employ the framework as a tool to further detail and analyze mathematics teachers' knowledge in various phases of planning, conducting, observing, and reflecting on teaching in one case-study cycle of LS (see Fig. 3).

6.3 Context and methodology. Eight generalist grade 3–4 primary school teachers from the Lausanne region, French-speaking part of Switzerland, were introduced to LS and conducted four LS cycles over two school years. The group was facilitated by two university teacher educators, one specialist in teaching and learning and one specialist in mathematics didactics. All meetings (37 of an average of 90 minutes duration) and research lessons (8) were transcribed and coded in a qualitative analysis software (NVivo). Student work, teachers observations during lessons, and lesson plans were also recorded and coded Clivaz [2016]. In the comprehensive analysis, see Clivaz and Ni Shuilleabhain [2019] we detail the different forms of Mathematical Knowledge for Teaching Ball, Thames, and Phelps [2008] and Levels of Teacher Activity Margolinas, Coulange, and Bessot [2005] that occur over each phase of the LS cycle. Through the use of quotes and graphical data we explicate teachers' knowledge recorded in their participation in LS.

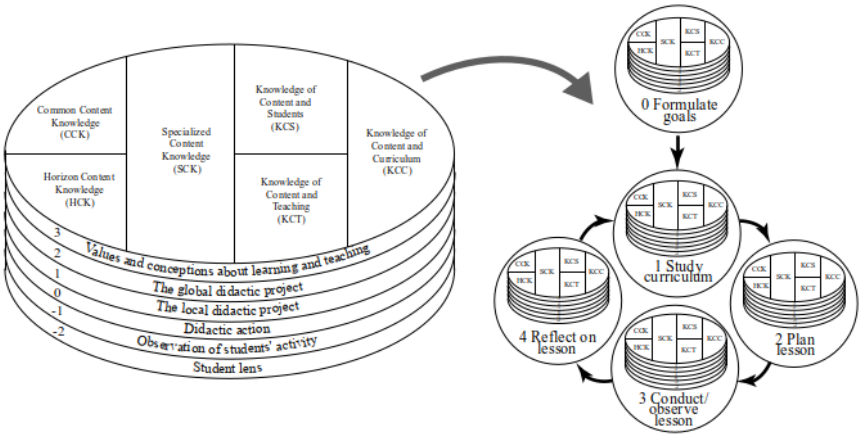

Figure 3: MKT and levels of teacher activity in a cycle of LS.

6.4 Some results. Several graphical representations in this chapter demonstrate the repartition of each of the components of MKT and Levels of Teacher Activity across a cycle of LS. Our research demonstrates that all levels of teacher activity, from the values and conceptions about learning and teaching to seeing mathematics through the eyes of the student, are afforded opportunities of articulation in teachers' participation in the collaborative work of LS, making implicit teacher knowledge explicit Fujii [2018] and Warwick, Vrikki, Vermunt, Mercer, and van Halem [2016]. Analysis of the data also evidences the presence of all categories of MKT across the phases of the cycle, particularly those of KCS and KCT (types of pedagogical content knowledge) and SCK (a form of subject matter knowledge) (see Fig. 4).

Combining both frameworks, our data shows a prevalence of KCT within the LS cycle, which may support other research findings which have demonstrated changes to teachers' classroom practices as a result of their participation in LS e.g. Batteau [2017], L. T. Goldsmith, Doerr, and Lewis [2014], Shuilleabhain and Seery [2018], and Takahashi and McDougal [2018]. Furthermore, our graphical representations (for example, Fig. 4), depict the benefit of participation in LS, where teachers have opportunity to utilize almost all elements of their MKT across each phase of the cycle and across all levels of teacher activity. An advantage of participating in LS may be the fact that it encourages teachers to incorporate, draw on, and potentially develop their knowledge at various levels, by openly articulating their knowledge through active participation across each of the phases.

In analyzing the types of knowledge occurring across each of the LS meetings, our research demonstrates that the phases of LS do not necessarily occur in a strict chronological or sequential order, but rather take place at varied points throughout the cycle. This may be an important finding in facilitating and analyzing LS, where teachers can

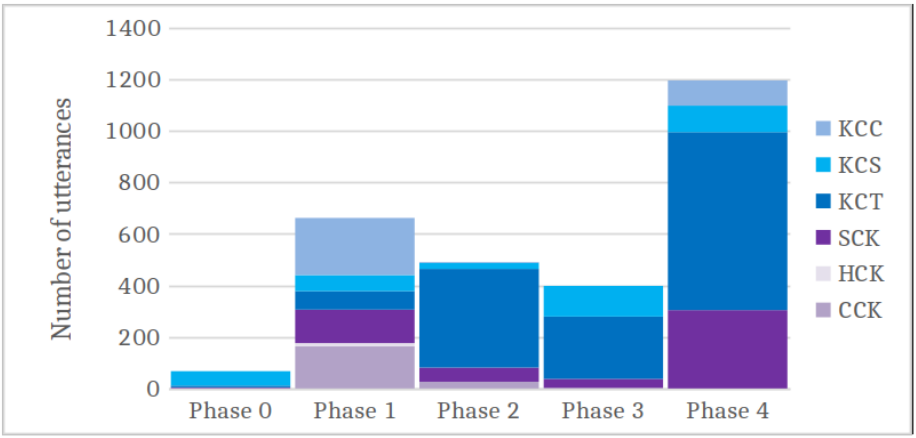

Figure 4: MKT expressed in one cycle of LS.

articulate goals, student learning, or subject topics at all point throughout their LS conversations.

This articulation of teacher knowledge in LS, and the way that teachers are propelled and compelled teachers to express this professional knowledge, is a key element of this model of professional development. Further investigation and theorization of teacher knowledge and teacher learning is required in order to sustain LS as a professional situation where teachers can develop their knowledge in and for teaching.

7 Further directions of LS

Based on literature reviews, the selected case studies, and the panel debates during the conference, the discussions about the future directions of LS are organized into three aspects: (1) conceptualization of LS; (2) research on LS; and (3) adaptation of LS.

7.1 Conceptualization of LS. Based on Japanese LS and its adaptation globally, Takahashi and McDougal [2018] proposed a collaborative lesson research (CLR) model (see section 2). The model includes key elements of LS: (1) A clear research purpose, (2)*Kyouzai kenkyuu*, the "study of teaching materials", (3) A written research proposal; (4) A live research lesson and discussion; (5) Knowledgeable others; (6) Sharing of results. Along these key components, the panel discussed the following questions: (1) Does Lesson studies need specific instructional guiding theories? (2) Is repeated teaching of the same lesson optional or necessary? (3) Is perfecting a research lesson not one of the goals of LS? (4) Does knowledgeable other need intervene the process of LS? (5) What is the specificity of LS in terms of teacher learning? Debates surrounding these questions could advance the conceptualization of LS.

Regarding the question if guiding theories for conducting LS are needed or not, there are different interpretations. Typically, in Japanese LS, there is not a general

learning theory as guiding principle. However, in learning study, the theory of variation is explicitly claimed as the only guiding instructional principle Marton and Pang [2006]. In Malawi, multiple theories could be used to guide LS. For example, during lesson plan stage, MKT Ball, Thames, and Phelps [2008] could be helpful to identify what kind of knowledge teachers need to have in order to develop a lesson plan effectively. While, evaluating research lesson, the theories about discourse analysis Adler and Ronda [2015] could be very helpful to analyze the nature of interaction between teacher and students. In Thailand, the community of practice Wenger [1998], metacognitive theory, and open approach of teaching Becker and Shimada [1997] were used to guide LS. In China, both learning trajectory Simon [1995] and teaching through variation are used to guide LS as well Huang, Gong, and Han [2016]. However, if we classify theories into local theories (how teachers teach a lesson based on their local tradition and cultural value, such as Japanese structured problem solving approach Stigler and Hiebert [1999], and Chinese teaching through variation F. Gu, Huang, and L. Gu [2017] and global theories (general theories about teaching and learning such as situated learning theory Lave [1988] and learning trajectory Simon [1995]), then all LS are guided by certain theories. If knowledgeable others who are mathematics specialists focus on improving teaching practice only, then, local theories are often used implicitly; if knowledgeable others who are mathematics educators from university, focus on both improving teaching practice and developing some theoretical knowledge, then, some grand theories may be used explicitly. So, whether theories are used explicitly, it depends on the "research purpose".

With regard to if repeated teaching is necessary, there are different opinions. Japanese LS does not require repeated teaching of the same lesson. However, LS in China, UK, and Sweden prefer repeated teaching. In China and Sweden, repeated teaching of the same content to different groups of students is required. Chen [2017] explained this specification from a cultural perspective and argued that it is related to the Chinese philosophy of unity of knowing and doing, and learning through deliberate practice. Practically, teachers want to see if suggested changes work in their classes. In Sweden, learning study is a combination of Japanese LS and design-research Cobb, Jackson, and Dunlap [2016], so the orientation of design research requires repeated teaching to testify any changes over the process. In Switzerland, repeated teaching is option, but it is realized that repeated teaching with careful redesign could provide valuable learning opportunities for participating teachers. Thus, it is clear that if specific guiding theories are used and repeated teaching is required, then, LS could be conducted as a design research with the purpose of developing the connection between theory and practice. But, if LS is only focused on improving teaching practice its self, it could be seen as an action research Elliott [2012] with a group of teachers. So, a LS could be located between action research (without any explicit guiding theory and repeated teaching) and design research (both guiding theories and repeated teaching are required).

Perfecting research lesson as a goal during a LS is debatable. In Japanese LS, perfecting research lesson is not the goal of LS. Yet, in China, developing exemplary lessons (product) is the major goal of LS. In Malawi and Thailand, good lesson plans and/or

videotaped lessons developed through LS often are for pre-service teachers preparation program. In Sweden, learning study emphasizes developing shareable instructional products (annotated lesson plan, students work, lesson videotaped lesson and theoretical analysis documents). In Switzerland, reaching a perfect lesson is clearly not the goal. Nevertheless designing a lesson that helps the most students to learn is the apparent goal whereas the real goal is to acquire professional knowledge. The Chinese practice of emphasizing development of an exemplary lesson is closely related to teachers' learning from "examples" X. Li [2019]. The positive effect of emphasizing developing good lessons is the possibility of developing sharable instructional products Runesson, Lövström, and Hellquist [2018] and then further improving teacher learning on larger scale. However, it may mislead teachers that there are perfect lessons, or that there is a best way to teach a certain topic. Thus, the LS will be misled to focus on teaching performance rather than student thinking and learning. Ideally, balancing the goals of perfecting a lesson (product) and maximizing teachers' learning opportunity (process) will be important.

Regarding the roles of knowledgeable others during LS, it is agreed that having knowledgeable others' involvement during LS is important. In China, the promoting system ensures there are master teachers in each school and that there are teaching research officers who are responsible for teaching research activity in each district. So, there are quite a number of knowledgeable others. However, in many countries, LS is new and there are not so many knowledgeable others available. For example, in Thailand, knowledgeable others are normally university professors. They not only take responsibility for administrating LS (aliasing with school principals, forming LS groups, and scheduling meetings) but also facilitating the LS process (content and pedagogy aspects). Thus, the role of the knowledgeable other is important in Thailand. Similarly, in Malawi, LS is new. Teachers need to learn what LS is, and learn how to conduct LS from knowledgeable others. In all situations, how to facilitate post-lesson debriefs is crucial. Often, teachers focus on superficial aspects with praise for teachers, rather than focus on critical aspects of student learning. Training on facilitating skills is needed. Thus, it is a challenging task to train qualified knowledgeable others for facilitating LS.

7.2 Research on LS. With regard to research on LS, both theoretical perspectives and research methodologies are critical. As section 6 illustrated, many theories such as knowledge integration environment, self-determination theory, self-efficacy theory, cultural-historical activity theory deliberate practice, interconnected model of professional growth, and Theory of Didactical Situation have been explored to examine LS as a system or a construction of individual components. In addition to case study, both qualitative and quantitative, and mixed methods are used to study LS. For those theoretical perspectives, more empirical studies are needed to examine the strengths and constraints of each theory when used to examine LS.

In return, LS provides an extraordinary field of research for qualitative studies about teachers in mathematics education. The observation of LS group provides insight on

how teachers think about mathematics, on how they prepare lessons, on how they observe students, on how they reflect about a lesson, on how they use past experiences or on how they incorporate external (theoretical or practical) resources; all these observations being in a quasi-natural environment. Moreover, these topics are interconnected in teachers' talk and these connections can be studied in LS, presumably better than in other research design Clivaz [2016] and Stigler and Hiebert [2016]. Last but not least, research on LS is part of LS as a research. The two types of research are interconnected and they influence each other. There is a risk that these two types of research would intermingle, and this has to be avoided. If such a confusion is avoided, the connection of academic research on LS and teacher research is a promising way of bridging the research gap between academics and teachers.

7.3 Strategies for adaptation of LS. For the country where LS is new, it is critical for knowledgeable others to talk to different departments, and convince administrators that LS is a powerful and rewarding professional development approach to invest. The support from education administration is decisive for the success of LS. It is also useful to have a long-term plan and strategies to scale up LS. For example, several universities could work together to train Ph.D. Students to serve as knowledgeable others. It is important to get government funding. In addition, to have demo lessons given by master teachers, and invite potential participating teachers to watch demo lessons and participate in post-lesson debriefs facilitated by knowledgeable others to help teachers understand the LS process and obtain a preliminary experience in LS.

Overall, it is hard to identify research question for teachers. So, specific attention needs to be paid to developing teachers' awareness of and ability of identifying research questions. Moreover, challenging issues include how to use LS with poverty students and how to address equity when conducting LS. Much more research in this area is needed.

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INTERNATIONAL MATHEMATICAL KNOWLEDGE TRUST IMKT: AN UPDATE ON THE GLOBAL DIGITAL MATHEMATICS LIBRARY

Patrick D. F. Ion, Thierry Bouche, Gadadhar Misra,
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Abstract

A Panel at the 2018 International Congress of Mathematicians concerning the efforts to realize the dream of a Global Digital Mathematics Library consisted of Thierry Bouche, Gadadhar Misra, Alf A. Onshuus, Stephen M. Watt and Liu Zheng and was moderated by the writer of the lines below as recorder. This report contains a description setting the stage for the panel, summaries of the panelists' statements and of some questions and answers at the session as well as at a later open opportunity for further discussion.

1 Intent of the Panel

This panel was arranged by the Committee on Electronic Information and Communication (CEIC) of the International Mathematical Union (IMU) through its Global Digital Mathematics Library Working Group (GDML WG). This panel follows up on a previous panel in Seoul at ICM 2014 with the title "The World Digital Mathematics Library" which coincided with the inception of the GDML WG. That working group came to the conclusion that a significant step to realizing to dream of a World Digital Mathematics Library would be to set up a legal entity, the International Mathematical Knowledge Trust, initially based in Canada, and hence the name of this panel.

A recording of the panel session may be able to be seen at the the YouTube collection of ICM 2018 like the Seoul 2014 panel.

The objective of the session was to provide the community with information about further strengthening of access to our mathematical literature and its preservation as a global public good. This is part of building a World Digital Mathematical Library, as endorsed by the IMU as long ago as 2006.

It is necessary to increase the commitment of our community to what it being done and what needs to be done to preserve our mathematical heritage, and it was hoped the variety of relevant issues that can be discussed could improve both support for WDML and the effectiveness of the movement setting it up.

Issues that might have been discussed include:

MSC2010: primary 65Z05; secondary 52C23.
Keywords: Machine proof.

- The structure of a GDML and its services — why we set up an International Mathematical Knowledge Trust
- Opportunities for worldwide collaboration
- The state of mathematical publishing in the digital age
- Approaches to getting more out of the presently available digital literature by data mining and statistical methods
- The design of a semantic markup language for mathematics
- The present tools for the digital handling of mathematics and how they interact: e.g. TEX, MathML, MathJax, Word, computer algebra systems and proof assistants,
- Issues of reproducibility of results when computing is involved
- The need for authority trails in a reliable literature

A difficulty with developing a GDML is the breadth of relevant issues as shown in the list above, itself by no means an exhaustive one. The actual content of the panel emphasized, rather than technical discussions the global nature of the effort involved by the diversity of panelist origins and experience, and, in particular, by the representation of the largest populations centres of the world.

It seems worth repeating some historical references mentioned at the 2014 panel on GDML efforts. At the first ICM in 1897, in Zürich, there was a session including pasigraphy, under the chairmanship of G. Peano, concerned with questions of how to encode mathematical knowledge. (It was in connection with such efforts that Peano developed his axioms for the natural numbers.) Schröder said there he was sure that pasigraphy would take its rightful place on the agenda of all succeeding ICMs. (Pasigraphy is the study of universal languages of symbols intended to encapsulate semantics and to provide a basis for calculational ratiocination.) This clearly did not happen.

At the 1928 ICM, in Bologna, there was active discussion of how to provide comprehensive bibliographic resources for mathematics to everyone. They were to be based on publishing the exhaustive collection of about a quarter of a million catalogue slips assembled by Georg Valentin. These slips were eventually destroyed in 1942 by fires resulting from bombs dropped on the Prussian State Library on Unter den Linden in Berlin.

Earlier, in the decades from about 1900 on, the Belgians Paul Otlet and Henri La Fontaine planned, and started to create, a master bibliography of all the world's published knowledge. La Fontaine used his Nobel Prize money for their cause. In 1934, Otlet sketched plans for a global network of "electric telescopes", as he called them, that would allow people to search and browse through millions of interlinked documents, images, audio and video recordings. He described how people would use the devices to send messages to one another, to ask queries of the central repository and receive answers, to share information and even to get together electrically connected social networks. The whole system was termed a "réseau". The Second World War killed this effort as well, though there is a Mundaneum museum and center today to visit in Mons.

The point of these glances to the past is to emphasize that a GDML effort must not give up, even if it takes significant time to mobilize the resources needed.

1.1 Panelists. The panelists, listed alphabetically below, were chosen to represent the global intent of the effort to realize access to the legacy literature of mathematics as well as for their expertise with modern digital technology. Their present affiliations are given in parentheses,

- Thierry Bouche (Université Grenoble Alpes, Grenoble, France)
- Patrick D. F. Ion (GDML WG, University of Michigan, Ann Arbor MI, USA) — moderator
- Gadadhar Misra (Indian Institute of Science, Bangalore, India)
- Alf A. Onshuus (Universidad de los Andes, Bogotá, Columbia)
- Stephen M. Watt (University of Waterloo, Waterloo ON, Canada)
- Liu Zheng (National Science Library, Chinese Academy of Sciences, Beijing, China)

2 Statements from the panel

The order of speaking was not alphabetical. The initial paragraphs provide short panelist biographies just mentioning details relevant here rather than their much larger CVs. The bracketed initials show the handles used in reporting discussion. Slide sets from the panelists, which are more extensive than those which could be shown during the panel session itself, due to time pressures can be found at the IMKT Projects site.

Stephen M. Watt [SMW] is now Dean of Mathematics at the University of Waterloo, Ontario, Canada. The Waterloo Faculty of Mathematics includes four Mathematics Departments and Computer Science with more than 8,000 graduate and undergraduate students, 240 full-time professors, and 300 courses in mathematics, statistics and computer science. Previously Watt held positions at Western University (Ontario), IBM T.J. Watson Research Center, INRIA and the University of Nice. Among his main interests have been computer algebra. In particular, he has been a co-author of the CA languages Maple and Axiom, and Aldor. Aside from research publications, he has been a co-author of the World Wide Web Consortium (W3C) markup language standards MathML and InkML. He was Co-Principal Investigator with Ingrid Daubechies on the Alfred P. Sloan Grant that enabled setting up the IMKT based in Waterloo ON.

Stephen Watt addressed particularly IMKT and GDML efforts. He began by noting that Mathematical Knowledge *can* be of permanent relevance, represented exactly, and treated mechanically. This led to envisioning a Global Digital Math Library. Such a notion has been a longstanding vision with many advocates over time, likely including some in the present room. It had been often thought of, in general terms, as linked digitization of (much or) all of past mathematical literature in enhanced, open article

repositories. The scope would then be of the order of 10^6 to 10^7 items. It was the task of the IMU GDML Working Group whose activity was initiated by Ingrid Daubechies at the Seoul ICM to begin to turn this into some sort of reality. The group membership was

- Thierry Bouche (U Grenoble, France)
- Bruno Buchberger (RISC Linz, Austria)
- Patrick Ion (Math Reviews, USA) chair
- Michael Kohlhase (FAU Erlangen-Nürnberg, Germany)
- Jim Pitman (UC Berkeley, USA)
- Olaf Teschke (zbMATH, Germany)
- Eric Weisstein (Wolfram Research, USA)

The GDML WG was charged with the tasks of

- designing a road map for the practical next steps towards the GDML
- determining its organizational structure
- prioritizing the different requirements for its implementation
- estimating an incremental budget, both start-up and sustaining funds
- fostering the writing of proposals to funding organizations

To carry out its remit the WG engaged in the following activities

- extensive regular tele-conferences
- organizing workshops, sessions and panels (such as this)
- specific presentations and conference presentations
- gaining a seed funding grant
- founding a not-for-profit, the International Mathematical Knowledge Trust (IMKT) in Canada
- encouraging seed projects

More about this history can be found at the new IMKT web presence. There can also be found an explanation of the governance of the IMKT, its Board Membership and legal status.

Watt next passed to "What the Future Was". In an extreme form this could be summed up in the simple question "A linked repository of articles is nice, but don't we have this already?", which suggests there's nothing more to be done.

In the present day there are already all sorts of (conventional) article repositories such as publisher archives — e.g. Springer — and public ones such as JSTOR, arXiv, EuDML, NUMDAM, Göttinger Digitalisierungszentrum, RusDML, valuable private lists like Ulf Rehmann's, and many more.

In addition there are indexes, review collections, and author databases such as Inspec, zbMATH, MathReviews/MathSciNet, the Math Genealogy Project, and the broad access method Google Scholar.

Furthermore, there are General Summary Works such as the Wikipedia Mathematics Project, MathWorld, the Wolfram Functions Site, the NIST Digital Library of Mathematical Functions, and the Digital Repository of Math Formulae.

There are Specialized Tools and Databases such as the Online Encyclopedia of Integer Sequences, the Dynamic Dictionary of Mathematical Functions, the Online Integral Calculator, the Inverse Symbolic Calculator, Atlas of Finite Simple Groups, the L-functions and modular forms database, the Combinatorial Statistic Finder, A Catalogue of Lattices, and the Encyclopedia of Triangle Centers.

There are Proof Libraries, such as those devoted to Big Proofs of the Four Colour Theorem, the Feit–Thompson Theorem, the Kepler Conjecture, aka the Hales–Ferguson Theorem, and so on. This was discussed in the previous panel at the Rio ICM. But there's an enormous amount of knowledge in the Mizar Mathematical Library, the Archive of Formal Proofs, and the MetaMath Proof Explorer.

Finally there are even more miscellaneous mathematical projects and activities online such as the Encyclopedia of Continued Fractions, the Maple Function Advisor, Maple implementations of Exact Solutions of the Einstein Field Equations, the GAP Package `liealgdb`, Matrix Market, EqWorld, Formulasheet, and SymbolicData site.

With all this material actively online it is clear what the current direction is. We are passing from digitized page images to the results of parsing pages, then on to semantic markup of pages, and finally to corpus analysis, once material has passed through the earlier stages or been created with a higher degree of preparation.

A full use of digital technology for mathematics will have to provide support tools for dealing with the meaning of the content of mathematical works. Then we will be able to have a real Mathematical Digital Library, and/or a Mathematical Knowledge Base with Mathematical Knowledge Tools to explore and develop it. The fundamental point is mathematics can be likened to Pando, the Trembling Giant. This is a group of quaking aspen trees estimated to have 47,000 trunks on about 433,000 square meters, weighing a total of 6,000,000 Kg. It is now realized to be a single organism extensively connected by its roots below the ground, which have been growing together and expanding their cover for 80,000 years. Mathematics is perhaps not that old but it is remarkably interconnected as a subject. It too shows "the fundamental interconnectedness of all things".

To return to the more mundane Watt then mentioned the four basic Initiatives that IMKT has been trying to get started. More description of the intent of these initiatives and their progress will be found on the IMKT site.

Special Function Concordance The special functions (SF), are traditionally those of mathematical physics, but nowadays are widely applied as well, for instance, in

computer science, biology, statistics and finance. They are commonly considered well-known basic material. SF can be calculated with any of the common computer algebra systems or looked up in tables or online, to get both values and properties. However, the various systems and resources that are called upon for reference vary in details of conventions in their definitions. The DLMF from NIST (the thorough-going revision of the famous Abramowitz and Stegun handbook), the DDMF from France, the implementations of special functions by the hundreds in Wolfram's Mathematica and Alpha, or Maple, or in Sage (Pari, Gap, et al.) and even the libraries of definitions in automatic theorem provers and logic assistants such as Mizar, HOL Light, Coq and LEAN, all show variations in their handling of common functions, such as the inverse trigonometric, Bessel or elliptic functions. This leads to simple non-interoperability of results that are published, and a tendency for a particular research team to stick with one vendor so as to be at least internally consistent. Only occasionally do massive difficulties result from this Babel, but there's a good deal of time and effort lost in checking consistency when users employ more than one resource. The SFC initiative is getting representatives of groups with a stake in special functions, as mentioned above, together to find common ways to express their accords and their differences, publicly and in a machinable form. Already Bruce Miller of DLMF has developed OpenMath Content Dictionaries that may act as a springboard for further development.

FABstracts This project, suggested by Tom Hales and now running under a Sloan grant awarded to Hales and Jeremy Avigad, aims to provide formal abstracts for mathematical items from an initial sample of mathematics. It is using the language of the LEAN3 theorem prover in its initial efforts, and is international in that the initial teams are in Pittsburg, USA, and Hanoi, Vietnam, with other contributors from all over the world and a GitHub presence. Public results may be expected soon.

Formal Harmony There's need for harmonization similar to that for SFC above comes where formalization of mathematics is concerned. In fact, this is a sort of thing that practitioners of mathematical formalization can see as a useful project. There are by now many overlapping pieces of mathematics, such as basic pieces of Euclidean geometry and some fundamental lemmas and theorems of analysis that have had to be formalized in more than one system. The main systems where there is real overlap are HOL Light, Coq and Mizar. What has not been done is to make careful comparisons of the formulations of, say, the Jordan Curve Theorem, or the Fundamental Theorem of Calculus in the various contexts, and to see in what senses they really are the same and really may differ. This is not just a matter of the way the proofs proceed but may be about subtleties in the semantics resulting from different foundational bases.

Clarity should be of value to appliers of mathematics and also to educational efforts. What is required for a theorem's truth is often somewhat obscure. Researchers have expressed interest in exploring the ways systems are the same and

differ. As a practical result the search for applicable results may be extended to a federated search over different library collections of already formalized material with increased clarity over what's returned.

Mathematical Document Analysis and Classification There are several sorts of document analysis that are possible and have not been applied to the mathematical corpus, which has its own special features. Mathematics is arguably the natural language of a globally distributed people (those with enough training in mathematics to do science and engineering) but not a typical one. For a start, its written form is not truly linear, involves a large sign vocabulary, and sometimes different grammar. We do see a need for power tools for document digestion as the GDML effort ramps up, so Natural Language Processing (NLP) methods tried and tested elsewhere will need to be adapted to handling the mathematical corpus.

N-gram analysis is a relatively easy place to begin. NLP tools here are readily available for ordinary languages. Sample corpora, e.g., those of zbMATH and arXiv are readily available and their curators interested in such ideas. Some early experiments have been tried at zbMATH for keyword generation, and another small project was completed at ORCCA under Watt's direction that concentrated the N-grams on the mathematical formulas and found characteristics of different subjects could be detected.

This is a much more limited approach than deep learning and needs much fewer resources, but it is applicable in context finding and clustering tools that can also be deployed much more.

In addition, there are other projects underway using machine learning for the examination of the extant mathematical literature that hope to enhance the access to the semantics implicit in the publication data already collected through new search mechanisms and to facilitate input of mathematics through hand-writing and retrieval with math-aware search. The respective examples are being worked on by teams led by Charles S. Lafferty and David M. Blei, and by Richard Zanibbi and Lee Giles.

All these ultimately must involve bringing together several groups' work.

Watt finished with some 'slightly less obvious statements about getting something done'. First we can afford to hand-annotate or capture the semantics and special characteristics of a small subset of mathematics, but we must rely on improvements in technology and artificial intelligence to get further. The stages would seem to be 'by hand', closely supervised automation, automation allowed to run but then checked by humans, and finally full automation. It is true that approximation can go a long way in this process. However, as projects follow this path they must consciously future-proof their data, not, for instance, throwing away too much information collected at one stage that could be useful for later automation.

Finally, he remarked that mathematics has to solve the problems of mobilizing its community to more open participation and to do this there have to be provided new inducements as returns on effort. There are the examples ranging from the now almost classic Wikipedia, while the relatively closed (moderated) Nupedia couldn't take over and that should inform us.

Gadadhar Misra [GM] is a mathematics professor at the Indian Institute of Science, Bangalore. He is a Fellow of the Indian National Academy of Sciences and its Vice-President of the Publications and Informatics. His main field of mathematical research has been analysis and, in particular, operator theory. He has worked on the Indian efforts to provide an Open Access Repository of publications of Fellows of the Indian Academy of Sciences and and to make research open and actionable in a timely manner. India is the country with the second largest population and an example to developing nations.

Misra began his view from India by remarking that in a way the situation there was rather good. The infrastructure of the internet and access to web resources was good in a land with a successful information technology business sector and IT education. Concentrating on progress in the field of mathematics, but applying more generally to much of science, preprint archives provide a platform for permanently storing soft copies of scientific manuscripts, often before peer review, and allow open access to any interested person able to get on the internet. Indeed the NIH has published the statement "Scientists issue preprints to speed dissemination, establish priority, obtain feedback, and offset publication bias", and this is true in our subject as well.

To take a quick look at recent history we can note that one of the first, and popular, online preprint archives was **arXiv**, which in over 25 years of its existence, strongly influenced many publishers and impacted how science is disseminated.

The arXiv originated at Los Alamos (for physics) and at Duke University (for mathematics). These sites were then combined at the Los Alamos (LANL) centre and have since moved to Cornell University. After the service was widely used mirror sites were introduced.

The IMSc mirror at `https://in.arxiv.org` is a mirror site for arXiv that has been functional since the late nineties. Currently, that mirror is connected to the internet with a sufficiently high-speed link that it is up-to-date within about 20–30 minutes from the time that the primary site is updated. There is enough local processing power and disk-space to last for quite a while.

This sort of clear success has meant that a new initiative, Open Access India, is in the process of forming a steering committee to manage a Preprints Repository for India, **indiaRxiv**. This is an Open Access Repository of publications of Fellows of the Indian Academy of Sciences. It attempts to collect, preserve and disseminate the intellectual output of the Academy available as publications by its Fellows in peer-reviewed journals.

For the Indian Institute of Science (IISc), which is a country-wide distributed system, an **ePrints@IISc** repository collects, preserves and disseminates in digital format the research output created by the IISc research community. This enables the Institute community to deposit their preprints, postprints and other scholarly publications using a web interface, and organizes these publications for easy retrieval.

Going beyond the individual postings of preprints, or indeed of reprint and alternative material, that allow open and free access there is a new approach that can be taken to publishing that could be of value to mathematics.

EpiSciences.org is an innovative combination of the two routes of free access:

- the gold route provided by hosting journals in open access (overlay journals)

- the green route whereby articles are submitted to journals by depositing them in an open archive

The Hardy–Ramanujan Journal, a respected Indian publication since its inception in 1978 moved to publication on EpiSciences in 2014. (A diagram showing the revised workflow of publication resulting from this change is on a slide in Misra's presentation).

The growing interest, which is significant for policy making at a national level, in facilitating wide-spread access to mathematical results can be summarized by two recent public declarations by major bodies.

On 3 February 2018 the "Declaration By Open Access Communities & Attendees of OpenCon 2018, New Delhi" stated

> We will strive to publish our interim research outputs as preprints or postprints (e.g. Institutional Repositories) and encourage our peers and supervisors to do the same to make our research open and actionable in a timely manner. [See also http://openaccessindia.org/.]

On 2 June 2018 the "Recommendation By The Indian National Science Academy (INSA)" declared

> Various agencies/organizations in India that fund research should take cognizance of articles that have been deposited in established free open access Pre-Print Archives as a proof of prior-data.

Summarizing, Gadadhar Misra said that in many respects the practical details of access in India to the new mathematical literature through preprints, or indeed to the old insofar as it is generally available elsewhere, are largely solved, and there is a growing awareness of how this might be capitalized upon. However, he emphasized that there are important sociological aspects of society, and of the practice of mathematics and science in education, that still obstruct making use of what is becoming technically possible. Concomitant changes in the way mathematical work is judged for promotion and appointments are slow in coming while it is easy to use simpler older methods of counting pages or using other simple-minded statistics.

Alf A. Onshuus [AO] is a mathematics professor at Universidad de los Andes, Bogotá, Colombia. He is a member of the CEIC of the IMU, and also of the IMU's Committee on Developing Countries (CDC). As a mathematician his interest are logic and the foundations of mathematics, number theory and algebra. He has been involved in initiatives in Colombia to enhance access to mathematical literature and concerned with the details of open access as they impact developing countries.

Onshuus titled his remarks "Some ideas from developing countries for GDML". As an initial overview he offered, and then discussed, four main points:

- Most universities in developing countries, even research universities, don't subscribe to all the main mathematical journals.

- Article access is achieved through:

1. Personal contacts,
2. Library access of researchers through an Alma Mater,
3. Library agreements.

- arXiv has become a very mainstream tool.
- Internet access and speed have both become better in most places. Speed of downloading does not seem a significant obstacle, contrary to what was happening some years ago.

He then moved on to point out some initiatives in Colombia to help access to the mathematical literature; these he felt were a good start and can be examples to developing nations generally.

- Library agreements for sharing journal articles.
- Joint access to MathScinet from many small universities under a consortium agreement.

But there's a challenge we are all talking about. The "Gold Open Standard" initiative has been discussed. Changing the model from a "pay-to-read" to a "pay-to-publish" model can have complicated consequences for scientists in Colombia and the developing world who do not, in general, belong to institutions with large numbers of subscriptions to journals. This is a problem that has to be faced.

Onshuus then offered comments that should serve to emphasize that there are some details with serious consequences to be kept in mind in trying to move towards global digital access to mathematics. These resulted from an informal poll of colleagues he did.

A couple of quotations:

> "It would be great to have access to information at more basic levels than what is found in Arxiv. For example, to support a graduate student in Cambodia or Honduras."

> "I have not explored what is available but what I have seen is at the level of research that a student (and most teachers) in many developing countries could not read. Something that would be of special interest at the level of developing countries would be to have documents edited by experts on how one enters some areas of research."

A couple more replies:

> "This is a very interesting initiative. It should be open access. One thing to explore is to map mathematical knowledge, understanding for example how many written articles are in the interface of two subfields."

> "It would be important to discuss first the balance between novelty and soundness, between the number of publications and their quality, between open source and funding for this purpose."

Liu Zheng [LZ] is Deputy Director of the Collection & Knowledge Organization Center of the National Science Library at the Chinese Academy of Sciences, Beijing. She is by training a librarian and information specialist with, nowadays, a special interest in and responsibility for effecting the change from document retrieval to domain knowledge structure discovery using digital technologies. In particular, the mathematics collections of China, the world's most populous country, are within her purview.

Liu Zheng's views from China are conditioned by the very size of mathematics in China. She immediately offered a table of the official 2017 statistics estimates for mathematics degrees.

Type	Graduates	University and Research Institutions
Bachelor	66,100–71,500	649
Master	6000	246
Doctor	1000	75

Table 1: Mathematics in China

Further examination of mathematics research shows the number of papers published has increased year by year since 2009. Funding for mathematics research is at 500 million yuan from the National Natural Science Foundation of China (ca. 63,200,000 Eu or 73,500,000 US $). However, it is an official concern that perhaps the quality of mathematics research is still lower than it should be on average. Obviously there are some centers of excellence but the country is very large.

On the Digital Mathematics Library front in China, unfortunately there is no digital mathematics library as such. Much material is available through electronic database subscriptions to commercially published material. Efforts have gone into constructing mathematics library home pages and subject portals to aid in mathematics electronic journal navigation. There are some platform starts but very few mathematical papers are submitted to reprint platforms or an institutional repository. Zheng showed screen shots of some relevant examples (viewable on her slides).

In China this is a time for planning the future. We hope and suggest, Liu Zheng said, that introducing Chinese research results to international peers, and understanding the progress of similar research on a topic, will encourage mathematics research in China. We need to decrease the time required for retrieval and discovery of mathematics and hope that a GDML will be a helpful tool in this regard.

This is a time when we should be passing from document retrieval to discovery of domain knowledge structure using new digital capabilities.

It can be hoped that integrated and sharing of warehoused collections of, say, journals, proceedings of conferences, monographs, textbooks, tutorial papers etc., will lead to better networks of relationships for mathematical researchers, whether at a university, college or institution and regardless of whether a student, teacher, researcher, or just a user of mathematics.

We will need intelligent tools and assistants that are suited to particular subject domains. In early form they are the formularies, collections of mathematical modeling

available today, but it seems they will only be fully developed when more is understood of the foundations and philosophy of mathematics.

In summary, the CAS library is devoted to developing the digital libraries to provide better services to Chinese researchers. It will be happy to join in the GDML effort and work for mathematics.

Thierry Bouche [TB] is a mathematics professor at the Université Grenoble Alpes, in Grenoble, France, and also a member of the Institut Fourier there. He is Director of Cellule Mathdoc and responsible for the significant projects NUMDAM (math serials retrodigitisation), Mini-DML (OAI-PMH integrator for digital math articles), CEDRAM (publishing platform and portal for academic math journals), Gallica-Math (article-level index of volumes digitised at Gallica) and Centre Mersenne (extension of Cedram going beyond math and its core services). He was scientific coordinator for the successful EuDML (European Digital Mathematics Library) project and is Chair of the succeeding EuDML Initiative. He has been a member of the IMU's CEIC and of the GDML WG. His mathematical field is global analysis, but his interests and involvements have long also extended to digitisation, e-publishing and DML.

Bouche discussed European DML and Open Access developments. He started by looking at the basic notion of a Mathematics Library. The library's main functions are:

Selection	of collections by subject, document type
	Not necessarily formally published: theses, reports, preprints…
	Some grey literature needs to be archived
	or made accessible to patrons (e.g. Nash, Perelman…)
Acquisition	of actual documents (books, journals…)
Cataloguing	for collection maintenance and easy retrieval
Archiving	and Preservation of the curated collections
Access	provision maintained for patrons

EuDML tries to recreate this in the digital era with

- A curated digital archive
- Physically hosted at public (at least non-profit, science-driven) institutions
- Free to use (possibly after some moving wall)

EuDML is conceived as a regional integrator ready for worldwide integration, but is presently missing international partners.

- Worldwide content
- Enhanced with intelligent math-aware agents
- More accessible to humans with disabilities
- Free to mine (more accessible to machines)

The vision for The Digital Mathematics Library is really

- A global (distributed) facility dedicated to archive historical (digitised) material as well as newly published (born digital)
- An up-to-date registry of all available resources
- Mechanisms for interlinking the holdings with existing and future infrastructures
- Seamless navigation across the whole corpus
- Instant and perpetual access

Just let us note the size and special features of our mathematical literature. The reference mathematical corpus is

- About 4,5000,000 items (135×10^6 pages perhaps)
- Heavily multilingual
- Growing by about 140,000 new items a year (mostly English)

But Bouche also offered some warnings. Beware confusion of the following:

Digital Mathematics Library vs. repositories mixing flavours of eprints

$\longrightarrow$ We need reliable references

Trusted archive (original sources) vs. Encyclopedia

$\longrightarrow$ we need access to the original sources!

Mathematical corpus vs. math. knowledge as big data

$\longrightarrow$ True vs. plausible theorems?

However our main current problem is that we lack support to significantly enlarge the available content.

This leads us to the matter of Open Access. Mathematicians have an overall favorable attitude towards Open Access (OA), but not at any cost. There are
Good options:

- Eventual perpetual OA to all (through GDML/IMKT)
- Diamond OA, with some regulation (e.g. funded and monitored by a scientific organization, selective support by library groups. . .)
- Reasonably priced subscription journals are better than crap OA journals!

and **Concerns**:

- An obvious dangerous cost: APCs (Author Paid Publication)
- Putting societies' publications at risk

- Research integrity might be endangered by open access mandates that could encourage bad practices

On the subject of open access to scientific data and literature and the assessment of research by metrics, there is an important report developed by a Sub-Group of the International Council for Science (ICSU) Executive Board, with input from individuals attending an expert workshop, ICSU Members and the ICSU Secretariat.

> The International Council for Science advocates the following goals for open access. The scientific record should be:
>
> - free of financial barriers for any researcher to contribute to;
> - free of financial barriers for any user to access immediately on publication;
> - made available without restriction on reuse for any purpose, subject to proper attribution;
> - quality-assured and published in a timely manner; and
> - archived and made available in perpetuity.
>
> These goals apply both to peer-reviewed research publications, the data on which the results and conclusions of this research are based, and any software or code used in the course of the research.

Generally there is the question as to whether the mathematical literature is headed towards a not-for-profit system? A plot of the numbers of journal articles listed in Mathematical Reviews from commercial and non-commercial publishers, offered by D. E. McClure of the AMS in 2010, shows a change from 56.4% commercial and 43.6% non-commercial in 2000 to the reversed position of 55.8% non-commercial and 44.2% commercial in 2009 with a crossover point in 2006.

Another effect on mathematics, which does in fact provide open access, is that of the arXiv. Florian Müller and Olaf Teschke of zbMATH have already raised the question of "Everything on the arXiv?", in a article in the European Mathematical Society Newsletter **99** (2016). They plotted the percentage of arXiv coverage for the main MSC subject classes. They found a reasonably complete coverage of recent research in many classes but many of the articles generally will usually not be indexed in zbMATH. On the other hand, the scopes of arXiv and zbMATH differ (for instance, certain math-ph submissions may very well be beyond the scope of zbMATH), and there is also a possible contribution due to the necessary delay in indexing by zbMATH. These two effects can be quite significant. It is possible too that the arXiv seem to be approaching a level of saturation for some subjects. Another impression to be gained from the plots is that behaviours in specific subject area can also change quickly. The study was based on a sample that included 60,000 arXiv submissions with DOIs that matched 45,000 items in zbMATH.

Bouche then passed on to mentioning several different initiatives for Open Access. He had little time left to show slides at the end of his statement, so in fact skipped over the rest of his material below in the oral presentation.

In France there has been the Jussieu Call for Open science and bibliodiversity [Points 7 and 8]:

> the scientific communities must be able to access national and international infrastructures which guarantee the preservation and circulation of knowledge against any privatization of contents. Business models should be found which preserve their long-term continuity;
>
> priority should be given to business models that do not involve any payments, neither for authors to have their texts published nor for readers to access them. Many fair funding models exist and only require to be further developed and extended: institutional support, library contributions or subsidies, premium services, participatory funding or creation of open archives, etc.

There is also an effort called MathOA as part of a wider series of OA journal sites for different subjects that promulgate practical Fair OA principles and have begun 'flipping' mathematics journals to their form of OA.

There is a Free Journal Network with about 25 mathematical titles from all over the world as members. The Electronic Library of Mathematics, eLibM from FIZ Karlsruhe contains open access journals and proceedings volumes in all fields of mathematics, and lists 11 titles as of writing.

Bouche then passed to the brand-new Centre Mersenne, which he directs. The Centre Mersenne for Open Scientific Publishing aims at supporting and fostering open access scientific publishing. It offers tools and services for scholars and editorial teams of open access journals formatted with LaTeX. Recently it took on the well-known journals "Algebraic Combinatorics" and "Annales de l'Institut Fourier" to add tot he other 7 in its list.

An alternative route to gaining access to the mathematical literature has been the German DEAL Project which has been negotiating country-wide licenses for the offerings of major publishers, notably Elsevier.

Then for mathematics there may be much to be learned from the initiatives coming out of other subjects, such as SciPost from physics:

- SciPost is a complete scientific publication portal.
- It is purely online-based, and offers freely, openly, globally and perpetually accessible science.
- Being managed by professional scientists, and making use of editor-solicited and contributed reviews, its Journals aim at the highest achievable standards of refereeing.
- SciPost Commentaries allow Contributors to seamlessly comment on all existing literature.

There is also the approach of Peer Community in …. "A free recommendation process of scientific preprints (and published articles) based on peer-reviews". The humanities have the Open Library of Humanities, and there's the universal Knowledge

Unlatched that "offers free access to scholarly content for every reader across the world. Our online platform provides libraries worldwide with a central place to support Open Access models from leading publishing houses and new OA initiatives."

In summary, there are many ways to approach Open Access still. What will be best for mathematics remains to be seen. It is to be hoped that the GDML/IMKT efforts can contribute to and benefit from the currently changing publishing context.

Patrick D. F. Ion [PI] chaired the panel. He is a member of the IMU's CEIC and chair of the GDML WG. He spent a 30-year career as an editor at Mathematical Reviews (MR) run by the American Mathematical Society, but maintains an affiliation with the University of Michigan at Ann Arbor. Before that he had positions at the University of London (UK), Rijksuniversiteit Groningen (Netherlands), RIMS Kyoto (Japan) and Universität Heidelberg (Germany); on sabbaticals he spent years at Université Strasbourg (France) and IHES, Bures-sur-Yvette (France) and months at the University of Auckland (NZ). In addition to work as a mathematician editor he was closely involved in the change at MR to the use of TEX and other digital tools, and later to its transition to a database-driven operation and ultimately to MathSciNet. He took a leading role in the revisions of the Mathematics Subject Scheme (MSC) in 2000 and to the current MSC2010, and its modernization to a SKOS form of Linked Open Data. His involvement in mathematical knowledge management (MKM) is of long standing, and he is on the Executive Committee of the OpenMath Society. He was co-chair of the World Wide Web Consortium Math Working Group for 12 years, and an editor and author of the W3C specification MathML, now an ISO standard as MathML3 and part of the HTML suite, and of XML Entity Definitions for Characters. His mathematical interests have been in quantum field theory and statistical mechanics, quantum stochastic processes, noncommutative geometry, q-analogues of special functions and combinatorics, and the discrete Fourier transform in connection with elementary geometry.

As moderator, Ion tried to summarize the state of play as follows: Different panel members were able to address some of the different aspects of what's involved in setting up a global digital mathematics library. It is obvious the GDML/IMKT effort needs to learn more about community views and to serve them. We did have on the panel geographic diversity and representatives of the world's largest populations, but the hope has to be we are all willing to try to think globally as well.

What has happened since ICM 2014 is that, supported an Alfred P. Sloan Foundation grant, we have a legally established International Mathematical Knowledge Trust. The IMKT web presence is expected to be able soon to provide services to the mathematical community. The four initiatives IMKT set out are starting to be realized, with FABstracts well underway. The SFC, which could have been imagined to be the easiest front on which to make open-source progress, has seen a company, Wolfram Research, make the most headway with additions to the Wolfram Language (aka Mathematica) for mathematical entity handling. With the very recent progress at DLMF this should mean SFC moves forward further. There's work afoot on the other two initiatives mentioned, and on significant projects using machine learning supposed to build tools for mathematicians to use.

One obvious GDML subgoal still to be realized is direct enhancement of access to mathematical literature. We hope to be able to build from what EuDML has achieved, but that is still in the, hopefully near, future.

If the GDML WG from 2014 has clearly achieved something, it is that it has not let the effort to develop a GDML drop, which seems to have occurred more than once before in our subject's long history. This long-term commitment will continue with the IMKT.

3 Questions and answers—comments from the floor

Q–Marie Farge, ENS France There's a lot of digitally closed material out there: will you re-digitise?

A–SW Though there is a good deal already in public digital form, the "copyright aggregators" are indeed an issue. Springer, for instance, does claim to be in favour of Open Access but has a commercial basis. In practice many institutions too seem wedded to the current system. Mathematics has the slight advantage that it's not as profitable as some of the other subjects, e.g., Medicine. But the area of relationship with commercial publishing is very much "work in progress".

Q–Gerhard Paseman, USA There's a lot of work being done in the area of Open Access, e.g. Gowers' work. This wasn't really mentioned. Also, what about the really useful mathematical work that is posted in forums.

A–PI We're certainly not unaware of these issues, but we need to prioritise our work. A full discussion of both matters touched on would take a long time. But these are matters that the GDML WG has discussed, and the IMKT has to take into account.

Q–Marie Farge, ENS France For one article, we were ready to pay the APC (Article Processing Charge), but the publisher still insisted on retaining copyright. This is a problem for many of us.

A–SW IMKT is focused on technical issues, though there are legal ones. We have to deal with the current copyright aggregators, and being a copyright advocate is not IMKT's business.

A–AO There are ongoing discussions at IMU about this.

A–PI Yes, it's the IMU's CEIC that is trying to revise an earlier report on copyright issues for mathematicians, in the light of changing conditions.

Q–Unknown There are a lot of good lawyers working on copyright issues. But I note that the ICM 2018 copyright agreement, which speakers had to sign, is pretty draconian.

A–PI Clearly this should be noted, since the IMU and its CEIC will have to reach consistency.

Q–James Davenport, UK AO's presentation had mentioned the importance of arXiv, especially in developing countries. But a student and I recently noted a case where the paper on arXiv was very different (and silently so) from the published paper. Isn't this a problem?

A–SW But we should note that paper corrigenda are not perfect either.

James Davenport, UK But that's a fault of technology: this is human weakness.

Marie Farge, France We need a good practice statement from the IMU.

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Dai, Wanyang (China)
Dalmasso, Estefanía (Argentina)
Dalyan, Elif (Turkey)
Damasceno Cunha, Thauan (Brazil)
Damian, Florin (Moldova)
Daneshkhah, Ashraf (Iran)
Daouda, Sangaré (Ivory Coast)
Darafsheh, Mohammadreza (Iran)
Darda, Ratko (France)
Darmon, Henri (Canada)
Daskalakis, Constantinos (USA)
Das, Kalyan (India)
Dattori da Silva, Paulo (Brazil)
Daubechies, Ingrid (USA)
Davenport, James (UK)
Davison, Ben (UK)
Davitashvili, Tinatin (Georgia)
Daza Torres, Maria Luisa (Mexico)
Debnath, Shyamal (India)
Dechamps, Myriam (France)
Degla, Guy (Benin)
Degond, Pierre (UK)
Del Barco, Viviana (France)
Delfino, Adriano (Brazil)
Delgadino, Matias (UK)
Delle Donne, Diego (Argentina)
Del Magno, Gianluigi (Brazil)
Delort, Jean-Marc (France)
Del Pezzo, Leandro (Argentina)
DeMarco, Laura (USA)
Demeneghi, Paulinho (Brazil)
Demeter, Ciprian (USA)
Demidenko, Gennady (Russia)
Dencker, Nils (Sweden)
Denega, Iryna (Ukraine)
Desimoni, Juan (Brazil)
Desquith, Etienne (Ivory Coast)
Devilart, Guilherme (Brazil)
Devkota, Jyoti (Nepal)
Dhawan, Sharanjeet (India)
Dhayal, Rajesh (India)
Dhersin, Jean-Stephane (France)
Diallo, Abdoul Salam (Senegal)
Diallo, Ouateni (Mali)
Dias Alexiou, Carolina (Japan)
Dias de Ávila Rodrigues, Luciana M. (Brazil)
Dias, Michell (Brazil)
Dias, Pedro (Brazil)
Dias Vaz, Cristina Lúcia (Brazil)
Diaz, Lorenzo J. (Brazil)
Díaz Martín, Rocío Patricia (Argentina)
Dickenstein, Alicia (Argentina)
Didier, Gustavo (USA)
Dierings, Gláucia (Brazil)
Díez, Carlos (Colombia)
Di Fiore, Carlos (USA)
Di Francesco, Philippe (France)
Dimitrov, Dimitar (Brazil)
Dinar, Yassir (Oman)
Ding, Zhiyuan (USA)
Đình, Nho Hào (Vietnam)
Đình, Tien-Cuong (Singapore)
Diop, Mamadou Abdoul (Senegal)
Di Rocco, Sandra (Sweden)
Djitte, Ngalla (Senegal)
Djoric, Mirjana (Serbia)
Djurdjevac, Ana (Germany)
Đoàn, Trung Cường (Vietnam)
Domelevo, Komla (France)
Domiciano Lopes, Bruno (Brazil)
Domínguez Vázquez, Miguel (Spain)
Donaldson, Ana (USA)
Donaldson, Simon (USA)
Donoho, David L. (USA)
Dória, Cayo (Brazil)
Đỗ, Trọng Hoàng (Vietnam)
Doungmo Goufo, Emile Franc (South Africa)
Dowla, Arif (Bangladesh)
van den Dries, Lou (USA)
Drummond Mariani, Hermano (Brazil)

Drumond, Raphael (Brazil)
Dugin, Paula (Brazil)
Duminil-Copin, Hugo (France)
Dumortier, Freddy (Belgium)
Dunne, Edward (USA)
Du, Qiang (USA)
Duque Franco, Jorge Armando (Brazil)
Duraes, Marcelo (Brazil)
Durfee, Alan (USA)
Durham, Matthew (USA)
Durojaye, Mary (Nigeria)
Durojaye, Olayiwola (Nigeria)
D'Urso, Luíze (Brazil)
Dussan Angulo, Martha P. (Brazil)
Dutra, Magda (Brazil)
Dzhuraev, Abubakir (Kyrgyzstan)
Edeki, Sunday (Nigeria)
Eduardo Nascimento, Vinicius (Brazil)
Eegunjobi, Adetayo (Namibia)
Effio, Carolina (Brazil)
Ehbauer, Stefan (Brazil)
Eiermann, Michael (Germany)
Ekholm, Tobias (Sweden)
El Hassan, El Kinani (Marocco)
Elias, Luciana (Brazil)
El Karoui, Noureddine (France)
El-Khatib, Youssef (UAE)
Ellingson, Leif (USA)
Ellis, Eugenia (Uruguay)
Eloranta, Kari (Finland)
El Yacoubi, Nouzha (Marocco)
Endo, Pedro (Brazil)
Eneya, Levis (Malawi)
Engler, Hans (USA)
Epequin Chavez, Jesua Israel (France)
Epiphanio Galvão, Leonardo (Brazil)
Erappa, Shobha Monnanda (India)
Erhard, Dirk (Brazil)
Erkurşun Özcan, Nazife (Turkey)
Escorcia Tafur, Jose Manuel (Brazil)
Esedoglu, Selim (USA)
Eslami, Peyman (UK)
Esmerado, Caio (Brazil)
Esnault, Hélène (Germany)
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Esteban, Maria J. (France)
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Estevez, Gabriela (Brazil)
Eswara, A. T. (India)
Eudave-Muñoz, Mario (Mexico)
Evangelista, David (Saudi Arabia)
Exel, Ruy (Brazil)
Exner, Pavel (Czech Republic)
Eyrea Irazú, María Emma (Argentina)
Fadel, Daniel (Brazil)
Fadipe-Joseph, Olubunmi (Nigeria)
Falconer, Isobel (UK)
Fall, Mouhamed Moustapha (Senegal)
Familton, Johannes (USA)
Fang, Xiao (Hong Kong)
Farah, Ilijas (Canada)
Farge, Marie (France)
Fargues, Laurent (France)
Faria Marinho, João Jefferson (Brazil)
de Farias, João Bosco Bezerra (Brazil)
Farnik, Michal (Poland)
Fasolai Barbosa, Stela (Brazil)
Fassbender, Heike (Germany)
Fathi, Albert (USA)
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Faustino, Jocel (Brazil)
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Faye, Bernadette (Senegal)
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Ferreira, Jocirei Dias (Brazil)
Ferreira, Luan Alberto (Brazil)
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Fialho, Vinicius (Brazil)
Fiebig, Peter (Germany)
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Flores-Espinoza, Ruben (Mexico)
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Flores, Marcelo (Chile)
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Franco, Álvaro (Brazil)
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Freire Moreira, Luiz Paulo (Brazil)
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Fryz, Iryna (Ukraine)
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Funaya, Bokuro (Japan)
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Gaulhiac, Sylvain (France)
Gebresilasie, Semere (USA)
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Gelfert, Katrin (Brazil)
Genra, Naoki (Japan)
Ghahramani, Saeed (USA)
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Ghini Bettiol, Renato (Brazil)
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Ghosh, Nirupam (India)
Ghosh, Partha Pratim (India)
Ghys, Etienne (France)
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Giga, Yoshikazu (Japan)
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Glazyrin, Alexey (USA)
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Godsil, Chris (Canada)
Goel, Charu (India)
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Goldstein, Catherine (France)
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Gomes Pinheiro, Elizabethe (Brazil)
Gomes, Ricardo (Brazil)
Gomez, Adrian (Colombia)
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Gómez Gómez, Alirio (Brazil)
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Gonçalves de Oliveira, Leonardo (Brazil)
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Gong, Jianhua (UAE)
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Gonzalez, Eduardo (USA)
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Gooya, Zahra (Iran)
Gorty, V. R. Lakshmi (India)
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Gouezel, Sebastien (France)
Gouin, Etienne (France)
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Gourmelon, Nicolas (France)
Goussevskii, Nikolai (Brazil)
Goyal, Megha (India)
Gozzi, Francisco Jose (Brazil)
Graça, Dalson (Brazil)
Grama, Lino (Brazil)
Grantcharov, Dimitar (USA)
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Griffeth, Stephen Price (Chile)
Griffiths, Simon (Brazil)
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Grosse-Brauckmann, Isabell (Germany)
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Guardia, Gleison (Brazil)
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Guazzi, Erika (Brazil)
Gubinelli, Massimiliano (Germany)
Gudiño, Elias (Brazil)
Gudiño, Noemí (Argentina)
Guedes Bonthonneau, Yannick (France)
Guella, Jean Carlo (Brazil)
Guelman, Nancy (Uruguay)
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Guerrini Alves, Marcela (Brazil)
Guerville-Ballé, Benoît (Brazil)
Guesmia, Aissa (France)
Guezane-Laakoud, Assia (Algeria)
Guillarmou, Colin (France)
Guimaraes, Felippe (Brazil)
Gupta, Mahendra Kumar (India)
Gupta, Shiv (USA)
Gurjanow, Iwan (Germany)
Gutemberg, Antonio (Brazil)
Guterres, Robert (Brazil)
Gutierrez, Luis Angel (Mexico)
Gutu, Valeriu (Moldova)
Guy, Bouchitte (France)
Gu, Yoon Hoe (South Korea)
Guzmán Adán, Alí (Belgium)
Gyori, Ervin (Hungary)
Hacking, Paul (USA)
Hacon, Christopher (USA)
Hải, Bùi (Vietnam)
Hailu, Habtu (Botswana)
Hall, Bryon (Brazil)
Hall, Eric (USA)
Hamidou, Toure (Burkina Faso)
Hamoud, Cheikh (UAE)
Han, Jun (China)
Hanke, Bernhard (Germany)
Hansen, Pierre (Canada)
Haran, Shai (Israel)
Haria, Kalpesh Jayantilal (India)
Harris, Pamela (USA)
Hartmann, Luiz (Brazil)
Hartmann, Stefan (Germany)
Hartz, Michael (USA)
Hasnaa, Zidani (France)
Hassanzadeh, S. Hamid (Brazil)
Hassler, Malte (Germany)
Hayashi, Masayuki (Japan)
Hayat, Amaury (France)
Haydon, Richard (UK)
Hayes, Benjamin (USA)
He, Daoyin (China)
Heinrich, Lothar (Germany)
Helfgott, Harald (Germany)
Hellmann, Bernhard Eugen (Germany)
Hemakul, Wanida (Thailand)
Henderson, Anthony (Australia)
Henni, Abdelmoubine Amar (Brazil)
Henrique Brasil Ribeiro, Paulo (Brazil)
Henrique Perez, Otávio (Brazil)
Herbig, Hans-Christian (Brazil)
Herden, Daniel (USA)
Hermosillo Reyes, Oyuki Hayde (Mexico)
Hernandez Posada, Alffer Gustavo (Brazil)
Herrera Cortés, Silvia (Mexico)
He, Siming (USA)
Heuer, Ben (UK)
He, Xinyu (UK)
He, Xuhua (USA)
Hilário, Douglas (Brazil)
Hilario, Marcelo (Brazil)
Hilario Poma, Cesar Augusto (Brazil)

Hilsum, Michel (France)
Hindry, Marc (France)
Hinojosa, Gabriela (Mexico)
Hinz, Andreas M. (Germany)
Hiraide, Koichi (Japan)
Hoàng, Nam Dung (Vietnam)
Hoàng, Truong (Germany)
Hochman, Michael (Israel)
Hoch, Michael Karl (Germany)
van der Hoeven, Govert (France)
Holden, Helge (Norway)
Hopkins, Samuel (USA)
Horita, Vanderlei (Brazil)
Horiuchi, Toshio (Japan)
Hornik, Anna (Brazil)
Horoldagva, Batmend (Mongolia)
Horsin, Thierry (France)
Hounkonnou, Mahouton Norbert (Benin)
Hryn, Aliaksandr (Belarus)
Hryniewicz, Umberto (Brazil)
Huang, Rongjin (USA)
Huang, Yunqing (China)
Huggett, Stephen (UK)
Hughes, Kenneth (South Africa)
Huh, June (USA)
Huisken, Gerhard (Germany)
Hulek, Klaus (Germany)
Hunton, John (UK)
Hurlbert, Glenn (USA)
Hutchings, Michael (USA)
Hytönen, Tuomas (Finland)
Iacobelli, Giulio (Brazil)
Iarmosh, Olena (Ukraine)
Ibieta, Miguel (Brazil)
Ichihara, Kazuhiro (Japan)
Idowu Ademola, Osinuga (Nigeria)
Iftimie, Dragos (France)
Ignacio, Paul Samuel (USA)
Ih, Su-Ion (USA)
Ilolov, Mamadsho (Tajikistan)
Imafuku, Roberto Seidi (Brazil)
Im, Bokhee (South Korea)
Imran, Muhammad (Pakistan)
Indyk, Piotr (USA)
Inprasitha, Maitree (Thailand)
Ioana, Adrian (USA)
Ionescu, Cristodor (Romania)
Ion, Patrick (USA)
Iosipoi, Leonid (Russia)
Iovanov, Miodrag (USA)
Ipanaque Zapata, Cesar Augusto (Brazil)
Irineo, Telmo (Uruguay)
Irwansyah (Indonesia)
Isah, Ahmed Ibrahim (Nigeria)
Ishida, Atsuhide (Japan)
Itovich, Griselda (Argentina)
Iturriaga, Renato (Mexico)
Ivanova, Ivani (Brazil)
Iyama, Osamu (Japan)
Jabin, Pierre-Emmanuel (USA)
Jackson, Stephen (USA)
Jafari Rad, Nader (Iran)
Jafarzadeh, Nafisehsadat (Iran)
Jaffke, Lars (Norway)
Jaiani, George (Georgia)
Jain, Shilpi (India)
Jakhar, Anuj (India)
James, Richard (USA)
Jansen, Renier (South Africa)
Januszkiewicz, Lech Tadeusz (Poland)
Jara, Milton (Brazil)
Jardim, Manoel (Brazil)
Jardim, Marcos (Brazil)
Jarque, Xavier (Spain)
Jean-Louis, Verger-Gaugry (France)
Jeffrey, David J. (Canada)
Jelonek, Zbigniew (Poland)
Jeltsch, Rolf (Switzerland)
Jenaliyev, Muvasharkhan (Kazakhstan)
Jena, Pravat (India)
Jena, Susil Kumar (India)
Jenda, Overtoun (USA)
Jeronimo, Gabriela (Argentina)
Jesus, Marcus Vinícius (Brazil)
Jézéquel, Malo (France)
Jia, Qiuhao (USA)
Jimenez Rolland, Rita (Mexico)
Jin, Shi (USA)
Jitman, Somphong (Thailand)
João, Antonio (Brazil)
Jofre, Alejandro (Brazil)
Johannes, Girsch (Austria)
Johnson, William (USA)
Jolevska-Tuneska, Biljana (Macedonia)
Joly, Benoit (France)
Jones, Vaughan (USA)
Jordan, Michael (USA)
Jordão, Thaís (Brazil)
Jorge Perez, Victor Hugo (Brazil)
José Pereira Júnior, Valdir (Brazil)
Joshi, Nalini (Australia)
Joshua, Chollom (Nigeria)
Jovovic, Mirjana (USA)
Jowkar, Nasrin (Iran)

Juan Pineda, Daniel (Mexico)
Juchani, Mariana Eva (Argentina)
Julianelli, José Roberto (Brazil)
Jung, Il Hyo (South Korea)
Jung Spier, Thomás (Brazil)
Jung, Tacksun (South Korea)
Jung, Woochul (South Korea)
Junior, Alcides (Brazil)
Jurado Cerron, Liliana Olga (Peru)
Jury Giraldi, João Matheus (Brazil)
Juteau, Daniel (France)
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Kahouadji, Nabil (USA)
Kaiser, Tobias (Germany)
Kalai, Gil (Israel)
Kalai, Yael (USA)
Kalaj, David (Montenegro)
Kaleff, Ana Maria (Brazil)
Kalidass, Mathiyalagan (India)
Kalimoldayev, Maksat (Kazakhstan)
Kalmenov, Tynysbek (Kazakhstan)
Kanas, Stanisława (Poland)
Kania-Bartoszynska, Joanna (USA)
Kanuni Er, Müge (Turkey)
Kara Hansen, Ayse (Denmark)
Karakoç, Fatma (Turkey)
Kargin, Vladislav (USA)
Karjanto, Natanael (South Korea)
Karlsson, Anders (Switzerland)
Karlsson, Ulf (Sweden)
Karoly, Barbara (Brazil)
Karthikeyan, Shanmugasundaram (India)
Kashaev, Rinat (Switzerland)
Kashiwara, Masaki (Japan)
Kashuba, Iryna (Brazil)
Kasilingam, Ramesh (India)
Kassel, Fanny (France)
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Katona, Gyula O. H. (Hungary)
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Kaur, Dilpreet (India)
Kaus-Zampieron, Alan (Brazil)
Kawahigashi, Yasuyuki (Japan)
Kawai, Kotaro (Japan)
Kawamata, Yujiro (Japan)
Kawamoto, Masaki (Japan)
Kaya, Ezgi (Turkey)
Kayal, Neeraj (India)
Kaya Merdan, Songül (Turkey)
Kayar, Zeynep (Turkey)
Kazima, Mercy (Malawi)
Kedem, Rinat (USA)
Kedukodi, Babushri Srinivas (India)
Keevash, Peter (UK)
Keller, Thomas (USA)
Kenig, Carlos (USA)
Kenyon, Richard (USA)
Kerimbekov, Akylbek (Kyrgyzstan)
Kerz, Moritz (Germany)
Kestner, Charlotte (UK)
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Khairnar, Anil (India)
Khaldi, Rabah (Algeria)
Khalid, Sadia (Pakistan)
Khalique, Chaudry Masood (South Africa)
Khan Caires Ribeiro Santos, Ali (Brazil)
Khanduja, Sudesh Kaur (India)
Khanin, Konstantin (Canada)
Khartchenko, Vladislav (Mexico)
Khurdei, Yevheniia (Ukraine)
Kianpi, Maurice (Cameroon)
Kiefer, Frank (Germany)
Kim, Hoil (South Korea)
Kim, Hosoo (South Korea)
Kim, Hyun-Min (South Korea)
Kim, Jeong Han (South Korea)
Kim, Joonhyung (South Korea)
Kim, Myeungsu (Korea)
Kimura, Shunichi (Japan)
Kimura, Takashi (USA)
Kim, Young Rock (South Korea)
King, Timothy (UK)
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Kiwi, Marcos (Chile)
Klaus, Stephan (Germany)
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Koenigsmann, Jochen (UK)
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Kokubu, Hiroshi (Japan)
Kolawole Sunday, Adegbie (Nigeria)
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Kreutz, Alessandra (Brazil)
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Kucharz, Wojciech (Poland)
Kuku, Aderemi (USA)
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Kumarasamy, Sakthivel (India)
Kumar, Dinesh (India)
Kumar, Manish (India)
Kumar, Manoj (India)
Kumar, Mohit (India)
Kumar, Rakesh (India)
Kumar, Sachin (India)
Kumar, Shiv Datt (India)
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Lanard, Thomas (France)
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Lavor, Otavio (Brazil)
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Lee, Keonhee (South Korea)
Lee, Sekwon (South Korea)
Lee, Yong-Hoon (South Korea)
Lee, Yongnam (South Korea)
Leguil, Martin (Canada)
Leitão, Antonio (Brazil)
Leitão, Raimundo (Brazil)
Leite, Charlene Tereza da Silva (Brazil)
Leite, Ricardo (Brazil)
Lelis, Jean (Brazil)
Lemos de Oliveira, Carolina (Brazil)
Lemos, Diego (Brazil)
de Lemos R. Collares Neto, Maurício (Brazil)
Lenarduzzi, Fernando (Brazil)
Lentfer, Bjoern-Ole (Germany)
Le Ny, Arnaud (France)
Leonardi, Florencia (Brazil)
Leriche Vázquez, Renato (Mexico)
Lessa, Pablo (Uruguay)
Le, Thị Thành Nhan (Vietnam)
Levcovitz, Daniel (Brazil)
Levesque, Claude (Canada)
Levina, Elizaveta (USA)
Levinson, Jake (USA)
Lewanski, Danilo (Germany)
Lezama Serrano, José Oswaldo (Colombia)
Li, Chao (USA)
Liechti, Livio (Switzerland)
Liendo, Álvaro (Chile)

Lieven, Sandra (Brazil)
Li, Jiyou (China)
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Lima, Dahisy (Brazil)
Lima, Davi (Brazil)
Lima dos Santos, Giovanni (Brazil)
Lima, Francisco Vanderson (Brazil)
Lima, Hudson (Brazil)
de Lima, Ronaldo (Brazil)
Lima, Suely (Brazil)
Lim, Seongan (South Korea)
Linares, Felipe (Brazil)
Linowitz, Benjamin (USA)
Liu, Linyuan (France)
Liu, Liu (USA)
Liu, Xiaoyan (USA)
Liu, Ya-Feng (China)
Liu, Zheng (China)
Liu, Zixuan (China)
Liverani, Carlangelo (Italy)
Li, Weijia (China)
Li, Yachun (China)
Li, Zhigiang (USA)
Li, Zhiyuan (China)
Loeser, Francois (France)
Loewe, Benedikt (Germany)
Logachev, Dmitry (Brazil)
Lo, Gane Samb (Canada)
Logunov, Aleksandr (Russia)
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Lomonaco, Luciana Luna Anna (Brazil)
Longa, Eduardo (Brazil)
Longhi, Ignazio (Italy)
Longla, Martial (USA)
Longo, Roberto (Italy)
Long, Qi (USA)
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Lopes Filho, Milton (Brazil)
Lopes, Janice (Brazil)
Lopez Aguayo, Daniel (Mexico)
Lopez, Erik (Brazil)
López, Rubén (Chile)
Lopez, Salvador (Spain)
Lopez Valdez, Hiram H. (Mexico)
López Vereau, Charles Edgar (Brazil)
Lorenat, Jemma (USA)
Lorenzo, Pilar (Chile)
Loubeau, Eric (France)
Lovász, László (Hungary)
Lozada, Eleazar (Brazil)
Lozano Huerta, Cesar Adrian (Mexico)
Lozano, Luis (Brazil)
Lubawski, Nathan (Brazil)
Lubich, Christian (Germany)
Lubotzky, Alex (Israel)
Luca, Florian (South Africa)
Luca Tudorache, Rodica (Romania)
Lucena, Antonio (Brazil)
Luchesi, Jackson (Brazil)
Luciano de Souza, Gustavo (Brazil)
Lui, Ka Wai (Hong Kong)
Luisa S. Machado, Ana (Brazil)
Luiz, Melissa (Brazil)
Luna, Jose Alejandro (Argentina)
Lungu, Edward (Botswana)
Luo, Jun Ren (China)
Lu, Shannian (China)
Luu, Đức Hoàng (Vietnam)
Luzia, Nuno (Brazil)
Ly, Ibrahim (Burkina Faso)
Lyubich, Mikhail (USA)
Maass, Alejandro (Chile)
Macdonald, Ben (USA)
Macena, Maria Carolina (Brazil)
Madrid Padilla, Jose Ramon (Italy)
Madry, Aleksander (USA)
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Maekawa, Lucas (Brazil)
Mafi, Amir (Iran)
Maga, Péter (Hungary)
Magno, Bruna (Brazil)
Mahfouf, Remy (France)
Maia, Ricardo (Brazil)
Mai, Tien Long (France)
Maity, Chandan (India)
Majdoub, Mohamed (Tunisia)
Majumdar, Manjusha (India)
Makinde, Oluwole Daniel (South Africa)
Málaga Sabogal, Alba Marina (Peru)
Maldonado, Ana Carolina (Argentina)
Ma, Li (China)
Malinnikova, Eugenia (Norway)
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Maliutin, Kostiantyn (Ukraine)
Malliaris, Maryanthe (USA)
Mallik, Sudipta (USA)
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Mamani, Edhin (Brazil)
Mamouni, My Ismail (Marocco)
Mandrescu, Eugen (Israel)
Mango Magero, John (Uganda)
Manjarrez Gutierrez, Fabiola (Mexico)
Manna, Santanu (India)
Manolescu, Ciprian (USA)

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Mantilla-Soler, Guillermo (Colombia)
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Marcon Farias, Diego (Brazil)
Marcos, Aboubacar (Benin)
Marcum, Howard (USA)
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Marinho, Andrea (Brazil)
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Maroju, Prashanth (India)
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Massamba, Fortune (South Africa)
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Máthé, András (Hungary)
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Mathew, Omeike (Nigeria)
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Navas, Andrés (Chile)
Nawa, Victor (Zambia)
Nawaz, Rab (Pakistan)
Ndiaye, Babacar Mbaye (Senegal)
Ndiaye, Fagueye (Senegal)
Nechuiviter, Olesia (Ukraine)
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Nelli, Barbara (Italy)

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Németh, László (Hungary)
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Nesetrilova, Helena (Czech Republic)
Nesin, Ali (Turkey)
Netto, Clarice (Brazil)
Neves, Sergio (Brazil)
Neves, Veronica (Brazil)
Neves, Wladimir (Brazil)
Newlands, Arthur (Brazil)
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Nguyễn, Đình Cống (Vietnam)
Nguyễn, Hoi (USA)
Nguyễn, Loan (Vietnam)
Nguyễn Thị Ngọc, Oanh (Vietnam)
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Nicolini, Marina (Argentina)
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Nie, Zhaohu (USA)
Nikcevic Simic, Stana (Serbia)
Nikmehr, Mohammad Javad (Iran)
Nimbhorkar, Shriram (India)
Nina Zarate, Nestor (Brazil)
Nisoli, Isaia (Brazil)
Niu, Yi-Shuai (China)
Nkemzi, Boniface (Cameroon)
Nkou, Vann Borhen (DR of the Congo)
da Nóbrega, Filipe (Brazil)
Noh, Sunsook (South Korea)
Nomirovskii, Dmytro (Ukraine)
Nongxa, Loyiso (South Africa)
Nonnenmacher, Stéphane (France)
Noor, Waleed (Pakistan)
Noseda, Francesco (Brazil)
Noundjeu, Pierre (Cameroon)
Nouri, Fatma Zohra (Algeria)
Nourouzi, Kourosh (Iran)
Nunes Cardoso, Andre (Brazil)
Nunes, Daniel (Brazil)
Nunes, Giovanni (Brazil)
Nunes, Juliana (Brazil)
Nunes, Samantha (Brazil)
Nussenzveig Lopes, Helena (Brazil)
Octavia, Gaël (France)
Odabaşi, Sinem (Chile)
Odagiri, Shinsuke (Japan)
Odzijewicz, Tatiana (Poland)
Oeding, Luke (USA)
Offen, Elias (Botswana)
Ogana, Wandera (Kenya)
Ogoke, Uchenna (Nigeria)
Ogundare, Babatunde (Nigeria)
Ogundiran, Michael Oluniyi (Nigeria)
Ogunrinde, Roseline (Nigeria)
Oguntuase, James Adedayo (Nigeria)
Oh, Jeongseok (Korea)
Ohno, Masahiro (Japan)
Oh, Seungly (USA)
Okada, Sou (Brazil)
Okano, Keiji (Japan)
Okounkov, Andrei (Russia)
Olela Otafudu, Olivier (South Africa)
Oliveira Bodart, Raquel (Brazil)
Oliveira, Daniel (Brazil)
Oliveira de Abreu, Lia (Brazil)
Oliveira, Gonçalo (Brazil)
de Oliveira, Gustavo (Brazil)
Oliveira, Hermenegildo (Portugal)
Oliveira, Krerley (Brazil)
Oliveira, Lucas (Brazil)
Oliveira, Marielson (Brazil)
Oliveira, Mateus (Norway)
Oliveira, Roberto Imbuzeiro (Brazil)
de Oliveira Valcácio, Ewerson Boás (Brazil)
Oliveira, Vanessa (Brazil)
Oliveira Vieira, Martim Lourenço (Brazil)
de Oliveira, Yure (Brazil)
Olivo, Andrea Del Valle (Argentina)
Olteanu, Constanta (Sweden)
Olteanu, Lucian (Sweden)
Ono, Ken (USA)
Onshuus, Alf (Colombia)
Origlia, Marcos (Argentina)
Orsted, Bent (Denmark)
Orukari, Mercy (Nigeria)
O'Shea, Donal (USA)
Osin, Denis (USA)
Osorio Acevedo, Luis Eduardo (Brazil)
Ospina, Andrés (Brazil)
Ottaviani, Giorgio (Italy)
Otto Karl, Andreas Griewank (Ecuador)
Ouedraogo, Marie Françoise (Burkina Faso)
Oukouomi Noutchie, Suares C. (South Africa)
Ovando, Gabriela (Argentina)
Owusu-Ansah, Emmanuel De-Graft J. (Ghana)
Ozawa, Makoto (Japan)
Ozawa, Tohru (Japan)
Pachol, Anna (UK)
Pacifico, Maria Jose (Brazil)
Padilha, Guilherme (Brazil)

Paguio, Victoria May (Philippines)
Pahari, Narayan (Nepal)
Paim, Tatiana (Brazil)
Pait, Felipe Miguel (Brazil)
Paiva Scarparo, Eduardo (Brazil)
Paixão, João (Brazil)
Pak, Igor (USA)
Palathingal, Jeepamol (India)
Palchik, Vyacheslav (Israel)
Palis, Jacob (Brazil)
Palka, Karol (Poland)
Palmas, Oscar (Mexico)
Pampena, Simon (Australia)
Panackal, Harikrishnan (India)
Pandey, Shikha (India)
Pandharipande, Rahul (Switzerland)
Pandzic, Pavle (Croatia)
Panetta, Alex (France)
Pani, Amiya (India)
Panin, Ivan (Russia)
Pansu, Pierre (France)
Pan, Yi (France)
Pappas, Georgios (USA)
Parameswaran, Aryampilly Jayanthan (India)
Pardon, John Vincent (USA)
Parente, Roberto (Brazil)
Pariguan, Eddy (Colombia)
Park, Byeong U. (South Korea)
Park, Hyungju (South Korea)
Park, Jongil (South Korea)
Park, Jung-Tae (South Korea)
Park, Kiho (USA)
Park, Kyewon Koh (South Korea)
Parra Cardenas, Eliazar (Mexico)
Paseman, Gerhard (USA)
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Passoni, Ivan (Brazil)
Passos Passamani, Apoenã (Brazil)
Pastine, Adrian (Argentina)
Pastor, Verónica (Argentina)
Pastran Ramirez, Ricardo Ariel (Colombia)
Paternina Salgado, David A. (Brazil)
Patidar, Vinod (India)
Patrizio, Giorgio (Italy)
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de Paulo, Naiara (Brazil)
Peçanha, Daniel (Brazil)
Pechen, Alexander (Russia)
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Pedroza, Andrés (Mexico)
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Peltonen, Kirsi (Finland)
Peñafort, Guillermo (Spain)
Pentón Machado, Mariela (Brazil)
Pe Pereira, Maria (Spain)
Pereira, Alan (Brazil)
Pereira, Aldo (Brazil)
Pereira, Fernanda (Brazil)
Pereira, Jorge Vitório (Brazil)
Pereira, Marcone (Brazil)
Pereira, Mariana (Uruguay)
Pereira, Pedro (Brazil)
Perez-Garcia, Victor (Mexico)
Perez, John (Colombia)
Perlaza, Samir (France)
Perrucci, Daniel (Argentina)
Pershyna, Iuliia (Ukraine)
Persson, Ulf (Sweden)
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Pesco, Sinesio (Brazil)
Petermichl, Stefanie (France)
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Phùng, Hồ Hải (Vietnam)
Piazza, Carolina (Brazil)
Piccione, Paolo (Brazil)
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Pila, Jonathan (UK)
Pilloni, Vincent (France)
Pilod, Didier (Norway)
Pimenta, Marcos (Brazil)
Pimentel, Edgard (Brazil)
Pinciu, Val (USA)
Pineda Fortin, Angel (USA)
Pineda Villavicencio, Guillermo (Australia)
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Pinheiro, Vilton (Brazil)
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Pintarelli, María Beatriz (Argentina)
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Pinto, Aline (Brazil)
Pires Sandre, Gabriela (Brazil)
Pirzada, Shariefuddin (India)
Planas, Gabriela (Brazil)
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Poganj, Tibor (Croatia)
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Pollington, Andrew (USA)
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Ponce Dawson, Silvina (Argentina)
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Ponnusamy, Prakash (India)
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Pontes, Edel Guilherme Silva (Brazil)
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Popovic, Bozidar (Montenegro)
Postnikov, Alexander (USA)
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Poveda, Leonardo (Brazil)
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Quddus, Safdar (India)
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Radziwill, Maksym (Canada)
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Raja Sekhar, T. (India)
Rajendran, Mabel Lizzy (India)
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Rodero, Ana (Brazil)
Rodrigues, Christian (Brazil)
Rodrigues de Souza, Tatiana (Brazil)
Rodrigues Ferreira, Gustavo (Brazil)
Rodrigues, Job (Brazil)
Rodriguez Cardenas, Carlos Wilson (Brazil)
Rodriguez Garzón, Brayan Mauricio (Brazil)
Rodríguez, Juan Felipe (Colombia)
Rodriguez, Luis Angel (Venezuela)
Rodriguez Marroquin, Daniel (Brazil)
Rodriguez Miranda, Juan Carlos (Brazil)
Rodriguez Valerio, Piere Alexander (Peru)
Rodriguez Villegas, Fernando (Italy)
Roeckner, Michael (Germany)
Rojas, Santiago (Peru)
Roldan Daquilema, Mario Rodolfo (Brazil)
Romaña Ibarra, Sergio Augusto (Brazil)
Romano Carvalho, Yagor (Brazil)
Romero Rojas, Martha Judith (Colombia)
Roque, Marian (Philippines)
Roque Nunez, Roberto Alonso (Brazil)
Roque, Tatiana (Brazil)
Rordam, Mikael (Denmark)
Rosales-Mendez, Jennyffer (Mexico)
Rosamond, Frances (Norway)
Rosa, Romulo (Brazil)
Rossman, Benjamin (Canada)
Rossomakhina, Mariia (Russia)
Rousseau, Christiane (Canada)
Rout, Sudhansu Sekhar (India)
Rowe, David (Germany)
Rowley, Chris (Thailand)
Royer, Emmanuel (France)
Roy, Marie-Françoise (France)
Roy, Priti Kumar (India)
Roy, Rahul (India)
Rubio, Diana (Argentina)
Ruchjana, Budi Nurani (Indonesia)
Ruffino, Paulo (Brazil)
Rusu, Galina (Moldova)
Ruth, Lauren (USA)
Ryjacek, Zdenek (Czech Republic)
Ryu, Hwayeon (USA)
Šabanac, Zenan (Bosnia and Herzegovina)
Sadofschi Costa, Iván (Argentina)
Sadovsky, Shay (Israel)
Sadyrbaev, Felix (Latvia)
Sá Earp, Henrique (Brazil)
Saez, Mariel (Chile)
Safe, Martín Darío (Argentina)
Sagastizabal, Claudia (Brazil)
Sagnier, Aurélien (France)
Saha, Biswajyoti (India)
Saha, Ekata (India)
Sahoo, Binod Kumar (India)
Sahoo, Sueet Millon (India)
Saifullah, Khalid (Pakistan)
Sakue, Kazuhiro (Japan)
Salarinoghabi, Mostafa (Brazil)
Salazar Flores, Elizabeth Ruth (Brazil)
Salazar Pinzón, María Amelia (Brazil)
Salcedo Sora, Juan Carlos (Brazil)
Salerno, Adriana (USA)
Sales, Guilherme (Brazil)
Sales, Valdiane (Brazil)
Salgado, Cecília (Brazil)
Salgado, Luciana (Brazil)
Salinas, Zuly (Ecuador)
Saller Nornberg, Gabrielle (Brazil)
Salles, Mário Otávio (Brazil)
Salles-Neto, Luiz (Brazil)
Sal Moslehian, Mohammad (Iran)

Salomão, Pedro (Brazil)
Salomone, Leandro Martin (Argentina)
Salort, Ariel (Argentina)
Salur, Sema (USA)
Sambarino, Andrés (France)
Sambarino, Martin (Uruguay)
Samreen, Maria (Pakistan)
Sanabria Villalobos, Rafael Antonio (Brazil)
Sanches Angelo, Rodrigo (Brazil)
Sánchez Chavarría, Adriana Cristina (Brazil)
Sanchez, Hector (Mexico)
Sánchez, María Daniela (Argentina)
Sánchez-Nungaray, Armando (Mexico)
Sanchez Rubio, Yeison Alexander (Colombia)
Sánchez Saldaña, Luis Jorge (Mexico)
Sánchez Terraf, Pedro (Argentina)
Sandman, Christopher Patrick (France)
Sandoval Gonzalez, Nicolle (USA)
Sandoval Salazar, Diego Alfonso (Brazil)
Saneblidze, Samson (Georgia)
Sane, Sharad (India)
Sankovich, Natalia (Russia)
Santacesaria, Matteo (Italy)
de Santana Araújo, Roberto (Brazil)
Santana, José (Brazil)
Santana Martinez, Lina Jasbleidy (UK)
Santiago, Daniel Raom (Brazil)
de Santiago, Rolando (USA)
Santiago, Yolanda (Peru)
Santo, Nedir (Brazil)
dos Santos, Bruna (Brazil)
Santos, Carlos Alberto (Brazil)
dos Santos, Dalmi Gama (Brazil)
dos Santos Domingues, Tiago (Brazil)
Santos, Erikson (Brazil)
dos Santos Filho, José Ruidival (Brazil)
dos Santos Gouveia, Elionai (Brazil)
Santos Leite, Jose Marcelo (Brazil)
Santos, Luiz-Rafael (Brazil)
Santos, Makson (Brazil)
Santos, Rafaella (Brazil)
Santos, Sandro (Brazil)
dos Santos, Vanderlei (Brazil)
Santos, Vinicius (Brazil)
dos Santos, Vinicius (Brazil)
dos Santos, Vinícius Lima (Brazil)
Santos, Viviane (Brazil)
Santos, Walcy (Brazil)
Sanz-Solé, Marta (Spain)
Saraiva Tavares, Hugo (Brazil)
Saremi, Hero (Iran)
Sarkar, Jaydeb (India)
Sarkar, Sucharit (USA)
Sarnak, Peter (USA)
Satapathy, Purnima (India)
Satpati, Goutam (India)
Sauer, Lisandra (Brazil)
Savin, Gordan (USA)
Sawant, Anand (Germany)
Sawon, Justin (USA)
Saxena, Subhash (USA)
Sayago, Amilcar (Brazil)
Schaffhauser, Florent (Colombia)
Schapira, Pierre (France)
Scheiner, Thorsten (New Zealand)
Schiele, Saskia (Germany)
Schiffmann, Olivier (France)
Schleicher, Dierk (Germany)
Schlein, Benjamin (Switzerland)
Schlichenmaier, Martin (Luxembourg)
Schmid, Wolfgang (Germany)
Schneider, Alexsandro (Brazil)
Schneider, Eivind (Norway)
Schneider, Kai (France)
Scholze, Frank (Germany)
Scholze, Handel (Brazil)
Scholze, Marie Susanne (Germany)
Scholze, Peter (Germany)
Schreiber, Bertram (USA)
Schubring, Gert (Germany)
Schwartz, Catherine (USA)
Schwarz, João (Brazil)
Schweitzer, Paul Alexander (Brazil)
Seaton, Christopher (USA)
Secco, Matheus (Brazil)
Sécherre, Vincent (France)
Seck, Diaraf (Senegal)
Seggev, Itai (USA)
Selmane, Schehrazad (Algeria)
Selvamuthu, Dharmaraja (India)
Selvaraj, Marshal Anthoni (India)
Semenov, Nikita (Germany)
Semmler, Klaus-Dieter (Switzerland)
de Sena Dall'Agnol, Marcel (Brazil)
Sen, Anupam (India)
Sene, Abdou (Senegal)
Sengupta, Indranath (India)
Senti, Samuel (Brazil)
Seo, Jinwoo (South Korea)
Sepanski, Mark (USA)
Sepe, Daniele (Brazil)
Serfaty, Sylvia (USA)
Sergeichuk, Vladimir (Ukraine)
Series, Caroline (UK)

Serna Díaz, Raquel Inés (Brazil)
Severo, Franco (France)
Shabbir, Ayesha (Pakistan)
Shahidi, Farruh (USA)
Shah, Lubna (India)
Shah, Riddhi (India)
Shang, Yilun (China)
Shan, Mariia (Ukraine)
Shao, Qiman (Hong Kong)
Sharifi, Kamran (Iran)
Sharma, Anuradha (India)
Sharma, Bibhya (Fiji Republic)
Sharma, P. L. (India)
Sharma, Veena (India)
Shcheglov, Dmitry (Brazil)
Shcherbakov, Evgueni (Russia)
Shcherbina, Mariya (Ukraine)
Shikare, Maruti (India)
Shimizu, Senjo (Japan)
Shimomura, Katsunori (Japan)
Shin, Jaemin (South Korea)
Shinohara, Katsutoshi (Japan)
Shishikura, Mitsuhiro (Japan)
Shi, Zhenfeng (China)
Shlyk, Vladimir (Belarus)
Shmerkin, Pablo (Argentina)
Sholl, Erika (Brazil)
Shumyatsky, Pavel (Brazil)
Shu, Shi (China)
Shyshkanova, Ganna (Ukraine)
Sichinel, Valentino (Brazil)
Sidana, Tania (India)
Siegmund-Schultze, Reinhard (Norway)
Sierżęga, Mikołaj (Poland)
Sijong, Kwak (South Korea)
Siles Molina, Mercedes (Spain)
Sil, Subhankar (India)
da Silva, Adam (Brazil)
Silva, Analia Concepción (Argentina)
Silva, Danielle (Brazil)
Silva da Rocha, Viviane (Norway)
Silva, Davi (Brazil)
Silva, Debora (Brazil)
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Silva, Eduardo (Brazil)
Silva, Elaine (Brazil)
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Silva, Franciele (Brazil)
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Silva, Leonardo (Brazil)
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Silva Neto, Gregório (Brazil)
da Silva Nunes, Adilson (Brazil)
da Silva Reis, Lucas (Brazil)
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Silva, Tamires (Brazil)
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Silva Villamil, Christian Camilo (Brazil)
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Simanjuntak, Rinovia (Indonesia)
Simarra Canate, Alejandro (Brazil)
Simas, Fabio (Brazil)
Simon, Christopher-Lloyd (France)
Simon, Lois (South Korea)
Simon, Peter (Hungary)
Şimşek, Yılmaz (Turkey)
Sindi, Suzanne (USA)
Sing, Bernd (Barbados)
Singer, Amit (USA)
Singer, Raffael (UK)
Singh, Shekhar (India)
Singla, Komal (India)
Siqueira Lima, Carlos Alberto (Brazil)
Siqueira Rocha, Jaqueline (Brazil)
Sirakov, Boyan (Brazil)
Sirbu, Parascovia (Moldova)
Sirolli, Nicolás (Argentina)
Siskova, Katarina (Belgium)
Skalski, Adam (Poland)
Skipper, Rachel (USA)
Slavova, Angela (Bulgaria)
Sly, Allan (USA)
Smaniotto, Bruno (Brazil)
Smaoui, Kais (Tunisia)
Smirnov, Stanislav (Russia)
Smith, Graham (Brazil)
Smith, Ivan (UK)
Soares Campos, Marcelo (Brazil)
Soares, Dyego (Brazil)
Soares, Marcio (Brazil)
Soares Ruas, Maria Aparecida (Brazil)
Sobottka, Marcelo (Brazil)
Sofi, Amin (India)
Sologuren Paz, Santiago (Bolivia)
Solotar, Andrea (Argentina)
Sonawane, Ramdas (India)
Sonoda, Guilherme (Brazil)

Sood, Garima (India)
Sorensen, Claus (USA)
Sorio, Jennifer (Brazil)
Sotelo Castelblanco, Deissy M. (Colombia)
Sotomayor, Jorge (Brazil)
Soto Sánchez, José Ezequiel (Brazil)
Sottinen, Tommi (Finland)
Sousa, Antonio (Brazil)
Sousa da Silva Ferreira, Vinícius (Brazil)
Sousa Jr., Luiz Amancio (Brazil)
Sousa, Mateus (Argentina)
Sousa, Priscila (Brazil)
Sousa, Wellimar (Brazil)
Souto, Ginnara (Brazil)
Souza, André (Brazil)
Souza, Beethoven (Brazil)
Souza, Caio (Brazil)
de Souza, Claudino (Brazil)
Souza, Daniella (Brazil)
Souza, Fabricia (Brazil)
de Souza, Juliana (Brazil)
Souza, Manuela (Brazil)
Souza, Michael (Brazil)
de Souza Neto, João Valerio (Brazil)
Souza, Pammella (Brazil)
de Souza, Paulo Ney (USA)
Souza, Valdirene (Brazil)
Souza, Victor (Brazil)
Sow, Djiby (Senegal)
Spiegel, Beate (Germany)
Srimongkol, Sineenart (Thailand)
Srinivas, Vasudevan (India)
Sseviiri, David (Uganda)
Stafa, Mentor (USA)
Stagnaro, Ezio (Italy)
Staib, Erich (USA)
Stamatovic, Biljana (Montenegro)
Steffenon, Rogerio (Brazil)
Steindorf, Vanessa (Brazil)
Steinhorn, Charles (USA)
Steinmetz, Misja (UK)
Stenlund, David (Finland)
Stepanauskas, Gediminas (Lithuania)
Steurer, David (Switzerland)
Stoehr, Karl-Otto (Brazil)
Stojakovic, Milos (Serbia)
Stolnicki, David (Brazil)
Stoyanov, Jordan (Bulgaria)
Straume, Eldar (Norway)
Stuhl, Izabella (USA)
Stukow, Michał (Poland)
Sturm, Karl-Theodor (Germany)
Suárez Albanesi, Rocío Belén (Argentina)
Suarez Suarez, Hector Julio (Colombia)
Suarez Villagran, Campo Elias (Brazil)
Sugeng, Kiki (Indonesia)
Sulima, Anna (Poland)
Sun, Cong (China)
Sun, Song (USA)
Sun, Wenxiang (China)
Sun, Yeneng (Singapore)
Suparta, I. Nengah (Indonesia)
Suragan, Durvudkhan (Kazakhstan)
Suthar, Dayalal (Ethiopia)
Suvinthra, Murugan (India)
Suzuki, Shintaro (Brazil)
Svaiter, Benar (Brazil)
Svaldi, Roberto (UK)
Sviridova, Irina (Brazil)
Svrtan, Dragutin (Croatia)
Sweilam, Nasser (Egypt)
Switala, Nicholas (USA)
Sy, Mamadou (Senegal)
Sy, Polly (Philippines)
Szalay, Laszlo (Hungary)
Szegedy, Balazs (Hungary)
Tadano, Homare (Japan)
Tadmor, Eitan (USA)
Taduri, Srinivasa Siva Rama K. (India)
Tagami, Daisuke (Japan)
Taha, Ibrahim Mohammedelmojtaba (Oman)
Tahzibi, Ali (Brazil)
Takahashi, Akihiko (USA)
Talay, Denis (France)
Tamm de Araujo Moreira, Carlos G. (Brazil)
Tanaka, Noriko (Japan)
Tanbay, Betül (Turkey)
Tang, Taiman (China)
Tang, Tao (China)
Tan, Victor (Singapore)
Tan, Xintong (China)
Taranenko, Anna (Russia)
Tardos, Éva (USA)
Tardos, Gábor (Hungary)
Tari, Farid (Brazil)
Tavares Paes Lopes, Pedro (Brazil)
Tayara Almeida da Costa, Jessica (Brazil)
Taylor, Jonathan (USA)
Tcaciuc, Adi (Canada)
Tchoundja, Edgar Landry (Cameroon)
Teh, Wen Chean (Malaysia)
Teixeira, Augusto (Brazil)
Teixeira, Eduardo (USA)
Teixeira, Gustavo (Brazil)

Teixeira Paula, Gisele (Brazil)
Tellez, Duvan (Colombia)
Temirgaliyeva, Zhanerke (USA)
Temirgaliyev, Nurlan (Kazakhstan)
Tenenblat, Keti (Brazil)
Tenorio Dias, Maria de Lourdes (Brazil)
di Teodoro, Antonio (Ecuador)
Teo, Lee-Peng (Malaysia)
Terakawa, Hiroyuki (Japan)
de Teresa, Luz (Mexico)
Terzic, Svjetlana (Montenegro)
Tetunashvili, Shakro (Georgia)
Thakur, Mugdha (USA)
Thành An, Phan (Vietnam)
Thasari Murali, Rajalaxmi (India)
Thom, Andreas (Germany)
Thomas, Rekha (USA)
Thomaz da Silva, Rodrigo (Brazil)
Thome de Lima, Paloma (Norway)
Thompson, Lola (USA)
Thompson, Philip (France)
Thorne, Jack (UK)
Tian, Ling (China)
Tiep, Pham (USA)
Tikhonov, Sergey (Belarus)
Tilahun, Surafel Luleseged (South Africa)
Tillmann, Ulrike (UK)
Timerman, Gabriela (Brazil)
Timóteo da Silva, Telles (Brazil)
Todea, Constantin-Cosmin (Romania)
Todjihounde, Leonard (Benin)
Toint, Philippe (Belgium)
Tojeiro, Ruy (Brazil)
Tokieda, Tadashi (USA)
Toland, John (UK)
Toledo, Anna (Brazil)
Tomasi, Hiandra (Brazil)
Tomazella, João N. (Brazil)
Tomazella, Vera (Brazil)
Tomei, Carlos (Brazil)
Tondeur, Philippe (USA)
Toninelli, Fabio Lucio (France)
Toon, Eduard (Brazil)
Torain, David (USA)
Tor, Ali Hakan (Turkey)
Toranzo, Jorge Alberto (Brazil)
Tornaría, Gonzalo (Uruguay)
Tornberg, Anna-Karin (Sweden)
Tosun, Meral (Turkey)
Toth, Balint (Hungary)
Toure, Saliou (Ivory Coast)
Tranah, David (UK)
Trần, Nguyễn Thành Ha (Canada)
Trejos Olmos, Jonathan Augusto (Brazil)
Trelat, Emmanuel (France)
Triana, Juan (Colombia)
Trofymenko, Olha (Ukraine)
Trojovsky, Pavel (Czech Republic)
Tropnikau, Yury (Belarus)
Trotman, David (France)
Trujillo, Frank (France)
Trương, Xuân Đức Hà (Vietnam)
Tschinkel, Yuri (USA)
Tsend-Ayush, Selenge (Mongolia)
Tsimerman, Jacov (Canada)
Tsiporen, Tomer (Israel)
Tsuboi, Takashi (Japan)
Tsujii, Masato (Japan)
Tuân Hoa, Lê (Vietnam)
Tucker, Henry (USA)
Tucker, Warwick (Sweden)
Tucsnak, Marius (France)
Tuladhar, Bhadra Man (Nepal)
Tulli, Victor (Brazil)
Tumpach, Alice Barbora (France)
Tuneski, Nikola (Macedonia)
Türk, Önder (Turkey)
Turova Schmeling, Tatiana (Sweden)
Turunen, Ville (Finland)
Udaya Kovilakath, Anandavardhanan (India)
Ullmo, Emmanuel (France)
Unger, Ryan (USA)
Uppal, Surindar Mohan (Kenya)
Urbina-Romero, Wilfredo (USA)
Uribe, Bernardo (Colombia)
Ushirobira, Rosane (France)
Valdez, Ferran (Mexico)
Vale, Deivid (Brazil)
Valentim, Fábio (Brazil)
Valério, Viviane (Brazil)
Valle, Rolv Haakon (Norway)
Vanderlinde, André (Brazil)
V., Anu (India)
Varandas, Paulo (Brazil)
Varão, Régis (Brazil)
Vargas Vasquez, Xaab Nop (Mexico)
Varghese, Anu (India)
Vasquez Campos, Brian David (Colombia)
Vasquez, Cristhabel (Brazil)
Vasquez, Lautaro (Chile)
Vay, Cristian (Argentina)
Vaz, Sami (Brazil)
Vaz, Sergio (Brazil)
Vega, Luis (Spain)

Vega, Santiago (Argentina)
Velez-Santiago, Alejandro (Puerto Rico)
Velicu, Cornel Andrei (UK)
Velimirovic, Ana (Serbia)
Velimirovic, Ljubica (Serbia)
Veloso, Jose (Brazil)
Velozo Ruiz, Renato Adolfo (Chile)
Venkataraman, Geetha (India)
Venkatesan, Govindaraj (India)
Venkatesh, Akshay (USA)
Vera-Gajardo, Andrea (Chile)
Veras da Silva, Vicente Eudes (Brazil)
Verbitskiy, Mikhail (Russia)
Verchinine, Vladimir (France)
Vergara, Fabrizzio (Peru)
Verma, Rajkumar (Chile)
Verri, Alessandra Aparecida (Brazil)
Vertman, Boris (Germany)
Veruete, Mario (France)
Veselovska, Hanna (Ukraine)
V. Honnappa, Satheeshkumar (Brazil)
Viana, Marcelo (Brazil)
Vianna, Branca (Brazil)
Viazovska, Maryna (Switzerland)
Victor, Carlos (Brazil)
Viehmann, Eva (Germany)
Vieira, Sandoel (Brazil)
Vieitez, Jose (Uruguay)
Vigneaux, Juan Pablo (France)
Vigneras, Marie-France (France)
Vilches, Emilio (Chile)
Villaça, Túlio (Brazil)
Villaflor Loyola, Roberto (Brazil)
Villagra, Guillermo (Brazil)
Villamizar-Morales, Jorge (Colombia)
Villani, Cédric (Brazil)
Villavicencio Fernández, Helmuth (Peru)
Villicaña Molina, Yesenia (Mexico)
Vinet, Luc (Canada)
Vinhas Bertolini, Marcel (Brazil)
Visgueiro, Dion (Brazil)
Vitti, Mannaim Gennaro (Brazil)
Vittone, Francisco (Argentina)
Viu Sos, Juan (Brazil)
Viveiros, Antonio (Brazil)
Volkov, Iurii (Russia)
Voloch, Jose (New Zealand)
Wahyuni, Sri (Indonesia)
Waldschmidt, Michel (France)
Walo Omana, Rebecca (DR of the Congo)
Walsh, Miguel (Argentina)
Wang, Hua (France)
Wang, Jingjing (China)
Wang, Min (France)
Wang, Shin-Hwa (Taiwan)
Wang, Yue (China)
Wanless, Ian (Australia)
Warzel, Simone (Germany)
Wasadikar, Meenakshi (India)
Watanabe, Ricardo Augusto (Brazil)
Watari, Masahiro (Malaysia)
Watt, Stephen (Canada)
Weibel, Charles A. (USA)
Weinstein, Jared (USA)
Wen, Jianning (China)
Werner, Wendelin (Switzerland)
Westenberger, Rafael (Brazil)
Wetzlar, Ruth (Germany)
Wiegand, Sylvia (USA)
Wienhard, Anna (Germany)
Wijaya, Kristiana (Indonesia)
Wilkinson, Mark (UK)
Williams Ii, Dwight (USA)
Williamson, Geordie (Australia)
Williams, Virginia (USA)
Willwacher, Thomas (Switzerland)
Winklmeier, Monika (Colombia)
Winter, Wilhelm (Germany)
Wohlmuth, Barbara (Germany)
Wolfram, Stephen (USA)
Woosung, Yoon (South Korea)
Wormald, Nicholas (USA)
Wrazidlo, Dominik (Germany)
Wresh, Lukas (Germany)
Wright, Thomas John (USA)
Wu, Hao (France)
Wutzig, Michael (Belgium)
van Wyk, Leon (South Africa)
Wyrwas, Małgorzata (Poland)
Xambó-Descamps, Sebastià (Spain)
Xavier Freitas, Adriana (Brazil)
Xiang, Shengquan (France)
Xiang, Yangrui (China)
Xiaochuan, Tian (USA)
Xiaojun, Duan (China)
Xiao, Xia (China)
Xu, Chenyang (China)
Xu, Yanning (UK)
Yadav, Sangita (India)
Yafaev, Andrei (UK)
Yamagishi, Manabu (Japan)
Yamamoto, Hikaru (Japan)
Yamane, Hiroyuki (Japan)
Yang, Yin (China)

Yan, Jiali (UK)
Yan, Jianshi (China)
Yao, Guowu (China)
Yaparova, Natalia (Russia)
Yapman, Ömer (Turkey)
Yarza Acuña, Sergio Enrique (Mexico)
Yasinski, Yahor (Russia)
Yavicoli, Alexia (Argentina)
Ye, Fan (USA)
Yemini, Bruno (Chile)
Yeroshkin, Dmytro (USA)
Ye, Xiaofeng (USA)
Yie, Ikkwon (South Korea)
Yokoyama, Tomoo (Japan)
Yoneyama, Taisuke (Japan)
You, Jiangong (China)
Young, Lai-Sang (USA)
Yuan, Jinyun (Brazil)
Yuan, Yaxiang (China)
Yue, Hong (USA)
Yuhjtman, Sergio Andres (Argentina)
Yukimura, Daniel (Brazil)
Yun, Gabjin (South Korea)
Yun, Zhiwei (USA)
Yurttaş, Aysun (Turkey)
Zaban, Bruno (Brazil)
Zalesski, Pavel (Brazil)
Zárate, Melina (Argentina)
Zare Firoozabadi, Sanaz (Iran)
Zatitskii, Pavel (Russia)
Zayed, Ahmed (USA)
Zelenyuk, Yuliya (South Africa)
Zelerowicz, Agnieszka (USA)
Zhang, Lei (China)
Zhang, Lunchuan (China)
Zhang, Pingwen (China)
Zhang, Shaowu (USA)
Zhang, Wei (USA)
Zhang, Wenjing (USA)
Zhao, Tianfu (China)
Zhao, Tong (China)
Zhao, Yufei (USA)
Zhu, Jubo (China)
Zhunussova, Zhanat (Kazakhstan)
Zhu, Zuonong (China)
Ziebell, Juliana (Brazil)
Ziegler, Günter M. (Germany)
Zine El Abidine, Zagharide (Tunisia)
Zoo, Ho Sung (South Korea)
Zugliani, Giuliano (Brazil)

Membership by Country

Algeria	6
Argentina	93
Armenia	1
Australia	11
Austria	3
Bangladesh	2
Barbados	1
Belarus	8
Belgium	5
Benin	5
Bolivia	2
Bosnia and Herz.	2
Botswana	5
Brazil	792
Bulgaria	3
Burkina Faso	3
Cameroon	7
Canada	25
Chile	44
China	51
Colombia	46
Croatia	6
Cuba	1
Czech Republic	6
Dem Rep. Congo	2
Denmark	5
Ecuador	5
Egypt	4
Ethiopia	1
Fiji Republic	2
Finland	11
France	134
Gabon	1
Georgia	7
Germany	84
Ghana	5
Greece	1
Hong Kong	7
Hungary	16
India	138
Indonesia	12
Iran	28
Iraq	3
Ireland	1
Israel	10
Italy	20
Ivory Coast	3
Japan	50
Kazakhstan	7
Kenya	2
Cambodia	1
Korea	3
Kuwait	1
Kyrgyzstan	3
Latvia	2
Lesotho	1
Lithuania	1
Luxembourg	1
Macedonia	2
Madagascar	2
Malawi	2
Malaysia	4
Mali	1
Marocco	9
Mexico	42
Moldova	4
Mongolia	2
Montenegro	7
Mozambique	2
Namibia	1
Nepal	5
Netherlands	4
New Zealand	3
Nigeria	22
Norway	21
Oman	4
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