Chapter 18

The Emperor and His Money

For good ye are and bad, and like to coins,
Some true, some light, but every one of you
Stamp'd with the image of the King.
Alfred, Lord Tennyson, *The Idylls of the King, The Holy Grail,* l.25.

Figure 1. The Emperor's Declaration.

"...Emperor Nu took power by overthrowing the divisive My-Nus dynasty. The Nu régime introduced many positive reforms, and in particular abolished the old (An-Tsient) irrational currency, which had his predecessor's head on it, and introduced the Nu system. The masters of the Imperial Mint, Hi and Lo, were alternately to decide the value of each new denomination, and after each decision, sufficiently many coins of this value were to be struck. All went well until Hi ordered the striking of a coin of value one, so throwing the Workers of the Mint into unemployment. They rose in a body, and threw the unfortunate Hi from the tower at the quiet end of the capital, which has been known as the Hi Tower ever since."

My-Nus—Some Divisive Times.

SYLVER COINAGE

Had Hi and Lo read this book, they would have realized they were only playing a game, the game of **Sylver Coinage**. In this the players alternately name different numbers, but are not allowed to name *any* number that is a sum of previously named ones. So, if 3 and 5 have been named, for example, neither of the players is allowed to play any of the numbers

$$3, \quad 5, \quad 6 = 3+3, \quad 8 = 3+5, \quad 9 = 3+3+3, \quad 10 = 5+5, \quad 11 = 3+3+5, \quad \ldots$$

When will this game end? If neither player has played 1, 1 will still be playable. But, of course, as soon as 1 has been played, every number

$$1, \quad 2 = 1+1, \quad 3 = 1+1+1, \quad 4 = 1+1+1+1, \quad 5 = 1+1+1+1+1, \quad \ldots$$

is illegal, and so the game ends. Because the player who names 1 is declared the *loser*, Sylver Coinage is a *misère* game. (Skilful players won't spend much time on the normal play version!).

We had better point out that because the old currency had been rather irrational (with coins of value $\sqrt{2}$, e and π) the Emperor declared that there was to be a new monetary unit, the **You-Nit**, and the value of each coin was to be an integral number of You-Nits. (You can see the Emperor making this declaration in Fig. 1!).

And recalling how people were nonplussed by the great financial scandal of the My-Nus dynasty when they had to take away Teh Kah-Weh for issuing currency of negative value, Emperor Nu decided that each coin's value must be a *positive* number of You-Nits.

HOW LONG WILL IT LAST?

It might take quite a long time. To see that it can last for a thousand moves, we need only consider the game

$$1000, 999, 998, \ldots, 4, 3, 2, 1.$$

And of course a thousand can be replaced by any other number, so that the game is **unbounded**. Many other games have this property, for example Green Hackenbush (Chapter 2) played with an infinite snake, but are *boundedly* unbounded because after some fixed number of moves the end will be in sight. Thus after the first move in the Hackenbush game only a finite amount of snake is left.

But Sylver Coinage is not like that! No matter what number you choose, Hi and Lo can find a way to play that number of moves so that what's left of the game will still be unbounded. Their first thousand moves might be

$$2^{1000}, 2^{999}, 2^{998}, \ldots, 2^4, 2^3, 2^2, 2^1$$

and the rest of the game can still last as long as you like:

$$1000001, 999999, 999997, \ldots, 7, 5, 3, 1.$$

In other words Sylver Coinage is *unboundedly unbounded*. And this isn't all. It's *unboundedly unboundedly unbounded* and unboundedly like that, and so (unboundedly) on!

Nevertheless, it can't go on for ever; in the language of Chapter 11 it's an *ender*. It is because the little theorem which proves this is due to the famous mathematician J.J. Sylvester that we have called the game *Sylver* Coinage.

For, at any time after the first move, let g be the greatest common divisor (g.c.d.) of the moves made. Then it's not hard to see that only finitely many multiples of g are *not* expressible as sums of numbers already played. So after at most this known number of moves the g.c.d. must be reduced. Eventually we must arrive at a position with $g = 1$ and can bound the number of moves yet to be made. So although we may not be able to bound the game after any given number of moves, we *can* bound the number of moves it will take to reduce the g.c.d.

SOME OPENINGS ARE BAD

The proof we gave in the Extras to Chapter 2 shows that from any position in Sylver Coinage there *is* a winning strategy for one of the two players *but* because of the infinite nature of the game we cannot work through all positions and guarantee to find winning strategies when they exist. In fact we do not know of (and there may not exist) any way of working out in a finite time who wins from an arbitrarily given position. But we do know the answers for some easy positions.

If at any time you name 1, you lose by definition.

If you name 2, my reply will be 3 if it's still available, and then all larger numbers

$$4 = 2+2, \quad 5 = 2+3, \quad 6 = 2+2+2, \quad 7 = 2+2+3, \quad 8 = 2+2+2+2, \quad \ldots$$

are excluded and you will be forced to name 1.

If you name 3, then for the same reason, 2 is a good reply.

So whoever first names any of 1, 2 and 3 will lose. In particular the first three numbers are bad opening moves. What will you reply if I open with 4? Maybe 5? If so the g.c.d. becomes 1 and there will be only finitely many numbers left. We can find out which by arranging the numbers as in Fig. 2. The circled numbers are excluded because they're multiples of 5 and these exclude the lower numbers by adding 4's. So only 1, 2, 3, 6, 7, 11 remain.

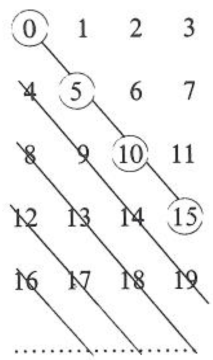


Figure 2. What's Left After {4, 5}.

I won't take 1, 2 or 3. If *I* say 6 or 7, you'll say the other, since these dismiss 11 and leave only 1, 2, 3 for me. So I'll say 11 and make *you* say 6 or 7 instead.

$$\boxed{\{4,5,11\} \text{ is a } \mathscr{P}\text{-position.}}$$

Here's what happens after 4 and 6.

$$\begin{array}{cccc}
0 & 1 & 2 & 3 \\
\cancel{4} & 5 & 6 & 7 \\
\cancel{8} & 9 & \cancel{10} & 11 \\
\cancel{12} & 13 & \cancel{14} & 15 \\
\cancel{16} & 17 & \cancel{18} & 19 \\
\end{array}$$

....

Since 5 and 7 exclude all large numbers, they kill each other. Similarly for 9 and 11, and for 13 and 15, and so on.

> After $\{4,6\}$ the pairs
> $(2,3), (5,7), (9,11), \ldots, (4k+1, 4k+3)$,
> for $k \geqslant 1$, are mates.

So if you open with 4, I shall respond with 6; if you open with 6, I shall respond with 4. A few similar strategies are known.

> After $\{8,12\}$ the pairs
> $(2,3), (5,7), (9,11), \ldots, (4k+1, 4k+3)$
> and
> $(4,6), (10,14), (18,22), \ldots, (8k+2, 8k+6)$,
> for $k \geqslant 1$, are mates.

There is a slightly more complicated strategy showing that another good reply to 6 is 9.

> After $\{6,9\}$ mate the pairs
> $(4,11), (5,8), (7,10)$ and $(3k+1, 3k+2)$ *for* $k \geqslant 4$,
> but *then*
> after 4,11 mate 5 with 7
> after 5,8 mate 4 with 7
> after 7,10 mate 4 with 5. 8 with 11.

We have proved that

{2,3} {4,6} {6,9} {8,12} are all $\mathscr{P}$-positions,

and so

{1} {2} {3} {4} {6} {8} {9} {12} are all $\mathscr{N}$-positions.

The numbers 1,2,3,4,6,8,9 and 12 are the only first moves for which explicit strategies have been found. You might expect that pairs (2,3), $(4k+1, 4k+3)$, (4,6), $(8k+2, 8k+6)$, (8,12), $(16k+4, 16k+12)$ provide a strategy after {16,24} but unfortunately 12 is *not* a legal move from the position {16,24,5,7,8}. On the other hand, for the strategies given above, both members of a pair are legal whenever one is. In fact 8 is a good reply to {16,24,5,7} because it makes 16 and 24 irrelevant and we shall soon see that

{5,7,8} *is a* $\mathscr{P}$-*position.*

We don't know whether 24 is a good reply to 16, nor even whether 16 *has* any good reply.

ARE ALL OPENINGS BAD?

If on observing the fate of 1, 2 and 3 you thought maybe that all openings were bad, then probably our discussions of 4, 6, 8, 9 and 12 have tended to confirm your suspicions. In this section we'll try to analyze 5 and 7. The discussion of possible replies is made a lot easier by the **clique technique**.

You've already seen some cliques: The number 1 forms a rather special clique all by itself; 2 and 3 form another because they exclude all larger numbers. In our discussion of {4,5}, 6 and 7 formed a clique since they excluded 11. Cliques have the property that any reply to a clique member must also be a clique member and these two numbers must together exclude all numbers outside the clique.

We illustrate the clique technique by discussing {6, 7} (Fig. 3).

```
0  1  2  3  4  5            (1)
   7  8  9  10 11           (2,3)
      14 15 16 17           (4,5)
         21 22 23           (8,10) (9,11)
            28 29           (15,23) (17,22) 16!
               35           29?
```

Figure 3. The Cliques after (6, 7).

As usual, we can disregard 1, 2 and 3 which form the innermost cliques in *every* position. Now 4 and 5 *together* exclude all larger numbers and so form a third clique. *No matter what larger numbers have been named*, 4 will answer 5 and 5 will answer 4. We can therefore afford to neglect them in discussing larger numbers.

Now we assert that 8, 9, 10 and 11 form the next clique, because 8 and 10 together exclude all but 9 and 11, and these together exclude all but 8 and 10. Even when some larger numbers have already been named, 8 will answer 10, 9 will answer 11, and vice versa, and we can dismiss all four from the subsequent discussion.

We now know that any good reply to any of the remaining numbers

$$\begin{array}{ccc} 15 & 16 & 17 \\ & 22 & 23 \\ & & 29 \end{array}$$

must be another of these. We see that 15 answers 23 and vice versa since these leave only 16 and 17. Similarly 17 and 22 are mates. But since 16 excludes *both* 22 and 23, leaving only 15 and 17, it's a good move by itself. These five numbers form a clique, since 29 is always excluded.

16 is the unique good reply to {6,7}.

Table 6 in the Extras exhibits complete strategies in a similar way for all the positions

$$\begin{array}{llll} \{4,5\}, & \{4,7\}, & \{4,9\}, & \\ \{5,6\}, & \{5,7\}, & \{5,8\}, & \{5,9\}, \\ \{6,7\}, & \{7,8\}, & \{7,9\}. & \end{array}$$

In particular it shows that

{4,5,11},	{4,7,13},	{4,9,19},
{5,6,19},	{5,7,8},	{5,9,31},
{6,7,16},	{7,9,19},	{7,9,24},
	are $\mathscr{P}$-positions.	

We deduce that any good reply to 5 or 7 must be at least a two-digit number. The smallest two-digit number, 10, isn't a legal answer to 5; is it a good answer to 7? No!

$\boxed{\{7,10,12\} \text{ is a } \mathscr{P}\text{-position.}}$

This is proved by Fig. 4. Since the clique technique isn't as helpful as it might have been, we've added extra notes for three of the pairs.

```
0  1  2  3  4  5  6            (1)
  8  9  10  11  12  13         (2,3)
                           (4,9) (5,8) (6,9)
  15  16  18      20         (11,16)  followed by (4,9) (5,13) (6,15) (8,13)
                             (13,15)  followed by (4,9) (5,8) (6,9) (8,11)] (16,18)
  22  23  25                 (13,18)  followed by (4,9) (5,8) (6,9) (8,11)] (15,16)
     30   32                 (23,25)
```

Figure 4. The Position {7, 10, 12}.

NOT ALL OPENINGS ARE BAD

R.L. Hutchings has proved that there can't be any good replies to 5 or 7! His main theorem is

> If a and b are coprime ($g = 1$) and $\{a,b\} \neq \{2,3\}$, then $\{a,b\}$ is an $\mathcal{N}$-position.

From this he deduces his ***p*-theorem**:

> If $p \geqslant 5$ is a prime number, $\{p\}$ is a $\mathcal{P}$-position,

p-positions are $\mathcal{P}$-positions.

(For any legal reply produces a position with a g.c.d. of 1.) And from the p-theorem he deduces in turn his ***n*-theorem**:

> If n is a composite number not of the form 2^a3^b, then $\{n\}$ is an $\mathcal{N}$-position,

n-positions are $\mathcal{N}$-positions.

(Since n has a prime divisor $p \geqslant 5$, which is a good reply.) Together these account for the first few missing numbers:

> $\{5\}, \{7\}, \{11\}, \{13\}, \{17\}, \ldots$ are $\mathcal{P}$-positions.
> $\{10\}, \{14\}, \{15\}, \{20\}, \{21\}, \ldots$ are $\mathcal{N}$-positions.

Our explicit strategies accounted for the eight smallest numbers 2^a3^b:

$\{1\}, \{2\}, \{3\}, \{4\}, \{6\}, \{8\}, \{9\}, \{12\}$
are $\mathcal{N}$-positions.

But

Nobody knows about
$\{16\}, \{18\}, \{24\}, \{27\}, \{32\}, \{36\}, \ldots$!

(We'd be glad to be proved wrong.)

STRATEGY STEALING

Hutchings proves his main theorem by a fine piece of strategy stealing. He considers the topmost number, t, that is not excluded by $\{a,b\}$ and proves that if t is *not* a good reply, then some other number *is*!

We shall call $\{a,b\}$ an **end-position** because, as we'll see in a moment, the topmost number is excluded by every other legal move.

Now let's ask:

Is t a good reply to $\{a,b\}$?

If the answer is "yes", then $\{a,b\}$ is an $\mathcal{N}$-position.

If the answer is "no", then either the game is over or there is a good reply s to $\{a, b, t\}$. But since a, b and s exclude t, s is itself a good reply to $\{a,b\}$. We can say that the player to move from $\{a,b\}$ finds his strategy by stealing the second player's strategy, if he has one, for $\{a,b,t\}$.

In some cases, e.g. $\{5,9\}$, t (here 31) is a good reply. But in others, e.g. $\{5,7\}$ (where $t = 23$) it *isn't*. The strategy stealing argument only tells us that good moves exist, not what they are. Theft is no substitute for honest toil!

In general,

An end position with $t > 1$
is an $\mathcal{N}$-position,

end-positions are $\mathcal{N}$-positions.

But the end-position $\{2,3\}$ is *not* an $\mathcal{N}$-position. This is because $t=1$ and the only legal move ends the whole game.

Why is $\{a,b\}$ an end-position if its g.c.d. is 1? In Fig. 5 we illustrate with $\{9,11\}$ for which the authors know no good reply.

$$
\begin{array}{ccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 \\
 & 19 & & 21 & 22 & 23 & 24 & 25 & 26 \\
 & 28 & & 30 & = s & 32 & 33 & 34 & 35 \\
 & 37 & & 39 & & 41 & & 43 & 44 \\
 & 46 & & 48 & & 50 & & 52 & \\
 & 55 & & 57 & & 59 & & 61 & \\
 & & & 66 & & 68 & & 70 & \\
 & & & & & 77 & & 79 = t = 30+3a+2b & \\
 & & & & & & & 88 &
\end{array}
$$

Figure 5. Hutchings's Theorem for $a = 9$, $b = 11$.

Writing the numbers in a columns, as is our wont, we see that in each column the *first excluded* (circled) number is a multiple of b, so the *last included* numbers must differ by multiples of b. Now from any legal move s we can get to the last legal number in its column by adding a's and from this we can get to t by adding b's showing that s excludes t (e.g. $s = 30$ in Fig. 5). The argument also provides a proof of Sylvester's well-known formula.

$$t = (a-1)b - a = ab - (a+b).$$

QUIET ENDS

Suppose Hi and Lo have named two coprime numbers a and b and Hi is considering making the move s. Then we know that the topmost number t will be obtainable using sufficiently many coins of values s, a and b. But our argument proved that only *one* copy of the new coin will be needed:

$$t = s + ma + nb.$$

More generally from a position $\{a,b,c,\ldots\}$ we shall say that s **quietly** excludes t if t can be made up using any numbers of $a,b,c,\ldots$ together with *just one* copy of s:

$$t = ma + nb + \ldots + s.$$

A **quiet end-position** is one in which the topmost legal move is *quietly* excluded by every number not already excluded.

> If a is coprime with each of b and b_1, then
> $$S = \{a, bc, bd, be, \ldots\}$$
> is a quiet end-position if and only if
> $$S_1 = \{a, b_1c, b_1d, b_1e, \ldots\}$$
> is.

THE QUIET END THEOREM

Thus

$$\{7,\ 1\times 3,\ 1\times 4\},$$

which is really the same position as $\{3,4\}$, is a quiet end-position, so that

$$\{7,9,12\} = \{7,\ 3\times 3,\ 3\times 4\}$$

and

$$\{7,15,20\} = \{7,\ 5\times 3,\ 5\times 4\}$$

are. In particular, these are end-positions and so are $\mathcal{N}$-positions by the strategy stealing argument. As usual we aren't told what the good replies are.

We shall use $\{7,9,12\}$ and $\{7,15,20\}$ to illustrate our proof of the quiet end theorem. Once again we write out the numbers in a (here 7) columns and circle the first excluded number in every column (Fig. 6). We assert that these numbers for the positions S and S_1 are in the proportion $b:b_1$ (3 :5 in the example; see Fig. 7).

0	1	2	3	4	5	6
	8	9	10	11	12	13
	15		17	18		20
	22		24			
						27
	29					
	36					

(a) $S = \{7,9,12\}$.

0	1	2	3	4	5	6
	8	9	10	11	12	13
	15	16	17	18	19	20
		23	24	25	26	
		30	31	32	33	
			38	39	40	
			45	46		
				53		
				60		

(b) $S_1 = \{7,15,20\}$.

Figure 6. Circled Numbers in Proportion.

								in the
		29	2	17	11	5	20	proportion
S:	0	36	9	24	18	12	27	3
								to
		53	8	33	23	13	38	
S_1:	0	60	15	40	30	20	45	5

Figure 7. The Circled Numbers Sorted.

We first see that the circled numbers for S really are multiples of b. Recall that we **circle** n for S if n is excluded by S, but $n-a$ is not. Since n is excluded, it has the form

$$n = ak + bm$$

where m would be excluded by $\{c,d,e,\ldots\}$. But if k were positive,

$$n - a = a(k - 1) + bm$$

would also be excluded by S, so $k=0$ and we have simply

$$n = bm.$$

Now our assertion is that bm is circled for S only if b_1m is circled for S_1. Now b_1m is certainly *excluded* by S_1 and so is circled unless

$$b_1m - a$$

is also excluded. But then we must have

$$b_1m - a = ak + b_1m'$$

for some m' excluded by $\{c,d,e,\ldots\}$, and

$$b_1m = a(k + 1) + b_1m',$$

showing that b_1 divides $k+1$ since it is coprime with a. We can now divide by b_1 and multiply by b to obtain

$$bm = ak' + bm'$$

for some positive number k', showing that

$$bm - a = a(k' - 1) + bm'$$

was excluded, and bm was *not* circled for S.

In its modest way, the quiet end theorem is quite powerful. It often gives the quietus to infinitely many replies with a single blow.

$$\boxed{\text{No odd number is a good reply to } \{16,24\}.}$$

For 1 clearly isn't and if a is any other odd number then $\{a,2,3\}$ is really the same as the quiet end position $\{2,3\}$. By the quiet end theorem $\{a,16,24\}$ is a quiet end-position and so an $\mathcal{N}$-position.

In a similar way it proves that $\{4,6\}$ and $\{6,9\}$ are $\mathcal{P}$-positions without bothering to provide a detailed strategy. Let's use it to discuss the position $\{8,10\}$. After $\{4,5\}$ we found that the only remaining moves were

$$1, \quad 2, \quad 3, \quad 6, \quad 7, \quad 11,$$

so after $\{8,10\}$ the only remaining even numbers will be twice these,

$$2, \quad 4, \quad 6, \quad 12, \quad 14, \quad 22.$$

The quiet end theorem enables us to say that any good reply to $\{8,10\}$ must be in one of these two sets, for otherwise it is an odd number a excluded by $\{4,5\}$ so $\{a,4,5\}$ and therefore $\{a,8,10\}$ will be quiet end-positions. Now,

1	loses instantly,
(2,3)	are mated as usual,
(4,6)	eliminate 8,10 and will mate, as will
(7,11)	(see $\{6,7\}$ in Table 6 in the Extras) and
(12,14)	by our strategy for $\{8,12\}$.

So 22 is the only hope for a good reply to $\{8,10\}$. We shall see later that

$\{8,10,22\}$ is a $\mathcal{P}$-position.

DOUBLING AND TRIPLING?

Note that the $\mathcal{P}$-position $\{8,10,22\}$ is the *double* of $\{4,5,11\}$. Our $\{8,12\}$ strategy shows that all $\mathcal{P}$-positions arising in the $\{4,6\}$ strategy have doubles that are also $\mathcal{P}$-positions. Maybe every $\mathcal{P}$-position doubles to another? No! For $\{5,6,19\}$ is $\mathcal{P}$, but $\{10,12,38\}$ is answered by 7 since $\{10,12,38,7\}$ is really the same as $\{7,10,12\}$.

Maybe the *triple* of every $\mathcal{P}$-position is another? No! This time $\{4,5,11\}$ is $\mathcal{P}$, but $\{12,15,33\}$ is answered by 5 since $\{5,12,33\}$ is a $\mathcal{P}$-position, as we'll soon see.

HALVING AND THIRDING?

Nevertheless there are many $\mathcal{P}$-positions whose doubles and triples are still $\mathcal{P}$. We conjecture:

¿If $\{2a,2b,2c,\ldots\}$ is $\mathcal{P}$ so is $\{a,b,c,\ldots\}$? and ¿If $\{3a,3b,3c,\ldots\}$ is $\mathcal{P}$ so is $\{a,b,c,\ldots\}$?

FINDING THE RIGHT COMBINATIONS

How should you start a game of Sylver Coinage? Now that you know so much you will perhaps name 5 for your first move. You now have a strategy for every move I might make and probably feel a little safe. But those stolen strategies are firmly locked inside that little safe you're feeling and more than sensitive fingers are needed to find the right combinations.

You know the first few: 1 needs no reply and you should make the pairs (2,3), (4,11), (6,19), (7,8) and (9,31). Is there any general rule? In trying to answer this question for you we went to a lot of trouble and eventually found a fairly efficient way of breaking open the safe. But the winning combinations it reveals (Fig. 8) suggest that there is no simple answer.

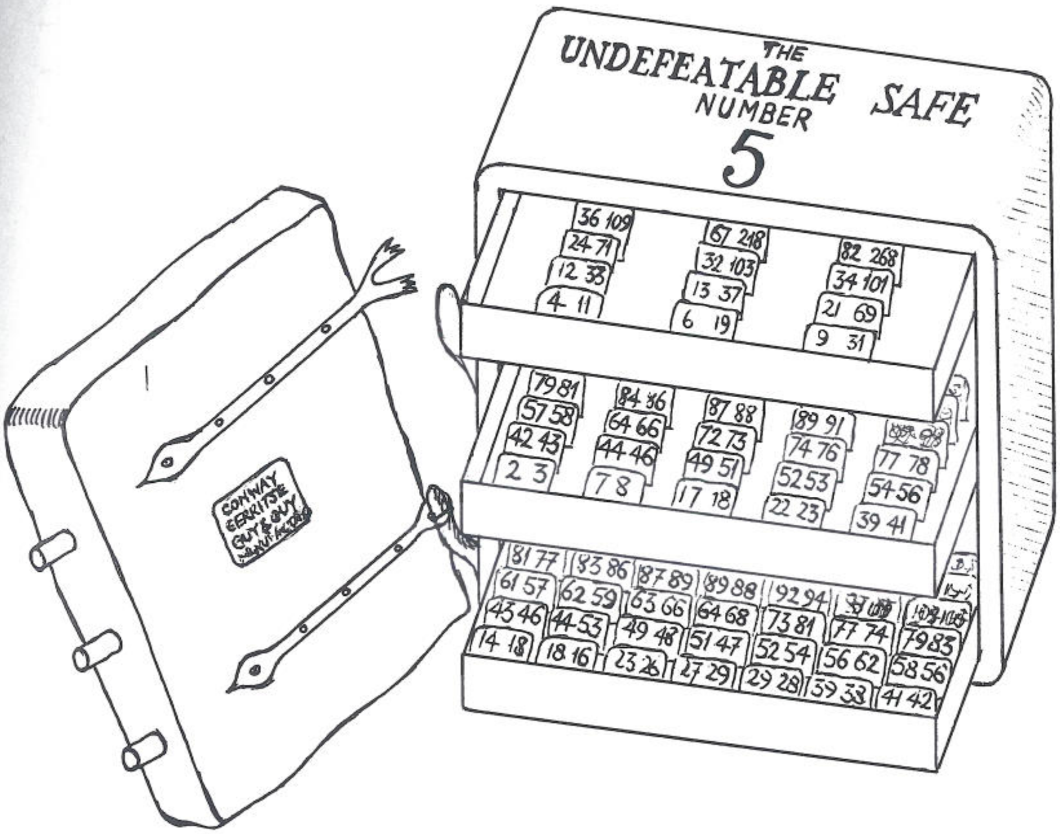


Figure 8. The Stolen Secrets of Safe Number 5.

Let's take a closer look at a position in which 5 and some other numbers have been named. If we were to write the numbers in five columns as usual we would circle 0 and just four other numbers a,b,c,d in the 1-,2-,3-,4-columns respectively, as in Fig. 9. We now make a three-dimensional table of $\mathscr{P}$-positions using just three of these numbers as headings and the fourth as an entry.

Table 1(a) shows the case in which a is the entry and b,c,d the row, column and layer headings. Tables 1(b,c,d) have b,c,d as entries.

$d = 4$

b \ c =	8
7	6
12	11

$d = 9$

b \ c =	8	13	18
7	11	16+	
12	6		16
17	6		21
22			26
27			31

$d = 14$

b \ c =	8	13	18	23	28
7	16+	11			
12		6	21	26+	
17		6	16		26
22		6	31	16	21
27		6	26	21	16
32			36+		
37					
42					

$d = 19$

b \ c =	8	13	18	23	28	33	38
7	16+	11					
12		16	6	6	6	6	6
17		21	6	6	6	6	6
22		26+	6	6	6	6	6
27			6	6	6	6	6
32			6	6	6	6	6
37			6	6	6	6	6
42				6	6	6	6
47					6	6	6
52						6	6
57							6

Table 1(a) Entries a for $\mathscr{P}$-positions $\{5,a,b,c,d\}$.

$c = 8$

d \ a =	6	11	16
4	7	12+	
9	12+	7	
14			7
19			7
24			7

$c = 13$

d \ a =	6	11	16	21	26
4	7	12+			
9			7	7	7
14	12+	7			
19		7	12	17	22
24		7	17	12	27
29			22	12	17
34				12	32
39					37

$c = 18$

d \ a =	6	11	16	21	26	31	36
4	7	12+					
9			12	17	22	27+	
14			17	12	27	22	32+
19	12+						
24			22	27	12	17	
29			27	22	12		
34			32+		12		17
39					12		17
44					12		17
49							17
54							17

Table 1(b). Entries b for $\mathscr{P}$-positions $\{5,a,b,c,d\}$.

$b = 7$

a \ d	4	9	14	
6	8+			
11		8	**13**	→
16		**13**	**8**	→
21		↓	**8**	→
26			↓	

$b = 12$

a \ d	4	9	14	19	24	
6		**8**	**13**	**18+**		
11	**8+**					
16		18+		13		
21			18		**13**	→
26			**23**		**18**	→
31			↓		**28**	→
36					**33**	→
41					↓	

$b = 17$

a \ d	4	9	14	19	24	29	34	
6		**8**	**13**	**18+**				
11	**8+**							
16			18		13	23	**28**	→
21		18+		13				
26			28+		23	**13**		
31					18	**13**	**23**	→
36					28	**13**	**18**	→
41					**38**	**13**	**18**	→
46					↓	**13**	**18**	→
51						↓	**18**	→
							↓	

Table 1(c). Entries c for $\mathscr{P}$-positions $\{5,a,b,c,d\}$

$a = 6$

c \ b	7	12	
8	**4**	**9**	→
13	**4**	**14**	→
18	↓	**19**	→
23		↓	

$a = 11$

c \ b	7	12	17	22	
8	9	**4**	**4**	**4**	→
13	14+	**4**	**4**	**4**	→
18		**4**	**4**	**4**	→
23		**4**	**4**	**4**	→
28		↓	**4**	**4**	→
33			↓	**4**	→
38				↓	

$a = 16$

c \ b	7	12	17	22	27	32	
8	14+						
13	**9**	19	24	29+			
18		**9**	14	24	29	34+	→
23		**9**	29	14	24		
28		**9**	34+		**14**	**24**	→
33		↓			**14**	**29**	→
38					**14**		
43					**14**		
48					↓		

Table 1(d). Entries d for $\mathscr{P}$-positions $\{5,a,b,c,d\}$.

Some positions will appear repeatedly because a heading is redundant. These are indicated by bold figures. For example

$$\{5,6,12,13,14\}, \quad \{5,6,17,13,14\}, \quad \{5,6,22,13,14\}, \quad \ldots$$

are really the same position because $12 = 6+6$ is redundant and so we have a column of 6's in layer 14 of Table 1(a). In $\{5,6,12,18,19\}$ both 12 and 18 are redundant, so the 19 layer of that table is almost entirely made up of 6's.

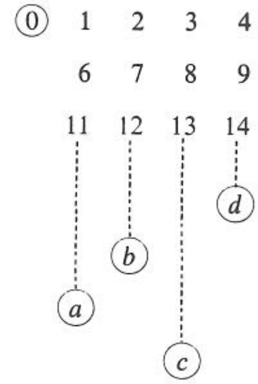


Figure 9. The General Position $\{5,x,y,...\}$.

In $\{5,16,7,13,9\}$ it is the *entry* $16 = 7+9$ that is redundant, so 16 can be replaced by any of

$$21, 26, 31, 36, 41, \ldots$$

and we have written $16+$ to indicate this. Really an entry $n+$ is short for infinitely many entries

$$n, n+5, n+10, n+15, n+20, \ldots .$$

The entries in Table 1(a) were computed in lexicographic order, by making due allowance for these repetitions and otherwise entering the least number $5k+1$ not appearing earlier in the same row, column or file.

You'll probably find the method easier to follow in Table 2 which deals with positions $\{4,a,b,c\}$ in a similar way. This time each entry is the smallest number $b = 4k+2$ which has not appeared earlier in its row or column and an entry $b+$, shorthand for

$$b, b+4, b+8, b+12, b+16, \ldots$$

is made when $b=2a$ or $2c$. It can be deduced from the quiet end theorem that other kinds of repetition will not appear.

Table 3 gives pairs $x<y$ for which $\{4,x,y\}$ is already a $\mathscr{P}$-position, extracted from an extended version of Table 2, kindly calculated for us by Richard Gerritse. It seems that the ratio y/x approaches 2·56....

As soon as 4 or 5 arises in your game you should refer to the appropriate one of these tables. If 6 turns up first, see the corresponding Table 7 in the Extras.

a \ c =	7	11	15	19	23	27	31	35	39	43	47	51	55	59	63	67	71	75	79	83	87	91
5	6	10+																				
9	10	6	14	18+																		
13	14+		6	10																		
17			10	6	14	18	22	26	30	34+												
21			18	14	6	10	26	22	34	30	38	42+										
25			22		10	6	14	18	26		30	34	38	42	46	50+						
29			26		18	14	6	10	22		34	30	42	38	50	46	54	58+				
33			30+		22	26	10	6	14	18												
37					26	22	18	14	6	10	42	38	30	34	54		46	50	58	62	66	70
41					30	34	38	42	10	6	14	18	22	26	58		50	46	54	66	62	74
45					34	30	42	38	18	14	6	10	26	22	62		58	54	46	50	70	66
49					38	42	30	34	46	22	10	6	14	18	26		62		50	54	58	78
53					42	38	34	30	50	26	18	14	6	10	22		66		62	46	54	58
57					46+				38		22	26	10	6	14	18	30	34	42			
61						46	50	54	42		26	22	18	14	6	10	34	30	38	58	74	62
65						50	46	58	54		62		34	30	10	6	14	18	22	26	38	42
69						54+		46			50				18	14	6	10	26	22	30	34
73							54	50	58		46		62	66	30	22	10	6	14	18	26	38
77							58	62	66		54		46	50	34	26	18	14	6	10	22	30
81							62+				58		50	46	38	30	22	26	10	6	14	18
85								66	62		70		54	58	42	34	26	22	18	14	6	10
89								70+			66		58	54		38	42		30	34	10	6
93									70		74		66	62	78	42	38		34	30	18	14
97									74		78		70	82	66		86	38	90	42	34	22
101									78+				74	70				42	66	38	46	26
105											82		78	74	70		90		86	94	42	46
109											86		82	78	74		70		94	90	50	54
113											90		86	94	82		74		70	78	98	50
117											94+		90	86			78		74	70	82	

Table 2. Values of b for which $\{4,a,b,c\}$ is a $\mathscr{P}$-position.

x	y	x	y	x	y	x	y	x	y	x	y	x	y
5	11	107	269	205	531	303	777	405	1043	501	1291	603	1549
7	13	109	279	207	529	305	783	407	1045	503	1289	605	1555
9	19	111	277	211	541	311	797	409	1051	505	1299	607	1557
15	33	113	287	213	547	313	807	411	1053	511	1309	609	1567
17	43	119	301	219	557	315	805	413	1063	513	1319	615	1577
21	51	121	307	221	567	317	819	415	1065	519	1329	617	1583
23	57	123	309	223	569	321	823	419	1077	521	1339	621	1595
25	67	125	319	227	585	323	829	421	1079	523	1341	623	1593
27	69	129	331	229	583	325	839	425	1095	525	1347	625	1603
29	75	131	333	231	593	327	841	427	1097	527	1349	627	1609
31	81	133	343	233	595	329	851	429	1103	533	1371	633	1627
35	89	135	345	237	611	335	857	431	1105	535	1373	635	1625
37	95	141	363	239	613	337	871	433	1115	537	1379	637	1635
39	101	143	365	241	619	339	869	439	1125	539	1381	639	1637
41	103	147	373	245	631	341	879	441	1135	543	1393	643	1649
45	115	149	379	247	629	347	885	447	1145	545	1395	645	1655
47	117	151	385	251	641	349	899	449	1151	549	1411	651	1665
49	127	153	391	253	647	351	901	451	1157	551	1413	653	1679
53	139	155	397	255	649	353	911	455	1165	553	1423	655	1681
55	137	157	399	257	659	355	909	457	1171	555	1425	657	1687
59	145	163	417	259	665	357	919	459	1177	559	1433	661	1699
61	159	165	423	261	671	359	917	461	1183	561	1443	663	1697
63	161	169	435	265	683	361	927	463	1189	563	1445	667	1709
65	167	171	437	267	685	367	941	465	1195	565	1451	669	1719
71	177	173	443	271	697	369	951	469	1207	571	1465	673	1731
73	183	175	445	273	699	371	953	471	1209	573	1471	675	1729
77	195	179	453	275	705	375	961	473	1215	575	1477	677	1739
79	193	181	467	281	723	377	971	479	1225	577	1483	679	1741
83	209	185	475	283	725	381	983	481	1235	579	1485	681	1751
85	215	187	477	285	731	383	981	485	1247	581	1495	687	1757
87	217	189	483	289	743	387	993	487	1245	587	1505	689	1771
91	225	191	489	291	745	389	1003	491	1257	589	1511	691	1769
93	235	197	507	293	755	393	1011	493	1267	591	1517	693	1783
97	243	199	509	295	757	395	1013	495	1269	597	1531	695	1781
99	249	201	515	297	767	401	1031	497	1279	599	1537	701	1803
105	263	203	517	299	765	403	1029	499	1277	601	1543	703	1801
												707	1813

Table 3. Pairs x,y for which $\{4,x,y\}$ is a $\mathscr{P}$-position.

WHAT SHALL I DO WHEN g IS TWO?

Example: {8,10,22}

Apparently we have to examine infinitely many possible replies. Fortunately there is a way of doing this in a finite time. A similar method will work for *any* position with $g=2$.

Let's see how the position will look after some play from {8,10,22}. If 1, 2 or 3 has been played, we know what to do. Otherwise the only even numbers that can have been played are

4, 6, 12, 14

and the even part of the position must look like one of

$$\boxed{\begin{array}{c}\{4,6\}\ \{4,10\}\ \{6,8,10\}\\ \{8,10,12,14\}\ \{8,10,12\}\ \{8,10,14\}\ \{8,10,22\}\end{array}}$$

What odd numbers have been played? If the least of them is n, then since $\{8,10,22\}$ excludes all of

$$16, 18, 20, 22, 24, \ldots$$

we can suppose that the only relevant odd numbers are among

$$n, n+2, n+4, n+6, n+12, n+14.$$

And if any even moves have been made they will restrict the possibilities still further. For instance if 6 has been played we can suppose that the odd numbers are one of the sets

$$n, n+2, n+4 \qquad n, n+2 \qquad n, n+4 \qquad n$$

Table 4 shows the status of the positions classified in this way. Since the last four columns repeat indefinitely, this finite table contains the information for every odd n. How was it computed and why is it periodic?

Let's take a typical entry:

$$\{8,10,14,n,n+6,n+12\}.$$

From this position there are three kinds of option:

(a) the *even* numbers 4, 6 or 12,
(b) the *small odd* numbers $m \leqslant n-14$,
(c) the *large odd* numbers $n-12, n-10, n-8, n-6, n-4, n-2, n+2, n+4$.

Case (a) leads to a position (in an earlier segment of the table) with even part

$$\{4,10\}, \{6,8,10\} \text{ or } \{8,10,12,14\}$$

and we can suppose that these have already been analyzed and found to be ultimately periodic in n.

A case (b) move leads to

$$\{8,10,14,m\}$$

since m excludes n and all larger odd numbers. If there is any odd m for which this is a $\mathscr{P}$-position, then $\{8, 10, 14, n, n+6, n+12\}$ will be an $\mathscr{N}$-position for all $n \geqslant m+14$. If not, we can reject moves in case (b).

Finally, case (c) moves either leave n unchanged or decrease it by at most 12. We conclude that the outcome of every position in the table is computed in a fixed way from

ultimately periodic information (case (a)),
ultimately constant information (case (b)), and
information in the last few columns (case (c));

it must therefore be ultimately periodic in n.

Position		5	7	9	11	13	15	17	19	21	23	25	27	29	31	33	35	37	39	41	43	45	47	49	51
{4,6} (P)	$n, n+2$	P	–	P	–	P	–	P	–	P	–	P	–	P	–	P	–	P	–	P	–	P	–	P	–
	n	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
{4,10} (N)	$n, n+2$	–	P	–	–	–	P	–	–	–	P	–	–	–	P	–	–	–	P	–	–	–	P	–	–
	$n, n+6$	P	–	–	–	P	–	–	–	P	–	–	–	P	–	–	–	P	–	–	–	P	–	–	–
	n	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
{6,8,10} (N)	$n, n+2, n+4$	–	P	–	P	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
	$n, n+2$	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
	$n, n+4$	P	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
	n	–	P	–	P	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
{8,10,12,14} (P)	$n, n+2, n+4, n+6$	P	–	P	–	P	–	P	–	P	–	P	–	P	–	P	–	P	–	P	–	P	–	P	–
	$n, n+2, n+4$	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
	$n, n+2, n+6$	–	–	–	–	–	–	–	P	–	–	–	P	–	–	–	P	–	–	–	P	–	–	–	P
	$n, n+2$	P	–	P	–	P	–	P	–	P	–	P	–	P	–	P	–	P	–	P	–	P	–	P	–
	$n, n+4, n+6$	–	P	–	P	–	–	–	P	–	–	–	P	–	–	–	P	–	–	–	P	–	–	–	P
	$n, n+4$	–	–	P	–	–	–	P	–	–	–	P	–	–	–	P	–	–	–	P	–	–	–	P	–
	$n, n+6$	–	–	P	–	–	–	P	–	–	–	P	–	–	–	P	–	–	–	P	–	–	–	P	–
	n	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
{8,10,12} (N)	$n, n+2, n+4, n+6$	P	–	–	–	–	–	–	–	–	–	–	–	–	–	–	P	–	–	–	P	–	–	–	P
	$n, n+2, n+4$	–	–	P	–	–	–	P	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
	$n, n+2, n+6$	–	–	P	–	–	–	–	–	–	–	–	–	–	–	–	–	P	–	–	–	P	–	–	–
	$n, n+2$	P	–	–	–	–	–	–	–	–	–	–	–	–	–	–	P	–	–	–	P	–	–	–	P
	$n, n+4, n+6$	–	P	P	–	–	P	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
	$n, n+4$	–	–	–	–	P	–	–	–	P	–	–	P	–	–	–	P	P	–	–	P	P	–	–	P
	$n, n+6$	–	–	–	–	–	–	–	P	–	–	–	P	–	–	–	P	–	–	–	P	–	–	–	P
	$n, n+14$	–	–	P	–	–	P	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
	n	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
{8,10,14} (N)	$n, n+2, n+4, n+6$	P	–	–	–	–	–	–	–	–	–	–	–	–	–	–	P	–	–	–	–	–	–	–	–
	$n, n+2, n+4$	–	–	P	–	–	–	–	–	–	–	–	–	P	–	P	–	–	–	–		–	–	–	–
	$n, n+2, n+6$	–	–	P	–	–	–	–	–	–	–	–	–	P	–	P	–	–	–	–	–	–	–	–	–
	$n, n+2$	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
	$n, n+4, n+6$	–	–	P	–	–	P	–	–	–	–	–	–	–	–	P	–	–	P	–	–	–	–	–	–
	$n, n+4$	–	P	–	–	–	–	–	P	–	–	–	–	–	–	–	–	–	–	–	P	–	–	–	–
	$n, n+6, n+12$	–	–	–	–	–	–	–	P	–	–	–	–	–	P	–	P	–	–	–	–	–	–	–	–
	$n, n+6$	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
	$n, n+12$	P	–	P	–	P	P	–	–	–	–	–	–	P	–	P	–	P	–	–	–	–	–	–	–
	n	–	–	–	–	–	–	–	–	–	–	–	–	–	P	–	P	–	–	–	–	–	–	–	–
{8,10,22} (P)	$n, n+2, n+4, n+6$	P	–	P	–	P	–	P	–	P	–	P	–	P	–	P	–	P	–	P	–	P	–	P	–
	$n, n+2, n+4$	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
	$n, n+2, n+6$	–	–	–	P	–	–	–	–	–	–	–	P	–	–	–	P	–	–	–	P	–	–	–	P
	$n, n+2, n+14$	–	–	P	–	P	–	P	–	P	–	P	–	P	–	P	–	P	–	P	–	P	–	P	–
	$n, n+2$	P	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
	$n, n+4, n+6$	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
	$n, n+4$	–	P	P	–	–	P	P	–	–	P	P	–	–	P	P	–	–	P	P	–	–	P	P	–
	$n, n+6, n+12$	–	–	P	–	–	–	P	–	–	–	P	–	–	–	P	–	–	–	P	–	–	–	P	–
	$n, n+6$	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
	$n, n+12, n+14$	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
	$n, n+12$	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
	$n, n+14$	P	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
	n	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–

Table 4. The Position {8,10,22}.

Every position with $g = 2$ can be handled in this way. When we have computed enough to verify the period, we can decide in particular whether there is any good reply. For $\{8,10,22\}$ there isn't one, so it is a $\mathscr{P}$-position.

THE GREAT UNKNOWN

We can best describe our knowledge in terms of the number g. When

$$\boxed{g = 1}$$

the position is bounded so you can find what to do by working through all positions. Of course this might take a long time even if one of our theorems already tells you the outcome. We know that there must a be good reply to $\{31,37\}$ but don't know any method which guarantees to find one in the next millenium. When

$$\boxed{g = 2}$$

the method we have just described will compute the outcome in a finite but probably even longer time. If

$$\boxed{g \text{ is divisible by a prime } p \geqslant 5}$$

then p is a good reply when it hasn't already been named, when of course there isn't any.

The authors have only been able to examine a few particular positions with other values of g. Table 8 in the Extras contains a complete discussion of $\{6,9\}$. Although this is a two-dimensional table, a periodicity develops which enables us to analyze the position to infinity. Maybe a similar thing happens for some other positions with $g = 3$. We computed a much larger three-dimensional table for $\{8, 12\}$ ($g = 4$), but could detect no structure outside the range covered by our explicit strategy.

16 is the first opening move whose status is in doubt. We don't know whether $\{16\}$ has a good reply nor even any way of finding out in any finite time. You might consider working upwards testing each possible reply in turn and hoping to detect some structure, but even this is impossible. We don't know any way to test the reply 24, say, in any finite time. We don't even know how to test 100, say, as a possible reply to $\{16,24\}$!

The quiet end theorem often eliminates infinitely many replies, for example all odd replies to $\{16\}$ or to $\{16,24\}$, but it never eliminates any reply that would be infinitely hard to analyze.

	{7,9,11}	{7,9}	{7,11}	{7}	{9,11}	{9}	{11}	{}
{6,8,10}	[]	[11]	[11]	[]	[4,5,7]	[5]	[]	[4,7,11]
{6,8}	[10]	[]	[]	[9,10,11]	[4,5]	[5,7]	[7,10]	[4]
{6,10}	[8]	[]	[15]	[8,9]	[4]	[7]	[8,13]	[4]
{6}	[]	[8,10,11]	[8,9]	[16]	[4,7]	[]	[26]	[4,9]
{8,10,12}	[4,5,6]	[4]	[13]	[5,6]	[13,14,15]	[23]	[6]	[14]
{8,12}	[5]	[6]	[6]	[5]	[]	[11,15]	[9,13]	[]
{10,12}	[4]	[4,6]	[16]	[]	[]	[11,13,14]	[9	[7
{12}	[6]	[15]	[27]	[10	[8,10]	[6]	[	[8
{8,10}	[4,5,6]	[4]	[]	[5,6,11]	[23]	[13,15]	[6,7]	[22]
{8}	[5]	[6]	[6,10]	[5]	[12]	[21	[	[12
{10}	[4]	[4,6]	[8]	[12]	[12]	[	[	[5
{}	[6]	[19,24]	[24,34]	[]	[	[6]	[]	[5,7,11,13,...

Table 5. Status of Subsets of {6,7,8,9,10,11,12} and Known Good Replies.

Even members of set at Left; Odd members at head. Bracket is closed when *all* good replies are known, so that [] indicates a $\mathscr{P}$-position, but [always indicates an $\mathscr{N}$-position for which no good reply is known. The last entry contains all primes greater than 3; and *may* contain some entries 2^a3^b.

Table 5 tells you the outcome and all the good replies we know to every position made from the numbers

6, 7, 8, 9, 10, 11, 12

(if 4 or 5 is involved, Tables 2, 3, 1 and Fig. 8 go much further). If you can add any more to this table or decide whether any number 2^a3^b is a good opening move we would like to hear from you.

ARE OUTCOMES COMPUTABLE?

We can prove that there *must be* a way of programming a computer to find the outcome of $\{n\}$ even though we don't know what that way *is*! The reason is:

There can only be finitely many good opening moves 2^a3^b.

For no one of these can divide any other, so that no two can have the same value of a or the same value of b. So if $2^{a_0}3^{b_0}$ is such a number with a_0 as small as possible, and 2^a3^b is any other, then we must have $b < b_0$ and so there are at most $b_0 + 1$ such numbers, say $n_1, n_2, \ldots, n_k$. We suspect there are none!

If you only knew what these numbers were, then you could program your machine with PORN (Fig. 10) and work out the outcome of any $\{n\}$. This argument shows that in the purely technical sense this is a computable function of n, even though we don't know what function it is.

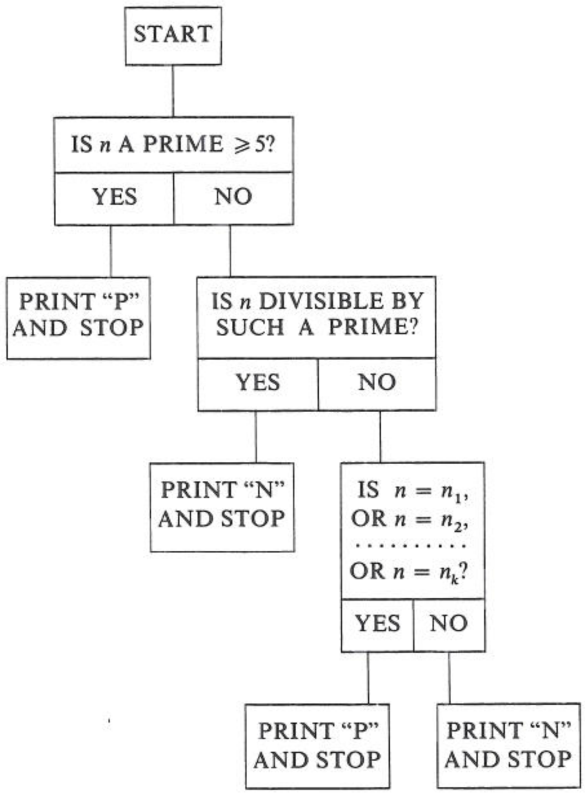


Figure 10. PORN, A Program Which Decides if $\{n\}$ is $\mathscr{P}$ or $\mathscr{N}$.

THE ETIQUETTE OF SYLVER COINAGE

Few Western readers can understand the subtleties of etiquette in the oriental country from which our game comes. But at least we can save you from the more obvious gaffes by pointing out that in Sylver Coinage it is customary for a player who knows he is winning to resign by naming 1, 2 or 3. This quaint custom is said to originate in the tradition that Hi, who could see much further than Lo, nobly took upon himself the fate that was about to befall his beloved brother.

When it's plain to all the world that you have a win, any move but 1 will insult your opponent, but in other cases we advise you to name 3 (2 is possible, but may be misunderstood). If your opponent concurs in your analysis, he will respond with 2, but you have allowed him to express another opinion by naming 1. (Replies to 3 other than 1 or 2 may also be available but their nuances are harder to interpret.)

Of course, one of the greatest insults you can offer is to name 1, 2 or 3 at the very start of the game, for this is the philosopher Hu Tchings' prerogative, at least until someone finds a new way to win.

EXTRAS

CHOMP

Here is a game with similar rules to Sylver Coinage. For some fixed number N, the players alternately name divisors of N which may not be multiples of previously named numbers. Whoever names 1 loses. If $N=432=2^4 3^3$, for example, a move is essentially to eat a square (e.g. 36) from the chocolate bar in Fig. 11, together with all squares below and/or to the right of it. Square number 1 is poisoned!

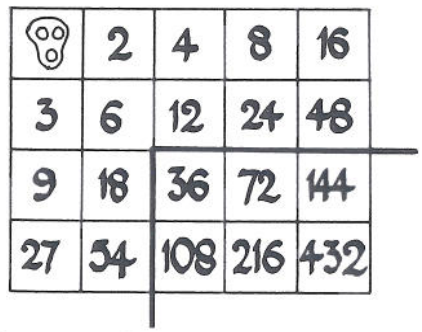


Figure 11. Chomping at a Chocolate Bar.

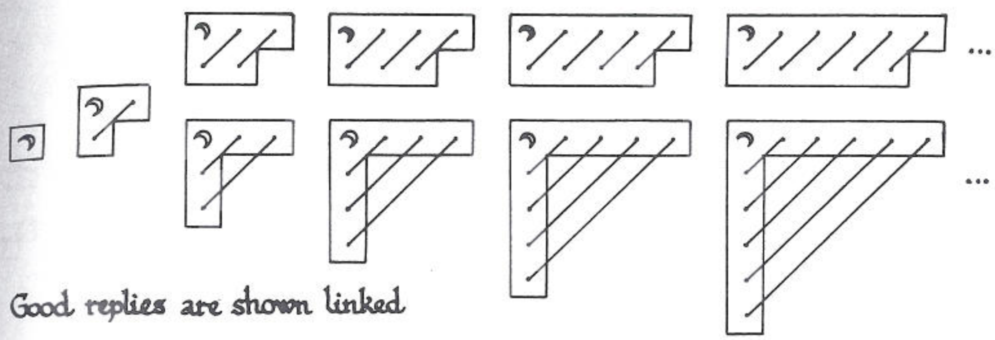
The first few $\mathscr{P}$-positions are shown in Fig. 12. Strategy stealing shows that rectangles larger than 1×1 are $\mathscr{N}$-positions; the replies are unique if either side is at most 3, but Ken Thompson found that 4×5 and 5×2 bites both answer 8×10.

The arithmetic form of the game is due to Fred. Schuh, the geometric one to David Gale.

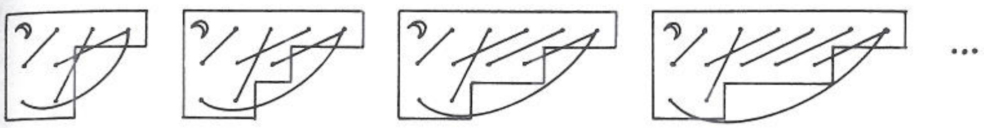
ZIG-ZAG

Two players alternately name distinct numbers (which are allowed to be fractional or negative) and the game ends as soon as the resulting sequence contains either an increasing subsequence (**zig**) of length a or a decreasing one (**zag**) of length b. The normal play $a+1$, $b+1$ game is really the same as the misère a, b game, and so we consider only the latter.

Zig-Zag, which was suggested to us by S. Fajtlowicz, sounds difficult to analyze, but fortunately there is a rather clever transformation into a geometrical game like Chomp. We regard square (r, s) in Fig. 13 as eaten if the number sequence so far contains a rising zig of length r and a sagging zag of length s that end with the same number. Then the moves are as in Chomp except that the first move may eat square (1,1) only, and the innermost square eaten on any subsequent move must be adjacent to a previously eaten square. The squares $(a,1)$ and $(1,b)$ are poisoned, so play really goes on inside the outlined $a-1$ by $b-1$ **chocolate bar** of Fig. 13.

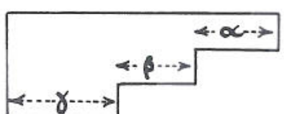


Good replies are shown linked



β \ γ	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
0	1	2	2	3	4	5	5	6	7	7	8	9	9	11	11	12	12	13	14	15	15	16	17	17	18	19	19
1	1	0	2	3	4	3	5	6	5	7	8	7	10	9	11	10	12	13	14	13	15	16	15	17	18	17	20
2				0	4	4	5	3	6	3	8	9	8	7	7	11	8	10	11	14	15	13	16	17	18	19	18
3					0	4	0	5	6	6	8	3	9	9	10	11	12	12	8	14	10	15	16	12	18	13	19
4								5	0	6	0	8	3	9	10	11	10	12	13	8	14	8	10	16	10	18	13
5												8	0	9	3	3	11	12	11	13	14	15	15	8	17	8	18
6												8		0	9	10	11	3	12	13	14	13	15	16	17	17	18
7												3			9	0	0	11	12	3	14	14	15	16	10	13	8
8												7						11	0	12	3	14	3	13	16	16	17
9												7								12	0	14	14	15	3	16	3
10																						0	14	0	15	16	16
11																									15	0	16

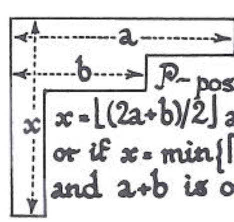
Values of α for which



is a $\mathcal{P}$-position

a \ b	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	$\mathcal{P}$	1×1	1×2	1×3	1×4	1×5	1×6	1×7	1×8	1×9	1×10	1×11	1×12	1×13	1×14	1×15
2	1×1	1×1	1×1	1×1	1×1	1×1	1×1	1×1	1×1	1×1	1×1	1×1	1×1	1×1	1×1	1×1
3	2×1	1×1	2×2	2×2	1×2	2×3	1×3	2×4	1×3	2×5	2×5	1×4	2×6	1×4	2×7	2×7
4	3×1	1×1	2×2	3×3	2×3	3×4	2×4	3×5								
5	4×1	1×1	2×1	3×2	4×4	3×3	4×5									
6	5×1	1×1	3×2	4×3	3×3	5×5										
7	6×1	1×1	3×1	4×2	5×4		6×6									
8	7×1	1×1	4×2	5×3				7×7		{4×5, 5×2}						

Winning bites from a×b rectangles



is a $\mathcal{P}$-position if $x = \lfloor (2a+b)/2 \rfloor$ and $a+b$ is even or if $x = \min\{\lceil (2a-b)/2 \rceil, \lceil 3(a-b)/2 \rceil\}$ and $a+b$ is odd.

Figure 12. $\mathcal{P}$-positions in Chomp.

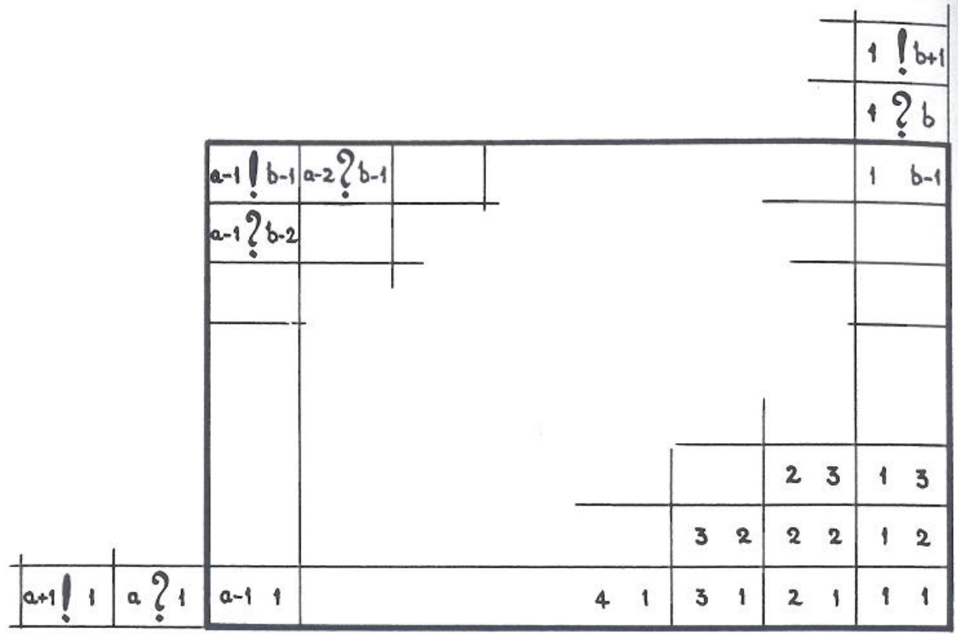


Figure 13. The Chocolate Bar Form of Zig-Zag.

If $a \geqslant 3$, $b \geqslant 3$ and $a+b \leqslant 17$, the first player wins the misère a-Zig, b-Zag game, because David Seal's calculations show that the corresponding $a-1$ by $b-1$ chocolate bars are $\mathscr{N}$-positions.

By assigning Heads to Horizontal edges and Tails to verTical ones we get an equivalent game with coin sequences, involving moves of a head rightwards over tails or a tail leftwards over heads, and Seal used this idea to compute Fig. 14 showing all $\mathscr{P}$- positions for which the uneaten part of the chocolate bar fits inside a 5×5 square.

To find $\mathscr{P}$-positions in both Chomp and Zig-Zag, we used the tabular technique of Chapter 15, and the Clique Technique of this one.

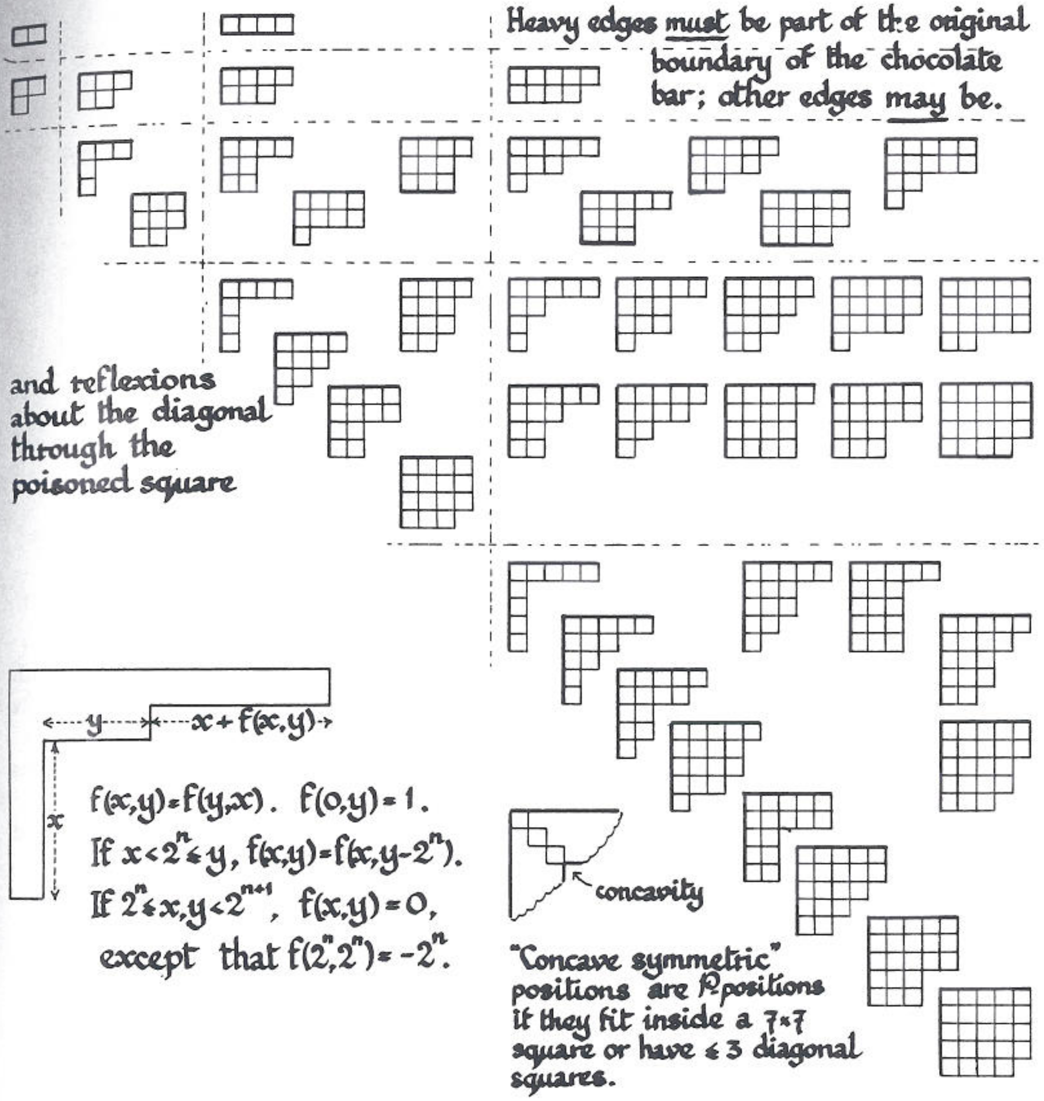


Figure 14. $\mathscr{P}$-positions for Zig-Zag.

MORE CLIQUES FOR SYLVER COINAGE

To follow the cliques in Table 6, we advise you to set out the remaining numbers as we did in Figs. 2, 3, 4 for the cases {4,5}, {6,7} and {7,10,12}. Numbers not mentioned are excluded by a good reply.

position	replies	strategy, with cliques indicated by]
{4,5}	11!	1?](2,3)](6,7)]11!
{4,7}	13!	1?](2,3)](5,6)](9,10)]13!
{4,9}	19!	1?](2,3)](5,11)(6,11)(7,10)](14,15)]19!
{5,6}	19!	1?](2,3)](4,7)](8,9)](13,14)]19!
{5,7}	8!	1?](2,3)](4,6)(9,13)(11,13)8!
{5,8}	7!	1?](2,3)](4,11)(6,9)7!
{5,9}	31!	1?](2,3)](4,11)(6,8)(7,13)](12,16)](17,21)](22,26)]31!
{6,7}	16!	1?](2,3)](4,5)](8,9)(8,10)(8,11)(9,10)(9,11)](15,23)(17,22)16!
{7,8}	5!	1?](2,3)](4,13)(6,9)(6,10)(6,11)5!
{7,9}	19!24!	1?](2,3)](4,10)(5,13)(6,8)(6,10)(6,11)](12,15)(17,20)(22,26)(29,33)19!24!
		after {7,9,22,26} (12,17)(19,24)(15,20)
		{7,9,29,33} (12,15)(17,20)(19,31)(22,26)(24,26) …
		{7,9,19} (12,15)(15,17)(15,20)](22,24)(29,31)
		{7,9,24} (12,15)(17,20)(19,22)](26,29)

Table 6. Some Complete Strategies for Sylver Coinage.

5-PAIRS

The safe combinations $\{5,x,y\}$ are of three types. In the top drawer in Fig. 8 are those with y so much larger than x that the coordinates $\{a,b,c,d\}$ are $\{x,2x,3x,y\}$. For these it seems that y/x tends to 3. But are there infinitely many numbers in the top drawer?

The middle drawer contains the remaining ones for which $x+y$ is a multiple of 5. It seems that for these, x and y always differ by 1 or 2.

In the bottom drawer we have arranged the pairs with coordinates $\{x, y, x+y, 2y\}$ where x and y are in the order given. It seems that here, as in the second drawer, y/x tends to 1.

POSITIONS CONTAINING 6

As in our other analyses, we write the numbers in six columns and circle 0 and five other numbers a,b,c,d,e, one in each of the 1-, 2-, 3-, 4- and 5-columns respectively. We tabulate $\mathscr{P}$-positions by entries c in a 4-dimensional table (Table 7) whose coordinates, a,b,d,e are congruent to 1,2,4,5, modulo 6.

Entries outside the areas enclosed by full lines are found by repeating entries according to the arrows, where appropriate. The tables for $b=8$, $d=4$ and $b=8$, $d=16$ can be extended indefinitely by repeating the portions between the pecked lines and increasing all entries by 12 or 60 respectively. The two tables with $d=10$, $b=8$ or 14 contain no further entries. All further entries in that for $b=14$, $d=16$ are **15**.

Entries are $c \equiv 3$, modulo 6.
Bold type indicates that the row or column heading is redundant.
Entries repeat indefinitely according to the arrows.

a	$e = 5$	11	17	23	29	35	41
7	9+						
13		**9**	**15** →				
19		**9**	**21**+				
25		↓		**21**	**27** →		
31				**21**	**33**+		
37				↓		**33**	**39** →
43						**33**	**45**+
						↓	

$b = 8$, $d = 4$.

a	$e = 5$	11	17	23
7		9	**15**+ →	
13	**9**	15		
19	↓	**21**+		
		↓		

$b = 8$, $d = 10$.

a	$e = 5$	11	17	23	29	35
7		**15**	**9** →			
13	**15**+					
19			15	21	**27** →	
25			21	**15**	33	**39**+ →
31			**27**+	**15**	**21**	
37			↓	**15**	**21**	
43				↓	**21**	
					↓	

$b = 14$, $d = 10$,

a	$e = 5$	11	17	23	29	35	41	47	53	59	65	71	77	83	89	95	101
7		**15**+	**9**	**9** →													
13	**9**		**21**+														
19	↓		**15**	**21**	**27**+												
25				**15**	**15**	**15**	**15** →										
31				**27**	**21**	**21**	**21**	**21** →									
37				↓		**39**	**33**	**33**	**33** →								
43						**45**	**39**	**51**+									
49						↓	**51**	**39**	**39**	**39**	**39** →						
55							↓			**57**	**63**+						
61									**57**	**63**		**69**+					
67									↓	**69**		**63**	**63**	**63** →			
73										↓			**81**+				
79													**75**	**81**	**87**+		
85														**75**	**75**	**75**	**75** →

$b = 8$, $d = 16$.

a	$e = 5$	11	17	23	29	35	41	47
7		**9**	**15**					
13	**15**+		9					
19		15		**9**	**9**	**9** →		
25		**21**	27	33	39+			
31		↓		39+	**15**	**15**	**15**	**15** →
37					**15**	**15**	**15**	**15**
43					**15**	**15**	**15**	**15**
49					↓	**15**	**15**	**15**
55						↓	**15**	**15**
61							↓	**15**
								↓

$b = 14$, $d = 16$.

Table 7. $\mathscr{P}$-Positions Containing 6,

	5,8 5	8,11 8	11,4 11	14,17 14	17,20 17	20,23 20	23,26 23	26,29 26	29,32 29	32,35 32	35,38 35	38,41 38	41,44 41	44,47 44	47,50 47	50,53 50	53,56 53	56,59 56	59,62 59	62,65 62	65,68 65	68,71 68	71,74 71
4,7 4	+ + − −	− − + −	− − + +	− − − −	− − − −	− − − −	− − − −	− − − −	− − − −	− − − −	− − − −	− − − −	− − − −	− − − −	− − − −	− − − −	− − − −	− − − −	− − − −	− − − −	− − − −	− − − −	− − − −
7,10 7	− − − −	+ − − +	− − + +	+ + − −	+ + − −	+ + − −	+ + − −	+ + − −	+ + − −	+ + − −	+ + − −	+ + − −	+ + − −	+ + − −	+ + − −	+ + − −	+ + − −	+ + − −	+ + − −	+ + − −	+ + − −	+ + − −	+ + − −
10,13 10	+ − + −	− − − −	− − − −	− − − −	− − − −	− − − −	− − − −	− − − −	− − − −	− − − −	− − − −	− − − −	− − − −	− − − −	− − − −	− − − −	− − − −	− − − −	− − − −	− − − −	− − − −	− − − −	− − − −
13,16 13	+ − + −	− − − −	− − − −	+ − − +	− − + −	+ − − −	− − − −	+ + − −	+ + − −	+ + − −	+ + − −	+ + − −	+ + − −	+ + − −	+ + − −	+ + − −	+ + − −	+ + − −	+ + − −	+ + − −	+ + − −	+ + − −	+ + − −
16,19 16	+ − + −	− − − −	− − − −	− + − −	+ − − +	− − − +	− − − −	− − − −	− − − −	− − − −	− − − −	− − − −	− − − −	− − − −	− − − −	− − − −	− − − −	− − − −	− − − −	− − − −	− − − −	− − − −	− − − −
19,22 19	+ − + −	− − − −	− − − −	+ − − −	− − − +	+ − − +	− − − −	+ − − −	− − − −	+ − − −	− − − −	+ + − −	+ + − −	+ + − −	+ + − −	+ + − −	+ + − −	+ + − −	+ + − −	+ + − −	+ + − −	+ + − −	+ + − −
22,25 22	+ − + −	− − − −	− − − −	− − − −	− − − −	− − − −	+ − − +	− − − +	+ − − −	− − − −	+ − − −	− − − −	− − − −	− − − −	− − − −	− − − −	− − − −	− − − −	− − − −	− − − −	− − − −	− − − −	− − − −
25,28 25	+ − + −	− − − −	− − − −	+ − + −	− − − −	+ − − −	− − − +	+ − − +	− − − −	+ − − −	− − − −	+ − − −	− − − −	+ − − −	− − − −	+ + − −	+ + − −	+ + − −	+ + − −	+ + − −	+ + − −	+ + − −	+ + − +
28,31 28	+ − + −	− − − −	− − − −	− − − −	− − − −	− − − −	+ − − −	− − − −	+ − − +	− − − +	+ − − −	− − − −	+ − − −	− − − −	+ − − −	− − − −	− − − −	− − − −	− − − −	− − − −	− − − −	− − − −	− − − −
31,34 31	+ − + −	− − − −	− − − −	− − − −	− − − −	+ − − −	− − − −	+ − − −	− − − +	+ − − +	− − − −	+ − − −	− − − −	+ − − −	− − − −	+ − − −	− − − −	+ − − −	− − − −	+ + − −	+ + − −	+ + − −	+ + − −

34,37	+ −	− −	− −	− −	− −	− −	+ −	− −	+ −	− −	+ −	− −	+ −	− −	+ −	− −	+ −	− −	+ −	− −	= −	− −	− −
34	+ −	− −	− −	− −	− −	− −	− −	− −	− −	− −	− +	− +	− −	− −	− −	− −	− −	− −	− −	− −	− −	− −	− −
37,40	+ −	− −	− −	− −	− −	+ −	− −	+ −	− −	+ −	− −	+ −	− −	+ −	− −	+ −	− −	+ −	− −	+ −	− −	+ −	− −
37	+ −	− −	− −	− −	− −	+ −	− −	− −	− −	− −	− +	− +	− −	− −	− −	− −	− −	− −	− −	− −	− −	− −	− −
40,43	+ −	− −	− −	− −	− −	− −	+ −	− −	+ −	− −	+ −	− −	+ −	− −	+ −	− −	+ −	− −	+ −	− −	+ −	− −	+ −
40	+ −	− −	− −	− −	− −	− −	− −	− −	− −	− −	− −	− −	− +	− +	− −	− −	− −	− −	− −	− −	− −	− −	− −
43,46	+ −	− −	− −	− −	− −	− −	− −	+ −	− −	+ −	− −	+ −	− −	+ −	− −	+ −	− −	+ −	− −	+ −	− −	+ −	
43	+ −	− −	− −	− −	− −	− −	− −	− −	− −	− −	− −	− −	− +	− +	− −	− −	− −	− −	− −	− −	− −	− −	
46,49	+ −	− −	− −	− −	− −	− −	+ −	− −	+ −	− −	+ −	− −	+ −	− −	+ −	− −	+ −	− −	+ −	− −	+ −		
46	+ −	− −	− −	− −	− −	− −	+ −	− −	− −	− −	− −	− −	− −	− −	− +	− +	− −	− −	− −	− −	− −		
49,52	+ −	− −	− −	− −	− −	− −	+ −	= −	− −	+ −	− −	+ −	− −	+ −	− −	+ −	− −	+ −	− −	+ −			
49	+ −	− −	− −	− −	− −	− −	+ −	− −	− −	− −	− −	− −	− −	− −	− +	− +	− −	− −	− −	− −			
52,55	+ −	− −	− −	− −	− −	− −	+ −	− −	+ −	− −	+ −	− −	+ −	− −	+ −	− −	+ −	− −	+ −				
52	+ −	− −	− −	− −	− −	− −	+ −	− −	− −	− −	− −	− −	− −	− −	− −	− −	− +	− +	− −				
55,58	+ −	− −	− −	− −	− −	− −	+ −	− −	− −	+ −	− −	+ −	− −	+ −	− −	+ −	− −	+ −					
55	+ −	− −	− −	− −	− −	− −	+ −	− −	− −	− −	− −	− −	− −	− −	− −	− −	− +	− +					
58,61	+ −	− −	− −	− −	− −	− −	+ −	− −	+ −	− −	+ −	− −	+ −	− −	+ −	− −	+ −						
58	+ −	− −	− −	− −	− −	− −	+ −	− −	+ −	− −	− −	− −	− −	− −	− −	− −	− −						
61,64	+ −	− −	− −	− −	− −	− −	+ −	− −	+ −	= −	− −	+ −	− −	+ −	− −	+ −							
61	+ −	− −	− −	− −	− −	− −	+ −	− −	+ −	− −	− −	− −	− −	− −	− −	− −							
64,67	+ −	− −	− −	− −	− −	− −	+ −	− −	+ −	− −	+ −	− −	+ −	− −	+ −								
64	+ −	− −	− −	− −	− −	− −	+ −	− −	+ −	− −	− −	− −	− −	− −	− −								

Table 8. A Complete Discussion of $\{6,9\}$.

The table represents a complete discussion of positions containing $\{6,9\}$. Reduced positions contain one, or possibly two neighboring, numbers of form $3k+1$, and of form $3k+2$; pairs of rows and columns refer to the latter and former possibilities. Positions represented by cells outside the crenellated line are not reduced. The pattern within the rectangular quadrant continues indefinitely. The minus signs denote $\mathscr{N}$-positions, the plus signs $\mathscr{P}$-positions and the "=" signs $\mathscr{N}$-positions which, at a casual glance at the pattern, might be mistaken for $\mathscr{P}$-positions.

SYLVER COINAGE HAS INFINITE NIM-VALUES

If we make naming 1 an *illegal* rather than a *stupid* move, Sylver Coinage becomes a normal play rather than a misère play game and we could consider adding it to other games using the Sprague–Grundy theory. However since some positions have infinitely many options, we can expect infinite nim-values and indeed they happen!

For example, $\mathscr{G}(2,2n+3)=n$ $(n\geqslant 0)$, so $\mathscr{G}(2)=\omega$. On the other hand, $\mathscr{G}(3,3n+1,3n+2)=1$ $(n\geqslant 1)$, so $\mathscr{G}(3)=1$. Here are some other nim-values:

$k =$	4	5	6	7	8	9	10	11	13	14	15	16	17	19	20	22	23	25	26	28	29	31	32
$\mathscr{G}(3,k) =$	2	3	1	4	6	1	7	8	9	11	1	12	14	15	15	17	19	20	21	23	24	26	27
$\mathscr{G}(4,k) =$	?	3	0	5	?	8	1	9	14	4	15	?	16	19	?								

$$\mathscr{G}(4,6,4n-1,4n+1)=1 \qquad \mathscr{G}(4,6,4n+1,4n+3)=0 \qquad (n\geqslant 1)$$

$$\mathscr{G}(5,6)=7,\ \mathscr{G}(5,7)=8,\ \mathscr{G}(5,8)=10, \qquad \mathscr{G}(6,7)=9, \qquad \mathscr{G}(3,3n-1,3n+1)=5\ (n\geqslant 6),$$

$$\mathscr{G}(3,9n-8,9n-4)=\mathscr{G}(3,9n+2,9n+7)=\mathscr{G}(3,9n+8,9n+13)=10 \qquad (n\geqslant 3).$$

A FEW FINAL QUESTIONS

Is there any effective technique for computing the outcome and all good replies for the *general position*?

If the game is played "between intelligent players", is the first person to make the game bounded the loser?

Is there a winning strategy of bounded length?

Is there an $\mathscr{N}$-position with $g>1$ for which *all* good replies lead to positions with $g=1$?

Is $\mathscr{G}(4)=\omega+1$? or is $\mathscr{G}(4)=6$, say?

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