THREE DIMES OF TOPOLOGY

A. Candel

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NOTE: These notes are being offered without warranty. Bibliography and references have not been collected. Chapter X does not exist.

CHAPTER I

TOPOLOGICAL SPACES

1. The Concept of Topological Space

Definition. A topological space is a pair $(X, \mathcal{O})$ consisting of a set X and a collection $\mathcal{O}$ of subsets of X (called "open sets"), such that the following axioms hold:

(1) Any union of open sets is an open set.
(2) The intersection of any two open sets is open.
(3) Both $\emptyset$ and X are open sets.

One also says that $\mathcal{O}$ is the topology of the topological space $(X, \mathcal{O})$, and usually we will drop $\mathcal{O}$ and speak of a topological space X.

Definition. Let X be a topological space.

(1) A subset F of X is called closed if $X \setminus F$ is open.
(2) A subset N of X is called a neighborhood of $x \in X$ if there is an open set U such that $x \in U \subset N$.
(3) Let Y be a subset of X. A point x in X is called an interior, exterior or boundary point of Y if Y, $X \setminus Y$ or neither is a neighborhood of X.
(4) The set Y° of the interior points of Y is called the interior of Y.
(5) The set Y^- of points of X which are not exterior points of Y is called the closure of Y.

Exercise. The interior (closure) of a set is the largest open set (smallest closed set) contained in it (which contains it).

The duality open-closed allows us to define a topology in terms of closed sets. The axioms are obtained from the ones above by means of Morgan's laws.

Axioms for closed sets. A topological space is a pair $(X, \mathcal{C})$ consisting of a set X and a family of subsets of X (called "closed sets") such that

(1) Any intersection of closed sets is closed.
(2) The union of any two closed sets is closed.
(3) The empty set and X are closed sets.

Originally the notion of topology was defined in terms of neighborhoods.

Axioms for Neighborhood. A topological space is a pair $(X, \mathfrak{N})$ consisting of a set X and a family $\mathfrak{N} = \{\mathfrak{N}_x\}_{x \in X}$ of sets $\mathfrak{N}_x$ of subsets of X (called "neighborhoods") such that:

(1) Each neighborhood of x contains x, and X is a neighborhood of each of its points.
(2) If $N \subset X$ contains a neighborhood of x, then N itself is a neighborhood of x.
(3) The intersection of two neighborhoods of x is a neighborhood of x.
(4) Each neighborhood of x contains a neighborhood of x which is also a neighborhood of each of its points.

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Kuratowski Closure axioms. A topological space is a pair $(X, ^-)$ consisting of a set X and a map $^- : \mathcal{P}X \to \mathcal{P}X$ from parts of X into itself such that:

(1) $\emptyset^- = \emptyset$.
(2) $A \subset A^-$ for all $A \subset X$.
(3) $A^{--} = A^-$.
(4) $(A \cup B)^- = A^- \cup B^-$.

Exercise. Formulate what the equivalence of this notions means and prove it.

Exercise. Kuratowski closure axioms can be replaced by only two: (1) $\emptyset^- = \emptyset$ and (2) $A \cup A^- \cup B^{--} \subset (A \cup B)^-$.

2. Bases and subbases

In several of the examples that we will discuss we see that we define open sets as union of a smaller collection of open sets. This is an important concept.

Base. Let X be a topological space. A collection $\mathfrak{B}$ of open sets is called a basis for the topology if every open set can be written as a union of sets in $\mathfrak{B}$.

A related concept is the following.

Subbase. Let X be a topological space. A collection $\mathfrak{S}$ of open sets is a subbasis for the topology if every open sets is a union of finite intersections of elements of $\mathfrak{S}$.

The collection of all open sets of a topological space is both a base and a subbase. Of course one is usually interested in smaller families. The importance of this definitions is the following.

Proposition. *Let X be a set and let $\mathfrak{B}$ be a collection of subsets of X whose union is X and such that for any B, B' in $\mathfrak{B}$ and $x \in B \cap B'$ there is $B'' \in \mathfrak{B}$ such that $x \in B'' \subset B \cap B'$. Then there is exactly one topology $\mathcal{O}(\mathfrak{B})$ on X for which $\mathfrak{B}$ is a basis.*

Proposition. *Let X be a set and let $\mathfrak{S}$ be an arbitrary collection of subsets of X. Then there is exactly one topology $\mathcal{O}(\mathfrak{S})$ on X for which $\mathfrak{S}$ is a subbasis.*

The only property that requires a comment is that $\emptyset$ and X are in $\mathcal{O}(\mathfrak{S})$. This hold by using the natural convention that the intersection of an empty family of sets is the whole space, the union of an empty family of sets is the empty set. One has to read the meaning of $x \in \cap_{i \in I} B_i$ as $x \in B_i$ for every $i \in I$, and of $x \in \cup_{i \in I} B_i$ as $x \in B_i$ for some $i \in I$.

These theorems are important because we usually want topologies satisfying certain properties, and we want these topologies to be as smaller as possible. The notion of " $\mathcal{O}$ is smaller than $\mathcal{O}'$ " refers to the partial relation $\mathcal{O} \subset \mathcal{O}'$ (the usual name is "coarser") or $\mathcal{O}'$ is finer that $\mathcal{O}$. There is a coarsest topology, namely the trivial one, and a finest one, also called discrete. In the typical situation the desired topology should be as coarse as possible, and contain at least the elements of $\mathfrak{S}$.

Neighborhood base. A neighborhood base at x in the topological space X is a subcollection $\mathfrak{B}_x$ of the neighborhood system $\mathfrak{N}_x$, having the property that each $N \in \mathfrak{N}_x$ contains some $V \in \mathfrak{B}_x$. That is $\mathfrak{N}_x$ is determined by:

$$\mathfrak{N}_x = \{N \subset X; V \subset N \text{ for some } V \in \mathfrak{B}_x\}.$$

For example, in any topological space the open neighborhoods of x form a neighborhood base at x. In a metric space, the balls centered at x with rational radius form a neighborhood base at x.

Definition. A topological space in which every point has a countable neighborhood base is said to be first countable. It is said to be second countable if it has a countable base for its topology.

A topology can also be described by giving a collection of basic neighborhoods at each point.

THEOREM. *Let X be a topological space and for each $x \in X$ let $\mathfrak{B}_x$ be a neighborhood base at x. Then:*

(1) *if $V \in \mathfrak{B}_x$, then $x \in V$,*
(2) *if $U, V \in \mathfrak{B}_x$, there exists $W \in \mathfrak{B}_x$ such that $W \subset U \cap V$,*
(3) *if $V \in \mathfrak{B}_x$, there is some $V_0 \in \mathfrak{B}_x$ such that if $y \in V_0$, then there is some $W \in \mathfrak{B}_y$ with $W \subset V$, and*
(4) *$U \subset X$ is open if and only if it contains a basic neighborhood of each of its points.*

Conversely, given a set X and a collection of subsets $\mathfrak{B}_x$ of X assigned to each of its points so as to satisfy (1), (2), (3) *above, there is a topology on X whose open sets are defined by* (4) *and which has $\mathfrak{B}_x$ as neighborhood base of each of its points.*

Rather than giving a proof of this, we describe some examples.

The Moore plane. Let M denote the closed upper half plane (x, y), $y \geq 0$. For a point in the open upper half plane basic neighborhoods would be the usual open discs (taken small enough so that they lie in M). For a point z in the x-axis the basic neighborhoods would be the sets $\{z\} \cup B$, where B is an open ball in the upper half plane tangent to the x-axis at z.

The slotted plane. At each point x in the plane, the basic nhoods at x are the sets $x \cup B$, where B is an open ball about x with a finite number of straight lines through x removed.

The looped line. At each point $x \neq 0$ of the real line the basic neighborhoods would be the open intervals centered at x. Basic neighborhoods of 0 would be the sets

$$(-\infty, -n) \cup (-\epsilon, \epsilon) \cup (n, \infty)$$

for all possible choices of $\epsilon > 0$, and positive integers n.

The scattered line. Define a topology on the real line as follows: a set is open if and only if it is of the form $U \cup V$, where U is a standard open set of the real line, and V is a subset of the irrationals. Describe an efficient neighborhood base at the irrational numbers and at the rationals.

3. Subspaces, unions, and hyperspaces

Subspaces. Let $(X, \mathcal{O})$ be a topological space and Y a subset of X. The collection

$$\mathcal{O}_Y = \{U \cap Y; U \in \mathcal{O}\}$$

is a topology on Y, called the induced or subspace topology. With this topology, Y is called a subspace of X. is a topology on Y, called the induced or subspace topology. With this topology, Y is called a subspace of X.

One should not confuse "open set in Y" with "open and in Y", since the first need not to be open (in X). There are situation in which they are.

Exercise. If U is open in Y and Y is open in X, then U is open in X.

Notation. It will be convenient to have an extra piece of notation for dealing with subsets of several spaces. So we will write, whenever there is danger of confusion, $\mathrm{Cl}_X A$ for the closure of A in X. Analogously, $\mathrm{Int}_X A$ denotes the interior of A in the space X.

Exercise. Let Y be a subset of the space X.

(1) $F \subset Y$ is closed in $Y \Leftrightarrow F = G \cap Y$, where G is closed in X.
(2) If $A \subset Y$, then $\mathrm{Cl}_Y A = Y \cap \mathrm{Cl}_X A$.
(3) If $\mathfrak{B}$ is a base for X, then $\{B \cap Y; B \in \mathfrak{B}\}$ is a base for Y.
(4) If $x \in Y$ and $\mathfrak{N}_x$ is a neighborhood base at x in X, then $\{N \cap Y; N \in \mathfrak{N}_x\}$ is a neighborhood base for x in Y.
(5) For $A \subset Y$, $A \cap \mathrm{Int}_X A \subset \mathrm{Int}_Y A$, and the containment may be strict.

Disjoint union. Let X, Y be topological spaces. Their disjoint union $X + Y$ is the set $X \times \{0\} \cup Y \times \{1\}$ with the topology whose open sets are those sets of the form $U + V$ with U open in X and V open in Y.

Note that if A is a subset of $X \coprod Y$, then $A = (A \cap X) + (A \cap Y)$, so that a set is open if and only if its intersection with X and with Y are open sets in X and in Y, respectively.

This can be done for an arbitrary family $\{X_\alpha\}_{\alpha \in A}$ of topological spaces. Their disjoint union $\sum_A X_\alpha$ is the set $\cup_A X_\alpha \times \{\alpha\}$ and whose open sets are those whose intersection with each X_α is open. This is the standard topology in the disjoint union.

There is another topology which agrees with the previous one in case of a finite family. The open sets are those subsets U of X such that U_α is open in X_α for every α and $U_\alpha = X_\alpha$ for almost very α. If the family is infinite, this topology is coarser that the previous one.

Hyperspaces. Let X be a topological space and denote by 2^X the collection of all nonempty closed subsets of X. If X is a metric space, then 2^X is metrizable via the Hausdorff metric. This topology is independent of the metric. Here is the definition: for an open subset U of X, let $\Gamma(U) = \{A \in 2^X; A \subset U\}$, and $\Lambda(U) = \{A; A \cap U \neq \emptyset\}$. If $U_1, \cdots U_n$ are open subsets of X, let

$$\langle U_1 \cdots U_n \rangle = \{A;\ A \subset \cup_{i=1}^n U_n,\ A \cap U_i \neq \emptyset,\ i = 1, \cdots, n\}.$$

Then

$$\mathfrak{B} = \{\langle U_1 \cdots U_n \rangle; U_i \text{ open in } X\}$$

is a base for a topology on X, and

$$\mathfrak{S} = \{\Gamma(U), \Lambda(U); U \text{ open in } X\}$$

is a subbase. If X is a compact metric space this topology coincides with the one given by the Hausdorff metric.

Products. Let X and Y be topological spaces. A subset W of the cartesian product $X \times Y$ is called open in the product topology if for each point (x, y) in W there are neighborhoods U of x in X and V of y in Y such that $U \times V \subset X \times Y$.

Subsets of the form $U \times V$ are called boxes, so that this topology is often called the box topology. Note that not every open subset of $X \times Y$ is an open box.

Quotients. Let X be a topological space and "$\sim$" be an equivalence relation on X. Let $X/\sim$ be the set of equivalence classes and denote by $\pi : X \to X/\sim$ the quotient map. We define a topology on $X/\sim$ by declaring a set U to be open if $\pi^{-1}(U)$ is open in X.

Examples. On the line $\mathbb{R}$ define a relation $x \sim y$ if $x - y$ is an integer. The quotient space is a circle.

Let X, Y be two copies of the real line. On the disjoint union $X + Y$ define an equivalence relation $(x, 0) \sim (x, 1)$ if and only if $x \neq 0$. The quotient space is the line with two origins.

4. Pseudometric and metric spaces

Definition. A pseudometric on a set X is a function d on $X \times X$ with values in the set of nonnegative real numbers such that for all points x, y and z in X

(1) $d(x,y) = d(y,x)$.
(2) $d(x,y) \leq d(x,z) + d(z,y)$,
(3) $d(x,y) = 0$ if $x = y$.

If furthermore d satisfies

(4) $d(x,y) = 0$ if and only if $x = y$

then (X,d) is called a metric space.

The r-ball centered at x is the set $B(x,r) = \{y \in X; d(x,y) < r\}$. A subset U of X is called open if for every x in U there exists an $r_x > 0$ such that $B(x,r_x) \subset U$.

Exercise. Let (X,d) be a pesudometric space. Let $\mathcal{O}(d)$ be the collection of open subsets of X. Prove that $(X, \mathcal{O}(d))$ is a topological space.

The metrization problem. Let $(X, \mathcal{O})$ be a topological space. Does there exist a metric d on X for which $\mathcal{O} = \mathcal{O}(d)$?

Clearly the answer is no in general. In a metric space two distinct points have disjoint neighborhoods, so that a topological space not having this property cannot be metrizable. But it is possible to have Hausdorff spaces which are not metrizable. The following example is a famous one, called the Sorgenfrey line. The underlying set X is the real line. The collection $\mathfrak{B}$ of all half-open intervals

$$[a,b) = \{x; a \leq x < b\}$$

is a basis for a topology on X. One notes that basis elements are both open and closed. The space X is connected. It is separable but has no countable basis. Every subspace of X is separable.

Now consider the product $Y = X \times X$ with the product topology. Then Y is separable, but the subspace formed by the points in the line $y = 1 - x$ is not.

5. The Order Topology

Let $(X,<)$ be a linearly ordered set. For x, y in X with $x \leq y$ we consider the following subsets of X, called intervals determined by x and y: $(x,y) = \{x \in X; x < z < y\}$, $(x,y] = \{x \in X; x < z \leq y\}$, $[x,y) = \{x \in X; x \leq z < y\}$, $[x,y] = \{x \in X; x \leq z \leq y\}$.

Definition. Let X be a linearly ordered set. Let $\mathfrak{B}$ denote the collection of all subsets of the following types:

(1) All open intervals (x,y) in X.
(2) All intervals of the form $[s,y)$, where s is the smallest element of X (if any).
(3) All intervals of the form $(x,m]$, where m is the maximal element of X (if any).

The collection $\mathfrak{B}$ is the basis for a topology of X, called the order topology.

Exercise. Define open rays and show that they define a subbasis for the order topology. The open rays are the sets of the form $\{x; x < a\}$ or $\{x; a < x\}$ for some $a \in X$.

Exercise. Let X be linearly ordered by $<$.

(1) The order topology is the coarsest topology for which the order is continuous in the following sense: if $a < b$ then there are neighborhoods U of a and V of b such that if $x \in U$ and $y \in V$ then $x < y$.
(2) If Y is a subset of X, then Y is also a linearly ordered set. But the order topology of Y may not be the same as the subspace topology. Find an example of this situation.

Dictionary order. If $(X, <)$ and $(Y, <)$ are linearly ordered we can define an order relation on the cartesian product $X \times Y$, called the dictionary order, by declaring $(x_1, y_1) < (x_2, y_2)$ if and only if either $x_1 < x_2$ or $x_1 = x_2$ and $y_1 < y_2$.

6. More examples

Hjalmar Ekdal topology. Let X be the set of positive integers with the topology whose open sets are those subsets of X which contain the successor of every odd integer in them. Thus a set F is closed in X if for each even n in F, $n - 1 \in F$.

Nested interval topology. Here $X = (0, 1)$ and the open sets are $U_n = (0, 1 - 1/n)$, $n = 1, 2, \cdots$, together with $\emptyset$ and X.

Long line. This is a very popular example. We need to recall a few facts from well-ordered sets. Let X be an uncountable set and well-order X into a well ordered set $X = \{x_1, x_2, \cdots, x_\alpha, \cdots\}$. Then either every element of X is preceded by at most a countable number of elements, or some element has an uncountable number of elements. If the first case occurs, let $A = X$. If the second case occurs, then, by the well-ordering property, the set of all those elements with an uncountable number of predecessors has a first element, say ω. Then let A denote the set of all predecessors of ω. In either case, A is a well-ordered set with the property that every element has countably many predecessors, but A is itself uncountable. Now consider $L = A \times [0, 1)$ with the order topology given by the dictionary order (the interval $[0, 1)$ has the standard order). This space L is called the long line (although it looks more like a long ray because it has a first element).

Cofinite topology. Let X be a set. The nonempty subsets of X are those $U \subset X$ with $X \setminus U$ finite.

Zariski topology. Let X be either $\mathbb{R}^n$ or $\mathbb{C}^n$. A set $F \subset X$ is closed if and only if there is a polynomial $P(x_1, \cdots, x_n)$ such that $P(x) = 0$ if and only if $x \in F$. This is the cofinite topology if $n = 1$, but not if $n > 1$.

Cocompact topology. Let X be the set of real numbers. A subset $U \subset X$ is open if either $U = \emptyset$ or $X \setminus U$ is a compact subset of $\mathbb{R}$ (with the usual topology).

Thomas' Corkscrew. Let $X = \cup_{i=1}^{\infty} L_i$ be the union of segments in the plane where $L_0 = \{(x, 0); 0 < x < 1\}$ and $L_i = \{(x, 1/i); 0 \leq x < 1\}$ for $i \geq 1$. If $i \geq 1$, each point of $L_i \setminus \{(0, 1/i)\}$ is open. A neighborhood base of $(0.1/i)$ is formed by the subsets of L_i with finite complement. The sets $N_i(x, 0) = \{(x, 0)\} \cup \{(x, 1/n); n > i\}$ for a neighborhood base at each point $(x, 0)$ of L_0.

CHAPTER II

CONTINUOUS MAPS

1. Continuous maps

Definition. (Continuous map) Let X, Y be topological spaces and let $f : X \to Y$. The f is continuous at $x \in X$ if for each neighborhood V of $f(x)$ there is a neighborhood U of x in X such that $F(U) \subset V$. We say that f is continuous if it is continuous at each point of X.

The definition is not altered if nhood is replace by basic nhood.

Proposition. *The following are equivalent for a map $f : X \to Y$ between topological spaces:*

(1) *the map f is continuous*
(2) *if V is open in Y, then $f^{-1}(V)$ is open in X*
(3) *if F is closed in Y, then $f^{-1}(F)$ is closed in X*
(4) *for each $E \subset X$, $f(E^-) \subset f(E)^-$*

Proof. (1) $\Rightarrow$ (2) If V is open in Y, then for each $x \in f^{-1}V$, V is a neighborhood of $f(x)$. By continuity of f, there is a neighborhood U of x such that $fU \subset V$. That is, $U \subset f^{-1}V$. Thus $f^{-1}V$ contains a neighborhood of each of its points.

(2) $\Rightarrow$ (3) For any subset B of Y we have $f^{-1}(Y \setminus B) = X \setminus f^{-1}(B)$.

(3) $\Rightarrow$ (4) Let K be closed in Y containing fE. By (3), $f^{-1}K$ is closed in X and contains E. Hence, $E^- \subset f^{-1}K$, so $f(E^-) \subset K$. This is true for any closed set K containing fE, thus (4).

(4) $\Rightarrow$ (1) Let x in X and V an open neighborhood of fx. Let $E = X \setminus f^{-1}V$, $U = X \setminus E^-$. Since $f(E^-) \subset f(E)^-$, we have $x \in U$. Also $f(U) \subset V$. Hence f is continuous.

(4) $\Rightarrow$ (3). Let K be closed in Y and let $E = f^{-1}(K)$. Then $f(E) \subset K$, so $f(E)^- \subset K$. If $x \in E^-$, $f(x) \in f(E^-) \subset f(E)^-$, so $f(x) \in K$, i.e., $x \in f^{-1}(K) = E$. Thus $E = E^-$ is closed

Exercise. We collect several properties of continuous maps. They are easy to prove, but important.

(1) The constant function is continuous.
(2) The identity map $\mathrm{id}_X : X \to X$ is continuous.
(3) Composition of continuous maps is continuous.
(4) If $f : X \to Y$ is continuous and X_0 is a subspace of X, then $f|_{X_0}$ is continuous.
(5) $f : X + Y \to Z$ is continuous if and only if $f|X$ and $f|Y$ are continuous.
(6) $f : Z \to X \times Y$ is continuous if and only if the composition with the projections is continuous.
(7) The quotient map $\pi : X \to X/\sim$ is continuous.
(8) If $Y \subset Z$ and $f : X \to Y$, then f is continuous as a map from X to Y if and only if it is continuous as a map from X to Z.

Exercise. The properties stated in (5), (6) and (7) characterize the corresponding topologies. More precisely, the product and sum topologies are the coarsest making the canonical maps continuous, the quotient is the finest.

Pasting continuous maps. The following is useful to put together continuous maps:

Proposition. *If $X = A \cup B$, where A and B are both open (or both closed), and if $f : X \to Y$ is a function such that both $f|A$ and $f|B$ are continuous, then f is continuous.*

Proof. Suppose A and B are open. If H is open in Y, then $f^{-1}(H) = (f|A)^{-1}(H) \cup (f|B)^{-1}(H)$, and each of the later is open in a subspace of X and so open on X.

The following is an extension of this proposition:

Proposition. *A family of subsets of a topological space is called locally finite if and only if each point of the space has a neighborhood meeting only finitely many members of the family.*

(1) *The union of any sub family from a locally finite family of closed sets id closed.*
(2) *If A_i is a locally finite collection of closed sets of X whose union is X, a function on X is continuous if and only if its restriction to each A_i is continuous.*
(3) *If B_i is a collection of open sets whose union is X, a function on X is continuous if and only if its restriction to each B_i is continuous.*

Homeomorphism. A bijection $f : X \to Y$ between topological spaces is called a homeomorphism when both maps f and f^{-1} are continuous. That is, U is open in X if and only if $f(U)$ is open in Y.

A map $f : X \to Y$ is an embedding if it induces a homeomorphism $f : X \to f(X)$, where $f(X)$ has the relative topology. A map which is continuous and injective is not necessarily an embedding: Let $f : [0, 2\pi) \to \mathbb{R}^2$ be $f(t) = (\cos t, \sin t)$. It is one-one and continuous, but the image of the open set $[0, 1/2)$ is not open in $f([0, 1))$.

Suppose that some topological property (i.e., one that can be formulated in terms of open sets) holds for X or some subset $A \subset X$. Then, if f is a homeomorphism, the same property must hold for Y or the subset $f(A)$. Some examples of topological properties of a space X are: X has countably many open sets, X has a countable dense subset, X is metrizable, X has a countable base. Being a subset of the reals, having a topology generated by the metric d, are not topological properties.

2. Continuity and convergence

The reader will notice the we do not state the usual characterization of continuity in terms of sequence as we did in the theory of metric spaces. The reason is that in general the topology of a space cannot be characterized in terms of limits of sequences. Say that a sequence $(x_n) = (x_1, x_2, \cdots)$ in X converges to x if any neighborhood of x contains all but a finite number of terms of the sequence (x_n). Then we have:

Proposition. *Let A be a subset of the topological space X. If there is a sequence of points of A converging to x, then $x \in A^-$. The converse holds if X is metrizable.*

Proof. The first part follows because $x \in A^-$ if every neighborhood of x meets A. The second part is from metric spaces.

It is easy to show that if f is continuous at x and $x_n \to x$, then $f(x_n) \to f(x)$. Indeed, if V is a neighborhood of $f(x)$, then there is a neighborhood U of x such that $f(U) \subset V$. So almost all terms of the sequence $f(x_n)$ are in V because almost all x_n are in U.

Example. Let $\mathbb{R}^\omega$ be the space of sequences of real numbers. The box topology for $\mathbb{R}^\omega$ has for open sets those of the form $\prod U_n$ where U_n is open in $\mathbb{R}$. Let A be the set of sequences $(x_n) \in \mathbb{R}^\omega$ all whose terms $x_n > 0$. In this topology the point 0 (the constant sequence 0) belongs to A^-, because if $U = \prod_n (a_n, b_n)$ is a basic open set containing 0, then $(b_n/2) \in A \cup U$. On the other hand, if $x_k = (x_{n,k})$ is a sequence of points in A, the neighborhood U of 0 defined by $U = \prod_n (-x_{n,n}, x_{n,n})$ contains no element of the sequence.

Example. Let X be an uncountable set with the cocountable topology. Then $x_n \to x$ if and only if $x_n = x$ eventually. One implication is obvious. If $x_n \to x$ but it is false that $x_n = x$ eventually, let $F = \{x_n; x_n \neq x\}$. This is a countable set and so $X \setminus F$ is a neighborhood of x, but x_n is not eventually in $X \setminus F$. On the other hand, X is not discrete: the one point sets $\{x\}$ are not open because $X \setminus \{x\}$ is uncountable.

CHAPTER III

PRODUCT AND QUOTIENT SPACES

1. The Product topology

Product space. Let $\{X_\alpha\}_{\alpha\in A}$ be a family of sets. Their product is the set of choice functions:

$$\prod_{\alpha\in A} X_\alpha = \{x : A \to \cup_{\alpha\in A} X_\alpha;\ x(\alpha) \in X_\alpha \text{ for every } \alpha \in A\}$$

By the Axiom of Choice, $\prod_A X_\alpha \neq \emptyset$ if all $X_\alpha \neq \emptyset$.

The maps

$$\pi_\beta : \prod_A X_\alpha \to X_\beta$$

defined by $\pi_\beta(x) = x(\beta)$ are called the projection maps.

If Y_α is a subset of X_α, then $\prod_A Y_\alpha \subset \prod_A X_\alpha$. If $X_\alpha = X$ for all $\alpha \in A$, the we write X^A instead of $\prod_A X_\alpha$.

Product topology. The product topology on $\prod_A X_\alpha$ has as basis of open sets those subsets of the form $\prod_A U_\alpha$, where

(1) U_α is open in X_α for every $\alpha \in A$, and
(2) $U_\alpha = X_\alpha$ for almost all α.

Note that if $\prod U_\alpha$ is such that $U_\alpha = X_\alpha$ for $\alpha \neq \alpha_1, \cdots, \alpha_n$, then

$$\prod U_\alpha = \pi_{\alpha_1}^{-1} U_{\alpha_1} \cap \cdots \cap \pi_{\alpha_n}^{-1} U_{\alpha_n}$$

so that the family of sets of the form $\pi_\alpha^{-1} U_\alpha$, with U_α open in X_α is a subbase for the product topology.

Exercise. The closure operation for the product topology can be described as follows. A point $x \in \prod X_\alpha$ is in A^- if and only if for any finite partition $A = A_1 \cup \cdots \cup A_n$ there exists A_k such that $\pi_\alpha(x)$ is in the closure of $\pi_\alpha(A_k)$ for every α.

There is another topology on $\prod X_\alpha$ which may seem more natural than the product topology. It is called the box topology, and its open sets are those satisfying only condition (1) above. It is finer than the product topology. They agree if the index set A is finite. We choose the product topology because it will allow us to prove theorems of the form: Every X_α has property P if and only if $\prod X_\alpha$ has property P.

Not every property will do here. For instance, it is false that the product is discrete if all X_α are.

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Example. Let $A = \mathbb{R}$ and $X_\alpha = \mathbb{R}$. The product space $\mathbb{R}^\mathbb{R}$ is the space of maps $\mathbb{R} \to \mathbb{R}$. A basic neighborhood of $f : \mathbb{R} \to \mathbb{R}$ is obtained by choosing $x_1, \cdots, x_n \in \mathbb{R}$, positive real numbers $\epsilon_1, \cdots, \epsilon_n$, and setting

$$U(f, x_1, \cdots, x_n; \epsilon_1, \cdots, \epsilon_n) = \{g;\ |g(x_i) - f(x_i)| < \epsilon_i, i = 1, \cdots, n\}$$

The sets

$$U(f, F, \epsilon) = \{g;\ |f(x) - g(x)| < \epsilon\},$$

where F is a finite subset of $\mathbb{R}$ and $\epsilon > 0$ define the same topology.

Exercise. The product topology on $\mathbb{R}^\mathbb{R}$ is the topology of pointwise convergence. Compare the box topology with uniform convergence. Consider also the product space $\mathbb{R}^\mathbb{I}$, where $\mathbb{I}$ is the closed unit interval.

Proposition. *The projection maps $\pi_\beta : \prod X_\alpha \to X_\beta$ are continuous and open. In general they are not closed.*

Proof. $\pi_\beta^{-1} U_\alpha$ is a subbasic element.

Proposition. *A map $f : Y \to \prod X_\alpha$ is continuous if and only if $\pi_\alpha \circ f$ is continuous for every α.*

Here is a useful characterization of the product topology.

Proposition. *The product topology is the coarsest topology on $\prod X_\alpha$ making all projection maps continuous.*

Proof. If $\mathcal{O}$ is a topology making every map π_β continuous and U_α is open in X_α, then $\pi_\beta^{-1} U_\beta$ is open in $\mathcal{O}$. Thus all elements of a subbasis of the product topology are in $\mathcal{O}$.

The characterization of the product topology just given suggests the following useful definition

Definition. Let X be a set and $f_\alpha : X \to X_\alpha$, $\alpha \in A$ be a collection of maps into topological spaces. The weak topology induced by the family of maps $\{f_\alpha\}$ is the coarsest topology on X for which all the maps f_α are continuous. It has a subbase consisting of the sets of the form $f_\alpha^{-1} U_\alpha$, where U_α is open in X_α.

It follows that the product topology on $\prod X_\alpha$ is the weak topology induced by the projection maps π_α.

Exercise. Let X have the weak topology induced by the maps $f_\alpha : X \to X_\alpha$. Then $g : Y \to X$ is continuous if and only if all the compositions $f_\alpha \circ g$ are continuous.

Exercise. Let $f : X \to X_\alpha$, $\alpha \in A$, be a family of maps. The evaluation map

$$e : X \to \prod X_\alpha$$

is defined by $[e(x)](\alpha) = f_\alpha(x)$. Show that if all f_α are continuous, then e is continuous.

2. Quotient topology

Dual to the notion of weak topology induced by a family of maps we have the notion of strong topology induced on Y by a collection of maps $g_\alpha : Y_\alpha \to Y$. This is the finest topology on Y making all the g_α's continuous. In the particular case when there is only one map $g : X \to Y$, the resulting topology on Y is called the quotient topology on Y induced by g.

Quotient topology. Let $g : X \to Y$ from a topological space X onto a set Y. The collection $\mathcal{O}_g$ of subsets of Y defined by

$$\mathcal{O}_g = \{V \subset Y;\, g^{-1}V \text{ is open in } \ X\}$$

is a topology on Y, called the quotient topology induced on Y by g. We also say that Y is a quotient space of X and g is the quotient map.

In general we can say very little about the quotient topology. Say that a map $f : X \to Y$ is open (closed) if the image of every open (closed) set is open (closed). There is no relation between open closed and continuous. The projection of the plane onto one of the axis is open and continuous, but not closed. Let $X = \{(x, y); xy = 0\}$ as a subset of the plane and Y the real line. The projection $p : (x, y) \mapsto y$. The image of a small interval around $(1, 0)$ maps to the point 0, which is not open. So p is not open, but it is closed. The restriction $p : X \setminus \{(0, 0)\} \to Y$ is neither open nor closed (the image of the closed set $\{(x, y); y \neq 0\}$ is not closed).

Proposition. *Let $f : X \to Y$ be a continuous onto map between topological spaces. If f is open (or closed) then Y has the quotient topology $\mathcal{O}_f$.*

Proof. Let $\mathcal{O}_Y$ denote the topology on Y. Then $\mathcal{O} \subset \mathcal{O}_f$, because f is continuous and $\mathcal{O}_f$ is the finest topology which makes f continuous. Suppose that f is open. If V is in $\mathcal{O}_f$, then $f^{-1}V$ is open in X, so $f(f^{-1}V) = V$ (f is onto) is in $\mathcal{O}$.

Proposition. *Let $f : X \to Y$ be onto and suppose that Y has the quotient topology. Then a map $g : Y \to Z$ is continuous if and only if $g \circ f$ is continuous.*

Proof. If W is open in Z and $g \circ f$ is continuous, then $(g \circ f)^{-1}W = (f^{-1} \circ g^{-1})W$ is open in X, so $g^{-1}W$ is open in Y by definition of the quotient topology.

Decompositions. The quotient topology and open and closed maps have little to do with the range space. Indeed, we can avoid it as follows. Suppose that $f : X \to Y$ is a surjective continuous map and that Y has the quotient topology. We can reconstruct Y from X and f as follows. Let $\mathcal{D}$ be the collection of all subsets of X of the form $f^{-1}(y)$, $y \in Y$. Let $p : X \to \mathcal{D}$ be the map $p : x \mapsto f^{-1}(f(x))$. There is a map $h : Y \to \mathcal{D}$ which takes $y \in Y$ to $f^{-1}(y)$. This map is a bijection and we have $h \circ f = p$ and $h^{-1} \circ p = f$. If we give $\mathcal{D}$ the quotient topology, then the previous proposition proves that h and h^{-1} are continuous (because $h \circ f = p$ and $h^{-1} \circ p = f$ are).

A collection $\mathcal{D}$ of disjoint nonempty subsets of X whose union is X is called a decomposition of X. An equivalent way of describing a partition of X is by means of equivalence relations. The elements of the partition are the equivalence classes of the relation.

If $\sim$ is an equivalence relation on X and $x \in X$ we denote by $[x]$ the subset of X consisting of all those $y \in X$ such that $y \sim x$. Similarly, if $A \subset X$, we denote by

$$[A] = \{x \in X;\, x \sim y \text{ for some } y \in A\} = \cup_{x \in A}[x].$$

The quotient space is denoted by $X/\sim$, with the quotient topology and projection map $p : X \to X/\sim$, $p(x) = [x]$.

If $B \subset X/\sim$, then $p^{-1}B = \cup\{A; A \subset B\}$. Thus B is open (closed) if and only if $\cup\{A; A \subset B\}$ is open (closed).

Proposition. *Let $p : X \to X/\sim$ be as above. The following are equivalent.*

(1) *The map p is open.*
(2) *If U is open in X, then $[U]$ is open.*
(3) *If A is closed in X, then the union of all elements of $X/\sim$ contained in A is closed.*

The statements obtained by interchanging "open" and "closed" are also equivalent.

Proof. (1) $\Leftrightarrow$ (2) For $A \subset X$, $[A] = p^{-1}pA$. If p is open and A is open then $[A]$ is open in X because p is continuous. If $p^{-1}pA$ is open then so is pA by definition of quotient topology, so p is open.

(2) $\Leftrightarrow$ (3) The union of all elements of $X/\sim$ contained in A is $X \setminus [X \setminus A]$. This set is closed for each closed set A if and only if $[X \setminus A]$ is open for each open set $X \setminus A$.

3. Cutting and pasting

Let X be a set. On the cartesian product $X \times X$ we have two operations. One is an involution defined by $(x, y)^{-1} = (y, x)$, and the other is a partially defined product: $(w, z) \circ (x, y)$ is defined if and only if $y = w$, and in this case it is equal to (x, z).

An equivalence relation $\sim$ on X can be represented by a subset R of $X \times X$ such that:

(1) R contains the diagonal $\Delta = \{(x, x); x \in X\}$.
(2) $R^{-1} = R$.
(3) $R \circ R \subset R$.

Exercise. Make sure this definition is equivalent to the one you know.

Let $[x]$ denote the equivalence class of a point x in X. If $\pi_1, \pi_2 : X \times X \to X$ are the projections onto the first and second factor, respectively, then you see that $[x] = \pi_2(\pi_1^{-1}(x) \cap R$.

Let X be a set and $\sim$ an equivalence relation on it. Then $\sim$ induces a decomposition of X whose elements are the equivalence classes of $\sim$.

Conversely, if $\mathcal{D}$ is a collections of nonempty mutually disjoint subsets of X whose union is X, we can define an equivalence relation whose equivalence classes are precisely the elements of $\mathcal{D}$. Simply set $x \sim y$ if and only if there exists $D \in \mathcal{D}$ such that $x, y \in D$.

Let X be a topological space and A be a nonempty subset of X. Then we can define a decomposition of X whose elements are: $\{x\}$ if $x \notin A$, and A. Let $\sim$ be the corresponding equivalence relation. The space $X/\sim$ with the quotient topology is denoted by X/A, and is said to be obtained from X by collapsing A to a point.

Exercise. Generalize this construction to an arbitrary collection $\{A_i\}_{i \in I}$ of nonempty disjoint subsets of X.

Exercise. Let $\mathbb{D}^n = \{x \in \mathbb{R}^n; |x| \le 1\}$, $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1}; |x| = 1\}$. Then $\mathbb{D}^n/\mathbb{S}^{n-1} = \mathbb{S}^n$.

Note that you could have started with any collection $\mathcal{D}$ of mutually disjoint subsets of X, not necessarily nonempty. Then you can define an equivalence relation as above, but the equivalence classes do not give you back $\mathcal{D}$, only those nonempty elements of $\mathcal{D}$.

Exercise. What would be $X/\emptyset$?

Cone over a space. Let X be a space. The cone over X is

$$CX = X \times [0, 1]/X \times \{1\}.$$

Suspension. For a topological space X, the space

$$SX = X \times [0, 1]/X \times 0, X \times 1$$

is called the suspension of X.

Exercise. SX is also called the double cone, because $CX/X \times 0 \cong SX$.

Projective spaces. The (real) projective space $\mathbb{P}^n$ is the quotient $\mathbb{S}^n/\sim$, where $\sim$ is the equivalence relation which identifies diametrically opposite points in $\mathbb{S}^n$. In other words, $\mathbb{P}^n$ is the space of lines in $\mathbb{R}^{n+1}$. A neighborhood of a line l would consist of those lines with slope close to that of l.

There are also complex projective spaces, defined as follows. In $\mathbb{C}^{n+1} \setminus 0$ define the equivalence relation

$$(z_0, \cdots, z_n) \sim (w_0, \cdots, w_n) \Leftrightarrow \text{there exists } \lambda \in \mathbb{C} \setminus 0 \text{ such that } \lambda x_k = w_k,\ k = 0, \cdots, n$$

Then $\mathbb{P}^n_{\mathbb{C}} = (\mathbb{C}^{n+1} \setminus 0)/\sim$. You can think of it as the space of complex lines in $\mathbb{C}^{n+1}$.

Exercise. The space $\mathbb{P}^1_{\mathbb{C}}$ is a familiar one.

Gluing spaces. Let X and Y be topological spaces, $A \subset X$, and $\varphi : A \to Y$ a continuous map. On the disjoint union $X + Y$ consider the equivalence relation whose equivalence classes are:

(1) $\{x\}$ for $x \in X \setminus A$.
(2) $\{y\}$ for $y \in Y \setminus \varphi A$, and
(3) $\{y\} \cup \varphi^{-1}(\{y\})$ for $y \in \varphi(A)$.

The quotient space $X + Y/\sim$ is denoted by $X \cup_\varphi Y$, and said to be obtained by gluing X to Y along φ.

Exercise. Suppose that A is closed, and let $p : X + Y \to X \cup_\varphi Y$ be the canonical projection. Then

(1) $p|Y$ is a homeomorphism and $p(Y)$ is closed in $X \cup_\varphi Y$.
(2) $p|(X \setminus A)$ is a homeomorphism and $p(X \setminus A)$ is open in $X \cup_\varphi Y$.

Exercise. Let $Y = \{*\}$ be a one-point space. Then $X \cup_\varphi \{*\} = X/A$, where $\varphi : A \to Y$ is the obvious map.

Exercise. Let $\varphi : X \to X$ be a homeomorphism. On $X \times [0, 1]$ identify the points $(x, 0)$ and $(\varphi(x), 1)$. The resulting quotient space is called the suspension of φ.

If $X = [0, 1]$ and φ is the identity, you obtain a cylinder. But if $\varphi(x) = 1 - x$ then you get a Mobius band.

If X is a circle and $\varphi(x) = -x$ is the antipodal map, you get a Klein bottle. If φ is the identity you get a torus.

Exercise. Show that you can get the Klein bottle by gluing two copies of a Mobius band along their boundary.

Exercise. Let A be the middle circle of the moebius band M. What is the space M/A?

CHAPTER IV

CONNECTED AND PATH-CONNECTED SPACES

1. Connected spaces

Definition. A topological space X is connected if it is not the union of two disjoint nonempty open subsets. A subset Y of X is connected if Y is connected as a topological space with the induced topology.

We say that $A|B$ is a partition of X if A and B are nonempty, open, disjoint and $X = A \cup B$. Thus a space is disconnected space if and only if it has a partition.

Examples. The empty set is connected. A trivial topological space is connected. Intervals in the real line are connected.

Example. The real line is connected. Let A be an open and closed subset of $\mathbb{R}$. Suppose that A and $\mathbb{R} \setminus A$ are nonempty. Let $x \in \mathbb{R} \setminus A$. Then one of the sets $A \cap [x, \infty)$, $A \cap (-\infty, x]$ is nonempty. Suppose $B = A \cap [x, \infty) \neq \emptyset$. Then B is closed and bounded below, so its has a smallest element, say b. But $B = A \cap (x, \infty)$ is also open, so there is an interval $(b - \epsilon, b + \epsilon) \subset B$, and this means that b cannot be the smallest element of B.

Connected subsets of the real line. They are the intervals, possibly degenerate. That open intervals are connected follows because they are homeomorphic to the line. For the others the the proof is like the one above.

Example. This example shows that one has to be very careful when dealing with relative topologies. Let X be the set of integers with the cofinite topology. Then X is connected. Let $Y = \{0, 1\} \subset X$. Then Y is not connected.

Here is a useful version of connectedness

PROPOSITION. *A space X is connected if an only if every continuous map $f : X \to 2$ is constant. Here $\mathbf{2} = \{0, 1\}$ with the discrete topology.*

PROOF. If $f : X \to \mathbf{2}$ is continuous and nonconstant, then $f^{-1}(0)|f^{-1}(1)$ is a partition of X. If $A|B$ is a partition of X, the quotient map $f : X \to \{A, B\}$ is continuous and nonconstant.

It has two easy consequences:

PROPOSITION. *Continuous maps take connected sets to connected sets*

PROPOSITION. *The closure of a connected set is connected.*

PROOF. Let $Y \subset X$ be a connected subset of Y. Let $f : Y^- \to \mathbf{2}$ be continuous. Since the restriction $f|Y$ is continuous, it is constant. Also, by continuity, $f(Y^-) \subset f(Y)^-$, so f must be constant.

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Mean value theorems. The following facts are standard applications of connectedness.

(1) Let X be a connected topological space, $f : X \to \mathbb{R}$ a continuous function, and a, b points of X. Then $f(X)$ takes on every value between $f(a)$ and $f(b)$.
(2) Every continuous map $f : \mathbb{I} \to \mathbb{I}$ has a fixed point.
(3) Every real polynomial of odd degree has a root.

Definition. We can rephrase the definition of connected subspace without passing to the relative topology. Say that two subsets A, B of X are mutually separated in X if:

$$A \cap B^- = A^- \cap B = \emptyset.$$

PROPOSITION. *A subspace Y of X is connected if and only if there are no nonempty mutually separated sets A and B of X such that $Y = A \cup B$.*

PROOF. If Y is disconnected by A and B, then

$$A \cap \mathrm{Cl}_X B = A \cap Y \cap \mathrm{Cl}_X B = A \cap \mathrm{Cl}_Y B = \emptyset.$$

Conversely, if A and B are mutually separated in X and $Y = A \cup B$, then

$$\mathrm{Cl}_Y A = Y \cap \mathrm{Cl}_X A = (A \cup B) \cap \mathrm{Cl}_X A = (A \cap \mathrm{Cl}_X A) \cup (B \cap \mathrm{Cl}_X A) = A$$

so A is closed in Y.

COROLLARY. *If A, B are mutually separated subsets of X and $Y \subset A \cup B$, then $Y \subset A$ or $Y \subset B$.*

PROOF. $A \cap Y$ and $B \cap Y$ are also mutually separated because

$$(A \cap Y)^- \cap B \cap Y \subset A^- \cap B = \emptyset.$$

PROPOSITION. *Let X be a topological space. Then X is connected if it satisfies one of the following conditions:*

(1) *$X = \cup_\alpha X_\alpha$ with each X_α connected and $\cap X_\alpha \neq \emptyset$.*
(2) *Every pair x, y in X lies in a connected set C_{xy}.*
(3) *$X = \cup_{n=1}^\infty X_n$, each X_n is connected and $X_n \cap X_{n+1} \neq \emptyset$.*

PROOF. (1) Let $f : X \to \mathbf{2}$ be a continuous map. Then $f|X_\alpha$ is continuous, hence constant. This constant must be the same for all α as the X_α have nonempty intersection.

(2) Fix $x \in X$. Then $X = \cup_{y \in X} C_{xy}$ satisfies (1).

(3) Let $Y_n = X_1 \cup \cdots \cup X_n$. Then Y_1 is connected and if Y_n is connected so is Y_{n+1}. We have $\cap_n Y_n = Y_1$ and $X = \cup_n Y_n$.

PROPOSITION. *A nonempty product space $\prod X_\alpha$ is connected if and only if each factor X_α is connected.*

PROOF. If the product is connected then each factor is because the projections are continuous and onto.

For the converse it is easier to consider first the case of the product of two connected spaces X and Y. Let (a, b) be a point in $X \times Y$. Then $X \times \{b\}$ is homeomorphic to X, so it is connected. The union $X \times \{b\} \cup \{a\} \times Y$ is also connected as the two pieces have nonempty intersection. Now

$$X \times Y = \cup_{x \in X} (X \times \{b\} \cup \{x\} \times Y)$$

is connected because it is the union of connected sets all having the point (a, b) in common.

The proof also works for any finite product of connected spaces by iteration.

Now consider an arbitrary product $\prod X_\alpha$. Let $p = (p(\alpha))$ be a point in the product. For each finite subset F of the index set A, let Y_F denote the subset of points $(x(\alpha)) \in \prod X_\alpha$ such that $x(\alpha) = p(\alpha)$ for $\alpha \in A \setminus F$. The space Y_F is homeomorphic to the finite product $\prod_{\alpha \in F} X_\alpha$, so it is connected.

Therefore the union $Y = \cup_F Y_F$ over all finite subsets $F \subset A$ is connected because all contain the point p. The difference with the finite case is that this union is not the whole product space.

We show that it is dense. Let $\prod U_\alpha$ be a basis element. Then $U_\alpha = X_\alpha$ except for indices α in a finite subset F of A. Let $(x(\alpha))$ be a point in U. Then the point $y(\alpha) = x(\alpha)$ if $\alpha \in F$ and $y(\alpha) = p(\alpha)$ if $\alpha \in A \setminus F$ is in $U \cap Y$.

Connected components. If $x \in X$, the largest connected subset $C(x)$ of X containing x is called the connected component of x. The components of points of X form a partition of X into maximal connected subsets. Indeed, if $x \neq y$ in X, then either $C(x) = C(y)$ or $C(x) \cap C(y) = \emptyset$, for otherwise $C(x) \cup C(y)$ would be a connected set containing x and Y and larger than $C(x)$ or $C(y)$.

PROPOSITION. *The components are closed sets.*

PROOF. We have seen that the closure of a connected set is connected. Thus if C is a connected component then $C^- \subset C$.

Example. The components need not be open. For instance, the components of the space $\mathbb{Q}$ of rational numbers are the points.

Applications. Connectedness provides a crude method of distinguishing between topological spaces. For instance, you should be able to show that $\mathbb{R}$ and $\mathbb{R}^n$ $(n > 1)$ are not homeomorphic, nor $[0, \infty)$ and $\mathbb{R}$, nor $[0, 1]$ and $\mathbb{S}^1$, nor $\mathbb{S}^1$ and $\mathbb{S}^n$ $(n > 1)$.

The typical argument involving connectedness runs as follows. If X is a connected space and $f : X \to Y$ is a locally constant map, then it is locally constant. Usually Y is a 'yes-no' space. For instance, let P be a property that points of X may or may not have, and suppose that we want to prove that all points of X have property P. Then it is enough to prove the following three assertions:

(1) There is at least one point with property P.
(2) If x has property P, the same applies to all points in a sufficiently small neighborhood.
(3) If x does not have property P then the same is true for points near x.

2. Path-connected spaces

Definition. A space X is path-connected if every two points $x, y \in X$ can be joined by a path, that is a continuous map $c : \mathbb{I} \to X$ such that $c(0) = x$ and $c(1) = y$.

Clearly, a path connected space is connected. The converse is not true. The typical example is the 'topologist's sine curve'. This is the subset of the plane given by

$$V = \{(x, \sin(1/x)); x > 0\} \cup \{(0, y); -1 \leq y \leq 1\}.$$

This example also shows that path-connectedness of a space does not imply that of its closure. The subspace $\{(x, \sin(1/x)\}$ is path connected, but not its closure.

The following proposition describes the behavior of path-connectedness under different operations.

PROPOSITION. (1) *Continuous images of path-connected spaces are path connected.*
(2) *Non-disjoint unions of path-connected spaces are path connected.*
(3) *Products of path-connected spaces are path connected.*

3. Local connectedness and path-connectedness

Definition. A space X is locally connected if each $x \in X$ has a neighborhood base consisting of connected sets.

Similarly, we say that X is locally path-connected if each point has a neighborhood base consisting of path-wise connected sets.

Examples. The topologist's sine curve is not locally path-connected. The space X in the plane consisting of the vertical lines $x = 0$ and $x = 1$ together with the segments $\{(x, 1/n); 0 \leq x \leq 1\}$ for $n = \pm 1, \pm 2, \cdots$ and the unit interval on the x-axis is connected and path-connected, but neither locally connected nor locally path-connected. The Sorgenfrey line is not locally connected. The space $[0, 1) \cup (1, 2]$ is locally connected but not connected.

Note also that local properties are not transfered by continuous maps, because continuous images of neighborhoods need not be neighborhoods. An example is the Warsaw circle as image of $[0, \infty)$.

Proposition. *A connected, locally path-connected space X is path-connected.*

Proof. Let $x \in X$, and let A be the set of all points of X that can be joined by a path to x. The set A is non-empty because $x \in A$. If we prove that it is both open and closed, then it must be all of X.

The set A is open. If $y \in A$ and U is a neighborhood of y which is path-connected, then we can join any point of U to y, and then y to x.

On the other hand, if $y \in A^-$, then any path-connected neighborhood U of y meets A, say $z \in A \cap U$. We can join z to y and then y to x.

Proposition. *A space is locally connected if and only if each component of each open set is open.*

Proof. Suppose that X is locally connected and $x \in C$, where C is a component of the open set $U \subset X$. By local connectedness, there is a connected neighborhood V of x with $V \subset U$. But then $V \subset C$, so C is open.

Conversely, assume that each component of each open set is open. If U is any open neighborhood of x in X, then the component of U containing x is an open connected neighborhood of x contained in U. Thus X is locally connected.

Corollary. *The components of a locally connected space are open and closed.*

Similar arguments provide proofs of the following:

Proposition. *A space is locally path connected if and only if each path component of each open set is open.*

The path-components of a space lie in the components. If the space is locally path-connected, then the components and the path-components are the same.

Proof. If P is the path component of a point, and C its component, then $P \subset C$ because P is connected. Suppose that the space is locally path-connected but $P \neq C$. Let Q denote the union of all path-components different from P which meet C. Each is contained in C, and so $C = P \cup Q$. Since C is connected, we must have $Q = \emptyset$, because the path-components of a locally path-connected space are open.

To conclude we mention the standard way in which local properties transfer to product spaces.

Proposition. *A nonempty product space is locally path-connected if and only if each factor is locally connected and almost all of them are connected.*

Proof. One way is easy because projection maps are continuous, onto and open. For the other way, let $x \in \prod X_\alpha$, and let $U = \prod U_\alpha$ be a neighborhood of x, where each U_α is open in X_α and

$U_\alpha = X_\alpha$ for almost all α, say $\alpha \in A \setminus F$. Enlarge F to a finite set so that it contains all indices α for which X_α is not connected. We can find connected neighborhoods V_α of x_α contained in U_α for $\alpha \in F$. Then $\prod_{\alpha \in F} \times \prod_{\alpha \notin F} X_\alpha$ is a connected neighborhood of x contained in U.

CHAPTER V

CONVERGENCE

1. Hausdorff spaces

Definition. A topological space is called hausdorff if any two different points have disjoint neighborhoods.

Examples. The typical example of hausdorff space is a metric space. Non-hausdorff spaces may seem rather weird, but their naturally appear in mathematics. The best-known example is the Zariski topology in algebraic geometry. Let a subset F of $\mathbb{C}^n$ be closed if there is a polynomial $P(z_1, \cdots, z_n)$ whose set of roots is precisely F, i.e., $P(x) = 0$ if and only if $x \in F$. This is the cofinite topology if $n = 1$, but it is different if $n > 1$.

Exercise. Subspaces, unions, disjoint unions and products of hausdorff spaces are hausdorff.

Example. Quotients of hausdorff spaces are not necessarily hausdorff. Neither continuous images.

The continuous and open image of a hausdorff space need not be hausdorff. For example the line with two origins as quotient of $\mathbb{R} \times \{0, 1\}$ by the equivalence relation $(x, 0) \sim (x, 1) \Leftrightarrow x \neq 0$.

Exercise. A space X is hausdorff if and only if the diagonal $\Delta = \{(x, x); x \in X\}$ is closed in $X \times X$.

Indeed, if $x \neq y$, the $(x, y) \notin \Delta$, so there is a basic open set $U \times V$ containing (x, y) and disjoint from Δ. The converse is similar.

Some nice properties of hausdorff spaces are collected in the following propositions

PROPOSITION. *In a hausdorff space a sequence can have at most one limit point.*

PROPOSITION. (1) *If $f : X \to Y$ is continuous and Y is hausdorff, then*

$$\Delta(f) = \{(x_1, x_2); f(x_1) = f(x_2)\}$$

is a closed subset of $X \times X$.

(2) *If f is an open map of X onto Y and $\Delta(f)$ is closed in $X \times X$, then Y is hausdorff.*

(3) *If f is a continuous open map of X onto Y, then Y is hausdorff if and only if $\Delta(f)$ is closed.*

PROOF. (1) If $(x_1, x_2) \notin \Delta(f)$, then there are disjoint neighborhoods U, V of $f(x_1)$, $f(x_2)$, respectively. Then $f^{-1}U \times f^{-1}V$ is a neighborhood of (x_1, x_2) which does not meet $\Delta(f)$.

(2) Suppose $f(x_1)$ and $f(x_2)$ are distinct points of Y. Then (x_1, x_2) is not in $\Delta(f)$, so there are neighborhoods U, V of x_1, x_2 such that $U \times V \cap \Delta(f) = \emptyset$. Then fU and fV are disjoint neighborhoods of $f(x_1)$, $f(x_2)$

(3) By (1) and (2).

2. Countability axioms

We have seen that sequences are adequate to describe the topology of a metric space, but they are not good enough in general. The reason for this is the nature of neighborhood bases of the points of a space. Here is the abstraction:

First countable. A space X is first-countable if each of its points has a countable neighborhood base.

Examples. Pseudometric spaces are first-countable. The Sorgenfrey line is first-countable: a neighborhood base at x consists of $[x, q)$ with q rational. An uncountable space with the cocountable topology is not. Neither it is the product space $\mathbb{R}^{\mathbb{R}}$.

Proposition. *Suppose that X is first-countable and $A \subset X$. Then $x \in A^-$ if and only if there is a sequence (x_n) in A with $x_n \to x$.*

Proof. The proof is like the one in metric spaces. If V_n, $n = 1, 2, \cdots$ are the elements of a countable neighborhood base at x, take $U_n = \cap_{i=1}^n V_n$. These U_n form a nested neighborhood base at x. Since $U_n \cap A \neq \emptyset$ for each n, we can find $x_n \in U_n \cap A$. This is a sequence in A which converges to x.

If (x_n) is a sequence in A which converges to x, the sequence lies eventually in every neighborhood of x, so every neighborhood of x meets A.

Example. Let $X = \mathbb{R}^{\mathbb{R}}$. Let A be the set of functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 0$ or 1, and $f(x) = 0$ for only finitely many x.

Let g be the constant function $g = 0$. A neighborhood U_g of g is determined by a finite set F and $\epsilon > 0$, in the form

$$U_g = \{h; |g(x) - h(x)| < \epsilon \,\mathrm{if}\, x \in F\}.$$

Every neighborhood of g meets A, simply take a function which is 0 on F and 1 everywhere else.

On the other hand, if (f_n) is a sequence in A, with $f_n = 0$ on the finite set F_n, any limit of f_n can be 0 at most on the countable set $\cup F_n$.

3. Filters

Definition. Let X be a set. A filter on X is a collection $\mathcal{F}$ of nonempty subsets of X satisfying the following conditions:

(1) if A, B are in $\mathcal{F}$, so is $A \cap B$, and
(2) if $A \subset B$ and $A \in \mathcal{F}$, then $B \in \mathcal{F}$

Note. It almost passes unnoticed, but it is very important to realize that for any sets A, B in a filter $A \cap B \neq \emptyset$.

A filter base on X is a collection $\mathcal{B}$ of nonempty subsets of X which satisfies the following condition: if $A, B \in \mathcal{B}$ then there exists $C \in \mathcal{B}$ such that $C \subset A \cap B$. We also say that $\mathcal{B}$ is a base for the filter $\mathcal{F}$ if each $F \in \mathcal{F}$ contains some $F' \in \mathcal{B}$.

Thus a filter is always a filter base, but not conversely. If $\mathcal{B}$ is a filterbase, then the collection of all subsets of X which contain an element of $\mathcal{B}$ is a filter which has $\mathcal{B}$ as a base. We call it the filter generated by $\mathcal{B}$.

Examples. If $A \subset X$ is a nonempty subset, then $\{F \subset X; A \subset F\}$ is a filter on X. A filter base is $\{A\}$.

If X is a topological space and $x \in X$, then $\mathfrak{N}_x$, the collection of all neighborhoods of x, is a filter on X. A neighborhood base at x is a base for $\mathcal{N}_x$.

The collection of cofinite subsets of $\mathbb{N}$ forms a filter on $\mathbb{N}$. The sets $S_n = \{n, n+1, \cdots\}$ form a base for this filter.

Let $Y \subset X$, $a \in Y^-$. Then $\{Y \cap U; U \in \mathfrak{N}_a\}$ is a filter.

Exercise. On a hausdorff space a filter converges at most to a point.

Convergence of filters. A filter $\mathcal{F}$ on a space X converges to a point x if $\mathfrak{N}_x \subset \mathcal{F}$. We write this as $\mathcal{F} \to x$.

We say that $x \in X$ is a cluster point of the filter $\mathcal{F}$ if each neighborhood of x and each element of $\mathcal{F}$ have nonempty intersection.

These definitions extend in the obvious way to filterbases: a filterbase $\mathcal{B}$ converges to x if the filter generated by $\mathcal{B}$ converges to x.

Exercise. Let A be a nonempty subset of the space X. The cluster points of the filter $\mathcal{F} = \{F \subset X; A \subset F\}$ contain each point of A^-. Add conditions to A or X so that $\mathcal{F}$ converges to some point.

Let $\mathcal{F}$ be the filter on $\mathbb{R}$ generated by the filterbase $\{(0,t); t > 0\}$. Then $\mathcal{F} \to 0$, although not every element of $\mathcal{F}$ contains 0.

If (x_n) is a sequence in X, the collection of subsets of X where the sequence eventually lies is a filter. This filter converges to x if and only if $x_n \to x$.

The relation between cluster point and limit point is the following:

PROPOSITION. *A filter $\mathcal{F}$ has x as a cluster point if and only if there is a filter $\mathcal{G} \supset \mathcal{F}$ such that $\mathcal{G} \to x$.*

PROOF. If $\mathcal{F}$ has x as cluster point, then $\mathcal{B} = \{U \cap F; U \in \mathfrak{N}_x\}$ is a filter base for a filter $\mathcal{G}$ which contains $\mathcal{F}$ and converges to x.

If $\mathcal{F} \subset \mathcal{G} \to x$, then each $F \in \mathcal{F}$ and each neighborhod U of x belong to $\mathcal{G}$, so they have nonempty intersection. Thus $\mathcal{F}$ clusters at x.

The next result shows that filter convergence is adequate to describe the topology.

PROPOSITION. *If $A \subset X$, then $x \in A^-$ if and only if there is a filter $\mathcal{F}$ such that $A \in \mathcal{F}$ and $\mathcal{F} \to x$.*

PROOF. If $x \in A^-$, then $\{U \cap A; U \in \mathfrak{N}_x\}$ is a filterbase. The filter it generates contains A and converges to x.

If $\mathcal{F} \to x$ and $A \in \mathcal{F}$, then x is a cluster point of $\mathcal{F}$, thus $x \in A^-$.

Filters and continuous maps. If $f : X \to Y$ is continuous and $\mathcal{F}$ is a filter on X, we denote by $f(\mathcal{F})$ the filter on Y generated by the sets $f(F)$, $F \in \mathcal{F}$.

PROPOSITION. *A map $f : X \to Y$ is continuous at $x \in X$ if whenever a filter $\mathcal{F} \to x$, then $f(\mathcal{F}) \to f(x)$.*

PROOF. Suppose f is continuous at x and $\mathcal{F} \to x$. If V is a neighborhood of $f(x)$. Then there is a neighborhood U of x with $f(U) \subset V$. But then $V \in f(\mathcal{F})$ because $U \in \mathcal{F}$.

For the converse, take $\mathcal{F} = \mathfrak{N}_x$. Then $\mathcal{F} \to x$, so $f(\mathcal{F}) \to f(x)$. Each neighborhood V of $f(x)$ is in $f(\mathcal{F})$, so there is a neighborhood U of x such that $f(U) \subset V$.

PROPOSITION. (1) *If $f, g : X \to Y$ are continuous and Y is hausdorff, then $\{x; f(x) = g(x)\}$ is closed in X.*

(2) *If f, g agree on a dense subset of X, then $f = g$.*

4. Ultrafilters

Definition. A filter which is not properly contained in another filter is called an ultrafilter.

Examples. Let A be a nonempty subset of X. Is the filter $\mathcal{F} = \{F \subset X; A \subset X\}$ an ultrafilter?

Let $\mathbb{N}$ be the positive integers. Then $\mathcal{F} = \{U \subset \mathbb{N}; \mathbb{N} \setminus U \text{ finite}\}$ is a filter, but not an ultrafilter. Can you construct an ultrafilter containing $\mathcal{F}$?

As a consequence of Zorn's lemma we have:

PROPOSITION. *Every filter is contained in a unique ultrafilter.*

Here is a remarkable property of ultrafilters

PROPOSITION. *Let $\mathcal{F}$ be an ultrafilter on X and A be a subset of X. Then exactly one of the sets A and $X \setminus A$ belongs to $\mathcal{F}$.*

PROOF. Both sets cannot be in $\mathcal{F}$ as their intersection is empty. Furthermore, one of the two sets has to intersect all the sets in the filter, otherwise by taking one filter set outside A and one outside $X \setminus A$ we contradict the definition of filter as their intersection would be empty. Suppose that A is the set that meets all elements of $\mathcal{F}$. Then the collection of all subsets G of X such that G contains some set of the form $F \cap A$, with $F \in \mathcal{F}$, is a filter which contains both $\mathcal{F}$ and $\{A\}$. By maximality, A must be in $\mathcal{F}$.

Exercise. If $f : X \to Y$ is continuous and onto, and $\mathcal{F}$ is an ultrafilter on X, then $f(\mathcal{F})$ is an ultrafilter.

CHAPTER VI

COMPACT SPACES

1. Compact spaces

Definition. A cover of a space X is a collection $\mathcal{U}$ of subsets of X whose union is X. It is called an open cover if every element of $\mathcal{U}$ is an open subset of X. A subcover of a cover $\mathcal{U}$ is a subfamily $\mathcal{V} \subset \mathcal{U}$ which still covers X.

Compact space. A space X is said to be compact if every open cover admits a finite subcover. A subset Y of X is compact if it is a compact space with the induced topology.

Exercise. To check whether a space is compact it is enough to consider covers consisting of elements of a base.

Alexander subbase theorem. While the last exercise is straightforward, one may wonder if it is possible to replace base by subbase. It can be done, but it is not as easy. This result is referred to as Alexander's subbase theorem.

Proposition. *Let $\mathfrak{S}$ be a subbase for the topology of a space X such that every cover of X by elements of $\mathfrak{S}$ has a finite subcover. Then X is compact.*

Proof. Let $\mathcal{L}$ be the collection of all open covers of X which have no finite subcover. If X is not compact, then $\mathcal{L}$ is nonempty. Order $\mathcal{L}$ by containment. If C is a chain in $\mathcal{L}$, then $\{U \subset X; U \in \mathcal{U}, \mathcal{U} \in C\}$ is an upper bound for C which belongs to $\mathcal{L}$. Thus we can apply Zorn's lemma to $\mathcal{L}$ to obtain a maximal element $\mathcal{M}$.

The key is the following property that $\mathcal{M}$ has: if $U \in \mathcal{M}$ and $V_1, \cdots V_n$ are open subsets of X such that $V_1 \cap \cdots \cap V_n \subset U$, then $V_k \in \mathcal{M}$ for some k. Indeed, if this was not the case, then for each k there would be subsets $U_{k_1}, \cdots, U_{k_{n_k}}$ in $\mathcal{M}$ such that together with V_k cover X, for otherwise $\mathcal{M}$ would not be maximal. But this implies that

$$X \subset (\cap_{k=1}^{n} V_k) \cup (\cup_{k,l} U_{k_l}) \subset U \cup (\cup_{k,l} U_{k_l})$$

so that $\mathcal{M}$ admits a finite subcover, contradicting the fact that $\mathcal{M} \in \mathcal{L}$.

Let $U \in \mathcal{M}$, and let $x \in U$. Since $\mathfrak{S}$ is a subbasis, $x \in B_1 \cap \cdots \cap B_n \subset U$ for some elements $B_i \in \mathfrak{S}$. By the key property of $\mathcal{M}$, some $B_k \in \mathcal{M}$. This implies that $\mathfrak{S} \cap \mathcal{M}$ is an open cover of X. But this is a contradiction: on one hand $\mathcal{M}$ admits no finite subcovers, on the other, every cover by elements of $\mathfrak{S}$ does.

Example. To show the usefulness of this result we prove that the interval $[0, 1]$ is compact. Its topology has a subbase consisting of the intervals $[0, a)$ and $(b, 1]$, with $a > 0$, $b < 1$. Any cover of $[0, 1]$ by elements of this subbase has a subcover consisting of exactly two elements.

Finite intersection property. An useful reformulation of compactness is the following. Say that a collection $\mathcal{E}$ of subsets of X has the finite intersection property if each finite collection of elements of $\mathcal{E}$ has nonempty intersection.

Proposition. *A space X is compact if and only if every family of closed subsets of X with finite intersection property has nonempty intersection.*

Proof. If $\mathcal{E} = \{F_{\mathfrak{a}}\}$ has the finite intersection property, but $\cap_{\mathfrak{a}} F_{\mathfrak{a}} = \emptyset$, then $\{X \setminus F_{\mathfrak{a}}\}_{\mathfrak{a}}$ is an open cover of X without finite subcover.

Compactness and filters. We know that a metric space is compact if and only if every sequence has a convergent subsequence. In general spaces this can be formulated in terms of convergence of filters.

Proposition. *A topological space X is compact if and only if every ultrafilter on X converges.*

Proof. Suppose that X is compact. Suppose that there is an ultrafilter $\mathcal{F}$ which converges to no point of X. Then for every $x \in X$ we can find an open neighborhood U_x of x such that $U_x \notin \mathcal{F}$, for if all open neighborhoods of x belong to $\mathcal{F}$, so would all neighborhoods, and the filter would converge to x. The cover $\{U_x\}_{x\in X}$ of X has a finite subcover $\{U_1, \cdots, U_n\}$. Since the sets U_k do not belong to $\mathcal{F}$, their complements must. But the intersection of the complements is empty, which contradicts one of the filter axioms.

For the reverse implication, suppose that there is an open cover $\{U_{\mathfrak{a}}\}_{\mathfrak{a}\in A}$ of X which has no finite subcover. That is, for every finite subset $I \subset A$ the set $X \setminus \cup_{\mathfrak{a}\in I} U_{\mathfrak{a}}$ is nonempty. Let $\mathcal{F}$ be the ultrafilter containing these sets. By hypothesis, the filter $\mathcal{F}$ converges to a point x of X. Hence x belongs to some U_x of the cover. Thus $U_x \in \mathcal{F}$ by definition of convergence. On the other hand $X \setminus U_x \in F$ by the construction of $\mathcal{F}$. This cannot happen in an ultrafilter.

Proposition. *A space is compact if and only if each filter on it has a cluster point.*

Proof. If $\mathcal{F}$ is a filter on X, then $\{F^-; F \in \mathcal{F}\}$ has the finite intersection property. If X is compact, there exists $x \in \cap_{\mathcal{F}} F^-$, which means that x is a cluster point of $\mathcal{F}$.

Conversely, suppose that every filter on X has a cluster point. Let $\mathcal{F}$ be an ultrafilter. That $\mathcal{F}$ has a cluster point x means that there is a filter $\mathcal{G} \supset \mathcal{F}$, such that $\mathcal{G} \to x$. But $\mathcal{F}$ is an ultrafilter, so $\mathcal{G} = \mathcal{F}$.

Closed and compact subsets. It is here that we see some differences with the behaviour of compact subsets of a metric space.

Proposition. (1) *If X is compact and $A \subset X$ is closed, then A is compact.*
(2) *If A is a compact subset of a hausdorff space X, then A is closed in X.*

Proof. (1) Let $\{V_{\mathfrak{a}}\}$ be an open cover of A. Then each $V_{\mathfrak{a}} = U_{\mathfrak{a}} \cap A$, with $U_{\mathfrak{a}}$ open in X. Thus $\{X \setminus A\} \cup \{U_{\mathfrak{a}}\}$ is an open cover of X. A finite subcover of it provides us with a finite subcover of the initial cover of A.

(2) We proof that $X \setminus A$ is open. Let $p \in X \setminus A$. For each $x \in A$ we can find disjoint neighborhoods U_x of p and V_x of x. Note that U_x does not meet $A \cap V_x$ (although it could meet A). Since A is compact, the cover $\{V_x \cap A\}_{x\in A}$ has a finite subcover. That is, there exist $x_1, \cdots, x_n$ such that $(V_{x_1} \cap A) \cup \cdots \cup (V_{x_n} \cap A) = A$. Hence $U_{x_1} \cap \cdots \cap U_{x_n}$ is a neighborhood of p which does not meet A.

Example. To see how (2) fails if X is not assumed to be hausdorff, take X to be the quotient of $\mathbb{R} \times \{0,1\}$ ($\{0,1\}$ dircrete) by the equivalence relation $(x,0) \sim (x,1)$ if $x \neq 0$. The image of $[-1,1] \times \{0\}$ in the quotient space is compact, but not closed.

Exercise. The intersection of two compact subsets of X need not be compact.

Exercise. The union of finitely many compact sets is compact. The intersection of closed compact sets is closed and compact. (Proof: one of them is compact, and the intersection is then closed in a compact set).

Exercise. Let $\mathcal{C}$ be a nonempty family of nonempty compact connected subsets of a hausdorff space. Suppose that whenever $C, C' \in \mathcal{C}$, there exists $C'' \in \mathcal{C}$ such that $C'' \subset C \cap C'$. Then the intersection of all sets in $\mathcal{C}$ is nonempty, compact, and connected.

Separation properties. Compact hausdorff spaces enjoy certain separation properties. The technique of proof is essentially the same as in the proposition above.

PROPOSITION. *In a compact hausdorff space X any pair of disjoint closed sets can be separated by disjoint open sets.*

PROOF. Suppose that H and K are closed in X. For each $x \in H$ and each $y \in K$ we can find disjoint open sets U_x and V_y which are neighborhoods of x and y, respectively. The collection $\{V_y\}_{y \in K}$ is an open cover of the compact set K, so finitely many of them are enough to do it, say $K \subset V_1 \cup \cdots \cup V_n = V'_x$. Then $U'_x = U_1 \cap \cdots \cap U_n$ is an open neighborhood of x disjoint from V'_x. Repeating the same process, but considering the cover $\{U'_x\}$ of H instead, we obtain two disjoint open sets separating H and K.

COROLLARY. *If U is open in X and $x \in U$, then there exists an open set V such that $x \in V \subset V^- \subset U$.*

PROPOSITION. *Let $A \times B$ be a compact subset contained in an open set W of the product space $X \times Y$. Then there are open sets U in X and V in Y such that $A \times B \subset U \times V \subset W$.*

PROOF. It consists in applying twice the technique of the previous proposition. Let $x \in A$. We cover the compact set $\{x\} \times B$ by base elements $U \times V \subset W$. Then there are finitely many of them, say $U_1 \times V_1, \cdots, U_n \times V_n$, such that $\{x\} \times B \subset \cup_{i=1}^n U_i \times V_i \subset W$. Let $U_x = \cap_{i=1}^n U_i$ and $V_x = \cup_{i=1}^n V_i$. Then $\{x\} \times B \subset U_x \times V_x \subset W$.

The sets $\{U_x \times V_x ; x \in A\}$ form an open cover of the compact set $A \times B$. Let $U_1 \times V_1, \cdots, U_n \times V_n$ form a finite subcover. Finally, take $U = \cup_{i=1}^n U_i$ and $V = \cap_{i=1}^n V_i$. Then $A \times B \subset U \times V \subset W$.

Continuity and compactness. The behavior is as expected

PROPOSITION. *The continuous image of a compact space is compact.*

We also have the following important result.

PROPOSITION. *Let $f : X \to Y$ be a continuous bijection. Suppose that X is compact and Y is hausdorff. Then f is a homeomorphisms.*

PROOF. If A is a closed subset of X, the it is compact. Its image fA is also compact, and since Y is hausdorff, it is closed in Y.

Exercise. If $f : \mathbb{I} \to X$ is continuous, onto and open, and X is a Hausdorff space with more than two points, then X and $\mathbb{I}$ are homeomorphic.

Exercise. Suppose that X is hausdorff and Y is compact and hausdorff. Then $f : X \to Y$ is continuous if and only if $\{(x, f(x)); x \in X\}$ is closed in $X \times Y$.

2. The Tichonov theorem

This is one of the great theorems of point set topology.

PROPOSITION. *A nonempty product $\prod_{\mathfrak{a}} X_{\mathfrak{a}}$ is compact if and only if each factor $X_{\mathfrak{a}}$ is compact.*

PROOF. If the product space is compact, so is each factor because the projections are continuous and onto.

Conversely, let $\mathcal{F}$ be an ultrafilter on the product space. Then $\pi_{\mathfrak{a}}\mathcal{F}$ is an ultrafilter on the space $X_{\mathfrak{a}}$, thus it converges to a point $x_{\mathfrak{a}} \in X_a$. Let $x = (x_{\mathfrak{a}}) \in \prod_{\mathfrak{a}} X_a$. To show that $\mathcal{F} \to x$, it is enough to show that every subbase element $\pi_{\mathfrak{a}}^{-1}U_{\mathfrak{a}}$, U_a open in $X_{\mathfrak{a}}$, is in $\mathcal{F}$. But $\pi_{\mathfrak{a}}\mathcal{F} \to x_a$, so $U_{\mathfrak{a}} \in \pi_{\mathfrak{a}}\mathcal{F}$, and by the 0-1 law for ultrafilters it must be that $\pi_{\mathfrak{a}}^{-1}U_{\mathfrak{a}} \in \mathcal{F}$.

Since Tichonov's theorem is so important, we may as well prove it twice. This second proof uses Alexander's subbase theorem.

Second proof. We consider the subbase $\mathfrak{S} = \{\pi_{\mathfrak{a}}^{-1}U_{\mathfrak{a}}; U_{\mathfrak{a}}\text{open in } X_{\mathfrak{a}}\}$ of the product space. Let $\mathcal{U}$ be an open cover of $\prod_{\mathfrak{a}} X_{\mathfrak{a}}$ by elements of $\mathfrak{S}$. For each index $\mathfrak{a}$, let $\mathcal{U}_{\mathfrak{a}}$ be the family of those open subsets U of $X_{\mathfrak{a}}$ such that $\pi_{\mathfrak{a}}^{-1}U \in \mathcal{U}$. It now follows that there exists an index β such that $\mathcal{U}_\beta$ is a cover of X_β. Indeed, if $U_{\mathfrak{a}}$ covers $X_{\mathfrak{a}}$ for no $\mathfrak{a}$, then we could pick a point $x_{\mathfrak{a}} \in X_{\mathfrak{a}}$ not belonging to any set in $\mathcal{U}_a$, and this would imply that the point $(x_{\mathfrak{a}}) \in \prod_{\mathfrak{a}} X\mathfrak{a}$ is not covered by $\mathcal{U}$.

Hence $\mathcal{U}_\beta$ is a cover of X_β, for some β. It has a finite subcover $U_1, \cdots, U_n$, and $\pi_\beta^{-1}U_1, \cdots, \pi_\beta^{-1}U_n$ is a finite subcover of $\mathcal{U}$.

Exercise. Tichonov's theorem does not hold if the product space is given the box topology.

Axiom of choice. In proving Tichonov's theorem one invokes the axiom of choice several times. An instructive exercise is to identify those places where the argument requires its use, and replace it by other topological hypothesis. For instance, can you prove that a product of finitely many compact hausdorff spaces is compact without using the axiom of choice?

It may perhaps be surprising that not only one needs the axiom of choice to prove the general version of Tichonov's theorem, but that one can deduce the axiom of choice from it. This is a theorem of Kelley.

Proposition. *Tichonov's theorem implies the axiom of choice.*

Proof. Let $\{X_\alpha\}$ be a collection of nonempty sets. Let ω be an object not in $\cup X_\alpha$. Let $Y_\alpha = X_\alpha \cup \{\omega\}$, and give it a topology which is cofinite on X_α and makes ω isolated. In $\prod Y_\alpha$, let $F_\alpha = \pi^{-1}(X_\alpha)$. This is a collection of closed sets with the finite intersection property, for if I is a finite subset of A, $x_\alpha \in X_\alpha$ a choice for $\alpha \in I$, $x_\alpha = \omega$ for $\alpha \notin I$, give a point of $\cap_I F_\alpha$. Thus $\cap F_\alpha \neq \emptyset$.

3. Local compactness

Definition. A space is said to be locally compact if each of its points has a neighborhood base formed by compact sets.

Examples. With the product topology, $\mathbb{R}^n$ is locally compact, but $\mathbb{R}^\omega$ is not. This last one is locally compact with the box topology.

Hilbert space $H = \{(x_n); x_n \in \mathbb{R}\,, \sum_n x_n^2 < \infty\}$ with the metric

$$d((x_n),(y_n)) = \sum_n (x_n - y_n)^2$$

is not locally compact. It is easy to see that a closed ball in hilbert space is not compact, using the sequential compactness version for metric spaces.

Exercise. The spaces $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are not locally compact.

Exercise. The Sorgenfrey line and the Moore plane are not locally compact.

Sometimes it is enough to find a compact neighborhood of a point to know there exists a compact neighborhood basis.

Proposition. *A hausdorff space is locally compact if and only if each point has a compact neighborhood.*

Proof. Let K be a compact neighborhood of x. Let U be any neighborhood of x. Then $V = (K \cap U)^\circ$ is an open neighborhood of x. The space V^- is compact and hausdorff and V is a neighborhood of x in V^-. Therefore, by the corollary above, we can find an open neighborhood W of x in V^- such that $\mathrm{Cl}_{V^-} W \subset V$. Since V is open, W is open in X. The set $\mathrm{Cl}_{V^-} W$ is closed in V^-, hence compact. Thus we found a compact neighborhood of x in X contained in U.

Corollary. *A compact hausdorff space is locally compact.*

Example. Let $X = \mathbb{Q} \cup \{\infty\}$. A subset U of X is open if either U is an open subset of $\mathbb{Q}$ or $X \setminus U$ is a compact subset of $\mathbb{Q}$. Then X is compact but not locally compact.

The following propositions describe the behaviour of local compactness under some standard constructions.

Proposition. *Continuous open onto maps preserve local compactness.*

Proof. Let $f : X \to Y$ be continuous, open and onto. Let $y \in Y$, and V a neighborhood of y. By continuity and local compactness, there exists a compact neighborhood K of $x \in f^{-1}(y)$ such that $fK \subset V$. Since $x \in K^\circ$ and f is open, $f(K^\circ)$ is a neighborhood of y, and so fK is a compact one.

Proposition. *A nonempty product $\prod X_\alpha$ is locally compact if and only if each factor is locally compact and almost all of them are compact.*

Proposition. *In a locally compact hausdorff space, the intersection of an open set with a closed set is locally compact. A locally compact subset of a hausdorff space is he intersection of a closed set and an open one. A dense subset of a compact hausdorff space is locally compact if and only if it is open.*

Proof. Suppose X is locally compact and hausdorff. If U is open in X and $x \in U$, then there is a compact neighborhood K of x contained in U. Thus U is also locally compact. If F is closed in X and $x \in F$, then x has a compact neighborhood K in X. But $K \cap F$ is a compact neighborhood of x in F, so it is locally compact. Since the intersection of two locally compact spaces in X is locally compact, the intersection of a closed and a open in X is locally compact.

4. Compactification

Definition. A compactification of a space X is a pair (Y, f), where Y is a compact space and $f : X \to Y$ is an embedding onto a dense subset.

Examples. $\mathbb{S}^1$ and $[0, 1]$ are compactifications of $(0, 1)$.
If X is a subset of a compact space Y, then (X^-, i) is a compactification of X.

One-point compactification. Also called Alexander compactification. Let X be a space, $X^* = X \cup \{\infty\}$, where ∞ is a point not in X. A subset U of X^* is open if either: (1) $U \subset X$ is open in the topology of X, or (2) $X^* \setminus U$ is a closed compact subset of X.

Exercise. If X is compact, ∞ is an isolated point of X^*.

Proposition. *X^* is a compact topological space, and X is a subspace of X^*. If X is not compact, then X is dense in X^*.*

Proof. Let U, V be open subsets of X^*. If U, V are open in X, so is $U \cap V$. If $H = X^* \setminus U$ and $K = X \setminus V$ are closed compact subsets of X, then $X^* \setminus (U \cap V) = H \cup K$ is closed and compact. If U is open in X and $K = X^* \setminus V$ is closed and compact, then $U \cap V = U \cap (X \setminus K)$ is open in X.

Let U_α, $\alpha \in A$ be a family of open subsets of X^*. If $\infty \notin U_\alpha$, the $U = \cup U_\alpha$ is open in X. Otherwise, $\infty \in U_0$, and $X^* \setminus U = X \setminus U = \cap_\alpha (X \setminus U_\alpha)$ is closed and compact, as is a closed subset of the compact space $X \setminus U_0$.

Next, X is a subspace of X^*. Indeed, if U is open in X, then $U = X^* \cap U$ is also open in $X^a st$. If U is open in X^* and $\infty \in U$, then $U \cap X$ is the complement in X of a closed set.

That X^* is compact is easy. If U_α is an open cover of X^*, then one U_0 contains ∞, and the other members of the covering form a cover of the compact complement of U_0.

Finally, X is dense in X^* if X is not compact. Indeed, if U is a neighborhood of ∞, then $X^* \setminus U$ is a compact subset of X, hence not equal to X. Thus $U \cap X \neq \emptyset$.

It may happen that the space X^* does not enjoy some of the properties that X has. Being hausdorff is one of them.

Example. Let $\mathbb{Q}^*$ be the one-point compactification of the rational numbers $\mathbb{Q}$. Although $\mathbb{Q}$ is hausdorff, $\mathbb{Q}^*$ is not because $\mathbb{Q}$ is not locally compact. However $\mathbb{Q}^*$ is T_1. Two other properties of $\mathbb{Q}^*$: it is connected, every sequence has a convergent subsequence.

Proposition. *The one-point compactification X^* of a locally compact hausdorff space X is a compact hausdorff space.*

Proof. We need to show that X^* is hausdorff. Let x, y be two distinct points of X^*. If both lie in X, then they can be separated because X is hausdorff. The other possibility is that $x \in X$ and $y = \infty$. By the local compactness of X, the point X has a compact neighborhood K, and this is disjoint of the neighborhood $X^* \setminus K$ of ∞.

Exercise. Let X be a compact hausorff space and $Y \subset X$. Then $(X \setminus Y)^*$ and the quotient space X/Y are homeomorphic.

Components of a compact hausdorff space

Definition. A subset A of a space X is quasiconnected in X if whenever $X = U \cup V$, the union of two disjoint open sets, then either $A \subset U$ or $A \subset V$.

Exercise. Connected subsets are quasiconnected, but not conversely. For instance, let $X = R \cup S$ be the subset of the plane union of the sets $R = \{(x, 1/n); -1 \leq x \leq 1, n = 1, 2, \cdots\}$ and $S = \{(x, 0); -1 \leq x \leq 1\}$. The subset $A = \{(x, 0); x \neq 0\}$ of X is quasiconnected in X but not connected.

Quasicomponents. The quasicomponent of a point x of a space X is the largest quasiconnected subset of X containing X. Thus a quasicomponent is the intersection of all subsets containing it which are both open and closed.

The quasicomponent of a point contains the component, but the example above shows they need not be equal. The set A is a quasicomponent of $R \cup A$, but not a component.

Proposition. *Let $\{A_i\}_{i \in I}$ be a collection of compact connected hausdorff subsets of a compact hausdorff space X, which is linearly ordered by inclusion. The $\cap_I A_i$ is compact and connected.*

Proof. The set $A = \cap_I A_i$ is compact because it is a closed subset of the compact hausdorff space X. In fact, it is not necessary to assume X to be compact and hausdorff, as it can be replaced by one of the A_i's. Suppose that A is not connected, so that $A = H \cup K$, the union of two disjoint nonempty closed subsets of A. Then H and K are also closed in X, and so they can be separated by disjoint open sets U and V. Since each A_i is connected, no A_i can be contained in $U \cup V$ (if both are nonempty). Because the A_i's are linearly ordered by inclusion, the family of nonempty closed sets $\{A_i \setminus (U \cup V)\}$ has the finite intersection property. Therefore, $\cap_i (A_i \setminus (U \cup V)) \neq \emptyset$. On the other hand

$$\cap_i (A_i \setminus (U \cup V)) = (\cap_i A_i) \setminus (U \cup V) = \emptyset,$$

a contradiction.

Exactly the same reasoning proves the following

Proposition. *Let $\{A_i\}_{i \in I}$ be a collection of compact connected hausdorff subsets of a compact hausdorff space X, which is linearly ordered by inclusion. The $\cap_I A_i$ is compact and quasiconnected in X.*

Proposition. *Let X be compact and hausdorff. Each component of X is the intersection of all open and closed sets containing it.*

Proof. Let Q be the quasi-component of a point $x \in X$. We know $Q = \cap F_\alpha$, where the F_α are the open and closed sets containing x. If Q is not connected, then $Q = H \cup K$ where H and K are disjoint nonempty closed subsets of Q, and $x \in H$. We can find disjoint open sets $U \supset H$ and $V \supset K$. The space $X \setminus (U \cup V)$ is covered by the open and closed sets $X \setminus F_\alpha$, and by compactness, finitely many of them are enough to cover it. Thus we have $F_1, \cdots, F_n$ such that $Q \subset F_1 \cap \cdots \cap F_n = F \subset U \cup V$. The set F is open and closed, so $F \cap U$ is open and contains x, and also

$$(F \cap U)^- \subset F \cap U^- = F \cap (U \cup V) \cap U^- = F \cap U$$

so that $F \cap U$ is also closed.

Totally disconnected. A space is totally disconnected if its components are the points.

Examples. The rational $\mathbb{Q}$, the irrational $\mathbb{R} \setminus \mathbb{Q}$, and Cantor's ternary set $\mathbb{G}$ are examples of totally disconnected spaces.

Knaster-Kuratowski space. This is a connected space K which has a point p such that $K \setminus \{p\}$ is totally disconnected. Recall that the Cantor set $\mathbb{G}$ is obtained by removing from the unit interval $\mathbb{I}$ a countable collection of open intervals. Let $E \subset \mathbb{G}$ be the set of endpoints of those intervals, and $F = \mathbb{G} \setminus E$. Let $p \in \mathbb{R}^2$ be the point $(1/2, 1/2)$. For each $x \in \mathbb{G}$, denote by S_x the straight line segment joining x and p. Let

$$T_x = \{(x_1, x_2) \in S_x; x_2 \,\text{rational}\,\} \qquad \text{if } x \in E,$$

$$T_x = \{(x_1, x_2) \in S_x; x_2 \,\text{irrational}\,\} \qquad \text{if } x \in F.$$

Then the subspace $K = \cup_{x \in \mathbb{G}} T_x$ of $\mathbb{R}^2$ is connected but $K \setminus \{p\}$ is totally disconnected.

Zero-dimensional spaces. A space X is zero-dimensional if each of its points has a neighborhood base consisting of sets which are both open and closed.

Exercise. X is zero-dimensional if and only if for each $x \in X$ and a closed set F not containing x there is an open-closed set containing x and disjoint from F.

Exercise. A zero-dimensional T_1-space is totally disconnected.

Exercise. A compact hausdorff space is totally disconnected if and only if each pair of distinct points can be separated by disjoint open-closed sets.

Exercise. A locally compact, totally disconnected hausdorff space is zero-dimensional.

Exercise. Let K, p be the Knaster-Kuratowski space and point described above. Then $K \setminus \{p\}$ is totally disconnected but not one dimensional.

Extremely disconnected spaces. A space is extremely disconnected if the closure of every open set in X is open.

Examples. Discrete spaces, cofinite spaces and cocountable spaces are extremely disconnected. A metric space is extremely disconnected if and only if it is discrete.

Exercise. X is extremely disconnected if and only if every two disjoint open subsets of X have disjoint closures.

Exercise. Dense subspaces and open subspaces of an extremely disconnected space are extremely disconnected. However, closed subspaces and products need not be.

Exercise. The only convergent sequences in an extremely disconnected space are those which are eventually constant.

CHAPTER VII

THE URYSOHN LEMMA

1. Regular and normal spaces

Axiom T_3. A topological space X is T_3 if whenever F is a closed subset of X and $x \notin F$, then there are disjoint open sets $U \ni x$ and $V \supset F$. In short, a closed set and a point not belonging to it can be separated by disjoint open sets.

Although it may look as if T_3 is stronger than hausdorff, that is not the case: a trivial space is T_3 but not hausdorff if it has more than one point.

Proposition. *The following are equivalent for a space X:*

(1) *X is T_3*
(2) *If U is open in X and $x \in U$, there exists an open set V such that $x \in V^- \subset U$.*
(3) *each point has a neighborhood base consisting of closed sets.*

Proof. (1) $\Rightarrow$ (2) $X \setminus U$ is a closed set not containing X.

(3) $\Rightarrow$ (1) If F is a closed set not containing x, then $X \setminus F$ is a neighborhood of x, so there is a closed neighborhood A of x with $A \subset X \setminus F$. Then A° and $X \setminus A$ are open sets separating x and F.

Regular spaces. Spaces which are T_1 (points are closed) and T_3 are called regular.

Regular spaces are hausdorff, but not conversely: Let X be the real line where every nonzero point has the standard neighborhood base. The neighborhoods of 0 being of the form $U \setminus F$, where U is a standard neighborhood and $F = \{1/n; n = 1, 2, \cdots\}$. This is a hausdorff space, but F is a closed set that cannot be separated from 0.

With the help of compactness it is true:

Proposition. *Compact hausdorff spaces are regular.*

Axiom T_4. A space X is T_4 if for each pair of disjoint closed subsets A, B of X there are disjoint open sets $U \supset A$ and $V \supset B$.

Normal spaces. A space which is T_1 and T_4 is called normal.

There is no relation between axioms T_3 and T_4. Let X be the real line with the topology which has the intervals (a, ∞) $(a \in \mathbb{R})$, as open sets. Then X is T_4 because any two nonempty closed sets intersect. On the other hand, the point 1 cannot be separated from the closed set $(-\infty, 0]$.

On the other hand, every normal space is regular.

Exercise. Find regular but not normal spaces.

Proposition. *Let X be a space in which every open cover has a countable subcover. Then if X is T_3 it is also T_4.*

Proof. Let A and B be disjoint closed subsets of X. For each $x \in A$ and each $y \in B$ we can find open neighborhoods U_x of x and V_y of y such that $U_x^- \cap B = V_y^- \cap A = \emptyset$.

The family of sets $\{U_x, V_y, X \setminus (A \cup B); x \in A, y \in B\}$ is an open cover of X, and by hypothesis it has a countable subcover. Thus we obtain countable families $\{U_n\}_n$ covering A and $\{V_n\}_n$ covering B.

Let $U'_n = U_n \setminus (V_1 \cup \cdots V_n)^-$ and $V'_n = V_n \setminus (U_1 \cup \cdots \cup U_n)^-$. Then $\{U'_n\}$ is still an open covering of A, for if $x \in A$,then $x \in U_n$ for some n, and x belongs to no V_k^-. Similarly, $\{V'_n\}$ is an open covering of B.

Furthermore, since $U'_n \cap V_m = \emptyset$ if $m \leq n$, we have $U'_n \cap V'_m = \emptyset$ if $m \leq n$. By reversing the roles of U's and V's, we obtain $U'_n \cap V'_m = \emptyset$ for all n, m. Thus $\cup_n U'_n$ and $\cup_m V'_m$ are disjoint open neighborhoods of A and B respectively.

We note the following condition implying the hypothesis of the proposition.

Proposition. *If the space X has a countable base $\mathfrak{B}$, then every open cover of X admits a countable subcover.*

Proof. Let $\mathfrak{U}$ be an open cover of X. For each $B \in \mathfrak{B}$, let $U_B \in \mathfrak{U}$ be such that $B \subset U$ (if it exists). The subfamily $\{U_B; B \in \mathfrak{B}\}$ is countable and still covers X, as every open set is a union of elements of $\mathfrak{B}$,

Second countable and lindelöf spaces. A space which has a countable base for its topology is said to be second countable. A space in which every open cover admits a countable subcover is called a lindelöf space.

Axiom $T_{3\frac{1}{2}}$. A space X satisfies the axion $T_{3\frac{1}{2}}$ if whenever F is a closed subset of X and $x \notin F$, there exists a continuous function $f : X \to \mathbb{I}$ such that $f(x) = 1$ and $f|F = 0$. Sometimes it may be more convenient to use the following equivalent definition: given $x \in X$ and a neighborhood U of x, there is a continuous function $F : X \to \mathbb{I}$ such that $f(x) = 0$ and $f(X \setminus U) = 1$.

Tichonov or Completely regular spaces. A space is completely regular or Tichonov if it is T_1 and $T_{3\frac{1}{2}}$.

Tichonov spaces are regular, but not conversely. In fact, there are T_3 spaces on which every continuous function is constant. However these examples are not easy to describe.

Example. The Moore plane M is tichonov. Let $x \in M$ and U a base neighborhood of x. That is, U is an open disc centered at x if x is in the open upper half plane, or $U = V \cup \{x\}$, V a disc tangent at x in the other case. Define f to be 0 at x and 1 on $X \setminus U$, and then extending linearly along the line segments joining x to the points on the boundary of U.

The space M is not normal, but the proof of this fact will be given after the Urysohn lemma.

Example. Every pseudometric space is completely regular. Indeed, $f(y) = d(y, F)$ is zero on F and $\neq 0$ on x.

2. Urhyson and Tietze extension lemmas

We will use the following version of regularity of a space X. Suppose that A, B are subsets of X with $A^- \subset B^\circ$. Then there is $C \subset X$ such that $A^- \subset C^\circ \subset C^- \subset B^\circ$.

Urysohn's lemma is one of the great theorems of topology. Not only it is simple to state, but also its proof if beautiful. Furthermore, its consequences and applications are numerous. Here it is:

Proposition. *Suppose that X is T_4. Then for every pair of disjoint closed (nonempty) subsets there exists a continuous function $f : X \to \mathbb{I}$ which takes the value 0 on one set and the value 1 on the other.*

Proof. Let A, B denote disjoint closed subsets of X. The idea to construct the continuous function $f : X \to \mathbb{I}$ with $f|A = 1$ and $f|B = 0$ is to consider a limit of step functions that increase from A to B. A family $\mathcal{A} = (A_0, A_1, \cdots, A_n)$ of subsets of X such that

$$A = A_0 \subset A_1 \subset \cdots \subset A_n = X \setminus B$$

and $A_{s-1}^- \subset A_s^\circ$ for all s will be called a chain from A to B. We call $A_{-1} = \emptyset$ and $A_{n+1} = X$

Given a chain $\mathcal{A}$, we consider the step function $f_{\mathcal{A}} : X \to \mathbb{I}$ which takes the value s/n on $A_s \setminus A_{s-1}$, $s = 0, \cdots, n$, and the value 1 outside A_n. The open sets $A_{s+1}^\circ \setminus A_{s-1}^-$, $s = 0, \cdots, n$, are called the step domains of $f_{\mathcal{A}}$. They form an open covering of the whole space because $A_s^- \setminus A_{s-1}^- \subset A_{s+1}^\circ \setminus A_{s-1}^-$. Also note that for each x, y in the same step domain of $\mathcal{A}$, the function $f_{\mathcal{A}}$ satisfies $|f_{\mathcal{A}}(x) \leq -f_{\mathcal{A}}(y)| \leq 1/n$.

The last concept we need is that of refinement of a chain $(A_0, \cdots, A_n)$. By this we mean a chain of the form $(A_0, A_0', A_1, \cdots, A_{n-1}', A_n)$.

Start with the chain $\mathcal{A}_0 = (A, X \setminus B)$, and let $\mathcal{A}_{n+1}$ be a refinement of $\mathcal{A}_n$ for each n. Let f_n be the step function of $\mathcal{A}_n$. Then the sequence of functions (f_n) is pointwise monotonically decreasing and bounded below by 0. Thus it is pointwise convergent and the limit function $f : X \to \mathbb{I}$ satisfies $f|A = 0$ and $f|B = 1$. It remains to show that f is continuous.

Since $|f(x) - f_n(x)| \leq 1/2^n$, and f_n does not fluctuate more than $1/2^n$ on each step domain of $\mathcal{A}_n$, it follows that f does not fluctuate more than $2/2^n$ on each step domain of $\mathcal{A}_n$. This implies continuity: given $x \in X$ and $\epsilon > 0$, choose n such that $2/2^n < \epsilon$, and the whole step domain of $\mathcal{A}_n$ containing x, which is an open neighborhood of x, will be mapped into $(f(x) - \epsilon, f(x) + \epsilon)$.

The Urysohn lemma has many important consequences and applications. The first one that is usually mentioned is Tietze extension lemma, although it was discovered earlier.

Proposition. *Let X satisfy the hypothesis of Urysohn lemma. Then every continuous function $f : A \to [a, b]$ defined on a closed set can be extended to a continuous function $F : X \to [a, b]$.*

Proof. To begin with, we observe that if $\phi : A \to \mathbb{R}$ is a function such that $|\phi(x)| \leq c$ for all $x \in A$, then there exists a continuous function $\Phi : X \to \mathbb{R}$ such that $|\Phi(x)| \leq \frac{1}{3}c$ for all $x \in X$ and $|\phi(x) - \Phi(x)| \leq \frac{2}{3}c$ for all $x \in A$. We call such a function Φ a $\frac{1}{3}$-approximate extension of ϕ.

Indeed, since the sets $H = \phi^{-1}[-c, -\frac{1}{3}c]$ and $K = \phi^{-1}[\frac{1}{3}c, c]$ are disjoint and closed in A, they are closed in X, and by the Urysohn lemma there exists a continuous function $g : X \to \mathbb{I}$ such that $g|H = 0$ and $g|K = 1$. Then we take $\Phi(x) = \frac{2}{3}c(g(x) - \frac{1}{2})$.

We now construct the extension F of f. We may assume that $[a, b] = [-1, 1]$. We first choose a $\frac{1}{3}$-approximate extension F_1 of f, and inductively, choose a $\frac{1}{3}$-approximate extension F_{n+1} of $f - (F_1 + \cdots + F_n)|A$. Then we have $|f(x) - \sum_{i=1}^n F_i(x)| \leq (\frac{2}{3})^n$ for all $x \in A$, and $|F_{n+1}(x)| \leq \frac{1}{3}(\frac{2}{3})^n$ for all $x \in X$. Therefore the series $\sum_{i=1} F_i$ converges uniformly to the continuous extension $F : X \to [-1, 1]$.

The Tietze extension lemma also holds if we replace the interval $[-1, 1]$ by the reals $\mathbb{R}$.

Proposition. *Let X be as above. Then every continuous function $f : A \to \mathbb{R}$ defined on a closed set $A \subset X$ can be extended to X.*

Proof. Denote by $i : \mathbb{R} \to [-1, 1]$ the embedding $i(x) = x/(1 + |x|)$. By what we just proved, there is a continuous extension $F_1 : X \to [-1, 1]$ of if. Clearly, $B = F_1^{-1}(\{-1, 1\})$ is a closed subset of X disjoint from A. Let $h : X \to [0, 1]$ be a continuous function such that $h|A = 1$ and $h|B = 0$. Then the function $F_2 = hF_1$ is also an extension of if with $F_2(X) \subset i(\mathbb{R}) = (-1, 1)$. Thus $F = i^{-1} \circ F_2$ is the required continuous extension of f.

Example. The Moore plane is hausdorff but not normal. This is the set of points in the upper half plane $\{(x, y); y \geq 0\}$. Let Y denote the x-axis. Its induced topology is discrete. The points (p, q), where q and q are rational and $q > 0$ is a countable dense subset D. This implies that there are at most $\mathfrak{c}^{\aleph_0} = \mathfrak{c}$ continuous functions on D. Since Y is discrete with cardinal $\mathfrak{c}$, there are at least $2^{\mathfrak{c}}$ continuous functions on Y. Since $\mathfrak{c} < 2^{\mathfrak{c}}$, not all of them can be extended to X.

Exercise. Let X be a space which has a dense subset D and a closed relatively discrete subset Y such that $2^{\mathfrak{o}(D)} \leq \mathfrak{o}(Y)$. Then X is not T_4.

3. Embeddings

Suppose that $\mathcal{A}$ is a family of continuous functions $f : X \to Y_f$. Then there is a natural map $e : X \to \prod_f Y_f$ whose value at x is the element $e(x)$ of the product whose f-coordinate is $f(x)$. This is called the evaluation map. It is continuous if all f are continuous. It will be an embedding if $\mathcal{A}$ contains sufficiently many functions.

We say that the family $\mathcal{A}$ separates points of X if for any pair $x \neq y$ in X there exists $f \in \mathcal{A}$ such that $f(x) \neq f(y)$. We say that $\mathcal{A}$ separates points of X from closed sets if for each closed subset F of X and each point $x \notin F$, there exists $f \in \mathcal{A}$ such that $f(x) \notin f(F)^-$.

Proposition. *Let $\mathcal{A} = \{f : X \to Y_f\}$ be a family of continuous functions. Then*

1. *The evaluation map $e : X \to Y = \prod Y_f$ is continuous.*
2. *If $\mathcal{A}$ separates points, then e is injective.*
3. *If $\mathcal{A}$ separates points from closed sets, then the map $X \to e(X)$ is open.*

Proof. The map e is continuous because its compositions with the projections π_f are continuous: $\pi_f e(x) = f(x)$. Part (2) is obvious.

To prove (3) we have to show that the image $e(U)$ of an open neighborhood $U \subset X$ of a point x contains the intersection of $e(X)$ with a neighborhood of $e(x)$ in the product space. Let $f \in \mathcal{A}$ be such that $f(x) \notin f(X \setminus U)^-$. Then $\pi_f^{-1}(Y_f \setminus f(X \setminus U))$ is an open set in Y containing $e(x)$, and its intersection with $e(X)$ is contained in $e(U)$.

A consequence of this proposition is that a tichonov space is homeomorphic to a subspace of a cube. A cube is a space homeomorphic to a product of closed intervals. We show that this actually characterizes tichonov spaces. We first need the following:

Proposition. *The product of $T_{3\frac{1}{2}}$-spaces is a $T_{3\frac{1}{2}}$-space.*

Proof. We say that a continuous function $f : X \to \mathbb{I}$ works for the pair (x, U) if U is a neighborhood of x and $f(x) = 0$, $f|(X \setminus U) = 1$. If $f_1, \cdots, f_n$ are functions that work for $(x, U_1), \cdots, (x, U_n)$, and if $g(x) = \sup\{f_i(x); 1 \leq i \leq n\}$, then g works for the pair $(x, \cap_{i=1}^n U_i)$.

Therefore, to show that a space is $T_{3\frac{1}{2}}$ it is enough to show that for each x and each neighborhood U of x belonging to a subbase for the topology, there is a function that works for (x, U).

Let $X = \prod_\alpha X_\alpha$ be a product of $T_{3\frac{1}{2}}$-spaces, and let $x \in X$. Let U_α be a neighborhood of $x(\alpha)$ in X_α. If f works for $(x(\alpha), U_\alpha)$, then $f \circ \pi_{\mathfrak{a}}$ works for $(x, \pi_\alpha^{-1} U_\alpha)$.

Proposition. *A space is tichonov if and only if it is homeomorphic to a subspace of a cube.*

Proof. Since the interval $\mathbb{I}$ is tichonov, a cube, being a product of closed intervals, is also tichonov. Every subspace of a tichonov space is tichonov: if $A \subset X$ and $x \in A \setminus B$, where B is a closed subset of A, then we write $B = X \cap F$, F closed in X. Since $x \not{i}n F$, we can find a continuous $f : X \to \mathbb{I}$ separating x and F. Thus $f|A$ separates x from B.

On the other hand, the family $\mathcal{A}$ of all continuous functions $X \to \mathbb{I}$ separates points and points from closed sets. Therefore the evaluation map $e : X \to \mathbb{I}^{\mathcal{A}}$ is an embedding.

As a consequence of the previous discussions we have

Corollary. *A locally compact hausdorff space is a tichonov space.*

4. Stone-Čech compactification

Recall that a compactification of a space X is a pair (Y, f) consisting of a compact space Y and an embedding $f : X \to Y$ with dense image. We say that (Y, f) is a hausdorff compactification if Y is hausdorff.

In the family of all compactifications of a space X we define the following relation: $(Y, f) \leq (Z, g)$ if and only if there is a continuous function $g : Z \to Y$ such that $h \circ g = f$. In other words, the function $f \circ g^{-1} : g(X) \to Y$ admits a continuous extension to Y.

If h can be taken to be a homeomorphism, then we say that (Y, f) and (Z, g) are topologically equivalent. In this case, $(Y, f) \leq (Z, g)$ and $(Z, g) \leq (Y, f)$.

Proposition. *The relation $\leq$ is a partial order in the set of compactifications of a space X. If hausdorff compactifications (Y, f) and (Z, g) of X satisfy $(Y, f) \leq (Z, g) \leq (Y, f)$, then they are topologically equivalent.*

Proof. If $(W, h) \leq (Z, g) \leq (Y, f)$ are compactifications of X, then there are functions $k : Y \to Z$ and $l : Z \to W$ such that $kf = g$ and $lg = h$. Thus $lk : Y \to Z$ satisfies $lkf = h$, so $(W, h) \leq (Y, f)$.

If $(Y, f) \leq (Z, g) \leq (Y, f)$ for hausdorff compactifications, then there are functions $k : Y \to Z$ and $l : Z \to Y$ such that $kf = g$ and $lg = f$. The function $lk : Y \to Z$ is the identity on $f(X) \subset Y$. Since this is a dense subspace and Y is hausdorff, it must be the identity on Y. Similarly, kl is the identity on Z. Thus k, l are homeomorphisms.

Exercise. Let $X = (0, 1)$. Then $(Y = [0, 1], f(x) = x)$ and $(Z = \mathbb{S}^1, g(x) = e^{2\pi x})$ are compactifications of X such that $(Z, g) \leq (Y, f)$, but not conversely.

Exercise. Compare the compactifications $\mathbb{Q}^*$ and $[-\infty, \infty]$ of the rationals $\mathbb{Q}$.

The smallest compactification of a compact space X is X itself. One may expect that the one-point compactification of a noncompact space would be the minimum for the partial order $\leq$. This is not true in general, although it is true if we consider only hausdorff compactifications.

Exercise. Let X be a locally compact hausdorff space. Then (X^*, i) is the minimum among all hausdorff compactifications of X.

The proof uses a previous exercise: If Z is a compact hausdorff space and $D \subset Z$ is a dense locally compact subspace, then D is open in Z. Using this it is easy to check that the map $h : Y \to X^*$ defined by $h(y) = \infty$ if $y \in Y \setminus f(X)$ and $h(y) = x$ if $y = f(x)$ is continuous and $hf = i$.

Stone-Čech compactification. If the space X admits a hausdorff compactification (so X is tichonov), then there is a maximal compactification among the hausdorff ones. (Note that the fact that a space admits a hausdorff compactication does not imply that its one point compactification is hausdorff. That is, subspaces of compact hausdorff spaces need not be locally compact.)

Let $B = B(X)$ denote the family of all continuous maps $X \to \mathbb{I}$. Then $\mathbb{I}^B$ is a compact hausdorff space and the evaluation map $e : X \to \mathbb{I}^B$ is an embedding if and only if X is a tichonov space. The Stone-Čech compactification of X is the pair $(\beta X, e)$, where βX is the closure of $e(X)$ in $\mathbb{I}^B$.

Proposition. *Let X be a tichonov space and let $f : X \to Y$ be a continuous map into a compact hausdorff space Y. Then $f \circ e^{-1}$ admits a continuous extension to βX.*

Proof. Let $B(Y)$ denote the set of continuous functions $Y \to \mathbb{I}$. Then f induces a map $f^* : B(Y) \to B$ by $f^*(g) = gf$. This in turn induces a map $f^B : \mathbb{I}^B \to \mathbb{I}^{B(Y)}$ by $f^B(q) = qf^*$. Let $\epsilon : Y \to \mathbb{I}^{B(Y)}$. We have the following diagram of continuous maps:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle \epsilon} \\ \beta X \subset \mathbb{I}^B & \xrightarrow{f^B} & \mathbb{I}^{B(Y)} \supset \beta Y \end{array}$$

The map $g : Y \to \beta Y$ is a homeomorphism because Y is compact and hausdorff. Since the map f^B is continuous, we would be done if we show that the diagram commutes ($f^B e = \epsilon f$), because then $\epsilon^{-1} f^B$ would be the required extension of fe^{-1}.

If $x \in X$ then $(f^B \circ e)(x) = f^B(e(x)) = e(x) \circ f^*$, and $(\epsilon \circ f)(x) = \epsilon(f(x))$. Let $h \in B(Y)$ and compute the h-coordinate of these two elements of $\mathbb{I}^{B(Y)}$:

$$\pi_h((f^B \circ e)(x)) = e(x)(h \circ f) = h \circ f(x) = (\epsilon \circ f)(x)(h) = \pi_h((\epsilon \circ f)(x)).$$

Exercise. This extension property of the Stone-Čech compactification with respect to compact hausdorff spaces does in fact characterize it among hausdorff compactifications. More precisely, suppose that Y is a hausdorff compactification of X with the property that every continuous map form X into a compact hausdorff space extends to Y. Then Y and βX are topologically equivalent compactifications of X.

This characterization allows you to decide where some space is or is not the Stone-Čech compactification of a familiar one. For instance, $\mathbb{I}$ is not the Stone-Čech compactification of $(0, 1)$, for the function $\sin(1/x)$ has no continuous extension to $\mathbb{I}$.

Size of $\beta\mathbb{N}$. . The product space $\mathbb{I}^{\mathfrak{c}}$ has a countable dense subset D. Any map $f : \mathbb{N} \to D$ is continuous, so it has a continuous extension $F : \beta\mathbb{N} \to \mathbb{I}^{\mathfrak{c}}$. Furthermore, if f is surjective, so is F, hence $\mathfrak{o}(\beta\mathbb{N}) \geq \mathfrak{o}(\mathbb{I}^{\mathfrak{c}}) = 2^{\mathfrak{c}}$. On the other hand, $\mathfrak{o}(B(\mathbb{N})) = \mathfrak{o}(\mathbb{I}^{\mathbb{N}}) = \mathfrak{c}$, so $\beta\mathbb{N} \subset \mathbb{I}^{\mathfrak{c}}$. Thus $\mathfrak{o}(\beta\mathbb{N}) = 2^{\mathfrak{c}}$.

Using the extension property of the Stone-Čech compactification one shows that $\beta\mathbb{R}$ and $\beta\mathbb{Q}$ also have cardinal number $2^{\mathfrak{c}}$.

Filter description of βX. The Stone-Čech compactification admits a description in terms of a special class of filters. It may be more appropriate to understand the structure of βX.

Zero sets. A set $A \subset X$ is called a zero set if there is a continuous function $f : X \to \mathbb{I}$ such that $A = f^{-1}(\{0\})$.

Thus, if X is a tichonov space, its zero sets form a base for the closed subsets, that is, every closed subset of X is an intersection of zero sets.

***z*-filters.** Let $\mathcal{Z}(X)$ denote the collection of zero sets of a tichonov space X. A nonempty family $\mathcal{F} \subset \mathcal{Z}(X)$ is called a z-filter if

(1) $\emptyset \notin \mathcal{F}$.
(2) If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.
(3) If $A \subset B$ are zero sets and $A \in \mathcal{F}$, then $B \in \mathcal{F}$.

In other words, a z-filter $\mathcal{F}$ is of the form $\mathcal{F}' \cap \mathcal{Z}(X)$, where $\mathcal{F}'$ is a filter on X.

A z-ultrafilter is a maximal z-filter. Let ζX denote the collection of all z-ultrafilters on X. A topology on ζX is defined by taking as base for closed sets all sets of the form $C_A = \{F \in \zeta X; A \in F\}$, where A is any zero sets. The function $h : X \to \zeta X$ which takes $x \in X$ to the z-ultrafilter of all zero sets containing x is an embedding with dense image. Furthermore, ζX is compact because it satisfies the finite intersection property.

To see that βX is homeomorphic to ζX, we verify that ζX has the universal extension property that characterizes the Stone-Čech compactification. So let $g : X \to Y$ be a continuous map into a compact hausdorff space Y. If $F \in \zeta X$, let $\mathcal{F}$ denote the collection of zero sets A in Y such that $g^{-1}A \in F$. Then $\mathcal{F}$ is a filter on Y, and since Y is compact, $\cap_{\mathcal{F}} A \neq \emptyset$. Furthermore, if A and B are zero sets such that $A \cup B \in \mathcal{F}$, then either A or B belongs to $\mathcal{F}$. Suppose that $x, y \in \cap_{\mathcal{F}} A$. If $x \neq y$, let U, V be disjoint neighborhoods of x and y in Y whose complements are zero sets. Since $(Y \setminus U) \cup (Y \setminus V) \in \mathcal{F}$, either $Y \setminus U$ or $Y \setminus V$ is in $\mathcal{F}$, which is a contradiction. Thus $\cap_{\mathcal{F}} A$ consists of exactly one point. We call this point $G(F)$. Then $G : \zeta X \to Y$ is an extension of g. To see that it is continuous, note that if A is a zero set in Y, then $G^{-1}(A) = C_{f^{-1}A}$.

5. Metrization

We know that metric spaces satisfy most of the properties of topological spaces that we have been discussing. They are normal and first countable. A theorem of Urysohn characterizes those second countable metric spaces.

PROPOSITION. *A countable product of metrizable spaces is metrizable.*

PROOF. Let X_n, $n = 1, 2, \cdots$, be a sequence of metrizable spaces, and let $X = \prod_x X_n$. Let d'_n be a metric on X_n inducing its topology. We replace d'_n by the bounded metric $d_n = \max\{1, d'_n\}$, which also induced the topology of X_n. Then, if $x = (x_n)$ and $y = (y_n)$ are points in X,

$$d(x, y) = \sum_{n=1}^{\infty} 2^{-n} d_n(x_n, y_n)$$

is a metric on X as one easily checks.

We show that this metric gives the product topology on X. Let $x = (x_n)$ be a point on X. A basic neighborhood U of x in the product topology is of the form

$$U = B_1(x_1, r_1) \times \cdots \times B_n(x_n, r_n) \times \prod_{k=n+1} X_k.$$

Choose $r = \min\{r_i/2^i; i = 1, \cdots, n\}$. One checks that if $d(x, y) < r$ then $d_i(x_i, y_i) < r_i$ for each $i = 1, \cdots, x_n$, so that $B(x, r) \subset U$.

Conversely, given $r > 0$, let n be large enough so that $\sum_{k=n} 2^k < r$. Then

$$B_1(x_1, r/2n) \times \cdots \times B_n(x_n, r_n/2n) \times \prod_{k=n+1} X_k$$

is a product basic neighborhood of x contained in the ball $B(x, r)$.

PROPOSITION. *For a space X the following are equivalent:*

(1) *X is regular and second countable,*
(2) *X is homeomorphic to a subspace of the cube $\mathbb{I}^{\mathbb{N}}$,*
(3) *X is separable and metrizable.*

PROOF. (1) $\Rightarrow$ (2) Let $\mathfrak{B}$ be a countable base for X. The hypothesis of (1) imply that X is normal, so for each pair of base elements $U, V \in \mathfrak{B}$ with $U^- \subset V$ there exists a continuous function $f_{UV} : X \to \mathbb{I}$ such that $f_{UV}|U^- = 0$ and $f_{UV}|(X \setminus V) = 1$. The family $\mathcal{A}$ of all these functions f_{UV} separates points and points from closed sets (because X is T_1 and $\mathfrak{B}$ is a base). Thus the evaluation map $e : X \to \mathbb{I}^{\mathcal{A}}$ is an embedding. Since $\mathfrak{B}$ is countable, so is $\mathcal{A}$, hence $\mathbb{I}^{\mathbb{N}}$ and $\mathbb{I}^{\mathcal{A}}$ are homeomorphic.

(2) $\Rightarrow$ (3) Being separable and metrizable are properties inherited by subspaces.

(3) $\Rightarrow$ (1) We already know this.

CHAPTER VIII

PARACOMPACT SPACES

1. Paracompact spaces

Refinements. Let X be a space and $\mathcal{U}$ be a cover of X. We say that a cover $\mathcal{V}$ of X is a refinement of $\mathcal{U}$ if each $V \in \mathcal{V}$ is contained in some $U \in \mathcal{U}$. We write $\mathcal{V} \prec \mathcal{U}$.

Locally finite families. A collection $\mathcal{U}$ of subsets of X is locally finite if each $x \in X$ has a neighborhood meeting only finitely many $U \in \mathcal{U}$.

Example. Under what conditions on X is the cover $\{\{x\}\}_{x\in X}$ locally finite?

Exercise. Let $\{A_\mathfrak{a}\}$ be a locally finite family of subsets of X. Then $\{A_\mathfrak{a}^-\}$ is also a locally finite family. Moreover, $\cup_\mathfrak{a} A_\mathfrak{a}^- = (\cup_\mathfrak{a} A_\mathfrak{a})^-$, so that the union of a locally finite family of closed sets is closed.

Paracompact. A space X is said to be paracompact if it is hausdorff and if each of its open covers has a locally finite open refinement.

Example. Compact hausdorff spaces are paracompact.

Exercise. A closed subset of a paracompact space is paracompact.

PROPOSITION. *A regular lindelóf space X is paracompact.*

PROOF. Let $\mathcal{U}$ be an open cover of X. For each $x \in X$, choose $U_x \in \mathcal{U}$ containing it. By regularity we can find open neighborhoods V_x, W_x of x such that $V_x \subset V_x^- \subset W_x \subset W_x^- \subset U_x$.

By the lindelöf property, we can find a countable subcover $\{V_1, V_2, \cdots\}$ of the open cover $\{V_x\}_{x\in X}$ of X. Let $T_1 = W_1$ and, for $n > 1$, let $T_n = W_n \cap (X \setminus V_1^-) \cap \cdots \cap (X \setminus V_{n-1}^-)$.

Then $\{T_n\}_{n=1}^\infty$ is an open refinement of $\mathcal{U}$, and a cover of X, for if $x \in X$, then there is a smallest n such that $x \in W_n$. Thus $x \notin W_1 \cup \cdots \cup W_{n-1} \supset V_1^- \cup \cdots \cup V_{n-1}^-$. Hence $x \in T_n$.

It is also locally finite, for if $x \in X$, then $x \in V_n$ for some n, and so $x \notin T_m$ if $m > n$.

Exercise. A regular second, countable space is paracompact.

Exercise. The Sorgenfrey line is paracompact, but $S \times S$ is not.

Exercise. If X is a compact hausdorff space and Y is paracompact, the $X \times Y$ is paracompact.

Exercise. If $X \times Y$ is paracompact, then both X and Y are paracompact.

Exercise. Continuous images of paracompact space need not be paracompact. For instance, every discrete space is paracompact, and every space is the continuous image of a discrete one.

PROPOSITION. *Every paracompact space is regular.*

PROOF. Suppose that F is a closed subset of a paracompact space X, and that $x \notin X$. For each $y \in F$, let V_y be an open neighborhood of y whose closure does not contain x (this is possible because X is hausdorff). Then $\{V_y\}_{y\in F} \cup \{X \setminus F\}$ is an open cover of X. Let $\mathcal{U}$ be an open locally finite refinement. Then $W = \cup\{U \in \mathcal{U}; U \cap F \neq \emptyset\}$ is an open set containing F. Since $\mathcal{U}$ is locally finite, $W^- = \cup\{U^-; U \cap F \neq \emptyset\}$, and so $x \notin W^-$.

Exercise. Every paracompact space is normal.

Stone's theorem. The following important result is due to Stone.

Proposition. *Metric spaces are paracompact.*

σ-compact. Given a property P of spaces, there is the σ-P property: A space X is σ-P is it is a countable union of spaces, each one having property P. We only discuss one possibility.

A space X is σ-compact if it can be expressed as a countable union of compact subsets.

Proposition. *If X is a locally compact, hausdorff, σ-compact space, then $X = \cup_{n=1}^{\infty} U_n$, where U_n are open subsets of X with compact closure, and such that for each n, $U_n^- \subset U_{n+1}$.*

Proof. By σ-compactness, we can write $X = \cup_{n=1}^{\infty} K_n$, a countable union of compact sets. For each $x \in K_1$, let V_x be an open neighborhood of X with compact closure. Then $\{V_x\}_{x \in K_1}$ is an open cover of K_1, from which we extract a finite subcover. Let U_1 be the union of the members of this finite subcover. Then U_1^- is compact. Apply the same process to obtain an open set U_2 with compact closure containing the compact set $K_2 \cup U_1^-$, and so on.

Proposition. *If X is a σ-compact space, then it is lindelöf.*

Proof. Let $\mathcal{U}$ be an open cover of X. Write $X = \cup_n K_n$, a countable union of compact subsets. For each n, let $U_{1,n}, \cdots, U_{k(n),n}$ be a finite number of members of $\mathcal{U}$ which cover the compact set K_n. It follows that $\{U_{i,n}; i = 1, \cdots, k(n)\}_n$ is a countable subcover of $\mathcal{U}$.

Manifolds. A manifold (of dimension n) is a second countable hausdorff space each point of which has a neighborhood homeomorphic to an open ball in $\mathbb{R}^n$.

Exercise. A manifold is a paracompact space. More generally, a hausdorff, locally compact and σ-compact space is paracompact.

Proposition. *If X is a compact manifold, then X can be embedded into some euclidean space $\mathbb{R}^N$.*

Proof. First note that if U is an open subset of X homeomorphic to a ball in euclidean space, then the quotient space $X/(X \setminus U)$ is homeomorphic to the n-sphere $\mathbb{S}^n$, where n is the dimension of X.

By covering X with a finite number of open sets like U above, we obtain a map from X into a product of spheres, and it is easily seen to be an embedding. Finally, embed each sphere in a euclidean space.

2. Partitions of unity

Partition of unity. A partition of unity of a space X is a family of continuous functions $\Phi = \{\varphi : X \to [0,1]\}$ such that

(1) it is locally finite, that is, each point $x \in X$ has a neighborhood on which only finitely many $\varphi \in \Phi$ do not vanish, and
(2) for each $x \in X$, $\sum_{\varphi \in \Phi} \varphi(x) = 1$.

A partition of unity Φ is subordinate to an open cover $\mathcal{U}$ of X if for each $\varphi \in \Phi$ there exists $U \in \mathcal{U}$ such that $\text{supp}(\varphi) \subset U$. Here $\text{supp}(\varphi) = \text{Cl}\{x \in X; \varphi(x) \neq 0\}$.

PROPOSITION. *Every open cover of a paracompact space admits a partition of unity subordinate to it.*

PROOF. Let $\mathcal{U} = \{U_{\mathfrak{a}}\}_{\mathfrak{a}\in A}$ be an open cover of the paracompact space X, which we assume to be locally finite.

First we show that there is an open cover $\{V_{\mathfrak{a}}\}$ of X such that $V_{\mathfrak{a}}^- \subset U_a$ for each $\mathfrak{a} \in A$. For each $x \in X$ we can find (using normality of paracompact spaces) an open neighborhood W_x of x such that $W_x \subset U$ for some $U \in \mathcal{U}$. Let $\{W_\beta\}$ be a locally finite refinement of the open cover $\{W_x\}_{x\in X}$. Let $V_\alpha = \cup_\beta\{V_\beta; W_\beta \subset U_{\mathfrak{a}}$. Then $\{V_\alpha\}$ is an open cover of X, and from the locally finiteness it follows that $V_\alpha^- \subset U_{\mathfrak{a}}$, for each $\mathfrak{a}$.

Applying this process twice, we find locally finite open covers $\{W_{\mathfrak{a}}\}$ and $\{V_{\mathfrak{a}}\}$ of X such that for each $\mathfrak{a}$, $W_{\mathfrak{a}}^- \subset V_{\mathfrak{a}} \subset V_{\mathfrak{a}}^- \subset U_{\mathfrak{a}}$. Then we can find continuous functions $\phi_{\mathfrak{a}} : X \to [0,1]$ such that $\phi_{\mathfrak{a}}|W_{\mathfrak{a}}^- = 1$ and $\phi_{\mathfrak{a}}|(X \setminus V_a) = 0$.

The function $\phi(x) = \sum_{\mathfrak{a}} \phi_{\mathfrak{a}}(x)$ is continuous and strictly positive on X. Let $\varphi_{\mathfrak{a}} = \phi_a/\phi$. Then $\{\varphi_{\mathfrak{a}}\}$ is the desired partition of unity.

Relabeling trick. The proof above produces a partition of unity $\{\varphi_{\mathfrak{a}}\}$ subordinated to a locally finite open cover $\{U_{\mathfrak{a}}\}$, with the same index set. In general, if you start with an arbitrary open cover $\mathcal{U}$, the construction will produce a partition of unity subordinate to a refinement $\mathcal{V}$ of $\mathcal{U}$, and there is no reason why the index sets of $\mathcal{U}$ and $\mathcal{V}$ should be the same. However, this mismatch problem is easily resolved with the following argument.

Suppose that $\{U_{\mathfrak{a}}\}_{\mathfrak{a}\in A}$ is an open cover of X and that $\{W_i\}_{i\in I}$ is an open, locally finite refinement of it. For each $i \in I$, choose $\mathfrak{a}(i) \in A$ such that $W_i \subset U_{\mathfrak{a}(i)}$. For each $\mathfrak{a} \in A$, let $V_{\mathfrak{a}} = \cup\{W_i; \mathfrak{a}(i) = \mathfrak{a}\}$. Then $\{V_{\mathfrak{a}}\}$ is a locally finite open cover of X such that $V_{\mathfrak{a}} \subset U_a$ for each $\mathfrak{a} \in A$. Of course, some of the $V_{\mathfrak{a}}$ may be empty.

Exercise. If $\mathcal{U}$ is a finite open cover of a normal space, then there is a partition of unity subordinate to $\mathcal{U}$.

Partitions of unity are useful because they allow us to glue local objects, usually functions. They are used very often in the study of manifolds.

Here is one of their applications. Say that a function $f : X \to \mathbb{R}$ is lower semicontinuous (resp. upper semicontinuous) if $f^{-1}(a,\infty)$ (resp. $f^{-1}(-\infty,a)$) is open in X for each $a \in \mathbb{R}$.

Thus a function $f : X \to \mathbb{R}$ is continuous if and only if it is is both upper and lower semicontinuous.

PROPOSITION. *Let f and g be lower and upper semicontinuous functions, respectively, on a paracompact space X, and such that $f(x) < g(x)$ for each $x \in X$. Then there is a continuous function h on X such that $f(x) < h(x) < g(x)$ for each $x \in X$.*

PROOF. For each $r \in \mathbb{R}$, let $U_r = f^{-1}(r,\infty) \cap g^{-1}(-\infty,r)$. Then $\mathcal{U} = \{U_r\}$ is an open covering of X. After passing to a locally finite open refinement and using the relabeling trick above, we obtain a partition of unity $\{\varphi_r\}$ subordinate to $\mathcal{U}$. Then the function $h = \sum_{r\in\mathbb{R}} r\varphi_r$ is continuous and has the required interpolation property.

CHAPTER IX

FUNCTION SPACES

1. Topologies on function spaces

Definition. Let Y be a topological space, X a set, $F \subset Y^X$ a non-empty family of functions, and $\Phi \subset \mathcal{P}(X)$ a non-empty family of subsets of X. For $K \in \Phi$ and $U \subset Y$ open, set

$$[K, U] = \{f \in F; f(K) \subset U\}.$$

The collection $\{[K, U]; K \in \Phi, U \subset Y \text{ open}\}$ is a subbase for the Φ-open topology on F.

Point-open topology. If $\mathcal{F} = Y^X$ and $\Phi = \{\{x\}; x \in X\}$, then the Φ-open topology is the product topology. It is also called the pointwise convergence topology.

Compact-open topology. If X is a space, $\mathcal{F} = \mathcal{C}(X, Y)$ the set of continuous functions, and $\Phi = \{K \subset X; K \text{ compact}\}$, the Φ-open topology is called the compact-open topology.

Example. These two topologies agree if X is discrete, but in general the compact-open topology is strictly finer.

Exercise. Let Y^X have the compact open topology. Then Y^X is T_0, T_1 or T_2 if and only if Y is.

Compact convergence topology. Let X be a space and let (Y, d) be a metric space. For $f \in Y^X$, $\epsilon > 0$, and a compact $K \subset X$, define

$$B(f, \epsilon, K) = \{g \in Y^X; \sup_{x \in K} d(f(x), g(x)) < \epsilon\}.$$

The collection $\{B(f, \epsilon, K); f \in Y^X, \epsilon > 0, K \subset X \text{ compact}\}$ is a base for compact convergence topology on Y^X, also called the topology of uniform convergence on compact sets.

Proposition. *A sequence $f_n : X \to Y$ converges to f in the compact convergence topology if and only if for each compact subset $K \subset X$, the sequence $f_n|K$ converges uniformly to $f|K$.*

k-spaces. We say the X is a k-space (or compactly generated space) if it satisfies the following condition: A subset $U \subset X$ is open if and only if $U \cap K$ is open in K for each compact subset K of X. Note that if we replace 'open' by 'closed' we obtain an equivalent definition.

Exercise. Let X be a T_1 space in which every compact set is finite. Then X is a k-space if and only if X is discrete.

Exercise. Let $X = \{0, 1, 2, \cdots\}$. A subset U is open if $0 \notin U$ or if $0 \in U$ and $\lim_{n\to\infty} \mathfrak{o}(U \cap [1, n])/n = 1$. Then X is not a k-space.

PROPOSITION. (1) *Every locally compact space is a k-space.*
(2) *Every first countable space is a k-space.*

PROOF. Suppose that $U \cap K$ is open in K for each compact K in X. Let $x \in U$, and let V be an open neighborhood of x with compact closure. Then $U \cap V^-$ is open in V^-, so $U \cap V = (U \cap V^-) \cap V$ is open in V, hence in X.

Suppose that X is first countable, and $F \cap K$ is closed in K for each compact K. If $x \in F^-$, then there is a sequence (x_n) in F which converges to x. But $\{x, x_n\}_n$ is compact, so $F \cap \{x, x_n\}$ is closed in $\{x, x_n\}$. Thus $x \in F$.

Exercise. Let X be a k-space. Then $f : X \to Y$ is continuous if and only if $f|K : K \to Y$ is continuous for each compact $K \subset X$.

PROPOSITION. *Let X be a k-space, (Y, d) a metric space. Then $\mathcal{C}(X, Y)$ is closed in Y^X in the topology of compact convergence.*

PROOF. Let $f \in Y^X$ be a limit point of $\mathcal{C}(X, Y)$. To show that f is continuous, it suffices to show that $f|K$ is continuous on each compact $K \subset X$.

For $n \in \mathbb{N}$, let $f_n \in B(f, 1/n, K) \cap \mathcal{C}(X, Y)$. Then $f_n|K \to f|K$ uniformly on K, so $f|K$ is continuous.

Exercise. Let X and Y be as above. If $f_n : X \to Y$ is a sequence of continuous functions which converges to f in the compact convergence topology, then f is continuous.

PROPOSITION. *Let X be a space and let (Y, d) be a metric space. Then the compact-open topology on $\mathcal{C}(X, Y)$ agrees with the compact convergence topology.*

PROOF. Let $[K, U]$ be a subbase element for the compact open topology, and let $f \in [K, U]$ be a continuous function. Then $f(K)$ is a compact subset of U, so there is an $\epsilon > 0$ such that $D_\epsilon[f(K)] \subset U$. Hence $B(f, \epsilon, K) \subset [K, U]$, and so the compact convergence topology is finer then the compact open topology.

Conversely, let $B(f, \epsilon, K)$ be a base element of the compact convergence topology. Then each $x \in K$ has a neighborhood U_x such that $f(U_x^-) \subset D_\epsilon[f(x)] = V_x$. Cover K by finitely many $U_1, \cdots U_n$, and let $K_i = K \cap U_i^-$. Then K_i is compact and $f \in [K_1, V_1] \cap \cdots \cap [K_n, V_n] \subset B(f, \epsilon, K)$.

2. The evaluation map

PROPOSITION. *Let X be a locally compact hausdorff space; let $\mathcal{C}(X, Y)$ have the compact-open topology. Then the map*

$$e : X \times \mathcal{C}(X, Y) \to Y$$

defined by $e(x, f) = f(x)$ is continuous.

PROOF. Let $(x, f) \in X \times \mathcal{C}(X, Y)$ and let V be an open neighborhood of $f(x) = e(x, f)$ in Y. Let U be an open neighborhood of x with compact closure such that $f(U^-) \subset V$. Then $U \times [U^-, V]$ is a neighborhood of (x, f) and $e(U \times [U^-, V]) \subset V$.

PROPOSITION. *Let X, $\mathcal{C}(X, Y)$ be as above, and let Z be a space. Then a map $f : X \times Z \to Y$ is continuous if and only if the map*

$$\widehat{f} : Z \to \mathcal{C}(X, Y)$$

defined by $[\widehat{f}(z)](x) = f(x, z)$ is continuous.

PROOF. Suppose that f is continuous and that $z \in Z$. Let $[K, U]$ be a neighborhood of $\widehat{f}(z)$ in the compact open topology. We have to find an open neighborhood W of z in Z such that

$\widehat{f}(W) \subset [K, U]$. This is equivalent to finding an open set W in Z such that $\widehat{f}(w)(K) \subset U$ for each $w \in W$, or what is the same, $f(K \times W) \subset U$.

Since $z \in \widehat{f}^{-1}([K, U])$, we have that $K \times \{z\} \subset f^{-1}(U)$. Hence there is an open neighborhood W of z such that $K \times W \subset F^{-1}(U)$. Therefore, $\widehat{f}(W) \subset [K, U]$.

If $\widehat{f}$ is continuous, then the map $g : X \times Z \to X \times \mathcal{C}(Z, Y)$ defined by $g(x, z) = (x, \widehat{f}(z))$ is also continuous. Then f is continuous because $e \circ g = f$.

3. Uniform spaces

Notation. Let S be a set. Let $\Delta = \{(x, x); x \in S\} \subset S \times S$ be the diagonal.

If U is a subset of $S \times S$, then let U^{-1} denote the set of all pairs (y, x) of $S \times S$ such that $(x, y) \in U$. If U and V are subset of $S \times S$, then $U \circ V$ is the set of all pairs (x, z) in $S \times S$ such that for some $y \in S$ we have $(x, y) \in V$ and $(y, z) \in U$.

If $x \in S$ and $U \subset S \times S$, let $U[x]$ denote the set of all points $y \in S$ such that $(x, y) \in U$. If A is a subset of S, then $U[A] = \{y \in S; (x, y) \in U \text{ for some } x \in A\} = \cup_{x \in A} U[x]$.

Let (Y, d) be a metric space. For each $\epsilon > 0$, let $D_\epsilon = \{(a, b) \in Y \times Y; d(a, b) < \epsilon\}$. Let $\mathcal{D}_Y$ denote the collection of all these subsets D_ϵ of $Y \times Y$ ($\epsilon > 0$), and let $\mathcal{U}_Y$ be the collection of all subsets of $Y \times Y$ such that $U \supset D_\epsilon$ for some $\epsilon > 0$. Thus $\mathcal{U}_Y$ is a filter in $Y \times Y$ with base $\mathcal{D}_Y$.

Note that if $y \in Y$ and $U \in \mathcal{U}_Y$, then $U[y]$ is a neighborhood of y. Similarly, if $K \subset Y$, then $U[K]$ is a neighborhood of K.

The purpose of this fancy notation is that it allows us to abstract the notion of metric space and to introduce concepts, like uniform continuity, without referring to the real numbers. Here is one example.

Proposition. *Let X, Y be metric spaces. Then $f : X \to Y$ is uniformly continuous if and only if for each $V \in \mathcal{U}_Y$ there exists $U \in \mathcal{U}_X$ such that $f(U[x]) \subset V[f(x)]$ for each $x \in U$.*

Uniform spaces. Here is the abstraction of the notation introduced above.

Let X be a set. A uniformity for X is a collection $\mathcal{U}$ of subsets of $X \times X$ such that:

(1) $\mathcal{U}$ is a filter,
(2) if $U \in \mathcal{U}$, then $\Delta \subset U$,
(3) if $U \in \mathcal{U}$, then $U^{-1} \in \mathcal{U}$,
(4) if $U \in \mathcal{D}$, then there exists $V \in \mathcal{U}$ such that $V \circ V \subset U$.

The pair $(X, \mathcal{U})$ is called a uniform space. Any subcollection $\mathcal{D} \subset \mathcal{U}$ such that every member of $\mathcal{U}$ contains one of $\mathcal{D}$ is called a base for the uniformity $\mathcal{U}$.

Example. A set D in a uniformity is called symmetric if $D = D^{-1}$. Then the symmetric sets $D \in \mathcal{U}$ form a base. Indeed, if $E \in \mathcal{D}$, then $E^{-1} \in \mathcal{D}$, and $D = E \cap E^{-1}$ is symmetric.

Example. For a metric or pesudometric space Y, the family $\mathcal{D}_Y$ is a base for a uniformity on Y, called the metric uniformity.

Topology and uniformity. The uniformity $\mathcal{U}$ defines a topology on X. A set $U \subset X$ is open if each $x \in U$ there is $D \in \mathcal{D}$ such that $D[x] \subset U$. Here $D[A]$ is the set of points $y \in X$ such that $(x, y) \in D$ for some $x \in A$.

Thus, for a metric or pseudometric space, the topology induced by the metric is the same as the one induced by the uniformity.

Exercise. Let $(X, \mathcal{U})$ be a uniform space, and let $\mathcal{D}$ be a base for $\mathcal{U}$. If $x \in X$, then $\{D[x]; D \in \mathcal{D}\}$ is a neighborhood base at x.

Exercise. Let $(X, \mathcal{D})$. If A is a subset of X, then $A^- = \cap\{U[A]; U \in \mathcal{U}\}$.

Let $x \in A^-$. For each $U \in \mathcal{U}$ choose a symmetric $D \subset U$. Then $D[x]$ is a neighborhood of x, so it meets A. If $y \in A \cap D[x]$, then $x \in D[y] \subset U[y] \subset U[A]$.

Conversely, if $x \notin A^-$, there is a symmetric $U \in \mathcal{U}$ such that $U[x] \cap A = \emptyset$, that is, $x \notin U[A]$.

Exercise. Every uniform space is a T_3-space. If $x \in X$ and $U \in \mathcal{U}$, let $V \in \mathcal{U}$ such that $V \circ V \subset U$. Then $(V[x])^- \subset V[V[x]] \subset U[x]$. Thus x has a base of closed neighborhoods.

4. Ascoli's theorem

Let X be a space, and let (Y, d) be a metric space. We use the notation $\mathcal{D}_Y$ introduced above for the family of ϵ-neighborhoods of the diagonal $\Delta \subset Y \times Y$.

Equicontinuous families. Let $\mathcal{F}$ be a subset of Y^X. Say that $\mathcal{F}$ is equicontinuous at $x \in X$ if for each $D \in \mathcal{D}_Y$ there is a neighborhood U of x such that $f(U) \subset D[f(x)]$, for each $f \in \mathcal{F}$. We say that $\mathcal{F}$ is equicontinuous if it is equicontinuous at each point of X. Note that every element of an equicontinuous family is a continuous function.

Proposition. *Let $\mathcal{F} \subset \mathcal{C}(X, Y)$ be an equicontinuous family of functions. Then its closure $\mathcal{F}'$ in the point-open topology is also equicontinuous.*

Proof. Given $x \in X$ and $E \in \mathcal{D}_Y$, let $D \in \mathcal{D}_Y$ be such that $D \circ D \circ D \subset E$ (that is, if E is an ϵ-neighborhood, take D to be an $\epsilon/3$-neighborhood). Let U a neighborhood of x such that $f(U) \subset D[f(x)]$ for all $f \in \mathcal{F}$. Let $g \in \mathcal{F}'$. We show that $g(U) \subset E[g(x)]$.

For each $y \in U$, the set $V_y = [\{x\}, D[g(x)]] \cap [\{y\}, D[g(y)]]$ is a neighborhood of g in the point-open topology, so there is $f \in V_y \cap \mathcal{F}$. This means that $(f(x), g(x)) \in D$, $(f(y), g(y)) \in D$. Since also $(f(y), f(x)) \in D$, we obtain $(g(x), g(y)) \in E$. Thus $g(y) \in E[g(x)]$ for all $y \in U$.

Proposition. *If $\mathcal{F} \subset \mathcal{C}(X, Y)$ is an equicontinuous family, then the compact-open topology coincides with the point-open topology.*

Proof. Let $[K, U]$ be a subbase element of the compact-open topology which contains f. Since f is continuous, $f(K)$ is a compact subset of U, so we can find $E \in \mathcal{D}_Y$ such that $E[f(K)] \subset U$. Let $D \in \mathcal{D}_Y$ be such that $D \circ D \subset E$.

By equicontinuity, each $x \in K$ has a neighborhood V_x such that $g(V_x) \in D[g(x)]$ for all $g \in \mathcal{F}$. The sets V_x cover K, and we pick a finite subcover $V_1, \cdots, V_n$. We show that $\cap_{i=1}^n [x_i, D[f(K)]] \subset [K, U]$.

Let $x \in K$. Then $x \in V_i$ for some i, and thus $g(V_i) \subset D[g(x_i)]$, so that $(g(x), g(x_i)) \in D$. But if $g \in \cap_{i=1}^n [x_i, D[f(K)]]$, then $g(x_i) \in D[f(K)]$. Thus there is $z \in K$ such that $(g(x_i), f(z)) \in D$. Hence $(g(x), f(x)) \in D \circ D$, and so $g(x) \in E[f(K)] \subset U$. This holds for all $x \in K$, so $g \in [K, U]$

Compactness in the point-open topology. Ascoli's theorem characterizes compactness of families of continuous functions in the compact-open topology. Compactness in the point-open topology is a consequence of Tichonov's theorem.

Proposition. *Let Y be a hausdorff space. A set of functions $\mathcal{F} \subset Y^X$ is compact in the point-open topology if and only if*

(1) *$\mathcal{F}$ is closed in Y^X,*
(2) *for each $x \in X$, $\mathcal{F}(x)$ has compact closure in Y.*

Exercise. Let Y be the function space $\mathbb{I}^{\mathbb{I}}$ with the point-open topology. Which of the following subspaces of Y is compact?

(1) $\{f \in Y; f(0) = 0\}$.
(2) $\{f \in Y; f \text{ continuous}, f(0) = 0\}$.
(3) $\{f \in Y; f \text{ differentiable}, |f'(x)| \leq 1 \text{ for all } x \in \mathbb{I}\}$.

Compactness in the compact-open topology. And now to Ascoli's theorem.

Proposition. *Let X be a hausdorff k-space, Y a metric space. A family of functions $\mathcal{F} \subset \mathcal{C}(X, Y)$ is compact in the compact-open topology if and only if*

(1) *$\mathcal{F}$ is closed in the point-open topology,*
(2) *for each $x \in X$, the orbit $\mathcal{F}(x) = \{f(x), f \in \mathcal{F}\}$ is compact,*
(3) *$\mathcal{F}$ is equicontinuous on each compact subset of X.*

Proof. If $\mathcal{F}$ is compact in the compact-open topology, it is also compact in the point-open topology, so the first two conditions are necessary by the previous proposition. Let K be any compact subset of X. Then the family $\mathcal{F}_K = \{f|K; f \in \mathcal{F}\}$ is compact in the compact-open topology on $\mathcal{C}(K, Y)$. This is so because the restriction map

$$r : \mathcal{C}(X, Y) \to \mathcal{C}(K, Y)$$

given by $r(f) = f|K$, is continuous in the compact-open topology, and $r(\mathcal{F}) = \mathcal{F}_K$.

Let $x \in K$, $D, D' \in \mathcal{D}_Y$ with $D \circ D \subset D'$. The space K is compact and hausdorff, so we can find a neighborhood U_f of x in K such that $f(U_f^-) \subset D[f(x)]$. The set $[U_f^-, D[f(x)]]$ is a neighborhood of f in the compact-open topology of $\mathcal{C}(K, Y)$. Thus the resulting cover of $\mathcal{F}_K$ has a finite subcover $[U_i^-, D[f_i(x)]]$, $i = 1, \cdots, n$.

Let $U = U_1 \cap \cdots \cap U_n$. If $f \in \mathcal{F}_K$, then $f \in [U_i^- D[f_i(x)]$ for some i, hence $f(U) \subset f(U_i^-) \subset D[f_i(x)]$. It follows that $f(U) \subset (D \circ D)[f(x)] \subset D'[f(x)]$, hence $\mathcal{F}_K$ is equicontinuous at x.

To prove sufficiency, note that (3) implies that the compact-open topology reduces to the point-open topology on $\mathcal{F}_K$, for each compact subset $K \subset X$. Let $[K, U]$ be any subbase element in the compact-open topology on X. Let $[K, U]_K = \{f : K \to Y; f(K) \subset U\}$ be the subbase element in the compact-open topology of $\mathcal{C}(K, Y)$. It is also open in the point-open topology. Furthermore, $[K, U]_K \cap \mathcal{F}_K = \{f|K, f \in [K, U] \cap \mathcal{F}\}$. The map r is continuous for the point-open topology, so $r^{-1}([K, U]_K \cap \mathcal{F}) = [K, U] \cap \mathcal{F})$ is open in the point-open topology. Thus the compact-open topology reduces to the point-open topology on $\mathcal{F}$, and compactness follows from Tichonov's theorem.

There is another version of Ascoli's theorem in which the condition on equicontinuity on compacts is replaced by global equicontinuity.

Proposition. *Let X be a locally compact hausdorff space and let (Y, d) be a metric space. A family of functions $\mathcal{F} \subset \mathcal{C}(X, Y)$ is compact in the compact open topology if and only if*

(1) *$\mathcal{F}$ is closed in the point-open topology.*
(2) *for each $x \in X$, the orbit $\mathcal{F}(x) = \{f(x); f \in \mathcal{F}\}$ is compact.*
(3) *$\mathcal{F}$ is equicontinuous.*

Proof. The proof is almost the same as the one above. The only difference appears in proving that if $\mathcal{F}$ is closed in the compact-open topology then $\mathcal{F}$ is equicontinuous and the orbits have compact closure. This uses the continuity of the evaluation map

$$e : X \times \mathcal{C}(X, Y) \to Y.$$

Let $x \in X$. The set $\{x\} \times \mathcal{F}$ is compact in $X \times \mathcal{C}(X, Y)$. Therefore, $e(x \times \mathcal{F}) = \mathcal{F}(x)$ is compact.

To prove that $\mathcal{F}$ is equicontinuous at $x \in X$, let $E, D \in \mathcal{D}_Y$ such that $D \circ D \subset E$. For each $f \in \mathcal{F}$, let $N_f = [K_f, U_f]$ be a neighborhood of f in the compact open topology and V_f a neighborhood of x such that $e(U_f \times N_f) \subset D[f(x)]$.

Let $N_1, \cdots N_r$ be a finite subcover of $\mathcal{F}$. Let $U = \cap_{i=1}^n U_i$. Equicontinuity will follow if we show that $f(U) \subset E[f(x)]$ for each $f \in \mathcal{F}$. Let $f \in \mathcal{F}$ and $y \in U$. Then $y \in U_i$ for some i, so $f(y) \in D[f_i(x)]$. Also, $f(x) \in D[f_i(x)]$ since $x \in U_i$. Hence $(f(y), f(x)) \in D \circ D \subset E$.

If $\mathcal{F}$ is equicontinuous we know that compact-open topology reduces to the point-open topology. If it is closed and the orbits are compact, then $\mathcal{F}$ is compact in Y^X in the point-open topology, hence also in the compact-open topology.

Exercise. Let $\mathbf{m}$ denote the space of bounded sequences of real numbers, $\mathbf{c}$ the set of all convergent sequences from $\mathbf{m}$, $\mathbf{c}_0$ the set all sequence which converge to 0. Give $\mathbf{m}$ the topology given by the metric

$$d((x_n),(y_n)) = \sup_n |x_n - y_n|$$

Them $\mathbf{m}$ is the space of bounded continuous functions $\mathbb{N} \to \mathbb{R}$. It is not compact because orbits are not compact.

What about $\mathbf{c}$? $\mathbf{c}_0$?

Exercise. A family of continuous functions in $\mathcal{C}(\mathbb{I},\mathbb{R})$ is compact in the compact open topology if and only if it is equicontinuous and uniformly bounded.

Exercise. Let X be a compact hausdorff space. Let $\mathcal{F} \subset \mathcal{C}(X,\mathbb{R})$ be a closed set. Then $\mathcal{F}$ is compact if and only if it is equicontinuous and pointwise bounded.

MATH 262. HOMEWORK 2 . DUE: 1/18/96

1. For each positive integer n, let $S_n = \{n, n+1, \dots\}$. The collection of all subsets of $\mathbb{N}$ which contain some S_n is a base for a topology on $\mathbb{N}$.

Describe the closure operation of this topological space.

2. Let X be the set of positive integers $n \geq 2$. Show that the sets $U_n = \{x \in X; x \,\text{divides}\, n\}$, $n \geq 2$, form a base for a topology on X. Find the closure of the one-point sets $\{x\}$, $x \in X$, and of the set of prime numbers.

3. (Sorgenfrey line) Show that the sets $[a, b)$, a, b real numbers, form a base for a topology on the real line. Determine which of the following subsets of X are open and which ones are closed: $(-\infty, a)$, $[a, b)$, $[a, \infty)$, (a, b), (a, ∞), $(-\infty, a]$, $[a, b]$, $\{a\}$.

4. Let X be the slotted plane. Describe the topology induced on a straight line, and the one induced on a circle.

5. Prove or disprove: The intersection of an arbitrary family of topologies on X is a topology on X. The union of two topologies on X is a topology on X.

6. (Exercise 5, §2-2) If $\mathfrak{B}$ is a base for a topology on X, then the topology generated by $\mathfrak{B}$ equals the intersection of all topologies on X that contain $\mathfrak{B}$.

7. (Small neighborhoods make large topologies.) For each $x \in X$, let $\mathfrak{B}^1_x$ and $\mathfrak{B}^2_x$ be neighborhood bases at x for topologies $\mathcal{O}^1$ and $\mathcal{O}^2$ on X. Then $\mathcal{O}^1$ is coarser than $\mathcal{O}^2$ (i.e., $\mathcal{O}^1 \subset \mathcal{O}^2$) if and only if at each $x \in X$, given $B^1 \in \mathfrak{B}^1_x$, there is some $B^2 \in \mathfrak{B}^2_x$ such that $B^2 \subset B^1$.

8. (Exercise 8, §2-5) Show that the dictionary order topology on the set $\mathbb{R} \times \mathbb{R}$ is the same as the product topology $\mathbb{R}_d \times \mathbb{R}$, where $\mathbb{R}_d$ denotes the set of real numbers with the discrete topology. Compare this topology with the standard topology of $\mathbb{R}^2$.

9. X has the discrete topology if and only if whenever Y is a topological space and $f : X \to Y$, then f is continuous.

X has the trivial topology if and only if whenever Y is a topological space and $f : Y \to X$, then f is continuous

10. A map $f : X \to Y$ is said to be closed if it takes closed subsets of X to closed subsets of Y. Show that f is continuous and closed if and only if $f(A^-) = f(A)^-$ for every subset A of X.

MATH 262. HOMEWORK 3. DUE: 1/25/96

1. Let X be an infinite set with the cofinite topology, and let (x_n) be an injective sequence in X (i.e., $x_n \neq x_m$ if $n \neq m$). Show that $x_n \to x$ for all $x \in X$.

2. Let X_α, $\alpha \in A$ be a family of topological spaces. Show that if F_α is closed in X_α for each α, then the product $\prod F_\alpha$ is closed in $\prod X_\alpha$. Is this also true in the box topology?

3. Let A and B be subsets of topological spaces X and Y, respectively. Prove that in the product $X \times Y$:

(1) $(A \times B)^- = A^- \times B^-$.
(2) $(A \times B)^\circ = A^\circ \times B^\circ$.

Do these results extend to arbitrary products?

4. Let $X = \prod_A X_\alpha$ and $Y = \prod_A Y_\alpha$ be two product spaces over the same index set A, and let $f_\alpha : X_\alpha \to Y_\alpha$ be continuous for each α in A. Then the map $f : X \to Y$ defined by $f(\{x(\alpha)\}) = \{f_\alpha(x(\alpha))\}$ is continuous.

5. Let $\sim$ be an equivalence relation on the space X and give $X/\sim$ the quotient topology. Let $p : X \to X/\sim$ be the quotient map. The following are equivalent:

(1) The map p is open.
(2) If A is open in X, then $[A]$ is open.
(3) If A is closed in X, then the union of all elements of $X/\sim$ contained in A is closed.

Note. Recall that by $[x]$ we denote the set $\{y; y \sim x\}$, and the corresponding point of $X/\sim$. For $A \subset X$, $[A]$ is the set of all those $y \in X$ equivalent to some point in A, that is, $[A] = \cup_{x \in A}[x]$. The same symbol represents a subset of $X/\sim$.

6. Let $\mathbb{C}$ denote the complex plane. Say that a function $f : \mathbb{C} \to \mathbb{C}$ satisfies the maximum modulus principle if the following holds: if $K \subset \mathbb{C}$ and $a \in K$ is such that $|f(z)| \leq |f(a)|$ for all $z \in K$ then a is in the boundary of K. Probably you know that nonconstant analytic functions satisfy this principle. Show that open maps $f : \mathbb{C} \to \mathbb{C}$ satisfy the maximum modulus principle.

(The boundary of a subset Y of a space X is $bY = Y^- \cap (X \setminus Y)^-$.)

7. Let $\mathbb{R}^\omega$ be the space of sequences of real numbers. Let $\mathbb{R}^\infty$ the subset of $\mathbb{R}^\omega$ consisting of all sequences (x_n) such that $x_n \neq 0$ for only finitely many values of n. What is the closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ in the product and box topologies?

8. Let X be the euclidean plane $\mathbb{R}^2$ with the standard topology. Let A be the x-axis. Show that the projection $p : X \to X/A$ is a closed map. Show that, if m is a positive integer, then the sequence (x_n) with $x_n = (m, 1/(n+1))$ converges to the point A in X/A. Does the sequence $(n, 1/(n+1))$ converge to A?

9. Let f be a continuous map from X onto Y and give Y the quotient topology. Then f is a homeomorphism if and only if it is one-one.

10. A subset D of a space X is said to be dense (in X) if $D^- = X$

(1) If D is dense in X if and only if D meets every nonempty open subset of X.

(2) If D is dense in X and $Y \subset X$, is it true that $D \cap Y$ is dense in Y?

(3) If X_α is a nonempty space and $D_\alpha \subset X_\alpha$, for each $\alpha \in A$, then $\prod D_\alpha$ is dense in $\prod X_\alpha$ if and only if D_α is dense in X_α for each α.

(4) Let $\mathfrak{B}$, $\mathfrak{S}$ be a base and a subbase for X, respectively. Are these statements true: D is dense if and only if D meets every nonempty element of $\mathfrak{B}$ (of $\mathfrak{S}$).

MATH 262. HOMEWORK 4. DUE: 2/1/96

1. Connected or not: $\mathbb{R}^\omega$ with the box topology (look at the space of bounded sequences). $[0,1]\times[0,1]$ with the order topology given by the dictionary order. $[0,1)\times[0,1)$ with the order topology given by the dictionary order.

2. True or false: A subset Y of X is not connected if and only if there are disjoint open subsets A and B in X, each meeting Y, and $Y \subset A \cup B$.

3. Let $(X,<)$ be a linearly ordered set (i.e., a chain) and give X the order topology. We say that X is order complete if every subset of X which has an upper bound has a supremum.

(1) If X is connected, then it is order-complete.
(2) If there are $a<b$ in X but no c such that $a<c<b$, we say that X has a jump. Show that X is connected if and only if it is order-complete and has no jumps.

4. Let X be a space. The path-components of X are the equivalence classes of the equivalence relation $x \sim y$ if there is a path joining x and y. That is, the path-component $P(x)$ containing $x \in X$ is the largest path-connected subset of X containing x.

(1) Show that X is locally path-connected if and only if each path-component of each open set is open.
(2) If X is locally path-connected, then the path-components are both open and closed.

5. Let $\mathbf{2}$ be the discrete space with two points, and let $C = \mathbf{2}^{\mathbb{N}}$ with the product topology. Then the components of C are the points.

6. The spaces $[0,1)\cup(2,3)$ and $(0,1)\cup(2,3)$ are not homeomorphic.

7. A map $f : X \to Y$ is locally constant if each $x \in X$ has a neighborhood U such that $f|U$ is constant. Show that a space is connected if and only if every locally constant map is constant.

8. Prove the following:

(1) If $f : X \to Y$ is continuous and Y is Hausdorff, then

$$\Delta(f) = \{(x_1,x_2); f(x_1) = f(x_2)\}$$

is a closed subset of $X \times X$.
(2) If f is an open map of X onto Y and $\Delta(f)$ is closed in $X \times X$, then Y is Hausdorff.
(3) If f is a continuous open map of X onto Y, then Y is Hausdorff if and only if $\Delta(f)$ is closed.

Definition. A topological space X is called a T_0-space (resp., T_1 space) if whenever x and y are two distinct points of X, there is an open set containing one and not the other (resp. there is a neighbrohood of each not containing the other). Hausdorff spaces are also called T_2-spaces. Clearly $T_2 \Rightarrow T_1 \Rightarrow T_0$, and you should be able to find pertinent examples showing that the converse implications need not be true. The T comes from *Trennungsaxiome*, the German version of *Separation axioms*.

9. Let X be a space. Define $x \sim y$ if $\{x\}^- = \{y\}^-$. Show that $\sim$ is an equivalence relation on X and the the quotient space $X/\sim$ is T_0.

10. For a space X the following are equivalent:

(1) X is T_1.
(2) One-point sets are closed.
(3) If $A \subset X$, then A is the intersection of all open sets containing it.

MATH 262. HOMEWORK 5. DUE: 2/8/96

1. Suppose that $\mathcal{F}$ is an ultrafilter on a set X.

(1) $\cap_{\mathcal{F}} F$ consists of at most one point.
(2) If $\cap_{\mathcal{F}} F = \{x\}$, then $\mathcal{F} = \{F \subset X; x \in F\}$.

2. Let $\mathcal{F}$ be a filter on a complete metric space (M, d). Show that $\mathcal{F}$ converges to some point of M if and only if for each $\epsilon > 0$ there is $F \in \mathcal{F}$ such that $\operatorname{diam}(F) < \epsilon$.

3. Show that if a filter $\mathcal{F}$ on a space X converges to a point x, then $x \in \cap_{\mathcal{F}} F^{-}$.

4. Show that the neighborhood filter of a point $x \in X$ is an ultrafilter if and only if $\{x\}$ is open (i.e., x is an isolated point).

5. A filter $\mathcal{F}$ on a product space $\prod X_\alpha$ converges to $x = (x(\alpha))$ if and only if $\pi_\alpha(\mathcal{F})$ converges to x_α for every α.

6. Use Alexander's subbase theorem to give a proof of Tychonoff's theorem without using ultrafilters.

7. Suppose that $A \times B$ is a compact subset of a product space $X \times Y$ and W is an open subset of $X \times Y$ which contains $A \times B$. Then there are open sets U in X and V in Y such that $A \times B \subset U \times V \subset W$.

8. Prove that in a compact hausdorff space X, every connected component is the intersection of all sets containing it which are both open and closed.

9. Let X be a compact hausdorff space. Prove that X has a base consisting of sets which are both open and closed if and only if every component of X is a simgle point.

10. Let X be a compact space and $\mathcal{A}$ be a family of continuous functions from X into $\mathbb{I}$ such that

(1) if $f, g \in \mathcal{A}$, then $f - g \in \mathcal{A}$, and
(2) for each $x \in X$ there is a neighborhood U_x of x and $f \in \mathcal{A}$ such that $f|U_x = 0$.

Show that $f(x) = 0$ for each $x \in X$ and each $f \in \mathcal{A}$.

MATH 262. HOMEWORK 6. DUE: 2/15/96

1. Let X be a hausdorff space and suppose that $\{K_n\}_{n=1}^{\infty}$ is a countable family of compact subsets of X such that $K_n \subset K_{n-1}$. Show that if there exists an open set U containing the intersection $\cap_{n=1}^{\infty} K_n$, then there exists an integer N such that $K_n \subset U$ if $n \geq N$.

2. True or false: The closure of a compact set is compact.

3. A compact locally connected space has only a finite number of components.

4. Let X be a first countable space. Then X is hausdorff if and only if every compact subset is closed.

5. Which of the following spaces is locally compact?

(1) The slotted plane.
(2) $\mathbb{R}$ with the cofinite topology.
(3) $A \cup B$, where $A = \{(x, 1/n); 0 \leq x \leq 1, n = 1, 2, \cdots\}$ and $B = \{(x, 0); 0 \leq x \leq 1\}$. The topology is that induced from the plane.
(4) $A \cup C$, where A is as above and $C = \{(0,0), (1,0)\}$. The topology is the induced one.

6. A subspace of a regular space is regular. A nonempty product space is regular if and only if each factor is regular.

7. Find a hausdorff space which is not regular.

8. The one-point compactification of $\mathbb{N}$ is homeomorphic to the subspace $\{1, 1/2, \cdots, 1/n, \cdots, 0\}$ of $\mathbb{R}$.

9. The one-point compactification of the rationals is T_1 but not T_2.

10. Suppose that X is a locally compact hausdorff space and $f : X \to \mathbb{R}$ a continuous map. Then f can be extended to a continuous map $f^* : X^* \to \mathbb{R}$ if and only if for each $\epsilon > 0$ there exists a compact set $K_\epsilon \subset X$ such that $|f(x) - f(y)| < \epsilon$ whenever $x, y \notin K_\epsilon$.

MATH 262. HOMEWORK 7. DUE: 2/22/96

1. Let X be a set and $A \subset X$ be a subset. The family of all subsets of X which contain A, together with the empty set, is a topology. Under what conditions is T_3? $T_{3\frac{1}{2}}$? T_4?

2. Let X be a $T_{3\frac{1}{2}}$ space, A a compact subset, and B a closed subset disjoint from A. Then there is a continuous function $f : X \to [0,1]$ such that $f|A = 1$ and $f|B = 0$.

3. A subset of a space is said to be a G_δ-set if it is a countable intersection of open sets.

(1) A one-point set on a first countable T_1-space is a G_δ-set.
(2) If f is a real valued continuous function on a space X, then $f^{-1}(0)$ is a G_δ-set.
(3) If A is a closed G_δ-set of a T_4-space X, then there is a continuous real valued function f on X such that $f^{-1}(0) = A$.

4. (1) Closed subspaces of normal (or T_4) spaces are normal (resp. T_4).
(2) The continuous image of a normal (or T_4) space under a closed map is normal (resp. T_4).
(3) The continuous open image of a normal space need not be normal.

5. (1) The Moore plane M (p.3 in notes) is a tichonov space.
(2) Subspaces of normal spaces need not be normal.

6. Let X be an uncountable set and Y and infinite set, both with the discrete topology. Let $X^* = X \cup \{\infty_X\}$ and $Y^* = Y \cup \{\infty_Y\}$ be their one point compactifications. Let W be the complement of (∞_X, ∞_Y) in $X^* \times Y^*$. Are the spaces $X^* \times Y^*$ and W normal? regular?

7. A space is lindelöf if each of its open covers of has a countable subcover.

(1) Every uncountable subset of a lindelöf space has a limit point.
(2) A T_3-space is lindelöf if each open cover has a countable subfamily whose closures cover.
(3) Let X be a space and 0 be a point not in X. Let $Y = X \cup \{0\}$, and take sets of the form $L \cup \{0\}$, $X \setminus L$ a lindelöf subspace of X, as neighborhoods of 0. Conclude that any space X can be embedded as a dense subset of a lindelöf space.

8. A space is separable if it has a countable dense subset.

(1) The continuous image of a separable space is separable.
(2) Let $\{X_\alpha; \alpha \in A\}$ be a collection of hausdorff spaces, each with at least two points. Then the product space $X = \prod_\alpha X_\alpha$ is separable if and only if each factor is separable and the index set A has cardinal number $\leq \mathfrak{c}$.

9. Let S denote the Sorgenfrey line (i.e., the set of real numbers with the topology having the half-open intervals $[a, b)$ as base). It is proved in the book that S is normal and lindelöf.

(1) $X = S \times S$ is neither normal nor lindelöf. (There is a proof of the former fact in the book, but you can do better.)
(2) True or false: separable, first countable spaces are second countable.

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10. A subspace A of a space X is called a retract of X if there is a continuous function $r : X \to A$ such that $r(a) = a$ for all $a \in A$. The map r is called a retraction.

(1) If $x \in \mathbb{R}^n$, then the closed ball $A = \{y \in \mathbb{R}^n; |x - y| \leq 1\}$ is a retract of $\mathbb{R}^n$.
(2) A retract in a hausdorff space is a closed set.
(3) $A \subset X$ is a retract of X if and only if every continuous function $f : A \to Z$ has a continuous extension $F : X \to Z$.

MATH 262. HOMEWORK 8. DUE: 2/29/96.

1. Suppose that (Y, f) is a hausdorff compactification of X such that it satisfies the following extension property: if $g : X \to \mathbb{R}$ is a bounded continuous function, then there is a continuous function $G : Y \to \mathbb{R}$ such that $G \circ f = g$. Then (Y, f) is equivalent to the Stone-Čech compactification $(\beta X, e)$.

2. (1) If X is a tichonov space, then no point of $\beta X \setminus X$ has a countable neighborhood base.
(2) If X is normal, then no point of $\beta X \setminus X$ is the limit point of a sequence in X.

Therefore βX cannot be metrizable unless X is already compact and metrizable.

3. Let X be an infinite discrete space.

(1) Any two disjoint open subsets of βX have disjoint closures.
(2) $\beta(X \times X)$ and $\beta X \times \beta X$ are not homeomorphic (look at the closure of the open set $\{(x, x); x \in X\}$).

4. (1) Let X be a tichonov space. then X is connected if and only if βX is connected.
(2) $\beta\mathbb{R}$ is a continuous image of $\beta\mathbb{N}$, but not conversely.

5. In class we saw that $\mathfrak{o}(\beta\mathbb{N}) = 2^{\mathfrak{c}}$. Show that $\beta\mathbb{Q}$ and $\beta\mathbb{R}$ also have cardinal number $2^{\mathfrak{c}}$.

6. Let X be locally compact metric space. The following are equivalent:

(1) The one-point compactification X^* of X is metrizable.
(2) $X = \cup_{n=1}^{\infty} U_n$, where each U_n is open with U_n^- compact and $U_n^- \subset U_{n+1}$.
(3) X is second countable.

7. Are the following spaces metrizable?

(1) $[0, 1] \times [0, 1]$ with the dictionary order topology.
(2) The Sorgenfrey line.

8. Let $f : X \to Y$ be a continuous map from a compact metric space X onto a hausdorff space Y. Then Y is metrizable.

9. Give examples of:

(1) a regular lindelöf space which is not metrizable,
(2) a hausdorf second countable space which is not metrizable.

10. Let $\{A_i\}_{i \in I}$ be a locally finite family of subsets of a space X (i.e., each point of X has a neigborhood meeting only finitely many A_i's).

(1) The family $\{A_i^-\}$ of closures is also locally finite.
(2) $\cup_i A_i^- = (\cup_i A_i)^-$.

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MATH 262. FIRST MIDTERM. 1/30/96

Work out 4 problems from Part I and 3 from Part II.

Part I

1. Let A be a nonempty collection of continuous functions from a topological space X into the closed interval $\mathbb{I} = [0,1]$. For $x \in X$, define $e(x) \in \mathbb{I}^A$ by $\pi_f e(x) = f(x)$ for $f \in A$. Show that the map $e : X \to \mathbb{I}^A$ is continuous.

2. Let X denote the set of real numbers with the cofinite topology. Show that X is path-connected.

3. Let X be the set of real numbers with the following topology: a neighborhood base of $x \neq 0$ is given by the usual open intervals centered at x. A neighborhood base of the origin is formed by the sets $(-\infty, -n) \cup (-\epsilon, \epsilon) \cup (n, \infty)$, for all choices of $\epsilon > 0$ and positive integers n. Show that the filter $\mathcal{F}$ generated by the filterbase $\{(a, \infty); a > 0\}$ converges to 0.

4. Let X be the real numbers with the topology which has the sets (a, ∞), $a \in \mathbb{R}$, as a base. Which sequences converge to which points? What is the closure of $(-\infty, 0)$?

5. Let $A \subset X$. Show that the family of all subsets of X which contain A, together with $\emptyset$, is a topology on X. Describe the closure and interior operations of this topology.

Part II

6. Let $\mathbb{N}$ be the set of positive integers and let $X = \mathbb{N} \cup \{\infty\}$.

(1) The finite subsets of $\mathbb{N}$, together with X, are the closed sets for a topology on X.
(2) X is T_0 but not T_1.
(3) X is path-connected and locally path-connected.

7. Let $X = \mathbb{R}^{\mathbb{N}}$ be the space of sequences of real numbers with the box topology.

(1) X is not first-countable.
(2) The component of $a \in X$ is the set of points $x \in X$ such that $\{n \in \mathbb{N}; a_n \neq x_n\}$ is finite.

8. Let I be an infinite set and a, b be two points not in I. Define a topology on $X = I \cup \{a, b\}$ as follows. Any subset of I is open, and a subset containing a or b is open if it contains all but a finite number of points of I.

(1) X is T_1 but not hausdorff.
(2) The components of X are the points.

9. Let X be the set of pairs of nonnegative integers with the following topology: each point (m, n), except $(0, 0)$, is open. A set U is a neighborhood of $(0, 0)$ if the sets $\{n; (m, n) \notin U\}$ are finite, except

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for finitely many integers m. (Thus, viewing X in the plane, a neighborhood of $(0,0)$ contains almost all the points of almost all the columns.)

(1) Show that no sequence (x_k) in $X \setminus \{(0,0)\}$ converges to $(0,0)$. Conclude that X is not first-countable.

(2) X is neither connected nor locally connected.

MATH 262. LAST MIDTERM. DUE: 3/5/96

1. Noetherian spaces

A topological space is called noetherian if every ascending sequence of open subsets is eventually constant, that is, if $U_1 \subset U_2 \subset \cdots$ are open subsets of X, then there exists n such that $U_n = U_{n+1} = \cdots$. A space is called artinian if every descending sequence of open sets is eventually constant.

1.1. A space is noetherian if and only if every subspace is compact.

1.2. A hausdorff noetherian space is finite.

1.3. Find an artinian space which is not noetherian, and a noetherian one which is not artinian.

2. Paracompact spaces

A function $f : X \to \mathbb{R}$ is lower semicontinuous (resp. upper semicontinuous) if and only if for each $a \in \mathbb{R}$, $f^{-1}(a, \infty)$ (resp. $f^{-1}(-\infty, a)$) is open in X.

2.1. Suppose that X is paracompact and that $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are upper and lower semicontinuous functions, respectively, such that $f(x) < g(x)$ for each $x \in X$. Show that there exists a continuous function $h : X \to \mathbb{R}$ such that $f(x) < h(x) < g(x)$ for each $x \in \mathbb{R}$.

A space is said to be σ-compact if it is locally compact, hausdorff, and can be expressed as a countable union of compact subsets.

2.2. If a space X is σ-compact, then there is a sequence $U_1, U_2, \cdots$ of open subsets of X, each with compact closure, with $U_n^- \subset U_{n+1}$, and such that $X = \cup_{n=1}^{\infty} U_n$.

2.3. If X is a σ-compact space, then X is lindelöf.

3. Filter description of βX

Let X be a tichonov space. Let ζX denote the collection of all z-ultrafilters on X. A topology on ζX is defined by taking as base for closed sets all sets of the form $C_A = \{F \in \zeta X; A \in F\}$, where A is any zero set.

3.1. The function $h : X \to \zeta X$ which takes $x \in X$ to the z-ultrafilter of all zero sets containing x is an embedding with dense image.

3.2. ζX is compact.

3.3. The Stone-Čech compactification βX is homeomorphic to ζX.

4. Universal spaces

The Cantor set $\mathbb{G}$ is the product space $\{0,1\}^{\mathbb{N}}$, where $\{0,1\}$ has the discrete topology.

4.1. If A is a closed subset of $\mathbb{G}$, then there is a continuous map $r : \mathbb{G} \to A$ such that $r(x) = x$ for each $x \in A$.

Let X be a second countable compact hausdorff space. Let $U_1, U_2, \cdots$ be a base for the topology of X, and define $f_n : \{0,1\} \to \mathcal{P}(X)$ by $f_n(0) = U_n^-$, $f(1) = X \setminus U_n$. If $x = (x_n) \in \mathbb{G}$, then $\cap_n f_n(x_n)$ is an intersection of closed subsets of X, hence it is either empty or else it contains a single point. In the latter case, denote this single point by $\phi(x)$.

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4.2. The map ϕ is defined on a closed subset A of $\mathbb{G}$. Furthermore, $\phi : A \to X$ is a continuous surjection. Therefore the composition $\phi \circ r : \mathbb{G} \to X$ is a continuous surjection.

4.3. If X is a second countable compact hausdorff space, then there is a continuous surjection $\beta\mathbb{N} \to X$.

4.4. Is there a continuous surjection $\mathbb{G} \to \beta\mathbb{N}$?

MATH 262. FINAL EXAM. 3/14/96

Work out 4 problems from Part I and 3 from Part II. Each problem of the first part is worth 5 points, and 6 points each one of the second.

Part I

1. Let X be a T_1-space. Then every connected subset of X containing more than one point is infinite.

2. True or false? Let $f : (0,1) \to \mathbb{R}$ be the function $f(x) = 1/x$. Then f extends to a continuous function $F : \beta(0,1) \to \mathbb{R}$.

3. A connected normal space having more than one point is uncountable.

4. Let $D \subset [0,1]^{\mathbb{N}}$ be the subspace of nondecreasing sequences. If $\mathbb{N}$ has the discrete topology, is D compact in the compact-open topology? What if $\mathbb{N}$ has the cofinite topology?

5. Show that every second countable space is separable, but the converse is false.

Part II

6. Suppose that X and Y are topological spaces. Let $\mathcal{C}(X,Y)$ be the space of continuous functions from X into Y with the compact-open topology. For each $y \in Y$, let $c_y : X \to Y$ denote the constant function $c_y(x) = y$. Let $j : Y \to \mathcal{C}(X,Y)$ be the map $j(y) = c_y$.

(1) The map j is an embedding of Y into $\mathcal{C}(X,Y)$.
(2) If Y is hausdorff, then $j(Y)$ is closed.

7. A space is called irreducible if it is non-empty and it is not the union of two proper closed subsets. The irreducible components of a topological space are the maximal irreducible subspaces. (A subspace is irreducible if it is irreducible as a space with the induced topology.)

(1) A topological space $X \neq \emptyset$ is irreducible if and only if every nonempty open subspace is dense. A subspace Y is irreducible if and only if Y^- is irreducible.
(2) The irreducible components of a space X are closed and cover X. What are the irreducible components of a hausdorff space?

8. Let X be the set of positive integers. A subset of X is open if it contains the successor of every odd integer in it.

(1) Every open cover of X has an open locally finite refinement. X is not compact, but it is locally compact.
(2) X is locally connected and locally path-connected. What are the components of X?

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9. Let X be an infinite set and let 0 be a particular point of X. Define a topology on X by declaring open any set whose complement either is finite or contains 0.

(1) If A, B are subsets of X such that $A \cap B^- = B \cap A^- = \emptyset$, then they can be separated by disjoint open sets.

(2) X is metrizable if and only if it is a countable set.

MATH 262. FINAL EXAM. FREE SAMPLE. 0/00/00

Work out 4 problems from Part I and 3 from Part II.

Part I

1. True or false: if $A \subset X$ is path-connected, then so is A^-.

2. True or false: If $A_1 \supset A_2 \supset \cdots$ is a collection of closed connected subsets of the plane, then $\cap_{n=1}^{\infty} A_n$ is connected.

3. Suppose that X is a hausdorff space and $D \subset X$ is a dense locally compact subspace. Then D is open.

4. The inclusion $\mathbb{N} \subset \mathbb{Q}$ induces and inclusion $\mathbb{N} \subset \beta\mathbb{Q}$. Show that $\mathrm{Cl}_{\beta\mathbb{Q}}\mathbb{N} = \beta\mathbb{N}$.

5. Let $X = (0,1) \cup (1,2) \cup (2,3) \cup (5,7)$ with the topology induced from $\mathbb{R}$. Describe the one-point compactification of X.

Part II

6. (1) True or false: Every subspace of a k-space is a k-space.
 (2) The space $\mathbb{R}^{\mathbb{I}}$ is not a k-space.
 (3) Let X be a k-space. Then $f : X \to Y$ is continuous if and only if $f|K : K \to Y$ is continuous for each compact $K \subset X$.

7. Let Y be a metric space. Let $f_n : X \to Y$ be a sequence of continuous functions, and let $f : X \to Y$ be a function (not necessarely continuous).

(1) If $\{f_n\}$ is equicontinuous and $f_n \to f$ in the point-open topology, then $f_n \to f$ in the compact-open topology.
(2) Prove the converse of (1) in case X is a hausdorff k-space.
(3) In either case, f is continuous.

8. Let H be the subspace of $\mathbb{I}^{\mathbb{I}}$ consisting of all non-decreasing functions.

(1) H is compact.
(2) H is separable.
(3) H is not metrizable.

9. Let X be the set of pairs of nonnegative integers with the following topology: each point (m,n), except $(0,0)$, is open. A set U is a neighborhood of $(0,0)$ if the sets $\{n; (m,n) \notin U\}$ are finite, except for finitely many integers m. (Thus, viewing X in the plane, a neighborhood of $(0,0)$ contains almost all the points of almost all the columns.)

(1) Show that no sequence (x_k) in $X \setminus \{(0,0)\}$ converges to $(0,0)$. Conclude that X is not first-countable.
(2) X is neither connected nor locally connected.
(3) X is lindelöf.

MATH 262. FINAL EXAM. 4/22/96

Work out 4 problems from Part I and 3 from Part II. Each problem of the first part is worth 5 points, and 6 points each one of the second.

Part I

1. Let X be a T_1-space. Then every connected subset of X containing more than one point is infinite.

2. True or false? Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = e^x$. Then f extends to a continuous function $F : \beta\mathbb{R} \to \mathbb{R}$.

3. A connected normal space having more than one point is uncountable.

4. Let $D \subset [0,1]^{\mathbb{N}}$ be the subspace of nondecreasing sequences. If $\mathbb{N}$ has the discrete topology, is D compact in the compact-open topology?

5. Show that every second countable space is separable, but the converse is false.

Part II

6. Suppose that X and Y are topological spaces. Let $\mathcal{C}(X,Y)$ be the space of continuous functions from X into Y with the compact-open topology. For each $y \in Y$, let $c_y : X \to Y$ denote the constant function $c_y(x) = y$. Let $j : Y \to \mathcal{C}(X,Y)$ be the map $j(y) = c_y$.

(1) The map j is an embedding of Y into $\mathcal{C}(X,Y)$.
(2) If Y is hausdorff, then $j(Y)$ is closed.

7. A space X is called symmetric if $x \in \{y\}^-$ implies $y \in \{x\}^-$.

(1) Every T_3-space is symmetric.
(2) A symmetric T_4-space must be T_3.

8. Let X be the interval $[-1,1]$ with the following topology. A subset of X is open if and only if either does not contain $\{0\}$ or does contain $(-1,1)$.

(1) X is compact. Is it locally compact?
(2) X is locally connected and locally path-connected. What are the components of X?

9. Let X be an infinite set and let 0 be a particular point of X. Define a topology on X by declaring open any set whose complement either is finite or contains 0.

(1) If A, B are subsets of X such that $A \cap B^- = B \cap A^- = \emptyset$, then they can be separated by disjoint open sets.
(2) X is metrizable if and only if it is a countable set.