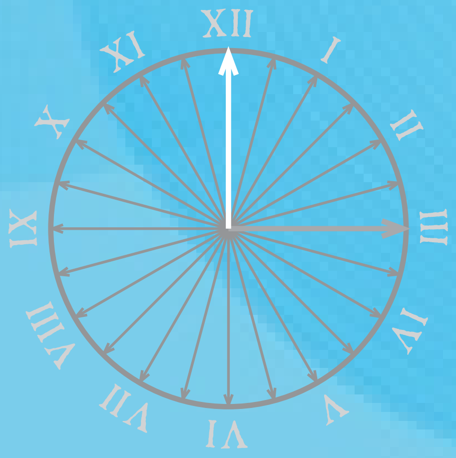
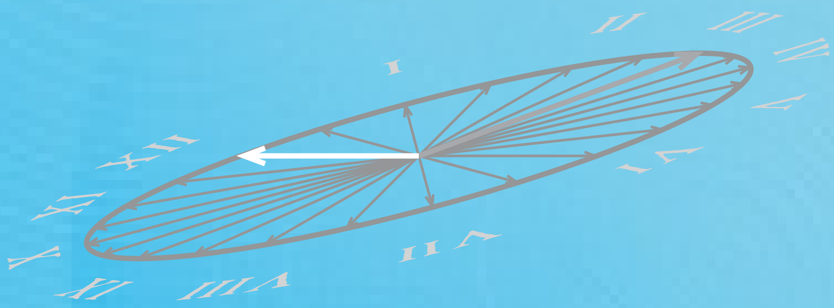
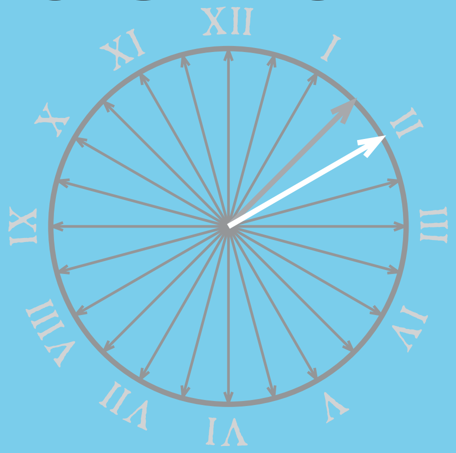
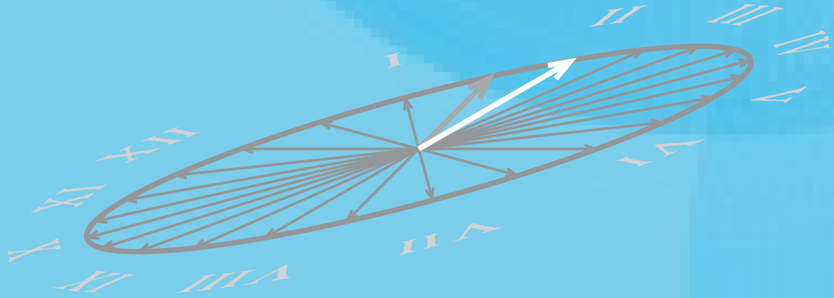




STUDENT'S SOLUTIONS MANUAL

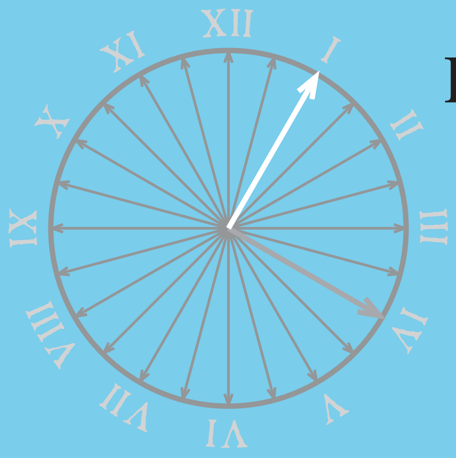


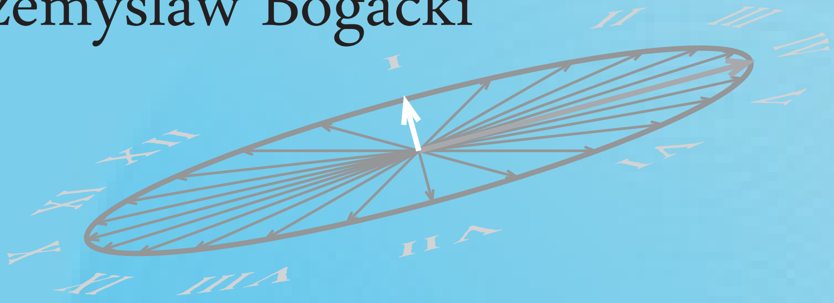


LINEAR ALGEBRA

Concepts and Applications

Przemyslaw Bogacki





CONTENTS

Updated: January 15, 2019

1 Vectors and Matrices

1.1 Vectors

1. $\overrightarrow{P_1P_2} = \begin{bmatrix} -1-2 \\ 3-5 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$; $\left\|\overrightarrow{P_1P_2}\right\| = \sqrt{(-3)^2+(-2)^2} = \sqrt{13}$.

3. $\overrightarrow{Q_2Q_1} = \begin{bmatrix} 0-2 \\ -2-2 \\ 1-1 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \\ 0 \end{bmatrix}$; $\left\|\overrightarrow{Q_2Q_1}\right\| = \sqrt{(-2)^2+(-4)^2+0^2} = \sqrt{20} = 2\sqrt{5}$

5. a. $\overrightarrow{u} + \overrightarrow{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

   b. $2\overrightarrow{v} = 2\begin{bmatrix} 6 \\ -4 \end{bmatrix} = \begin{bmatrix} 12 \\ -8 \end{bmatrix}$

   c. $-\overrightarrow{w} = -\begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$

   d. $\|\overrightarrow{w}\| = \sqrt{0^2+3^2} = 3$

7. $2\overrightarrow{u} - \overrightarrow{v} = 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 6 \\ -4 \end{bmatrix} = \begin{bmatrix} -4 \\ 8 \end{bmatrix}$

9. $-(3\overrightarrow{u} + \overrightarrow{v}) + \overrightarrow{w} = -(3\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 6 \\ -4 \end{bmatrix}) + \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} -9 \\ 1 \end{bmatrix}$

11. $\overrightarrow{u} \cdot \overrightarrow{v} = (1)(6) + (2)(-4) = -2.$

13. $\overrightarrow{w} \cdot \overrightarrow{w} = (0)(0) + (3)(3) = 9.$

15. a. $\overrightarrow{u} + \overrightarrow{w} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}$

    b. $2\overrightarrow{v} = 2\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ -6 \end{bmatrix}$

    c. $-\overrightarrow{w} = -\begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ -2 \end{bmatrix}$

    d. $\|\overrightarrow{w}\| = \sqrt{0^2+2^2+2^2} = \sqrt{8} = 2\sqrt{2}$

17. $\overrightarrow{w} + 4\overrightarrow{v} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} + 4\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \\ -10 \end{bmatrix}$

19. $4\overrightarrow{u} - (\overrightarrow{v} - \overrightarrow{w}) = 4\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} - (\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}) = 4\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ -5 \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \\ 13 \end{bmatrix}$

21. $\frac{1}{\|\overrightarrow{w}\|}\overrightarrow{w} = \frac{1}{\sqrt{0+4+4}}\begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} = \frac{1}{2\sqrt{2}}\begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

23. $\overrightarrow{u} \cdot \overrightarrow{w} = (-1)(0) + (0)(2) + (2)(2) = 4$

25. a. $\overrightarrow{u} + \overrightarrow{w} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 3 \\ 2 \end{bmatrix}$

b. $2\overrightarrow{v} = 2\begin{bmatrix} -1 \\ 3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \\ 2 \\ 4 \end{bmatrix}$

c. $-\overrightarrow{w} = -\begin{bmatrix} 2 \\ 0 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ -3 \\ 0 \end{bmatrix}$

d. $\|\overrightarrow{w}\| = \sqrt{2^2 + 0^2 + 3^2 + 0^2} = \sqrt{13}$

27. $-2\overrightarrow{u} + \overrightarrow{w} = -2\begin{bmatrix} 2 \\ 1 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ 3 \\ -4 \end{bmatrix}$

29. $\overrightarrow{v} \cdot \overrightarrow{v} = (-1)(-1) + (3)(3) + (1)(1) + (2)(2) = 15$

31. TRUE

Since $\|\overrightarrow{u}\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$, all components of $\overrightarrow{u}$ $(a_1, a_2, \ldots, a_n)$ must be zero for the sum of their squares to be zero. (In other words, the only vector with zero length is the zero vector.)

33. FALSE

$\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} = (1)(3) + (2)(0) + (-3)(-1) = 6 \neq 0.$

35. True for all vectors and scalars

$$\begin{aligned} c(\overrightarrow{u} - \overrightarrow{v}) &= c(\overrightarrow{u} + (-1)\overrightarrow{v}) \\ &= c\overrightarrow{u} + c((-1)\overrightarrow{v}) \text{ by Property 7 of Theorem 1.1} \\ &= c\overrightarrow{u} + (-c\overrightarrow{v}) \text{ by Property 9 of Theorem 1.1} \\ &= c\overrightarrow{u} - c\overrightarrow{v} \end{aligned}$$

37. Counterexample: $\overrightarrow{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $c = -3$.

$LHS = \left\| \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix} \right\| = 3$

$$RHS = -3\left\|\begin{bmatrix}1\\0\\0\end{bmatrix}\right\| = -3$$

$$LHS \neq RHS$$

39. Counterexample: $\overrightarrow{u} = \begin{bmatrix}3\\0\end{bmatrix}, \overrightarrow{v} = \begin{bmatrix}2\\0\end{bmatrix}, \overrightarrow{w} = \begin{bmatrix}1\\0\end{bmatrix}$

$$LHS = \begin{bmatrix}1\\0\end{bmatrix} - \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$RHS = \begin{bmatrix}3\\0\end{bmatrix} - \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix}$$

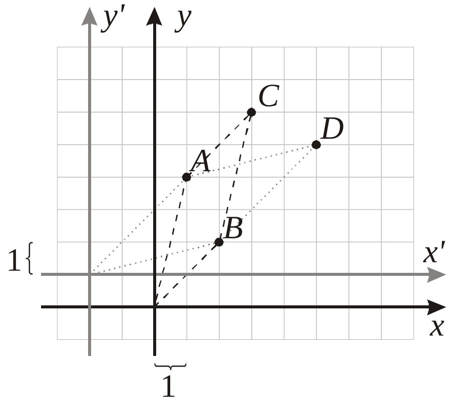
$$LHS \overset{\text{generally}}{\neq} RHS$$

45. a. $\begin{bmatrix}1\\1\\3\end{bmatrix} \times \begin{bmatrix}2\\1\\0\end{bmatrix} = \begin{bmatrix}(1)(0)-(3)(1)\\(3)(2)-(1)(0)\\(1)(1)-(1)(2)\end{bmatrix} = \begin{bmatrix}-3\\6\\-1\end{bmatrix}$

b. $\begin{bmatrix}3\\-1\\2\end{bmatrix} \times \begin{bmatrix}-6\\2\\-4\end{bmatrix} = \begin{bmatrix}(-1)(-4)-(2)(2)\\(2)(-6)-(3)(-4)\\(3)(2)-(-1)(-6)\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

51. a. $\overrightarrow{p} = \begin{bmatrix}1\\4\end{bmatrix}$; $\overrightarrow{q} = \begin{bmatrix}2\\2\end{bmatrix}$; $\overrightarrow{p} + \overrightarrow{q} = \begin{bmatrix}3\\6\end{bmatrix}$;

b. $\overrightarrow{u} = \begin{bmatrix}3\\3\end{bmatrix}$; $\overrightarrow{v} = \begin{bmatrix}4\\1\end{bmatrix}$; $\overrightarrow{u} + \overrightarrow{v} = \begin{bmatrix}7\\4\end{bmatrix}$



53. $\overrightarrow{a} = 0.40\overrightarrow{t} + 0.25\overrightarrow{q} + 0.35\overrightarrow{f}$

1.2 Matrices

1. a. $a_{21} = -2$,

   b. $a_{34} = -4$,

   c. $\text{col}_3 A = \begin{bmatrix} 0 \\ -3 \\ -5 \end{bmatrix}$,

   d. $\text{row}_2 A = \begin{bmatrix} -2 & 3 & -3 & 1 \end{bmatrix}$,

   e. $A^T = \begin{bmatrix} 2 & -2 & 4 \\ 1 & 3 & 5 \\ 0 & -3 & -5 \\ -1 & 1 & -4 \end{bmatrix}$.

3.

Matrix in Exercise #	K	L	M	N
a. a diagonal matrix	Yes	Yes	No	No
b. an upper triangular matrix	Yes	Yes	No	No
c. a lower tiangular matrix	Yes	Yes	Yes	No
d. a scalar matrix	No	Yes	No	No
e. an identity matrix	No	No	No	No
f. a symmetric matrix	Yes	Yes	No	No

5. a. $\begin{bmatrix} 3 & 0 \\ 5 & -2 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 3 & 0 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 8 & -2 \\ 0 & 3 \end{bmatrix}$

   b. $\begin{bmatrix} 10 & 5 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \end{bmatrix}$ cannot be evaluated since the two matrices have different dimensions

   c. $-3 \begin{bmatrix} 6 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -18 & -3 \\ -3 & -3 \\ 0 & 0 \end{bmatrix}$

7. $LHS = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 1 & 6 \end{bmatrix}$

   $RHS = \begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 1 & 6 \end{bmatrix}$

9. $\left(A^T\right)^T = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}^T = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix} = A$

11. $A = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & -1 \\ -3 & 1 & 0 \end{bmatrix}$ is skew-symmetric, since $A^T = \begin{bmatrix} 0 & 0 & -3 \\ 0 & 0 & 1 \\ 3 & -1 & 0 \end{bmatrix} = -A$.

    $B = \begin{bmatrix} 1 & 3 \\ -3 & -1 \end{bmatrix}$ is not skew-symmetric, since $B^T = \begin{bmatrix} 1 & -3 \\ 3 & -1 \end{bmatrix} \neq \begin{bmatrix} -1 & -3 \\ 3 & 1 \end{bmatrix} = -B$

$C = \begin{bmatrix} 0 & -4 \\ 4 & 0 \\ 0 & 0 \end{bmatrix}$ is not skew-symmetric, since $C^T = \begin{bmatrix} 0 & 4 & 0 \\ -4 & 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 4 \\ -4 & 0 \\ 0 & 0 \end{bmatrix} = -C$

$D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is skew-symmetric since $D^T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = -D.$

13. Note that

$$\begin{gathered} |i-j| > 1 \\ \Updownarrow \\ i-j > 1 \text{ or } i-j < -1 \\ \Updownarrow \\ i > j+1 \text{ or } i < j-1. \end{gathered}$$

$A = \begin{bmatrix} 1 & 3 & \boxed{0} & \boxed{0} \\ 2 & -1 & 5 & \boxed{0} \\ \boxed{0} & 4 & 1 & 7 \\ \boxed{0} & \boxed{0} & 6 & -1 \end{bmatrix}$ is tridiagonal since the entries where $i > j+1$ or $i < j-1$ (in boxes) are all 0;

$B = \begin{bmatrix} 0 & 1 & \boxed{0} & \boxed{0} & \boxed{0} \\ 0 & 0 & 0 & \boxed{0} & \boxed{0} \\ \boxed{0} & 1 & 0 & 1 & \boxed{0} \\ \boxed{0} & \boxed{0} & 0 & 0 & 0 \\ \boxed{0} & \boxed{0} & \boxed{0} & 1 & 0 \end{bmatrix}$ is tridiagonal since the entries where $i > j+1$ or $i < j-1$ (in boxes) are all 0;

$C = \begin{bmatrix} 1 & 0 & \boxed{1} \\ 0 & 1 & 0 \\ \boxed{0} & 0 & 1 \end{bmatrix}$ is not tridiagonal because $c_{13} = 1 \neq 0.$

15. TRUE

$$\underbrace{\underbrace{3A}_{2\times 3} + \underbrace{4B}_{2\times 3}}_{2\times 3}$$

17. TRUE

by Property 2 of Theorem 1.4

19. FALSE

$$\underbrace{\underbrace{A^T}_{4\times 3} + \underbrace{A}_{3\times 4}}_{\text{cannot evaluate}}$$

21. TRUE

If $m \neq n$ then an $m \times n$ matrix A and the $n \times m$ matrix A^T cannot possibly be equal.

23. Example: $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ (note: every scalar matrix is diagonal and every diagonal matrix is lower triangular)

25. Example: $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

1.3 Matrix Multiplication

1. a. $CD = \begin{bmatrix} 3 & 4 \\ 5 & -2 \end{bmatrix} \begin{bmatrix} 4 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}$

$$= \begin{bmatrix} (3)(4)+(4)(1) & (3)(1)+(4)(-1) & (3)(1)+(4)(-1) \\ (5)(4)+(-2)(1) & (5)(1)+(-2)(-1) & (5)(1)+(-2)(-1) \end{bmatrix} = \begin{bmatrix} 16 & -1 & -1 \\ 18 & 7 & 7 \end{bmatrix}$$

b. DC cannot be evaluated (the number of columns in D does not match the number of rows in C).

c. $AD = \begin{bmatrix} 2 & -1 \\ 3 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}$

$$= \begin{bmatrix} (2)(4)+(-1)(1) & (2)(1)+(-1)(-1) & (2)(1)+(-1)(-1) \\ (3)(4)+(0)(1) & (3)(1)+(0)(-1) & (3)(1)+(0)(-1) \\ (2)(4)+(1)(1) & (2)(1)+(1)(-1) & (2)(1)+(1)(-1) \end{bmatrix} = \begin{bmatrix} 7 & 3 & 3 \\ 12 & 3 & 3 \\ 9 & 1 & 1 \end{bmatrix}$$

d. $DA = \begin{bmatrix} 4 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 3 & 0 \\ 2 & 1 \end{bmatrix}$

$$= \begin{bmatrix} (4)(2)+(1)(3)+(1)(2) & (4)(-1)+(1)(0)+(1)(1) \\ (1)(2)+(-1)(3)+(-1)(2) & (1)(-1)+(-1)(0)+(-1)(1) \end{bmatrix} = \begin{bmatrix} 13 & -3 \\ -3 & -2 \end{bmatrix}$$

e. $CE = \begin{bmatrix} 3 & 4 \\ 5 & -2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} (3)(2)+(4)(0) & (3)(0)+(4)(2) \\ (5)(2)+(-2)(0) & (5)(0)+(-2)(2) \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & -4 \end{bmatrix}$

f. $EC = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 5 & -2 \end{bmatrix} = \begin{bmatrix} (2)(3)+(0)(5) & (2)(4)+(0)(-2) \\ (0)(3)+(2)(5) & (0)(4)+(2)(-2) \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & -4 \end{bmatrix}$

3. a. $(AC)^T = \left(\begin{bmatrix} 2 & -1 \\ 3 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 5 & -2 \end{bmatrix} \right)^T = \begin{bmatrix} (2)(3)+(-1)(5) & (2)(4)+(-1)(-2) \\ (3)(3)+(0)(5) & (3)(4)+(0)(-2) \\ (2)(3)+(1)(5) & (2)(4)+(1)(-2) \end{bmatrix}^T$

$$= \begin{bmatrix} 1 & 10 \\ 9 & 12 \\ 11 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 9 & 11 \\ 10 & 12 & 6 \end{bmatrix}$$

b. $A^T C^T$ cannot be evaluated (the number of columns in A^T does not match the number of rows in C^T).

c. $C^T A^T = \begin{bmatrix} 3 & 5 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 2 & 3 & 2 \\ -1 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} (3)(2)+(5)(-1) & (3)(3)+(5)(0) & (3)(2)+(5)(1) \\ (4)(2)+(-2)(-1) & (4)(3)+(-2)(0) & (4)(2)+(-2)(1) \end{bmatrix} = \begin{bmatrix} 1 & 9 & 11 \\ 10 & 12 & 6 \end{bmatrix}$$

d. $\left(B^T C\right)^T = \left(\begin{bmatrix} 4 & 2 \\ 0 & -2 \\ 0 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 5 & -2 \end{bmatrix} \right)^T = \begin{bmatrix} (4)(3)+(2)(5) & (4)(4)+(2)(-2) \\ (0)(3)+(-2)(5) & (0)(4)+(-2)(-2) \\ (0)(3)+(0)(5) & (0)(4)+(0)(-2) \\ (1)(3)+(3)(5) & (1)(4)+(3)(-2) \end{bmatrix}^T$

$$= \begin{bmatrix} 22 & 12 \\ -10 & 4 \\ 0 & 0 \\ 18 & -2 \end{bmatrix}^T = \begin{bmatrix} 22 & -10 & 0 & 18 \\ 12 & 4 & 0 & -2 \end{bmatrix}$$

e. $C^T B = \begin{bmatrix} 3 & 5 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 & 1 \\ 2 & -2 & 0 & 3 \end{bmatrix}$

$$= \begin{bmatrix} (3)(4)+(5)(2) & (3)(0)+(5)(-2) & (3)(0)+(5)(0) & (3)(1)+(5)(3) \\ (4)(4)+(-2)(2) & (4)(0)+(-2)(-2) & (4)(0)+(-2)(0) & (4)(1)+(-2)(3) \end{bmatrix}$$

$$= \begin{bmatrix} 22 & -10 & 0 & 18 \\ 12 & 4 & 0 & -2 \end{bmatrix}$$

5. a. $C^2 = CC = \begin{bmatrix} 3 & 4 \\ 5 & -2 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 5 & -2 \end{bmatrix}$

$$= \begin{bmatrix} (3)(3)+(4)(5) & (3)(4)+(4)(-2) \\ (5)(3)+(-2)(5) & (5)(4)+(-2)(-2) \end{bmatrix} = \begin{bmatrix} 29 & 4 \\ 5 & 24 \end{bmatrix}$$

b. $D^2 = DD$ cannot be evaluated (D is not a square matrix).

7. a. $ED + CA^T = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 4 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix} + \begin{bmatrix} 3 & 4 \\ 5 & -2 \end{bmatrix} \begin{bmatrix} 2 & 3 & 2 \\ -1 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} (2)(4)+(0)(1) & (2)(1)+(0)(-1) & (2)(1)+(0)(-1) \\ (0)(4)+(2)(1) & (0)(1)+(2)(-1) & (0)(1)+(2)(-1) \end{bmatrix}$$

$$+ \begin{bmatrix} (3)(2)+(4)(-1) & (3)(3)+(4)(0) & (3)(2)+(4)(1) \\ (5)(2)+(-2)(-1) & (5)(3)+(-2)(0) & (5)(2)+(-2)(1) \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 2 & 2 \\ 2 & -2 & -2 \end{bmatrix} + \begin{bmatrix} 2 & 9 & 10 \\ 12 & 15 & 8 \end{bmatrix} = \begin{bmatrix} 10 & 11 & 12 \\ 14 & 13 & 6 \end{bmatrix}$$

b. $(AC)\,D = \left(\begin{bmatrix} 2 & -1 \\ 3 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 5 & -2 \end{bmatrix} \right) \begin{bmatrix} 4 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}$

$$= \begin{bmatrix} (2)(3)+(-1)(5) & (2)(4)+(-1)(-2) \\ (3)(3)+(0)(5) & (3)(4)+(0)(-2) \\ (2)(3)+(1)(5) & (2)(4)+(1)(-2) \end{bmatrix} \begin{bmatrix} 4 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 10 \\ 9 & 12 \\ 11 & 6 \end{bmatrix} \begin{bmatrix} 4 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} (1)(4)+(10)(1) & (1)(1)+(10)(-1) & (1)(1)+(10)(-1) \\ (9)(4)+(12)(1) & (9)(1)+(12)(-1) & (9)(1)+(12)(-1) \\ (11)(4)+(6)(1) & (11)(1)+(6)(-1) & (11)(1)+(6)(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 14 & -9 & -9 \\ 48 & -3 & -3 \\ 50 & 5 & 5 \end{bmatrix}$$

(By Property 1 of Theorem 1.5, $A(CD)$ yields the same result.)

c. $\underbrace{\underbrace{C^2}_{2\times 2}\ \underbrace{E}_{2\times 2}\ \underbrace{D}_{2\times 3}}_{2\times 3} + \underbrace{\underbrace{A}_{3\times 2}\ \underbrace{B}_{2\times 4}}_{3\times 4}.$

The sum cannot be evaluated

9. $LHS = (AC)\,E = \left(\begin{bmatrix} 0 & 2 & -1 \\ 1 & 3 & -2 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}\right)\begin{bmatrix} 4 & 2 \end{bmatrix}$

$= \begin{bmatrix} 1 \\ 5 \end{bmatrix}\begin{bmatrix} 4 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 20 & 10 \end{bmatrix}$

$RHS = A\,(CE) = \begin{bmatrix} 0 & 2 & -1 \\ 1 & 3 & -2 \end{bmatrix}\left(\begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} 4 & 2 \end{bmatrix}\right)$

$= \begin{bmatrix} 0 & 2 & -1 \\ 1 & 3 & -2 \end{bmatrix}\begin{bmatrix} 16 & 8 \\ 4 & 2 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 20 & 10 \end{bmatrix}$

11. $LHS = (A+B)C = \left(\begin{bmatrix} 0 & 2 & -1 \\ 1 & 3 & -2 \end{bmatrix} + \begin{bmatrix} 2 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix}\right)\begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 2 & 4 & -1 \\ 1 & 4 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ 9 \end{bmatrix}$

$RHS = AC + BC = \begin{bmatrix} 0 & 2 & -1 \\ 1 & 3 & -2 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 1 \\ 5 \end{bmatrix} + \begin{bmatrix} 10 \\ 4 \end{bmatrix} = \begin{bmatrix} 11 \\ 9 \end{bmatrix}$

13. TRUE

$AB^2 = A(BB) = (AB)B$ by Property 1 of Theorem 1.5.

15. FALSE

$(A-B)(A+B) = AA + AB - BA - BB = A^2 + AB - BA - B^2 \overset{\text{generally}}{\neq} A^2 - B^2$ since generally AB is not equal to BA.

17. TRUE

$(A^2)^3 = (A^2)(A^2)(A^2) = (AA)(AA)(AA) = A^6$

19. FALSE

$(A-B)^2 = (A-B)(A-B) = A^2 - AB - BA + B^2 \overset{\text{generally}}{\neq} A^2 - 2AB + B^2$ since generally AB is not equal to BA.

21. TRUE

$(A^TB)^T = B^T(A^T)^T = B^TA$

follows from Property 5 of Theorem 1.5 and Property 3 of Theorem 1.4.

	CC	LS	PT	MV
Plant 1	100	30	0	0
Plant 2	0	0	100	80
Plant 3	0	30	60	40

23. Writing the entries from the table in the matrix form we obtain

$$A = \begin{bmatrix} 100 & 30 & 0 & 0 \\ 0 & 0 & 100 & 80 \\ 0 & 30 & 60 & 40 \end{bmatrix}.$$

	Price ($)	Profit ($/vehicle)
CC	12,000	500
LS	30,000	3,000
PT	18,000	1,500
MV	25,000	2,500

Likewise, the table corresponds to $B = \begin{bmatrix} 12000 & 500 \\ 30000 & 3000 \\ 18000 & 1500 \\ 25000 & 2500 \end{bmatrix}$.

Multiplying A by B we obtain

$$\begin{bmatrix} 100 & 30 & 0 & 0 \\ 0 & 0 & 100 & 80 \\ 0 & 30 & 60 & 40 \end{bmatrix} \begin{bmatrix} 12,000 & 500 \\ 30,000 & 3,000 \\ 18,000 & 1,500 \\ 25,000 & 2,500 \end{bmatrix} = \begin{bmatrix} 2,100,000 & 140,000 \\ 3,800,000 & 350,000 \\ 2,980,000 & 280,000 \end{bmatrix}$$

or, in the table form,

	Total revenue ($/day)	Total profit ($/day)
Plant 1	$2,100,000$	$140,000$
Plant 2	$3,800,000$	$350,000$
Plant 3	$2,980,000$	$280,000$

25. First, let us arrange the data on the duration of calls (in minutes per month) made by John and Kate to different countries in a table:

	Canada	France	Japan	Mexico	U.K.
John	30	40	40	0	0
Kate	60	30	0	20	60

with the corresponding matrix $A = \begin{bmatrix} 30 & 40 & 40 & 0 & 0 \\ 60 & 30 & 0 & 20 & 60 \end{bmatrix}$.

	D.E.	S.T.	T.L.
Canada	$0.05	$0.10	$0.05
France	$0.10	$0.10	$0.10
Japan	$0.20	$0.10	$0.30
Mexico	$0.05	$0.10	$0.15
U.K.	$0.10	$0.10	$0.05

The second table corresponds to the matrix $B = \begin{bmatrix} 0.05 & 0.10 & 0.05 \\ 0.10 & 0.10 & 0.10 \\ 0.20 & 0.10 & 0.30 \\ 0.05 & 0.10 & 0.15 \\ 0.10 & 0.10 & 0.05 \end{bmatrix}$.

The matrix multiplication yields

$$AB = \begin{bmatrix} 30 & 40 & 40 & 0 & 0 \\ 60 & 30 & 0 & 20 & 60 \end{bmatrix} \begin{bmatrix} 0.05 & 0.10 & 0.05 \\ 0.10 & 0.10 & 0.10 \\ 0.20 & 0.10 & 0.30 \\ 0.05 & 0.10 & 0.15 \\ 0.10 & 0.10 & 0.05 \end{bmatrix} = \begin{bmatrix} 13.5 & 11.0 & 17.5 \\ 13.0 & 17.0 & 12.0 \end{bmatrix}$$

resulting in the table

	D.E.	S.T.	T.L.
John	$13.50	$11.00	$17.50
Kate	$13.00	$17.00	$12.00

In order to minimize their monthly phone bill, John should choose Symmetric Telecom, while Kate ought to select Transpose Labs.

33. Direct multiplication: $\begin{bmatrix} 1 & -2 \\ 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 2 & 1 & -1 \\ 1 & -1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 4 & -1 & -7 \\ 1 & -1 & 1 & 3 \\ 3 & 1 & 5 & 7 \end{bmatrix}$

An example of partitioning: $C = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 3 \end{bmatrix}$, $E = \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix}$, $F = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$

$$\begin{bmatrix} C \\ --- \\ D \end{bmatrix} \begin{bmatrix} E & | & F \end{bmatrix} = \begin{bmatrix} CE & CF \\ DE & DF \end{bmatrix}$$

$$CE = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ 1 & -1 \end{bmatrix}$$

$$CF = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} -1 & -7 \\ 1 & 3 \end{bmatrix}$$

$$DE = \begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \end{bmatrix}$$

$$DF = \begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 7 \end{bmatrix}.$$

$$\begin{bmatrix} C \\ --- \\ D \end{bmatrix} \begin{bmatrix} E & | & F \end{bmatrix} \overset{\checkmark}{=} AB$$

1.4 Introduction to Linear Transformations

1. The matrix is $\left[F(\begin{bmatrix}1\\0\end{bmatrix}) \;\middle|\; F(\begin{bmatrix}0\\1\end{bmatrix}) \right] = \begin{bmatrix}1&\frac{1}{2}\\0&1\end{bmatrix}$. $F(\begin{bmatrix}3\\2\end{bmatrix}) = \begin{bmatrix}1&\frac{1}{2}\\0&1\end{bmatrix}\begin{bmatrix}3\\2\end{bmatrix} = \begin{bmatrix}4\\2\end{bmatrix}$.

3. From $F(\begin{bmatrix}1\\0\end{bmatrix}) = \begin{bmatrix}2\\0\end{bmatrix}$ and $F(\begin{bmatrix}0\\1\end{bmatrix}) = \begin{bmatrix}0\\-1\end{bmatrix}$ we obtain the matrix $\left[F(\begin{bmatrix}1\\0\end{bmatrix}) \;\middle|\; F(\begin{bmatrix}0\\1\end{bmatrix}) \right] = \begin{bmatrix}2&0\\0&-1\end{bmatrix}$.

5. From $F(\begin{bmatrix}1\\0\end{bmatrix}) = \begin{bmatrix}0\\0\end{bmatrix}$ and $F(\begin{bmatrix}0\\1\end{bmatrix}) = \begin{bmatrix}0\\1\end{bmatrix}$ we obtain the matrix $\left[F(\begin{bmatrix}1\\0\end{bmatrix}) \;\middle|\; F(\begin{bmatrix}0\\1\end{bmatrix}) \right] = \begin{bmatrix}0&0\\0&1\end{bmatrix}$.

7. $x_1\begin{bmatrix}2\\3\end{bmatrix} + x_2\begin{bmatrix}-1\\0\end{bmatrix} = \begin{bmatrix}2&-1\\3&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$; The matrix is $\begin{bmatrix}2&-1\\3&0\end{bmatrix}$.

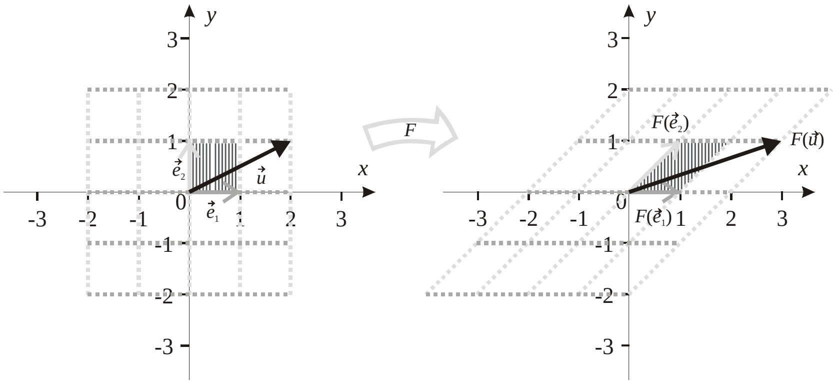
9. $2x_1\begin{bmatrix}-2\\8\\1\end{bmatrix} - x_3\begin{bmatrix}4\\0\\0\end{bmatrix} = x_1\begin{bmatrix}-4\\16\\2\end{bmatrix} + x_2\begin{bmatrix}0\\0\\0\end{bmatrix} + x_3\begin{bmatrix}-4\\0\\0\end{bmatrix} = \begin{bmatrix}-4&0&-4\\16&0&0\\2&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$.

The matrix is $\begin{bmatrix}-4&0&-4\\16&0&0\\2&0&0\end{bmatrix}$.

11. $x_1\begin{bmatrix}6\\1\end{bmatrix} + 3x_2\begin{bmatrix}3\\2\end{bmatrix} + x_4\begin{bmatrix}-2\\7\end{bmatrix} = x_1\begin{bmatrix}6\\1\end{bmatrix} + x_2\begin{bmatrix}9\\6\end{bmatrix} + x_3\begin{bmatrix}0\\0\end{bmatrix} + x_4\begin{bmatrix}-2\\7\end{bmatrix}$

$= \begin{bmatrix}6&9&0&-2\\1&6&0&7\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}$. The matrix is $\begin{bmatrix}6&9&0&-2\\1&6&0&7\end{bmatrix}$.

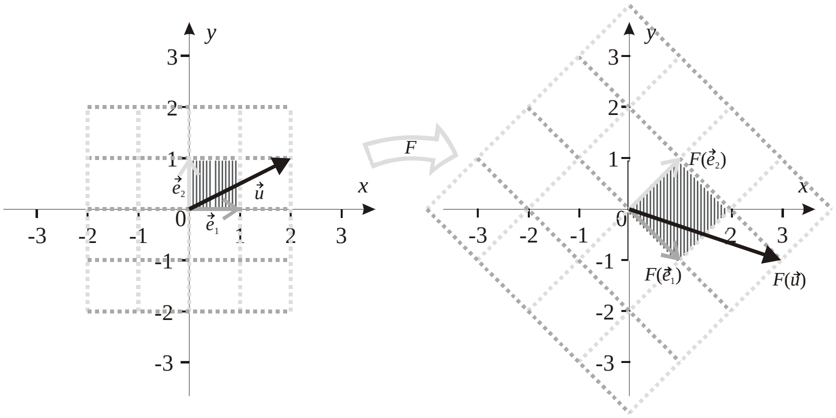
13. a.,b.



$F(\begin{bmatrix}2\\1\end{bmatrix}) = \begin{bmatrix}3\\1\end{bmatrix}$

c. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

15. a.,b.



$F(\begin{bmatrix} 2 \\ 1 \end{bmatrix}) = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

c. $F(\begin{bmatrix} 2 \\ 1 \end{bmatrix}) = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

17. a. Every horizontal vector $\begin{bmatrix} a \\ 0 \end{bmatrix}$ transforms into $\begin{bmatrix} -2a \\ 0 \end{bmatrix}$, therefore $F(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$.

Every vertical vector remains unchanged therefore $F(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

The matrix is $A = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$ (answer i).

b. Every horizontal vector is doubled $\Longrightarrow F(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

Every vertical vector $\begin{bmatrix} 0 \\ b \end{bmatrix}$ transforms into $\begin{bmatrix} b \\ b \end{bmatrix} \Longrightarrow F(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

The matrix is $A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$ (answer iii).

c. Every horizontal vector $\begin{bmatrix} a \\ 0 \end{bmatrix}$ remains unchanged $\Longrightarrow F(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Every vertical vector is doubled $\Longrightarrow F(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

The matrix is $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ (answer ii).

19. a. Every horizontal vector is negated, therefore $F(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

Every vertical vector $\begin{bmatrix} 0 \\ b \end{bmatrix}$ is transformed into $\begin{bmatrix} 0 \\ -2b \end{bmatrix}$, therefore $F(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$.

The matrix is $A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$.

b. Every horizontal vector $\begin{bmatrix} a \\ 0 \end{bmatrix}$ transforms into $\begin{bmatrix} -a \\ -a \end{bmatrix} \Longrightarrow F(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$.

Every vertical vector remains unchanged $\Longrightarrow F(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

The matrix is $A = \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}$.

c. Every horizontal vector $\begin{bmatrix} a \\ 0 \end{bmatrix}$ transforms into $\begin{bmatrix} 2a \\ a \end{bmatrix} \Longrightarrow F(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Every vertical vector remains unchanged $\Longrightarrow F(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

The matrix is $A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$.

21. $A = \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

$$A^3 = \left(\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \right) \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{4} - \frac{1}{4} & \frac{-\sqrt{3}-\sqrt{3}}{4} \\ \frac{\sqrt{3}+\sqrt{3}}{4} & \frac{-1+3}{4} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{3}-\sqrt{3}}{4} & \frac{-1-3}{4} \\ \frac{3+1}{4} & \frac{-\sqrt{3}+\sqrt{3}}{4} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

equals the matrix of rotation by 90 degrees clockwise.

23. Compose $F(G(H(\overrightarrow{x}))$ where

- F is the counterclockwise rotation by 45 degrees represented by
$$F(\overrightarrow{x}) = \begin{bmatrix} \cos(45^\circ) & -\sin(45^\circ) \\ \sin(45^\circ) & \cos(45^\circ) \end{bmatrix} \overrightarrow{x} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \overrightarrow{x}$$
- G is the projection onto the x axis $G(\overrightarrow{x}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \overrightarrow{x}$
- H is the clockwise rotation by 45 degrees (i.e., counterclockwise by -45°), represented by
$$H(\overrightarrow{x}) = \begin{bmatrix} \cos(-45^\circ) & -\sin(-45^\circ) \\ \sin(-45^\circ) & \cos(-45^\circ) \end{bmatrix} \overrightarrow{x} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \overrightarrow{x}$$

The resulting matrix is the product

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

31. a. $A = \begin{bmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{bmatrix}$

b. $\begin{bmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{bmatrix} \begin{bmatrix} 0.1 \\ 0.9 \end{bmatrix} = \begin{bmatrix} 0.36 \\ 0.64 \end{bmatrix}$

There is a 36% chance that the day after tomorrow will be sunny.

2 Linear Systems

2.1 Systems of Linear Equations

1. Augmented matrix $\begin{bmatrix} 1 & 6 & | & 0 \\ 0 & 3 & | & 1 \end{bmatrix}$; Coefficient matrix $\begin{bmatrix} 1 & 6 \\ 0 & 3 \end{bmatrix}$

3. Augmented matrix $\begin{bmatrix} 0 & 0 & 1 & | & 4 \\ 7 & 0 & 3 & | & 5 \\ 0 & 5 & 0 & | & -6 \\ 1 & -1 & 0 & | & 3 \end{bmatrix}$; Coefficient matrix $\begin{bmatrix} 0 & 0 & 1 \\ 7 & 0 & 3 \\ 0 & 5 & 0 \\ 1 & -1 & 0 \end{bmatrix}$

5. a.

$$\begin{array}{rcrcl} 2x & - & 3y & = & 4 \\ 6x & & & = & 0 \end{array}$$

b.

$$\begin{array}{rcrcrcl} x & + & 2y & + & z & = & 2 \\ 3x & & & - & z & = & 0 \\ & & y & + & z & = & 0 \\ & & & & 0 & = & 1 \end{array}$$

7. a. (i) both reduced row echelon form, and row echelon form
   b. (ii) row echelon form, but not in reduced row echelon form
   c. (iii) neither (zero row above nonzero row)
   d. (iii) neither (rows 2 and 3 do not follow the staircase pattern)

9. a. i. $\begin{bmatrix} \boxed{1} & 0 & | & 0 \\ 0 & \boxed{1} & | & 2 \end{bmatrix}$.
   ii. one solution
   iii. $x = 0, y = 2$
   b. i. $\begin{bmatrix} \boxed{1} & 0 & | & 0 \\ 0 & \boxed{1} & | & 0 \\ 0 & 0 & | & \boxed{1} \end{bmatrix}$.
   ii. no solution
   c. i. $\begin{bmatrix} \boxed{1} & 0 & -3 & | & 4 \\ 0 & \boxed{1} & 2 & | & 5 \end{bmatrix}$.
   ii. infinitely many solutions
   iii. $x = 3z + 4, y = -2z + 5, z-$arbitrary

11. a. i. $\begin{bmatrix} \boxed{1} & 3 & 0 & | & 0 \\ 0 & 0 & \boxed{1} & | & 0 \\ 0 & 0 & 0 & | & \boxed{1} \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$.
   ii. no solution
   b. i. $\begin{bmatrix} 0 & \boxed{1} & 0 & -5 & 0 & | & 0 \\ 0 & 0 & \boxed{1} & 3 & 0 & | & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & | & 0 \end{bmatrix}$.
   ii. infinitely many solutions

iii.

$$\begin{aligned} x_1 &= \text{arbitrary} \\ x_2 &= 5x_4 \\ x_3 &= -3x_4 \\ x_4 &= \text{arbitrary} \\ x_5 &= 0 \end{aligned}$$

13. a. $\begin{bmatrix} 1 & 5 & | & -2 \\ 5 & 2 & | & 13 \end{bmatrix}$

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2C738TV

b. r.e.f.: $\begin{bmatrix} 1 & 5 & | & -2 \\ 0 & 1 & | & -1 \end{bmatrix}$; r.r.e.f. $\begin{bmatrix} 1 & 0 & | & 3 \\ 0 & 1 & | & -1 \end{bmatrix}$

c.

$$\begin{aligned} y &= -1 \\ x &= -5y - 2 = 3 \end{aligned}$$

d. $x = 3, y = -1.$

e. $$\begin{array}{ccccc} 3 & + & 5(-1) & \overset{\checkmark}{=} & -2 \\ 5(3) & + & 2(-1) & \overset{\checkmark}{=} & 13 \end{array}$$

15. a. augmented matrix: $\begin{bmatrix} 2 & 4 & | & 3 \\ 1 & 2 & | & -1 \end{bmatrix}$

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2Fbe5Yj

b. r.e.f.: $\begin{bmatrix} 1 & 2 & | & \frac{3}{2} \\ 0 & 0 & | & 1 \end{bmatrix}$; r.r.e.f.: $\begin{bmatrix} 1 & 2 & | & 0 \\ 0 & 0 & | & 1 \end{bmatrix}$

c.$0 = 1 \Rightarrow$ No solution

d.$0 = 1 \Rightarrow$ No solution

17. a. augmented matrix: $\begin{bmatrix} 2 & 1 & 3 & -1 & | & 7 \\ -1 & 3 & 2 & 4 & | & 0 \end{bmatrix}$

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2Qw7Lfy

b. r.e.f.: $\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & \frac{-1}{2} & | & \frac{7}{2} \\ 0 & 1 & 1 & 1 & | & 1 \end{bmatrix}$; r.r.e.f.: $\begin{bmatrix} 1 & 0 & 1 & -1 & | & 3 \\ 0 & 1 & 1 & 1 & | & 1 \end{bmatrix}$

c.

$$\begin{aligned} x_2 &= -x_3 - x_4 + 1 \\ x_1 &= \frac{-1}{2}\left(-x_3 - x_4 + 1\right) - \frac{3}{2}x_3 + \frac{1}{2}x_4 + \frac{7}{2} \\ &= \frac{1}{2}x_3 + \frac{1}{2}x_4 - \frac{1}{2} - \frac{3}{2}x_3 + \frac{1}{2}x_4 + \frac{7}{2} \\ &= -x_3 + x_4 + 3 \\ x_3 &= arbitrary \\ x_4 &= arbitrary \end{aligned}$$

d.

$$\begin{aligned} x_1 &= -x_3 + x_4 + 3 \\ x_2 &= -x_3 - x_4 + 1 \\ x_3 &= arbitrary \\ x_4 &= arbitrary \end{aligned}$$

e.

$$\begin{array}{ccccccccc} 2\left(-x_3 + x_4 + 3\right) & + & \left(-x_3 - x_4 + 1\right) & + & 3x_3 & - & x_4 & \overset{\checkmark}{=} & 7 \\ -\left(-x_3 + x_4 + 3\right) & + & 3\left(-x_3 - x_4 + 1\right) & + & 2x_3 & + & 4x_4 & \overset{\checkmark}{=} & 0 \end{array}$$

19. a. augmented matrix $\left[\begin{array}{rrrcr} 3 & 0 & 1 & | & 2 \\ -1 & 1 & 0 & | & 1 \\ 4 & 2 & 1 & | & 4 \end{array}\right]$

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2LTs7yx

b. r.e.f.: $\left[\begin{array}{ccccc} 1 & 0 & \frac{1}{3} & | & \frac{2}{3} \\ 0 & 1 & \frac{1}{3} & | & \frac{5}{3} \\ 0 & 0 & 1 & | & 2 \end{array}\right]$; r.r.e.f.: $\left[\begin{array}{ccccc} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 1 \\ 0 & 0 & 1 & | & 2 \end{array}\right]$

c.

$$\begin{aligned} z &= 2 \\ y &= \frac{-1}{3}z + \frac{5}{3} = \frac{-2}{3} + \frac{5}{3} = 1 \\ x &= \frac{-1}{3}z + \frac{2}{3} = \frac{-2}{3} + \frac{2}{3} = 0 \end{aligned}$$

d. $x = 0, y = 1, z = 2$

e. $$\begin{array}{rcccccc} 3(0) & & & + & 2 & \overset{\checkmark}{=} & 2 \\ -(0) & + & 1 & & & \overset{\checkmark}{=} & 1 \\ 4(0) & + & 2 & + & 2 & \overset{\checkmark}{=} & 4 \end{array}$$

21. a. augmented matrix $\left[\begin{array}{rrrrcr} 1 & 2 & 3 & -3 & | & 0 \\ 2 & 1 & 3 & 0 & | & 6 \\ -1 & 1 & 0 & -3 & | & -6 \end{array}\right]$

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2VBxrev

b. r.e.f.: $\left[\begin{array}{rrrrcr} 1 & 2 & 3 & -3 & | & 0 \\ 0 & 1 & 1 & -2 & | & -2 \\ 0 & 0 & 0 & 0 & | & 0 \end{array}\right]$; r.r.e.f.: $\left[\begin{array}{rrrrcr} 1 & 0 & 1 & 1 & | & 4 \\ 0 & 1 & 1 & -2 & | & -2 \\ 0 & 0 & 0 & 0 & | & 0 \end{array}\right]$

c.

$$\begin{aligned} y &= -z + 2w - 2 \\ x &= -2y - 3z + 3w \\ &= -2\left(-z + 2w - 2\right) - 3z + 3w \\ &= -z - w + 4 \\ z &= arbitrary \\ w &= arbitrary \end{aligned}$$

d.

$$\begin{aligned} x &= -z - w + 4 \\ y &= -z + 2w - 2 \\ z &= arbitrary \\ w &= arbitrary \end{aligned}$$

e.

$$\begin{array}{rcrcrcrcr} (-z - w + 4) & + & 2\left(-z + 2w - 2\right) & + & 3z & - & 3w & \overset{\checkmark}{=} & 0 \\ 2\left(-z - w + 4\right) & + & \left(-z + 2w - 2\right) & + & 3z & & & \overset{\checkmark}{=} & 6 \\ -\left(-z - w + 4\right) & + & \left(-z + 2w - 2\right) & & & - & 3w & \overset{\checkmark}{=} & -6 \end{array}$$

23. a. augmented matrix $\begin{bmatrix} 1 & 0 & 0 & 5 & | & 1 \\ 1 & 1 & 0 & 1 & | & 0 \\ 2 & 3 & 1 & 0 & | & -3 \\ 0 & -1 & 2 & 0 & | & -3 \end{bmatrix}$

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2GTVNN0

b. r.e.f.: $\begin{bmatrix} 1 & 0 & 0 & 5 & | & 1 \\ 0 & 1 & 0 & -4 & | & -1 \\ 0 & 0 & 1 & 2 & | & -2 \\ 0 & 0 & 0 & 1 & | & 0 \end{bmatrix}$; r.r.e.f.: $\begin{bmatrix} 1 & 0 & 0 & 0 & | & 1 \\ 0 & 1 & 0 & 0 & | & -1 \\ 0 & 0 & 1 & 0 & | & -2 \\ 0 & 0 & 0 & 1 & | & 0 \end{bmatrix}$

c.

$$\begin{array}{rcl} w & = & 0 \\ z & = & -2w-2=-2 \\ y & = & 4w-1=-1 \\ x & = & -5w+1=1 \end{array}$$

d. $x=1, y=-1, z=-2, w=0.$

e.

$$\begin{array}{ccccccccc} 1 & & & & & + & 5(0) & \overset{\checkmark}{=} & 1 \\ 1 & + & (-1) & & & + & 0 & \overset{\checkmark}{=} & 0 \\ 2 & + & 3(-1) & + & (-2) & & & \overset{\checkmark}{=} & -3 \\ & - & (-1) & + & 2(-2) & & & \overset{\checkmark}{=} & -3 \end{array}$$

25. a. augmented matrix $\begin{bmatrix} 0 & 1 & 1 & 1 & -2 & | & 0 \\ 0 & -1 & -1 & -1 & 2 & | & 0 \\ 1 & -1 & 0 & -3 & 3 & | & 0 \\ 1 & 0 & 1 & -2 & 1 & | & 0 \end{bmatrix}$

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2FcyhJD

b. r.e.f.: $\begin{bmatrix} 1 & -1 & 0 & -3 & 3 & | & 0 \\ 0 & 1 & 1 & 1 & -2 & | & 0 \\ 0 & 0 & 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$; r.r.e.f.: $\begin{bmatrix} 1 & 0 & 1 & -2 & 1 & | & 0 \\ 0 & 1 & 1 & 1 & -2 & | & 0 \\ 0 & 0 & 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$

c.

$$\begin{array}{rcl} x_2 & = & -x_3-x_4+2x_5 \\ x_1 & = & x_2+3x_4-3x_5 \\ & = & -x_3-x_4+2x_5+3x_4-3x_5 \\ & = & -x_3+2x_4-x_5 \\ x_3 & = & arbitrary \\ x_4 & = & arbitrary \\ x_5 & = & arbitrary \end{array}$$

d.

$$\begin{array}{rcl} x_1 & = & -x_3+2x_4-x_5 \\ x_2 & = & -x_3-x_4+2x_5 \\ x_3 & = & arbitrary \\ x_4 & = & arbitrary \\ x_5 & = & arbitrary \end{array}$$

e.

$$\begin{array}{rrrrrrrrrrr} & & (-x_3 - x_4 + 2x_5) & + & x_3 & + & x_4 & - & 2x_5 & \overset{\checkmark}{=} & 0 \\ & - & (-x_3 - x_4 + 2x_5) & - & x_3 & - & x_4 & + & 2x_5 & \overset{\checkmark}{=} & 0 \\ (-x_3 + 2x_4 - x_5) & - & (-x_3 - x_4 + 2x_5) & & & - & 3x_4 & + & 3x_5 & \overset{\checkmark}{=} & 0 \\ (-x_3 + 2x_4 - x_5) & & & + & x_3 & - & 2x_4 & + & x_5 & \overset{\checkmark}{=} & 0 \end{array}$$

27. Such system does not exist: we need three leading entries in the first three columns of the r.r.e.f., but there are only two rows, making it impossible.

29. e.g., $\left[\begin{array}{ccccc} 1 & 0 & 0 & | & 3 \\ 0 & 0 & 1 & | & 5 \end{array}\right]$

31. e.g., $\left[\begin{array}{cccc} 1 & 0 & | & 0 \\ 0 & 0 & | & 1 \\ 0 & 0 & | & 0 \end{array}\right]$

33. FALSE

Counterexample: $\underbrace{\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right]}_{\text{r.r.e.f.}} + \underbrace{\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right]}_{\text{r.r.e.f.}} = \underbrace{\left[\begin{array}{cc} 2 & 0 \\ 0 & 2 \end{array}\right]}_{\text{not r.r.e.f.}}$

35. TRUE

The last row is $\left[\begin{array}{ccccc} 0 & 0 & 0 & | & 1 \end{array}\right]$

37. TRUE

If A is an $m \times n$ then to transform A to $3A$, we can multiply each of the m rows by 3.

2.2 Elementary Matrices and the Geometry of Linear Systems

1. a. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}$; b. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & -2 & 2 \\ -4 & 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & -2 & 2 \\ 0 & 11 & 1 & 6 \end{bmatrix}$

3. a. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$; b. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

5. a. $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$; b. $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 & -2 \\ 2 & 4 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 & -2 \\ 0 & 1 & 4 & 4 \end{bmatrix}$

7. a. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$; b. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 & -2 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 2 & -6 \\ 0 & 0 & 3 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 & -2 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 3 & 7 \end{bmatrix}$

9. a. $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$; b. $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 0 & 3 \\ 2 & 5 \\ -7 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 0 & 3 \\ 0 & 2 \\ -7 & 1 \end{bmatrix}$

11. Applying the corresponding elementary row operations to I_4 yields

a. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 5 & 0 & 0 & 1 \end{bmatrix}$; b. $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$; c. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

13. Applying the corresponding elementary row operations to I_5 yields

a. $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -6 \end{bmatrix}$; b. $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$; c. $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

17. FALSE

Counterexample: the system with augmented matrix $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right]$ has no solution (because of the second row)

19. TRUE

If some columns of C did not have leading entries, they would correspond to unknowns that are arbitrary, so that there would be infinitely many solutions.

2.3 Matrix Inverse

NOTE, For most solutions in this section, the individual row operations required can be obtained using the Linear Algebra Toolkit.

1. Since $\begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 1 & 6 \end{bmatrix} = \begin{bmatrix} 5 & 18 \\ 1 & 9 \end{bmatrix} \neq I_2$, we conclude that $\begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}$ is not the inverse of $\begin{bmatrix} 1 & 3 \\ 1 & 6 \end{bmatrix}$.

3. Since $\begin{bmatrix} -4 & 0 & -3 \\ 0 & 1 & 2 \\ 7 & 0 & 5 \end{bmatrix}\begin{bmatrix} 5 & 0 & 3 \\ 14 & 1 & 8 \\ -7 & 0 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$ and $\begin{bmatrix} 5 & 0 & 3 \\ 14 & 1 & 8 \\ -7 & 0 & -4 \end{bmatrix}\begin{bmatrix} -4 & 0 & -3 \\ 0 & 1 & 2 \\ 7 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$, we conclude that $\begin{bmatrix} -4 & 0 & -3 \\ 0 & 1 & 2 \\ 7 & 0 & 5 \end{bmatrix}$ is the inverse of $\begin{bmatrix} 5 & 0 & 3 \\ 14 & 1 & 8 \\ -7 & 0 & -4 \end{bmatrix}$.

Refer to the
Linear Algebra
Toolkit for details:
http://bit.ly/2RhzZ2U

5. a. $\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 5 & 0 & 1 \end{array}\right]$ has r.r.e.f. $\left[\begin{array}{cc|cc} 1 & 0 & -5 & 3 \\ 0 & 1 & 2 & -1 \end{array}\right]$

Inverse: $\begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$

Check: $\begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Refer to the
Linear Algebra
Toolkit for details:
http://bit.ly/2AxAew1

b. $\left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 2 & 5 & 0 & 1 \end{array}\right]$ has r.r.e.f. $\left[\begin{array}{cc|cc} 1 & 0 & \frac{5}{4} & -\frac{3}{4} \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} \end{array}\right]$

Inverse: $\begin{bmatrix} \frac{5}{4} & -\frac{3}{4} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$

Check: $\begin{bmatrix} \frac{5}{4} & -\frac{3}{4} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Refer to the
Linear Algebra
Toolkit for details:
http://bit.ly/2RaIH2F

c. $\left[\begin{array}{cc|cc} 3 & 6 & 1 & 0 \\ 2 & 4 & 0 & 1 \end{array}\right]$ has r.r.e.f. $\left[\begin{array}{cc|cc} 1 & 2 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{3}{2} \end{array}\right]$

No inverse (singular matrix)

Refer to the
Linear Algebra
Toolkit for details:
http://bit.ly/2QphfJb

7. a. $\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$ has r.r.e.f. $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -2 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$

Inverse: $\begin{bmatrix} 1 & 0 & -2 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Check: $\begin{bmatrix} 1 & 0 & -2 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2AwEOuy

b. $\left[\begin{array}{ccc|ccc} -2 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & -1 & 0 & 0 & 1 \end{array}\right]$ has r.r.e.f. $\left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array}\right]$

No inverse (singular matrix)

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2AwDDeS

c. $\left[\begin{array}{ccc|ccc} -3 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$ has r.r.e.f.: $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & 4 \\ 0 & 1 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 & 1 & 3 \end{array}\right]$

Inverse: $\begin{bmatrix} 1 & 2 & 4 \\ 1 & 2 & 3 \\ 1 & 1 & 3 \end{bmatrix}$

Check: $\begin{bmatrix} 1 & 2 & 4 \\ 1 & 2 & 3 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} -3 & 2 & 2 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2VxhyW6

9. a. $\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$ has r.r.e.f. $\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$

Inverse: $\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Check: $\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2C6mbxx

b. $\left[\begin{array}{cccc|cccc} 0 & 1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$ has r.r.e.f. $\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} & -1 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 \end{array}\right]$

No inverse - singular matrix

11. a. $\left[\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -3 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$

has r.r.e.f. $\left[\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2Vw6RDf

Inverse $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

Check $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \overset{\checkmark}{=} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

b. $\left[\begin{array}{cccccc|cccccc} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$

has r.r.e.f. $\left[\begin{array}{cccccc|cccccc} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2VAoL7P

Inverse: $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

Check: $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

Refer to the
Linear Algebra
Toolkit for details:
http://bit.ly/2LUYvAX

13. $\begin{bmatrix} 1 & -1 \\ -4 & 3 \end{bmatrix}$, inverse: $\begin{bmatrix} -3 & -1 \\ -4 & -1 \end{bmatrix}$

$$\begin{bmatrix} -3 & -1 \\ -4 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Refer to the
Linear Algebra
Toolkit for details:
http://bit.ly/2CSFwno

15. $\begin{bmatrix} 0 & 3 & 2 \\ -1 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, inverse: $\begin{bmatrix} 0 & 0 & 1 \\ -1 & 2 & 2 \\ 2 & -3 & -3 \end{bmatrix}$

$$\begin{bmatrix} 0 & 0 & 1 \\ -1 & 2 & 2 \\ 2 & -3 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \\ 6 \end{bmatrix}$$

17. Inverse transformation: reflection with respect to the y-axis (same as F)

$$A = A^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \overset{\checkmark}{=} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

19. Not invertible

21. a. $(AB)^{-1} = B^{-1}A^{-1} = \begin{bmatrix} 0 & 3 & 1 \\ 3 & 1 & 3 \\ 1 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 7 & 1 \\ 7 & 11 & 0 \\ 4 & 8 & -1 \end{bmatrix}$

b. $\left(A^T\right)^{-1} = (A^{-1})^T = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$

c. Premultiplying both sides of $A\overrightarrow{x} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$ by A^{-1} yields $A^{-1}A\overrightarrow{x} = A^{-1}\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$, therefore

$$\overrightarrow{x} = A^{-1}\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 5 \end{bmatrix}.$$

d. From the equivalent conditions, invertible $n \times n$ matrices must be row equivalent to I_n. Since A and B are both invertible, they are both row equivalent to I_n, and consequently are row equivalent to each other.

23. a. $(B^TA)^{-1} = A^{-1}(B^T)^{-1} = A^{-1}(B^{-1})^T = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 2 & 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 1 & 3 & 1 \\ 2 & 3 & 2 & 1 \\ 2 & 2 & 4 & 2 \end{bmatrix}.$

b. $((A^{-1})^{-1})^{-1} = A^{-1} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 2 & 0 & 2 & 0 \end{bmatrix}$

c. $\left(B^2\right)^{-1} = (BB)^{-1} = B^{-1}B^{-1} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & 2 & 3 & 2 \\ 2 & 3 & 2 & 1 \\ 1 & 1 & 2 & 1 \end{bmatrix}$

d. $(AB^{-1})^{-1}(BA^{-1})^{-1} = (B^{-1})^{-1}A^{-1}(A^{-1})^{-1}B^{-1} = BA^{-1}AB^{-1} = BI_4B^{-1} = BB^{-1}$

$= I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$

27. TRUE

$LHS = (A^{-1}B^{-1})^T = (B^{-1})^T(A^{-1})^T$

$RHS = (A^TB^T)^{-1} = (B^T)^{-1}(A^T)^{-1}$

29. FALSE

By the equivalent conditions, A is nonsingular if and only if $A\overrightarrow{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ has a unique solution.

31. TRUE

By the equivalent conditions, if A is row equivalent to I_3 then $A\overrightarrow{x} = \begin{bmatrix} 9 \\ 7 \\ 3 \end{bmatrix}$ has a unique solution (therefore, it must be consistent).

33. TRUE

By the equivalent conditions, all nonsingular 5×5 matrices are row equivalent to I_5, therefore, they are also row equivalent to each other.

2.4 Applications of Linear Systems and Matrix Factorizations

Refer to the Linear Algebra Toolkit for details: http://bit.ly/2AAGfIt

1. The augmented matrix $\left[\begin{array}{ccc|c} 2 & -1 & 0 & 0 \\ 5 & -2 & -2 & 0 \end{array}\right]$ has the r.r.e.f.: $\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & -4 & 0 \end{array}\right]$.

   $x_1 = 2x_3$, $x_2 = 4x_3$, x_3 is arbitrary.

   Let $x_3 = 1$: 2 $N_2O_5 \to$ 4 NO_2 + O_2

Refer to the Linear Algebra Toolkit for details: http://bit.ly/2GW6yyA

3. The augmented matrix $\left[\begin{array}{cccc|c} 10 & 0 & -1 & 0 & 0 \\ 16 & 0 & 0 & -1 & 0 \\ 0 & 2 & 0 & -1 & 0 \end{array}\right]$ has the r.r.e.f.: $\left[\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{1}{16} & 0 \\ 0 & 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{5}{8} & 0 \end{array}\right]$

   $x_1 = \frac{1}{16}x_4,\ x_2 = \frac{1}{2}x_4,\ x_3 = \frac{5}{8}x_4,\ x_4$ is arbitrary

   Let $x_4 = 16$ (any smaller positive integer would lead to at least some fractions):

   $C_{10}H_{16} + 8Cl_2 \to 10C + 16HCl$

Refer to the Linear Algebra Toolkit for details: http://bit.ly/2TuWHB9

5. The augmented matrix $\left[\begin{array}{ccc|c} 6 & -2 & -1 & 0 \\ 12 & -6 & 0 & 0 \\ 6 & -1 & -1 & 0 \end{array}\right]$ has the r.r.e.f.: $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$

   The system has only one solution: $x_1 = x_2 = x_3 = 0$.

   The reaction equation cannot be balanced.

   (It is rather fortunate that the process of fermentation of glucose produces ethanol and carbon **dioxide** CO_2, rather than carbon **monoxide** CO.)

Refer to the Linear Algebra Toolkit for details: http://bit.ly/2SMhUGw

7. The augmented matrix $\left[\begin{array}{cccc|c} 3 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 4 & 0 & 0 & -4 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 3 & -1 & 0 & 0 \end{array}\right]$ has the r.r.e.f.: $\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

   $x_1 = x_4,\ x_2 = x_4,\ x_3 = 3x_4,\ x_4$ is arbitrary

   Let $x_4 = 1$: $Rb_3PO_4 + CrCl_3 \to 3RbCl + CrPO_4$

9.

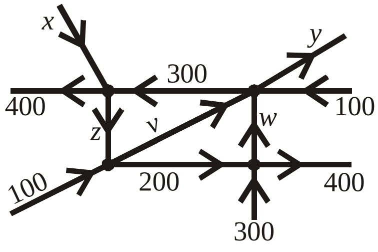


Figure for Exercise 9

There are four intersections in this network – each corresponds to one equation:

$$\begin{array}{rcrcrcrcrcl} x & & & - & z & & & & & = & 400-300 \\ & - & y & & & + & v & + & w & = & 300-100 \\ & & & & z & - & v & & & = & 200-100 \\ & & & & & & & - & w & = & 400-300-200 \end{array}$$

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2Fer0Im

The augmented matrix $\begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 100 \\ 0 & -1 & 0 & 1 & 1 & 200 \\ 0 & 0 & 1 & -1 & 0 & 100 \\ 0 & 0 & 0 & 0 & -1 & -100 \end{bmatrix}$ has the r.r.e.f.: $\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 200 \\ 0 & 1 & 0 & -1 & 0 & -100 \\ 0 & 0 & 1 & -1 & 0 & 100 \\ 0 & 0 & 0 & 0 & 1 & 100 \end{bmatrix}$.

Therefore, every solution satisfies

$$\begin{array}{rcl} x & = & 200+v \\ y & = & -100+v \\ z & = & 100+v \\ v & = & arbitrary \\ w & = & 100 \end{array}$$

There are infinitely many solutions, some of which involve negative values, which are not allowed in this model. Examples of solutions with all positive numbers can be constructed by taking any $v > 100$, e.g. for $v = 200$:

$$x = 400,\ y = 100,\ \ z = 300,\ v = 200,\ w = 100.$$

11.

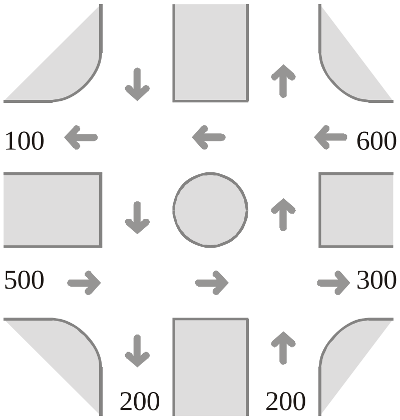
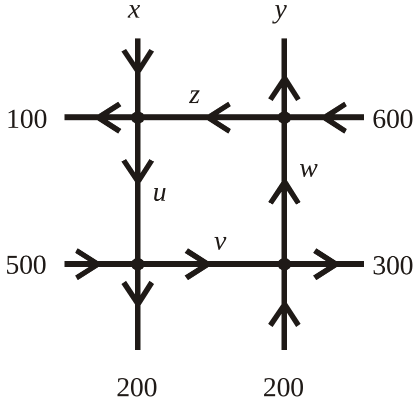


Figure for Exercise 11

The network can be expressed using oriented line segments as follows



Each of the four intersections yields a linear equation:

$$\begin{array}{rrrrrrrrrrrcr} x & & & + & z & - & u & & & & & = & 100 \\ & - & y & - & z & & & & & + & w & = & -600 \\ & & & & & & u & - & v & & & = & 200-500 \\ & & & & & & & & v & - & w & = & 300-200 \end{array}$$

Refer to the
Linear Algebra
Toolkit for details:

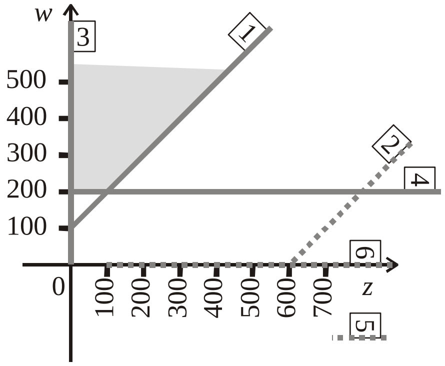
http://bit.ly/2RxWobd

The augmented matrix of this system $\begin{bmatrix} 1 & 0 & 1 & -1 & 0 & 0 & 100 \\ 0 & -1 & -1 & 0 & 0 & 1 & -600 \\ 0 & 0 & 0 & 1 & -1 & 0 & -300 \\ 0 & 0 & 0 & 0 & 1 & -1 & 100 \end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & -1 & -100 \\ 0 & 1 & 1 & 0 & 0 & -1 & 600 \\ 0 & 0 & 0 & 1 & 0 & -1 & -200 \\ 0 & 0 & 0 & 0 & 1 & -1 & 100 \end{bmatrix}$.

Every solution must satisfy

$$\begin{aligned} x &= -100 - z + w \\ y &= 600 - z + w \\ z &= arbitrary \\ u &= -200 + w \\ v &= 100 + w \\ w &= arbitrary \end{aligned}$$

Feasible solutions correspond to the following region in the $z - w$ plane:



An example of a solution with all positive values is obtained when. $z = 100, w = 300$:

$$x = 100,\ y = 800,\ z = 100,\ u = 100,\ v = 400,\ w = 300$$

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2AxvIxw

13. $\begin{bmatrix} 22/24 & 14/24 & 18/24 & 18/24 \\ 1/24 & 6/24 & 0 & 2/24 \\ 1/24 & 4/24 & 6/24 & 4/24 \end{bmatrix}$ has the reduced row echelon form: $\begin{bmatrix} 1 & 0 & 0 & \frac{2}{7} \\ 0 & 1 & 0 & \frac{2}{7} \\ 0 & 0 & 1 & \frac{3}{7} \end{bmatrix}$

By melting together two parts of alloy I, two parts of alloy II, and three parts of alloy III, alloy V is obtained.

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2SGbYz0

15. $\begin{bmatrix} 22/24 & 14/24 & 18/24 & 18/24 & 18/24 \\ 1/24 & 6/24 & 0 & 2/24 & 3/24 \\ 1/24 & 4/24 & 6/24 & 4/24 & 3/24 \end{bmatrix}$ has the reduced row echelon form: $\begin{bmatrix} 1 & 0 & 0 & \frac{2}{7} & \frac{3}{7} \\ 0 & 1 & 0 & \frac{2}{7} & \frac{3}{7} \\ 0 & 0 & 1 & \frac{3}{7} & \frac{1}{7} \end{bmatrix}$

Denoting the unknowns by x, y, z, and w, we have an arbitrary $w \geq 0$ such that $\frac{2}{7}w \leq \frac{3}{7}$ and $\frac{3}{7}w \leq \frac{1}{7}$ so that $0 \leq w \leq \frac{1}{3}$. An example of a feasible solution is obtained by taking $w = \frac{1}{4}$:

$$\begin{aligned} x &= \frac{3}{7} - \frac{2}{7} \cdot \frac{1}{4} = \frac{5}{14} \\ y &= \frac{3}{7} - \frac{2}{7} \cdot \frac{1}{4} = \frac{5}{14} \\ z &= \frac{1}{7} - \frac{3}{7} \cdot \frac{1}{4} = \frac{1}{28} \\ w &= \frac{1}{4} \end{aligned}$$

Therefore, one way to obtain alloy IV is by melting together 10 parts of alloy I, 10 parts of alloy II, 1 part of alloy III, and 7 parts of alloy V.

17\.

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2C6sSj9

- $\begin{bmatrix} 18/24 & 18/24 & 18/24 \\ 0 & 3/24 & 2/24 \\ 6/24 & 3/24 & 4/24 \end{bmatrix}$ has the reduced row echelon form: $\begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 0 \end{bmatrix}$
  adding 1 part of alloy III and 2 parts of alloy IV, we can obtain alloy V

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2C3Vg5r

- $\begin{bmatrix} 18/24 & 18/24 & 18/24 \\ 0 & 2/24 & 3/24 \\ 6/24 & 4/24 & 3/24 \end{bmatrix}$ has the reduced row echelon form: $\begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$
  alloy IV cannot be obtained by mixing alloy III and alloy V

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2LVM4oj

- $\begin{bmatrix} 18/24 & 18/24 & 18/24 \\ 3/24 & 2/24 & 0 \\ 3/24 & 4/24 & 6/24 \end{bmatrix}$ has the reduced row echelon form: $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$
  alloy III cannot be obtained by mixing alloy IV and alloy V.

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2C8FfeO

19\. We are looking for the polynomial $p(x) = a_0 + a_1x + a_2x^2$ such that $p(0) = 2$, $p(1) = 4$, and $p(2) = 0$. This leads to the system with augmented matrix $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 1 & 1 & 1 & 4 \\ 1 & 2 & 4 & 0 \end{bmatrix}$, with the reduced row echelon form: $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -3 \end{bmatrix}$. Therefore $p(x) = 2 + 5x - 3x^2$.

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2sbgZ7b

21\. We are looking for the polynomial $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ such that $p(0) = 0$, $p(1) = 2$, $p(2) = 2,$. and $p(3) = 0$. This leads to the system with augmented matrix $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 2 \\ 1 & 2 & 4 & 8 & 2 \\ 1 & 3 & 9 & 27 & 0 \end{bmatrix}$ with the reduced row echelon form: $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$. The Lagrange interpolating polynomial is $p(x) = 3x - x^2$.

Refer to the
Linear Algebra
Toolkit for details:
http://bit.ly/2LVMdIn

23. The augmented matrix of the system $\begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 3 & 0 \end{bmatrix}$ has the r.r.e.f.: $\begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix}$.

Therefore, $p(x) = 2 + x + 4x^2 - 3x^3$.

Refer to the
Linear Algebra
Toolkit for details:
http://bit.ly/2RBFVD8

25. The augmented matrix of the system $\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 4 & 12 & 0 \\ 1 & 3 & 9 & 27 & 2 \end{bmatrix}$ has the r.r.e.f.: $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

Therefore, there is no solution to this system so it is not possible to find a desired polynomial.

27. $p_0(1) = 3 : b_0 + c_0 + d_0 = 3 - 2$

$p_1(2) = 0 : b_1 + c_1 + d_1 = 0 - 3$

$p_0'(1) = p_1'(1) : b_0 + 2c_0 + 3d_0 = b_1$

$p_0''(1) = p_1''(1) : 2c_0 + 6d_0 = 2c_1$

$p_0''(0) = 0 : 2c_0 = 0$

$p_1''(2) = 0 : 2c_1 + 6d_1 = 0$

Refer to the
Linear Algebra
Toolkit for details:
http://bit.ly/2QpoQrb

$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & -3 \\ 1 & 2 & 3 & -1 & 0 & 0 & 0 \\ 0 & 2 & 6 & 0 & -2 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 6 & 0 \end{bmatrix}$ has the reduced row echelon form: $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$

$p_0(x) = 2 + 2x - x^3$

$p_1(x) = 3 - 1(x-1) - 3(x-1)^2 + 1(x-1)^3$

Check: $p_0(1) = 2 + 2 - 1 = 3$

$p_1(2) = 3 - 1 - 3 + 1 = 0$

$p_0'(x) = 2 - 3x^2$; $p_0'(1) = -1$; $p_1'(x) = -1 - 6(x-1) + 3(x-1)^2$; $p_1'(1) = -1$

$p_0''(x) = -6x$; $p_0''(1) = -6$; $p_1''(x) = -6 + 6(x-1)$; $p_1''(1) = -6$

$p_0''(0) = 0$

$p_1''(2) = -6 + 6 = 0$.

29. $r_2 - ar_1 \to r_2$

If $6 - 3a \neq 0$, i.e. $a \neq 2$, then the system has a unique solution (a leading entry corresponds to each unknown)

If $6 - 3a = 0$, i.e. $a = 2$, then the matrix becomes $\begin{bmatrix} 1 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, - the system has infinitely many solutions

(i) impossible

(ii) $a \neq 2$

(iii) $a = 2$

31. $r_1 \leftrightarrow r_2$

$$\begin{bmatrix} -1 & a & 0 \\ a & 1 & 0 \end{bmatrix}$$

$-r_1 \to r_1$

$$\begin{bmatrix} 1 & -a & 0 \\ a & 1 & 0 \end{bmatrix}$$

$r_2 - ar_1 \to r_2$

$$\begin{bmatrix} 1 & -a & 0 \\ 0 & 1+a^2 & 0 \end{bmatrix}$$

(i) impossible

(ii) for all real a values

(iii) impossible ($1 + a^2$ cannot equal 0 for any real a).

33. $r_2 - (a-1)r_1 \to r_2$

$$\begin{bmatrix} 1 & a+1 & 0 \\ 0 & 3-(a+1)(a-1) & b \end{bmatrix} = \begin{bmatrix} 1 & a+1 & 0 \\ 0 & 4-a^2 & b \end{bmatrix}$$

If $4 - a^2 \neq 0$, i.e. $a \neq \pm 2$ then the system has a unique solution (a leading entry corresponds to each unknown)

If $4 - a^2 = 0$, i.e. $a = \pm 2$ then

- if $b = 0$, the system has many solutions,
- if $b \neq 0$, the system has no solution.
  (i) $a = \pm 2$ and $b \neq 0$
  (ii) $a \neq \pm 2$
  (iii) $a = \pm 2$ and $b = 0$.

35.
- If $a = b = 0$ then there are infinitely many solutions.
- If $a = 0$ and $b \neq 0$ then
  $r_1 \leftrightarrow r_2$
  $$\begin{bmatrix} b & 0 & 0 \\ 0 & b & 0 \end{bmatrix}$$
  $\frac{1}{b}r_1 \to r_1$
  $\frac{1}{b}r_2 \to r_2$
  $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$
  Unique solution
- If $a \neq 0$ then
  $\frac{1}{a}r_1 \to r_1$
  $$\begin{bmatrix} 1 & \frac{b}{a} & 0 \\ b & a & 0 \end{bmatrix}$$
  $r_2 - br_1 \to r_2$
  $$\begin{bmatrix} 1 & \frac{b}{a} & 0 \\ 0 & a - \frac{b^2}{a} & 0 \end{bmatrix}$$
  If $a - \frac{b^2}{a} = 0$, i.e. $\frac{a^2-b^2}{a} = \frac{(a-b)(a+b)}{a} = 0$ then there are infinitely many solutions
  Otherwise, there is a unique solution.
  (i) impossible
  (ii) $a \neq b$ and $a \neq -b$,
  (iii) $a = b$ or $a = -b$

41. a. $L = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$, $U = \begin{bmatrix} 1 & 5 \\ 0 & -23 \end{bmatrix}$; $LU = \begin{bmatrix} 1 & 5 \\ 5 & 2 \end{bmatrix}$

b. Solve $\begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 13 \end{bmatrix}$ by forward substitution

$$\begin{aligned} y_1 &= -2 \\ y_2 &= 13 - 5y_1 = 23 \end{aligned}$$

Solve $\begin{bmatrix} 1 & 5 \\ 0 & -23 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 23 \end{bmatrix}$ by backsubstitution

$$\begin{aligned} x_2 &= -1 \\ x_1 &= -2 - 5x_2 = 3 \end{aligned}$$

43. a. $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 0 & -1 & 2 & 1 \end{bmatrix}$; $U = \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -8 \end{bmatrix}$; $LU = \begin{bmatrix} 1 & 0 & 0 & 5 \\ 1 & 1 & 0 & 1 \\ 2 & 3 & 1 & 0 \\ 0 & -1 & 2 & 0 \end{bmatrix}$

b. Solve $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 0 & -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -3 \\ -3 \end{bmatrix}$ by forward substitution

$$\begin{aligned} y_1 &= 1 \\ y_2 &= -y_1 = -1 \\ y_3 &= -3 - 2y_1 - 3y_2 = -2 \\ y_4 &= -3 + y_2 - 2y_3 = 0 \end{aligned}$$

Solve $\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -2 \\ 0 \end{bmatrix}$ by backsubstitution

$$\begin{aligned} x_4 &= 0 \\ x_3 &= -2 - 2x_4 = -2 \\ x_2 &= -1 + 4x_4 = -1 \\ x_1 &= 1 - 5x_4 = 1 \end{aligned}$$

45. $P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; $L = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -4 & 2 & 1 \end{bmatrix}$; $U = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & -1 \end{bmatrix}$

Verify: $PA = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 1 \\ -1 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 3 & 0 & 1 \\ 4 & 2 & 1 \end{bmatrix}$

$LU = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -4 & 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 3 & 0 & 1 \\ 4 & 2 & 1 \end{bmatrix}$

Solve the system $A\overrightarrow{x} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$ (followig the method of Example 2.27)

$$PA\overrightarrow{x} = P\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$$

$$LU\overrightarrow{x} = P\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$$

$$L\overrightarrow{y} = P\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

Solve $\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -4 & 2 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ for $\overrightarrow{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ from top to bottom:

$y_1 = 1$

$y_2 = 2 + 3y_1 = 2 + (3)(1) = 5$

$y_3 = 4 + 4y_1 - 2y_2 = 4 + (4)(1) - (2)(5) = -2$

Solve $U\overrightarrow{x} = \overrightarrow{y}$ for $\overrightarrow{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ by backsubstitution:

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ -2 \end{bmatrix}$$

$x_3 = \frac{-2}{-1} = 2$

$x_2 = \frac{5-x_3}{3} = \frac{5-2}{3} = 1$

$x_1 = \frac{1-x_2}{-1} = \frac{1-1}{-1} = 0$

This matches the solution that was obtained in Exercise 19 in Section 2.1 (although that exercise used x, y, and z instead of x_1, x_2, and x_3 as unknowns).

3 Determinants

3.1 Cofactor Expansions

1. a. $\det\begin{bmatrix}1 & 3\\ 2 & 1\end{bmatrix}$

i. Expand along the first row:
$\det A = (1)\det[1] - 3\det[2] = -5$
ii. Let $A = \begin{bmatrix}1 & 3\\ 2 & 1\end{bmatrix}$
$r_2 - 2r_1 \to r_2$
$A_1 = \begin{bmatrix}1 & 3\\ 0 & -5\end{bmatrix}$, $\det A_1 = \det A$
$\det A_1 = (1)(-5)$ by Theorem 3.3 and
$-5 = \det A_1 = \det A$ by Theorem 3.8.
Therefore, $\det A = -5$.
iii. $\det A = (1)(1) - (3)(2) = 1 - 6 = -5$.

b. $\det\begin{bmatrix}1 & 3 & 0\\ 0 & 3 & 5\\ 2 & 0 & 1\end{bmatrix}$

i. Expand along the first row:
$\det A = (-1)^{1+1}(1)\det\begin{bmatrix}3 & 5\\ 0 & 1\end{bmatrix} + (-1)^{1+2}(3)\det\begin{bmatrix}0 & 5\\ 2 & 1\end{bmatrix} + 0$
$= ((3)(1) - (5)(0)) - 3((0)(1) - (5)(2)) + 0$
$= 3 + 30 = 33$.
ii. Let $A = \begin{bmatrix}1 & 3 & 0\\ 0 & 3 & 5\\ 2 & 0 & 1\end{bmatrix}$
$r_3 - 2r_1 \to r_3$
$A_1 = \begin{bmatrix}1 & 3 & 0\\ 0 & 3 & 5\\ 0 & -6 & 1\end{bmatrix}$, $\det A_1 = \det A$
$r_3 + 2r_2 \to r_3$
$A_2 = \begin{bmatrix}1 & 3 & 0\\ 0 & 3 & 5\\ 0 & 0 & 11\end{bmatrix}$, $\det A_2 = \det A_1$
$\det A_2 = (1)(3)(11)$ by Theorem 3.3 and
$33 = \det A_2 = \det A_1 = \det A$ by Theorem 3.8.
Therefore, $\det A = 33$.
iii. $\det A = (1)(3)(1) + (3)(5)(2) + (0)(0)(0) - (0)(3)(2) - (1)(5)(0) - (3)(0)(1)$
$= 3 + 30 + 0 - 0 - 0 - 0 = 33$.

c. $\det\begin{bmatrix}9 & 7 & 2\\ 8 & 6 & 7\\ 0 & 0 & 0\end{bmatrix}$

i. Expand along the third row: $\det A = 0 + 0 + 0 = 0$.
ii. $r_2 - \frac{8}{9}r_1 \to r_2$
$A_1 = \begin{bmatrix}9 & 7 & 2\\ 0 & -\frac{2}{9} & \frac{47}{9}\\ 0 & 0 & 0\end{bmatrix}$
$\det A_1 = (9)(-\frac{2}{9})(0) = 0 = \det A$ by Theorems 3.3 and 3.8.

iii. $\det A = (9)(6)(0) + (7)(7)(0) + (2)(8)(0) - (2)(6)(0) - (9)(7)(0) - (7)(8)(0)$
$= 0 + 0 + 0 - 0 - 0 - 4 = 0.$

3. a. $\det \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 2 & 4 \\ -2 & 0 & 0 & 3 \end{bmatrix}$

i. Expand along the second row:

$$\det A = 0 + 0 + (-1)^{2+3}(1) \det \begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & 4 \\ -2 & 0 & 3 \end{bmatrix} + 0$$

expand the 3×3 determinant along the second row:

$$= (-1)[0 + (-1)^{2+2}(2) \det \begin{bmatrix} 1 & 3 \\ -2 & 3 \end{bmatrix} + (-1)^{2+3}(4) \det \begin{bmatrix} 1 & 1 \\ -2 & 0 \end{bmatrix}]$$

$= (-1)[2((1)(3) - (3)(-2)) - 4((1)(0) - (1)(-2))]$
$= -[2(9) - 4(2)] = -10$

ii. Let $A = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 2 & 4 \\ -2 & 0 & 0 & 3 \end{bmatrix}$

$r_4 + 2r_1 \to r_4$

$$A_1 = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 2 & 4 \\ 0 & 2 & 0 & 9 \end{bmatrix}, \det A_1 = \det A$$

$r_3 \leftrightarrow r_2$

$$A_2 = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 9 \end{bmatrix}, \det A_2 = -\det A_1$$

$r_4 - r_2 \to r_4$

$$A_3 = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 5 \end{bmatrix}, \det A_3 = \det A_2$$

$r_4 + 2r_3 \to r_4$

$$A_4 = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}, \det A_4 = \det A_3$$

$\det A_4 = (1)(2)(1)(5)$ by Theorem 3.3 and
$10 = \det A_4 = \det A_3 = \det A_2 = -\det A_1 = -\det A$ by Theorem 3.8.
Therefore, $\det A = -10$.

b. $\det \begin{bmatrix} 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & 1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 3 \\ 4 & 0 & 2 & 0 & 0 \end{bmatrix}$

i. Expand along the first row:

$$\det A = 0 + (-1)^{1+2}(2)\det\begin{bmatrix} 2 & 1 & 2 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 4 & 2 & 0 & 0 \end{bmatrix} + 0 + 0 + 0$$

Expand the 4×4 determinant along the second row:

$$\det A = -2(0 + (-1)^{2+2}(-1)\det\begin{bmatrix} 2 & 2 & 0 \\ 0 & 1 & 3 \\ 4 & 0 & 0 \end{bmatrix} + 0 + 0)$$

Expand the 3×3 determinant along the third row:

$$= 2[(-1)^{3+1}(4)\det\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} + 0 + 0] = (2)(4)[(2)(3) - (0)(1)] = 48$$

ii. Let $A = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & 1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 3 \\ 4 & 0 & 2 & 0 & 0 \end{bmatrix}$

$r_2 \leftrightarrow r_1$

$$A_1 = \begin{bmatrix} 2 & 0 & 1 & 2 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 3 \\ 4 & 0 & 2 & 0 & 0 \end{bmatrix}, \det A_1 = -\det A$$

$r_5 - 2r_1 \to r_5$

$$A_2 = \begin{bmatrix} 2 & 0 & 1 & 2 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 0 & -4 & 0 \end{bmatrix}, \det A_2 = \det A_1$$

$r_4 - \frac{1}{2}r_2 \to r_4$

$$A_3 = \begin{bmatrix} 2 & 0 & 1 & 2 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & -4 & 0 \end{bmatrix}, \det A_3 = \det A_2$$

$r_5 + 4r_4 \to r_5$

$$A_4 = \begin{bmatrix} 2 & 0 & 1 & 2 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 12 \end{bmatrix}, \det A_4 = \det A_3$$

$\det A_4 = (2)(2)(-1)(1)(12)$ by Theorem 3.3 and
$-48 = \det A_4 = \det A_3 = \det A_2 = \det A_1 = -\det A$ by Theorem 3.8.
Therefore, $\det A = 48$.

5. a. $\det\begin{bmatrix} 0 & -2 \\ 3 & -7 \end{bmatrix}$
$= (0)(-7) - (-2)(3) = 0 + 6 = 6.$

b. $\det\begin{bmatrix} 2 & 0 & 1 \\ -3 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$

Expand along the first row:

$= 2(-1)^{1+1}\det\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} + 0 + 1(-1)^{1+3}\det\begin{bmatrix} -3 & 1 \\ 0 & 2 \end{bmatrix}$

$= 2((1)(1) - (2)(2)) + 0 + ((-3)(2) - (1)(0))$

$= 2(-3) - 6$

$= -12$

c. $\det\begin{bmatrix} 2 & 1 & 3 \\ 2 & -1 & -2 \\ 1 & 1 & 2 \end{bmatrix}$

Expand along the first row:

$= 2(-1)^{1+1}\det\begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix} + 1(-1)^{1+2}\det\begin{bmatrix} 2 & -2 \\ 1 & 2 \end{bmatrix}$

$+3(-1)^{1+3}\det\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$

$= 2((-1)(2) - (-2)(1)) - 1((2)(2) - (-2)(1)) + 3((2)(1) - (-1)(1))$

$= 2(0) - 1(6) + 3(3)$

$= 3$

7. a. $\det\begin{bmatrix} 2 & 1 & 0 & 3 \\ 4 & 0 & 2 & 0 \\ 1 & -1 & 1 & 2 \\ 0 & 0 & -1 & 2 \end{bmatrix}$

Expand along the fourth row:

$= 0 + 0 + (-1)(-1)^{4+3}\det\begin{bmatrix} 2 & 1 & 3 \\ 4 & 0 & 0 \\ 1 & -1 & 2 \end{bmatrix} + 2(-1)^{4+4}\det\begin{bmatrix} 2 & 1 & 0 \\ 4 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}$

$= (-1)(-1)[(2)(0)(2) + (1)(0)(1) + (3)(4)(-1) - (3)(0)(1) - (2)(0)(-1) - (1)(4)(2)]$

$+2[(2)(0)(1) + (1)(2)(1) + (0)(4)(-1) - (0)(0)(1) - (2)(2)(-1) - (1)(4)(1)]$

$= [0 + 0 - 12 - 0 - 0 - 8] + 2[0 + 2 + 0 - 0 + 4 - 4]$

$= -20 + 2(2)$

$= -16$

b. $\det\begin{bmatrix} -1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 1 & 0 & 1 & 2 & -1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$

Expand along the fourth column:

$=0 + 0 + 2(-1)^{3+4}\det\begin{bmatrix} -1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} + 0 + 0$

now, expand along the first column:

$= -2((-1)(-1)^{1+1}\det\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix})$

and expand along the third row

$$\begin{aligned} &= -2(-1)[0 - 1 \det \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + 0] \\ &= -2(-1)[0 - 1(1-1) + 0] \\ &= -2(-1)(0) \\ &= 0 \end{aligned}$$

9.

a. Answer: 6 (by Theorem 3.8 adding a multiple of one row to another does not change the determinant)

b. Answer: -6 (by Theorem 3.8 interchanging rows 1 and 3 reverses the sign of the determinant)

c. Answer: 0 (by Corollary 3.2 a square matrix with a zero row has a zero determinant)

d. $\det(A^{-1}) = 1/\det A = 1/6$ by Corollary 3.12

e. By Theorem 3.8, every time a row is multiplied by a factor, the determinant is also multiplied by the same factor. Therefore,
$\det(2A) = (2)(2)(2)(6) = 48.$

11. a. $\det(A^T) = (2)(-1)(3) = -6$ by Theorem 3.3 (A^T is upper triangular).
We apply Theorems 3.7:
$\det A = \det(A^T) = -6.$

b. Answer: -6
(We applied Theorem 3.7 and used the result obtained in part a.)

c. $\det B^{-1} = (-1)(2)(-2) = 4$ by Theorem 3.3.
We apply Theorem 3.11 and Corollary 3.12.
$\det(AB) = \det A \det B = \det A \frac{1}{\det B^{-1}} = \frac{-6}{4} = \frac{-3}{2}$

13. a. We apply Theorems 3.11 and 3.7:
$\det(B^T A^2) = \det(B^T)\,(\det(A))^2 = \det(B)\,(\det(A))^2 = (5)(-3)^2 = 45$

b. We apply Theorem 3.11 and Corollary 3.12:
$\det(A^{-1}B) = \frac{1}{\det(A)} \det(B) = \left(\frac{1}{-3}\right)(5) = \frac{-5}{3}$

c. $\det(3A) = 3^3 \det(A) = (27)(-3) = -81$ (every time a row is multiplied by k, the determinant is also multiplied by k - since all three rows are scaled by k, by Theorem 3.8 the determinant is multiplied by $k \cdot k \cdot k = k^3$)

d. We apply Corollary 3.12 and Theorem 3.8:
$\det((2B)^{-1}) = \frac{1}{\det(2B)} = \frac{1}{2^3 \det(B)} = \frac{1}{(8)(5)} = \frac{1}{40}$

e. We apply Theorems 3.11 and 3.7 and Corollary 3.12:
$\det(A^T B A^{-1}) = \det(A^T)\det(B)\det(A^{-1}) = \det(A)\det(B)\left(\frac{1}{\det(A)}\right) = \det(B) = 5$

f. We apply Theorems 3.11 and 3.7 and Corollary 3.12:
$\det((A^{-1}B)^{-1}(BA)^T) = \det(B^{-1}AA^TB^T) = \frac{1}{\det(B)}\det(A)\det(A)\det(B) = (\det(A))^2 = 9$

15. FALSE

Counterexample: $\det \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = 1 - 4 = -3.$

17. TRUE

If the ith column of an $n \times n$ matrix A is a linear combination of the remaining columns, then the ith row of A^T is a linear combination of the remaining rows of A^T :

$\text{row}_i = c_1 \text{row}_1 + c_2 \text{row}_2 + \cdots + c_{i-1} \text{row}_{i-1} + c_{i+1} \text{row}_{i+1} + \cdots + c_n \text{row}_n$

Performing row operations

$\text{row}_i - c_1 \text{row}_1 \to \text{row}_i$

$\text{row}_i - c_2 \text{row}_2 \to \text{row}_i$

$\vdots$

$\text{row}_i - c_{i-1} \text{row}_{i-1} \to \text{row}_i$

$\text{row}_i - c_{i+1} \text{row}_{i+1} \to \text{row}_i$

$\vdots$

$\mathrm{row}_i - c_n \,\mathrm{row}_n \longrightarrow \mathrm{row}_i$

results in row_i becoming a zero row. By Corollary 3.2 as well as Theorems 3.8 and 3.7 it follows that the determinant of the original matrix is 0.

19. FALSE

Counterexample

$A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, B = \begin{bmatrix} 2 & 3 \end{bmatrix}.$

$\det(AB) = \det \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} = 0$ while

$\det(BA) = \det [8] = 8.$

21. TRUE

From the equivalent statements, rank $A = n$ is equivalent to A.being row equivalent to the identity matrix.

3.2 Applications of Determinants

1. $\det A = \det\begin{bmatrix} 3 & -1 \\ -2 & 3 \end{bmatrix} = (3)(3) - (-1)(-2) = 9 - 2 = 7 \neq 0$, therefore, the system has a unique solution, which can be found using Cramer's rule.

   $\det A_1 = \det\begin{bmatrix} 3 & -1 \\ 5 & 3 \end{bmatrix} = (3)(3) - (-1)(5) = 9 + 5 = 14.$

   $\det A_2 = \det\begin{bmatrix} 3 & 3 \\ -2 & 5 \end{bmatrix} = (3)(5) - (3)(-2) = 15 + 6 = 21.$

   Therefore,

   $x_1 = \frac{\det A_1}{\det A} = \frac{14}{7} = 2.$

   $x_2 = \frac{\det A_2}{\det A} = \frac{21}{7} = 3.$

3. $\det A = \det\begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix} = (2)(2) - (-1)(-4) = 4 - 4 = 0$, therefore, the system either has many solutions or none. Cramer's rule cannot be used here.

5. $\det A = \det\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

   $= (1)(1)(3) + (2)(0)(0) + (3)(1)(1) - (3)(1)(0) - (1)(0)(1) - (2)(1)(3)$

   $= 3 + 0 + 3 - 0 - 0 - 6 = 0$ therefore, the system either has many solutions or none. Cramer's rule cannot be used here.

7. $\det A = \det\begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & -2 \\ 2 & -1 & 1 \end{bmatrix}$

   $= (1)(3)(1) + (2)(-2)(2) + (-1)(0)(-1) - (-1)(3)(2) - (1)(-2)(-1) - (2)(0)(1)$

   $= 3 - 8 + 0 + 6 - 2 - 0 = -1 \neq 0$, therefore, the system has a unique solution, which can be found using Cramer's rule.

   $\det A_1 = \det\begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -2 \\ 1 & -1 & 1 \end{bmatrix}$

   $= (2)(3)(1) + (2)(-2)(1) + (-1)(1)(-1) - (-1)(3)(1) - (2)(-2)(-1) - (2)(1)(1)$

   $= 6 - 4 + 1 + 3 - 4 - 2 = 0.$

   $\det A_2 = \det\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -2 \\ 2 & 1 & 1 \end{bmatrix}$

   $= (1)(1)(1) + (2)(-2)(2) + (-1)(0)(1) - (-1)(1)(2) - (1)(-2)(1) - (2)(0)(1)$

   $= 1 - 8 + 0 + 2 + 2 - 0 = -3.$

   $\det A_3 = \det\begin{bmatrix} 1 & 2 & 2 \\ 0 & 3 & 1 \\ 2 & -1 & 1 \end{bmatrix}$

   $= (1)(3)(1) + (2)(1)(2) + (2)(0)(-1) - (2)(3)(2) - (1)(1)(-1) - (2)(0)(1)$

$= 3 + 4 + 0 - 12 + 1 - 0 = -4.$

Therefore,

$x_1 = \frac{\det A_1}{\det A} = \frac{0}{-1} = 0.$

$x_2 = \frac{\det A_2}{\det A} = \frac{-3}{-1} = 3.$

$x_3 = \frac{\det A_3}{\det A} = \frac{-4}{-1} = 4.$

9. a. $A_{11} = (-1)^{1+1} \det [1] = 1$

$A_{12} = (-1)^{1+2} \det [3] = -3$

$A_{21} = (-1)^{2+1} \det [2] = -2$

$A_{22} = (-1)^{2+2} \det [-1] = -1$

$$\text{adj } A = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -3 & -1 \end{bmatrix}.$$

$\det A = (1)(-1) - (-2)(-3) = -1 - 6 = -7$

$$A \,\text{adj } A = \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -3 & -1 \end{bmatrix} = \begin{bmatrix} -7 & 0 \\ 0 & -7 \end{bmatrix} = (\det A)\, I_2.$$

b. $A_{11} = (-1)^{1+1} \det \begin{bmatrix} 1 & -1 \\ 2 & -3 \end{bmatrix} = -1$

$A_{12} = (-1)^{1+2} \det \begin{bmatrix} -2 & -1 \\ 0 & -3 \end{bmatrix} = -6$

$A_{13} = (-1)^{1+3} \det \begin{bmatrix} -2 & 1 \\ 0 & 2 \end{bmatrix} = -4$

$A_{21} = (-1)^{2+1} \det \begin{bmatrix} 1 & 2 \\ 2 & -3 \end{bmatrix} = 7$

$A_{22} = (-1)^{2+2} \det \begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix} = -3$

$A_{23} = (-1)^{2+3} \det \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = -2$

$A_{31} = (-1)^{3+1} \det \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} = -3$

$A_{32} = (-1)^{3+2} \det \begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix} = -3$

$A_{33} = (-1)^{3+3} \det \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} = 3$

$$\text{adj } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} -1 & 7 & -3 \\ -6 & -3 & -3 \\ -4 & -2 & 3 \end{bmatrix}$$

$\det A = (1)(1)(-3) + (1)(-1)(0) + (2)(-2)(2) - (2)(1)(0) - (1)(-1)(2) - (1)(-2)(-3)$

$= -3 + 0 - 8 - 0 + 2 - 6 = -15$

$$A\,\text{adj}\,A = \begin{bmatrix} 1 & 1 & 2 \\ -2 & 1 & -1 \\ 0 & 2 & -3 \end{bmatrix} \begin{bmatrix} -1 & 7 & -3 \\ -6 & -3 & -3 \\ -4 & -2 & 3 \end{bmatrix} = \begin{bmatrix} -15 & 0 & 0 \\ 0 & -15 & 0 \\ 0 & 0 & -15 \end{bmatrix}$$

$= (\det A)\, I_3$

c. $A_{11} = (-1)^{1+1} \det \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = 2$

$$A_{12} = (-1)^{1+2} \det \begin{bmatrix} 0 & 0 \\ -4 & 2 \end{bmatrix} = 0$$

$$A_{13} = (-1)^{1+3} \det \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} = 4$$

$$A_{21} = (-1)^{2+1} \det \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix} = 0$$

$$A_{22} = (-1)^{2+2} \det \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix} = 0$$

$$A_{23} = (-1)^{2+3} \det \begin{bmatrix} 2 & 0 \\ -4 & 0 \end{bmatrix} = 0$$

$$A_{31} = (-1)^{3+1} \det \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = 1$$

$$A_{32} = (-1)^{3+2} \det \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} = 0$$

$$A_{33} = (-1)^{3+3} \det \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = 2$$

$$\text{adj}\,A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 4 & 0 & 2 \end{bmatrix}$$

$\det A = (2)(1)(2) + (0)(0)(-4) + (-1)(0)(0) - (-1)(1)(-4) - (2)(0)(0) - (0)(0)(2)$
$= 4 + 0 + 0 - 4 - 0 - 0 = 0$

$$A\,\text{adj}\,A = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 0 \\ -4 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 4 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = (\det A)\, I_3$$

11. $\overrightarrow{PQ} = \begin{bmatrix} -1-5 \\ 3-2 \end{bmatrix} = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$

$$\overrightarrow{PR} = \begin{bmatrix} 4-5 \\ 0-2 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

The triangle has the area:

$\frac{1}{2}\left|\det\begin{bmatrix} -6 & -1 \\ 1 & -2 \end{bmatrix}\right| = \frac{1}{2}\left|12+1\right| = \frac{13}{2}$.

13. (a) $\det = 4$ (each dimension is doubled) - matches $\det A_2$

    (b) $\det = 1$ (dimensions remain unchanged) - matches $\det A_1$

    (c) $\det = 1$ (the size of the parallelogram is the same as that of the original rectangle) - matches $\det A_3$.

15. TRUE

    If $i > j$ then A_{ij} involves a determinant of an $(n-1)\times(n-1)$ lower triangular matrix B for which $b_{jj} = 0$, therefore $\det B = 0$.

    Likewise, when $i < j$ then A_{ij} involves a determinant of an $(n-1)\times(n-1)$ upper triangular matrix B for which $b_{ii} = 0$, therefore $\det B = 0$.

17. FALSE

    Let A be invertible. Then adj $A = (\det A)A^{-1}$. Also, adj $(kA) = (\det kA)(kA)^{-1}$.

    However, $\det(kA) = k^n \det A$, and $(kA)^{-1} = \frac{1}{k}A^{-1}$. Therefore adj $(kA) = k^{n-1}$adj$A$.

19. FALSE

    Adjoint of every square matrix exists.

4 Vector Spaces

4.1 Vector Spaces

1. Condition 2.

   $LHS = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ b_2 \end{bmatrix}$

   $RHS = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} + \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ a_2 \end{bmatrix}$

   Generally, $LHS \neq RHS \Rightarrow$Condition does not hold.

   Condition 3.

   $LHS = (\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}) + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ b_2 \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ c_2 \end{bmatrix}$

   $RHS = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + (\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}) = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ c_2 \end{bmatrix}.$

   $LHS = RHS \Rightarrow$Condition holds

   Condition 7.

   $LHS = k(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}) = k\begin{bmatrix} a_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} ka_1 \\ kb_2 \end{bmatrix}$

   $RHS = k\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + k\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} ka_1 \\ ka_2 \end{bmatrix} + \begin{bmatrix} kb_1 \\ kb_2 \end{bmatrix} = \begin{bmatrix} ka_1 \\ kb_2 \end{bmatrix}$

   $LHS = RHS \Rightarrow$Condition holds

   Condition 8

   $LHS = (c+d)\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} (c+d)a_1 \\ (c+d)a_2 \end{bmatrix}$

   $RHS = c\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + d\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} ca_1 \\ ca_2 \end{bmatrix} + \begin{bmatrix} da_1 \\ da_2 \end{bmatrix} = \begin{bmatrix} ca_1 \\ da_2 \end{bmatrix}$

   Generally, $LHS \neq RHS \Rightarrow$Condition does not hold.

3. Condition 7.

   $LHS = k(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}) = k\begin{bmatrix} a_1+b_1 \\ a_2+b_2 \\ a_3+b_3 \end{bmatrix} = \begin{bmatrix} a_1+b_1 \\ a_2+b_2 \\ a_3+b_3 \end{bmatrix}$

   $RHS = k\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + k\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_1+b_1 \\ a_2+b_2 \\ a_3+b_3 \end{bmatrix}$

   $LHS = RHS \Rightarrow$Condition holds

   Condition 8.

   $LHS = (c+d)\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$

$$RHS = c\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + d\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 2a_1 \\ 2a_2 \\ 2a_3 \end{bmatrix}$$

Generally, $LHS \neq RHS \Rightarrow$Condition does not hold.

Condition 9.

$$LHS = (cd)\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$RHS = c(d\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}) = c\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$LHS = RHS \Rightarrow$Condition holds

Condition 10.

$$LHS = 1\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$LHS = RHS \Rightarrow$Condition holds

5. Condition 2.

$$LHS = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} + \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} a+b & 0 \\ 0 & a+b \end{bmatrix}$$

$$RHS = \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix} + \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = \begin{bmatrix} b+a & 0 \\ 0 & b+a \end{bmatrix}$$

$LHS = RHS \Rightarrow$Condition holds

Condition 3.

$$LHS = (\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} + \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix}) + \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} = \begin{bmatrix} a+b & 0 \\ 0 & a+b \end{bmatrix} + \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} = \begin{bmatrix} a+b+c & 0 \\ 0 & a+b+c \end{bmatrix}$$

$$RHS = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} + (\begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix} + \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}) = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} + \begin{bmatrix} b+c & 0 \\ 0 & b+c \end{bmatrix} = \begin{bmatrix} a+b+c & 0 \\ 0 & a+b+c \end{bmatrix}$$

$LHS = RHS \Rightarrow$Condition holds

Condition 4 holds with the zero vector $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Condition 9

$$LHS = (cd)\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = \begin{bmatrix} cda & 0 \\ 0 & cda \end{bmatrix}$$

$$RHS = c(d\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}) = c\begin{bmatrix} da & 0 \\ 0 & da \end{bmatrix} = \begin{bmatrix} cda & 0 \\ 0 & cda \end{bmatrix}$$

$LHS = RHS \Rightarrow$Condition holds

Condition 10.

$$1\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$$

$LHS = RHS \Rightarrow$Condition holds

7. Condition 4 fails: cannot find $\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ independent of a_1 and a_2 such that $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ for all a_1 and a_2.

   Also, Condition 10 fails (1 $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ 0 \end{bmatrix} \overset{\text{generally}}{\neq} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$)

9. All ten Conditions hold. (Using the theory in Section 4.2., this can be shown to be a subspace of R^2, therefore, it is a vector space.)

4.2 Subspaces

1. No, does not contain $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. (Violates Condition a.)

3. Yes, W =span$\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\}$ so we can apply Theorem 4.6 to justify that W is a subspace of R^2. (It can be shown that W satisfies all three conditions.)

5. No, violates Condition c.:

   e.g. $(-2)\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} -6 \\ 0 \end{bmatrix}$ is not in W.

7. Yes, W =span$\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\}$ so we can apply Theorem 4.6 to justify that W is a subspace of R^3. (It can be shown that W satisfies all three conditions.)

9. Yes, satisfies all three conditions.

   a. $0 + 2(0) - 0 \overset{\checkmark}{=} 0$

   b. Taking two vectors in W, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ such that

   $$x_1 + 2x_2 - x_3 = 0 \text{ and } y_1 + 2y_2 - y_3 = 0$$

   the sum $\begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix}$ satisfies

   $$(x_1 + y_1) + 2(x_2 + y_2) - (x_3 + y_3) = (x_1 + 2x_2 - x_3) + (y_1 + 2y_2 - y_3) = 0 + 0 \overset{\checkmark}{=} 0$$

   which means it is in W.

   c. Taking a scalar multiple of a vector in $W : c\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ with $x_1 + 2x_2 - x_3 = 0$, we obtain the vector $\begin{bmatrix} cx_1 \\ cx_2 \\ cx_3 \end{bmatrix}$ that satisfies

   $$cx_1 + 2cx_2 - cx_3 = c(x_1 + 2x_2 - x_3) = (c)(0) = 0.$$

   Note that this space is the solution space of the homogeneous equation $x_1 + 2x_2 - x_3 = 0$.

11. Yes, W =span$\{1 + t + t^2\}$ so we can apply Theorem 4.6 to justify that W is a subspace of P_2.

13. No, does not contain $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. (Violates Condition a.)

15. No, violates Condition b. , e.g. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$ is outside the set.

17. Yes. It satisfies the three conditions:
    - Condition a holds since the function $f(x) \equiv 0$ is continuous (it's in the set)
    - Condition b holds because a sum of two continuous functions, $f + g$ is also continuous.
    - Condition c holds because a scalar multiple of a continuous function f, cf, is also continuous for every scalar c.

19. No. It violates Condition c., e.g., while the function $f(x) = x$ is in the set, the scalar multiple $(-1)f$ is not.

Refer to the Linear Algebra Toolkit for details: http://bit.ly/2GJi3cu

21. Yes.

The equation $c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$ is equivalent to a linear system with the augmented matrix $\begin{bmatrix} 1 & 2 & | & d_1 \\ 1 & 1 & | & d_2 \end{bmatrix}$. The r.r.e.f., $\begin{bmatrix} 1 & 0 & | & -d_1 + 2d_2 \\ 0 & 1 & | & d_1 - d_2 \end{bmatrix}$ contains no row of the form $[\, 0 \cdots 0 \mid \text{nonzero}]$, therefore the system is consistent for all d_1 and d_2.

Refer to the Linear Algebra Toolkit for details: http://bit.ly/2ESvyEU

23. No.

The equation $c_1 \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$ is equivalent to a linear system with the augmented matrix $\begin{bmatrix} 1 & 0 & | & d_1 \\ 3 & 1 & | & d_2 \\ 0 & -2 & | & d_3 \end{bmatrix}$. After the row operations $r_2 - 3r_1 \to r_2$ and $r_3 + 2r_2 \to r_3$, $\begin{bmatrix} 1 & 0 & | & d_1 \\ 0 & 1 & | & -3d_1 + d_2 \\ 0 & 0 & | & -6d_1 + 2d_2 + d_3 \end{bmatrix}$ has the third row in the form $[\, 0 \cdots 0 \mid \text{nonzero}]$ for some d_1, d_2, and d_3, therefore for these d_1, d_2, d_3 values the system is inconsistent.

Refer to the Linear Algebra Toolkit for details: http://bit.ly/2CCf9lE

25. Yes.

The equation $c_1 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} + c_4 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$ is equivalent to a linear system with the augmented matrix $\begin{bmatrix} 0 & 1 & 0 & 1 & | & d_1 \\ 0 & 0 & 2 & 0 & | & d_2 \\ 0 & 2 & -1 & 0 & | & d_3 \end{bmatrix}$. The r.r.e.f., $\begin{bmatrix} 0 & 1 & 0 & 0 & | & \frac{1}{4}d_2 + \frac{1}{2}d_3 \\ 0 & 0 & 1 & 0 & | & \frac{1}{2}d_2 \\ 0 & 0 & 0 & 1 & | & d_1 - \frac{1}{4}d_2 - \frac{1}{2}d_3 \end{bmatrix}$ contains no row of the form $[\, 0 \cdots 0 \mid \text{nonzero}]$, therefore the system is consistent for all d_1, d_2, and d_3.

Refer to the Linear Algebra Toolkit for details: http://bit.ly/2BI3Gzt

27. Yes.

The equation

$$c_1 \left(1 - 3t^2\right) + c_2 t + c_3 \left(-1 + 2t^2\right) = d_1 + d_2 t + d_3 t^2$$

is equivalent to a linear system

$$\begin{array}{rcrcrcl} c_1 & & & - & c_3 & = & d_1 \\ & & c_2 & & & = & d_2 \\ -3c_1 & & & + & 2c_3 & = & d_3 \end{array}$$

with the augmented matrix $\begin{bmatrix} 1 & 0 & -1 & | & d_1 \\ 0 & 1 & 0 & | & d_2 \\ -3 & 0 & 2 & | & d_3 \end{bmatrix}$. The r.r.e.f., $\begin{bmatrix} 1 & 0 & 0 & -2d_1 - d_3 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 1 & -3d_1 - d_3 \end{bmatrix}$ contains no row of the form $[\, 0 \cdots 0 \mid \text{nonzero}]$, therefore the system is consistent for all d_1, d_2, and d_3.

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2VgZy22

29. No.

    The equation

    $$c_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} d_1 & d_2 \\ d_3 & d_4 \end{bmatrix}$$

    is equivalent to the linear system

    $$\begin{array}{ccccccc} c_1 & & & + & c_3 & = & d_1 \\ & & c_2 & & & = & d_2 \\ & & c_2 & + & c_3 & = & d_3 \\ c_1 & & & & & = & d_4 \end{array}$$

    with the augmented matrix $\begin{bmatrix} 1 & 0 & 1 & | & d_1 \\ 0 & 1 & 0 & | & d_2 \\ 0 & 1 & 1 & | & d_3 \\ 1 & 0 & 0 & | & d_4 \end{bmatrix}$.

    After the row operations $r_4 - r_1 \rightarrow r_4$; $r_3 - r_2 \rightarrow r_3$, and $r_4 + r_3 \rightarrow r_4$, we obtain the matrix $\begin{bmatrix} 1 & 0 & 1 & | & d_1 \\ 0 & 1 & 0 & | & d_2 \\ 0 & 0 & 1 & | & -d_2 + d_3 \\ 0 & 0 & 0 & | & -d_1 - d_2 + d_3 + d_4 \end{bmatrix}$, which has the fourth row in the form $[\ 0 \cdots 0 \mid \text{nonzero}]$ for some d_1, d_2, d_3, and d_4, therefore for these d_1, d_2, d_3, d_4 values the system is inconsistent.

31. FALSE: Every vector space has a subspace composed of just the zero vector. (Any other subspace, however, has infinitely many vectors in it.)

33. TRUE: span$\{\vec{u}, \vec{v}\}$ is spanned by vectors in span$\{\vec{u}, \vec{v}, \vec{w}\}$, therefore, it is a subspace of span$\{\vec{u}, \vec{v}, \vec{w}\}$

4.3 Linear Independence

NOTE: Refer to the Linear Algebra Toolkit for the details involved in solving these problems, including the individual elementary row operations.

1. a. $\overrightarrow{u_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \overrightarrow{u_2} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2rWmiHl

The set $\{\overrightarrow{u_1}, \overrightarrow{u_2}\}$ is linearly independent if and only if the homogeneous system

$$c_1\overrightarrow{u_1} + c_2\overrightarrow{u_2} = \overrightarrow{0}$$

has only the trivial solution $c_1 = c_2 = 0$. Otherwise, the set is linearly dependent.

Our homogeneous system has the augmented matrix $\begin{bmatrix} 1 & 2 & | & 0 \\ 0 & -1 & | & 0 \end{bmatrix}$ whose r.r.e.f. is $\begin{bmatrix} \boxed{1} & 0 & | & 0 \\ 0 & \boxed{1} & | & 0 \end{bmatrix}$.

Since each left hand side column of the r.r.e.f. contains a leading entry, the system has a unique (trivial) solution $c_1 = c_2 = 0$.

The vectors are linearly independent.

b. $\overrightarrow{u_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \overrightarrow{u_2} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \overrightarrow{u_3} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$.

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2VdtsED

The set $\{\overrightarrow{u_1}, \overrightarrow{u_2}, \overrightarrow{u_3}\}$ is linearly independent if and only if the homogeneous system

$$c_1\overrightarrow{u_1} + c_2\overrightarrow{u_2} + c_3\overrightarrow{u_3} = \overrightarrow{0}$$

has only the trivial solution $c_1 = c_2 = c_3 = 0$. Otherwise, the set is linearly dependent.

The homogeneous system's augmented matrix $\begin{bmatrix} 1 & 1 & 0 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & -1 & 1 & | & 0 \end{bmatrix}$

has the r.r.e.f. $\begin{bmatrix} \boxed{1} & 0 & 1 & | & 0 \\ 0 & \boxed{1} & -1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$.

The left hand side of the r.r.e.f. contains no leading entry in the third column, making c_3 arbitrary. Therefore, the system has many solutions, so that the vectors are linearly dependent.

To express one of them as a linear combination of the remaining ones, let us find a nontrivial solution by setting c_3 to some nonzero value, e.g.,

$$c_3 = 1.$$

It follows that

$$\begin{aligned} c_2 &= 1. \\ c_1 &= -1. \end{aligned}$$

Thus,

$$-1\overrightarrow{u_1} + 1\overrightarrow{u_2} + 1\overrightarrow{u_3} = \overrightarrow{0}.$$

We can express

$$\overrightarrow{u_3} = 1\overrightarrow{u_1} - 1\overrightarrow{u_2}.$$

c. $\overrightarrow{u_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \overrightarrow{u_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \overrightarrow{u_3} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2TdYESh

The set $\{\overrightarrow{u_1}, \overrightarrow{u_2}, \overrightarrow{u_3}\}$ is linearly independent if and only if the homogeneous system

$$c_1\overrightarrow{u_1} + c_2\overrightarrow{u_2} + c_3\overrightarrow{u_3} = \overrightarrow{0}$$

has only the trivial solution $c_1 = c_2 = c_3 = 0$. Otherwise, the set is linearly dependent.

Consider the augmented matrix of the homogeneous system: $\begin{bmatrix} 1 & 0 & 1 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 1 & 0 & 0 & | & 0 \end{bmatrix}$. The r.r.e.f. of this matrix is $\begin{bmatrix} \boxed{1} & 0 & 0 & | & 0 \\ 0 & \boxed{1} & 0 & | & 0 \\ 0 & 0 & \boxed{1} & | & 0 \end{bmatrix}$. Since each left hand side column of the r.r.e.f. contains a leading entry, the system has a unique (trivial) solution $c_1 = c_2 = c_3 = 0$.

The vectors are linearly independent.

3. a. $\overrightarrow{u_1} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \overrightarrow{u_2} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2TjTzIn

The set $\{\overrightarrow{u_1}, \overrightarrow{u_2}\}$ is linearly independent if and only if the homogeneous system

$$c_1\overrightarrow{u_1} + c_2\overrightarrow{u_2} = \overrightarrow{0}$$

has only the trivial solution $c_1 = c_2 = 0$. Otherwise, the set is linearly dependent.

The augmented matrix for our homogeneous system $\begin{bmatrix} 1 & 0 & | & 0 \\ 2 & 0 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$ obviously has the r.r.e.f. with no leading entry in the second column ($\begin{bmatrix} \boxed{1} & 0 & | & 0 \\ 0 & 0 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$). This makes the corresponding unknown (c_2) arbitrary, and $c_1 = 0$. The set $\{\overrightarrow{u_1}, \overrightarrow{u_2}\}$ is linearly dependent.

One of the nontrivial solutions is $c_1 = 0$, $c_2 = 1$, leading to

$$0\overrightarrow{u_1} + 1\overrightarrow{u_2} = \overrightarrow{0}$$

and

$$\overrightarrow{u_2} = 0\overrightarrow{u_1}$$

(SHORTCUT: It was quite obvious from the beginning, that a zero vector can be expressed as zero times any other vector, making the set linearly dependent.)

b. $\overrightarrow{u_1} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \overrightarrow{u_2} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \overrightarrow{u_3} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \overrightarrow{u_4} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2GGvLN2

The set $\{\overrightarrow{u_1}, \overrightarrow{u_2}, \overrightarrow{u_3}, \overrightarrow{u_4}\}$ is linearly independent if and only if the homogeneous system

$$c_1\overrightarrow{u_1} + c_2\overrightarrow{u_2} + c_3\overrightarrow{u_3} + c_4\overrightarrow{u_4} = \overrightarrow{0}$$

has only the trivial solution $c_1 = c_2 = c_3 = c_4 = 0$. Otherwise, the set is linearly dependent.

Our homogeneous system has the augmented matrix $\begin{bmatrix} 2 & 1 & 0 & 1 & | & 0 \\ 1 & 0 & 2 & 2 & | & 0 \\ 0 & 2 & 1 & 0 & | & 0 \end{bmatrix}$

with the r.r.e.f. $\begin{bmatrix} \boxed{1} & 0 & 0 & \frac{2}{3} & | & 0 \\ 0 & \boxed{1} & 0 & -\frac{1}{3} & | & 0 \\ 0 & 0 & \boxed{1} & \frac{2}{3} & | & 0 \end{bmatrix}$.

The fourth column contains no leading entry, so that c_4 is arbitrary.

Consequently our vectors are linearly dependent.

Finding an example of a nontrivial solution, let us set $c_4 = 3$, then calculate $c_1 = -2$, $c_2 = 1$, and $c_3 = -2$. We now have

$$-2\overrightarrow{u_1} + 1\overrightarrow{u_2} - 2\overrightarrow{u_3} + 3\overrightarrow{u_4} = \overrightarrow{0}$$

so that, e.g.

$$\overrightarrow{u_2} = 2\overrightarrow{u_1} + 2\overrightarrow{u_3} - 3\overrightarrow{u_4}$$

5. $\overrightarrow{u_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \overrightarrow{u_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \overrightarrow{u_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2GGhiRy

The set $\{\overrightarrow{u_1}, \overrightarrow{u_2}, \overrightarrow{u_3}\}$ is linearly independent if and only if the homogeneous system

$$c_1\overrightarrow{u_1} + c_2\overrightarrow{u_2} + c_3\overrightarrow{u_3} = \overrightarrow{0}$$

has only the trivial solution $c_1 = c_2 = c_3 = 0$. Otherwise, the set is linearly dependent.

Homogeneous system's augmented matrix $\begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix}$ is one elementary row operation away from r.r.e.f. $\begin{bmatrix} \boxed{1} & 0 & 0 & | & 0 \\ 0 & \boxed{1} & 0 & | & 0 \\ 0 & 0 & \boxed{1} & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$. Since every left hand side column contains a leading entry, the system has only one solution, $c_1 = c_2 = c_3 = 0$. Consequently, these vectors are linearly independent.

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2QSvVWs

7. The equation

$$c_1\left(1 + t^2\right) + c_2\left(2t + t^2\right) = 0$$

corresponds to a system with augmented matrix

$$\begin{bmatrix} 1 & 0 & | & 0 \\ 0 & 2 & | & 0 \\ 1 & 1 & | & 0 \end{bmatrix}.$$

The r.r.e.f. $\begin{bmatrix} \boxed{1} & 0 & | & 0 \\ 0 & \boxed{1} & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$ contains leading entries in all left hand side columns.

These vectors are L.I.

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2TfPswW

9. The equation

$$c_1\left(1 + t\right) + c_2\left(t + t^2\right) + c_3\left(1 + 2t + t^2\right) = 0$$

corresponds to a system with augmented matrix

$$\begin{bmatrix} 1 & 0 & 1 & | & 0 \\ 1 & 1 & 2 & | & 0 \\ 0 & 1 & 1 & | & 0 \end{bmatrix}.$$

The r.r.e.f. $\begin{bmatrix} \boxed{1} & 0 & 1 & | & 0 \\ 0 & \boxed{1} & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$ contains no leading entries in the third column: c_3 is arbitrary.

$$c_2 = -c_3; c_1 = -c_3.$$

These vectors are L.D.

Find a sample nontrivial solution, e.g., $c_3 = 1$, $c_2 = -1$, $c_1 = -1$.

$$-1\left(1 + t\right) - 1\left(t + t^2\right) + 1\left(1 + 2t + t^2\right) = 0$$

can be rewritten as

$$\left(1 + 2t + t^2\right) = \left(1 + t\right) + \left(t + t^2\right)$$

(the third polynomial is the sum of the first and the second)

Refer to the
Linear Algebra
Toolkit for details:
http://bit.ly/2GHVULB

11. The equation

$$c_1\left(1+t+t^2\right)+c_2\left(2t+2t^3\right)+c_3\left(1+t^2-t^3\right)=0$$

corresponds to a system with augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 \end{array}\right].$$

The r.r.e.f. $\left[\begin{array}{ccc|c} \boxed{1} & 0 & 1 & 0 \\ 0 & \boxed{1} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ contains no leading entries in the third column: c_3 is arbitrary.

$$c_2=\frac{1}{2}c_3; c_1=-c_3.$$

These vectors are L.D.

Find a sample nontrivial solution, e.g., $c_3=2, c_2=1, c_1=-2,$

$$-2\left(1+t+t^2\right)+1\left(2t+2t^3\right)+2\left(1+t^2-t^3\right)=0$$

can be rewritten as

$$\left(2t+2t^3\right)=2\left(1+t+t^2\right)-2\left(1+t^2-t^3\right)$$

(the second polynomial equals twice the first minus twice the third)

Refer to the
Linear Algebra
Toolkit for details:
http://bit.ly/2VccnLg

13. The equation

$$c_1\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}+c_2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}+c_3\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

corresponds to a system with augmented matrix

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right].$$

The r.r.e.f. $\left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ contains leading entries in all left hand side columns.

These vectors are L.I.

Refer to the
Linear Algebra
Toolkit for details:
http://bit.ly/2GKUnV4

15. L.D., since in

$$c_1\begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}+c_2\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

c_2 can be arbitrary.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}=0\begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}$$

(the second vector equals 0 times the first)

Refer to the
Linear Algebra
Toolkit for details:
http://bit.ly/2Q5Ufi9

19. The reduced row echelon form of the coefficient matrix $\begin{bmatrix} 22/24 & 14/24 & 18/24 \\ 1/24 & 6/24 & 0 \\ 1/24 & 4/24 & 6/24 \end{bmatrix}$ is $\begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix}$.

The three vectors are linearly independent. It is not possible to mix two of the alloys to obtain the third.

21. FALSE: $c_1 = c_2 = c_3 = 1$ is a nontrivial solution of $c_1\overrightarrow{u} + c_2\overrightarrow{v} + c_3\overrightarrow{w} = \overrightarrow{0}$

23. TRUE

If the subset were L.D., then one of its vectors would be expressible as a linear combination of the other vectors in the subset.

However, this would also be true for vectors in S, making it L.D. - a contradiction.

Consequently, the subset must be L.I.

4.4 Basis and Dimension

1. The number of vectors in the set, 2, matches $\dim R^2$. Therefore, by Theorem 4.16, it is sufficient to show that the set is L.I.

   Refer to the Linear Algebra Toolkit for details: http://bit.ly/2Rgtxs4

   The resulting homogeneous system has the augmented matrix $\left[\begin{array}{cc|c} 2 & -1 & 0 \\ 1 & 3 & 0 \end{array}\right]$ with the r.r.e.f. $\left[\begin{array}{cc|c} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \end{array}\right]$. There is a leading entry in each left hand side column, making the solution unique. The set is L.I.

   Therefore, (by Theorem 4.16) it is also a basis for R^2.

3. The number of vectors in the set, 3, does not match $\dim R^2 = 2$. The set cannot be a basis for R^2.

5. The number of vectors in the set, 2, does not match $\dim R^3 = 3$. The set cannot be a basis for R^3.

7. The number of vectors in the set, 3, matches $\dim R^3$. Therefore, by Theorem 4.16, it is sufficient to show that the set is L.I.

   Refer to the Linear Algebra Toolkit for details: http://bit.ly/2EUfm66

   The resulting homogeneous system has the augmented matrix $\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right]$ with the r.r.e.f. $\left[\begin{array}{ccc|c} \boxed{1} & 0 & -1 & 0 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$. There are many solutions, including some nontrivial ones, making the set L.D.

   Therefore, it is not a basis for R^3.

9. The number of vectors in the set, 4, matches $\dim R^4$. Therefore, by Theorem 4.16, it is sufficient to show that the set is L.I.

   Refer to the Linear Algebra Toolkit for details: http://bit.ly/2LHscp3

   The resulting homogeneous system has the augmented matrix $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{array}\right]$ with the r.r.e.f. $\left[\begin{array}{cccc|c} \boxed{1} & 0 & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 \end{array}\right]$. There is a leading entry in each left hand side column, making the solution unique. The set is L.I.

   Therefore, (by Theorem 4.16) it is also a basis for R^4.

11. The number of vectors in the set, 2, does not match $\dim P_2 = 3$. The set cannot be a basis for P_2.

13. The number of vectors in the set, 2, matches $\dim P_1$. Therefore, by Theorem 4.16, it is sufficient to show that the set is L.I.

    Refer to the Linear Algebra Toolkit for details: http://bit.ly/2ESz8Pc

    The equation
    $$c_1\,(2+t) + c_2(1+3t) = 0$$
    corresponds to the homogeneous system with the augmented matrix $\left[\begin{array}{cc|c} 2 & 1 & 0 \\ 1 & 3 & 0 \end{array}\right]$. The r.r.e.f. is $\left[\begin{array}{cc|c} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \end{array}\right]$. There is a leading entry in each left hand side column, making the solution unique.

The set is L.I.

Therefore, (by Theorem 4.16) it is also a basis for P_1.

15. The number of vectors in the set, 3, does not match $\dim P_3 = 4$. The set cannot be a basis for P_3.

17. The number of vectors in the set, 2, does not match $\dim M_{32} = 6$. The set cannot be a basis for M_{32}.

19. The number of vectors in the set, 4, matches $\dim M_{22}$. Therefore, by Theorem 4.16, it is sufficient to show that the set is L.I.

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2QRM4vq

The equation

$$c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

corresponds to the homogeneous system with the augmented matrix $\begin{bmatrix} \boxed{1} & 1 & 1 & 1 & | & 0 \\ 0 & \boxed{1} & 1 & 1 & | & 0 \\ 0 & 0 & \boxed{1} & 1 & | & 0 \\ 0 & 0 & 0 & \boxed{1} & | & 0 \end{bmatrix}$.

This matrix is already in row echelon form.

There is a leading entry in each left hand side column, making the solution unique. The set is L.I.

Therefore, (by Theorem 4.16) it is also a basis for M_{22}.

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2GJ81ba

21. a. The corresponding homogeneous system has augmented matrix

$$\begin{bmatrix} 1 & 2 & 1 & 1 & | & 0 \\ 2 & 4 & 1 & 4 & | & 0 \end{bmatrix}$$

with r.r.e.f.

$$\begin{bmatrix} \boxed{1} & 2 & 0 & 3 & | & 0 \\ 0 & 0 & \boxed{1} & -2 & | & 0 \end{bmatrix}.$$

The second and fourth vectors can be expressed as linear combinations of the remaining vectors (first and third), which are L.I. Therefore, a basis for span S is formed by the vectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

b. dim span $S = 2$ (since in part a we found a basis for span S containing two vectors)

c. the entire plane

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2AeqQNF

23. a. The corresponding homogeneous system has augmented matrix

$$\begin{bmatrix} 0 & 1 & -1 & | & 0 \\ 0 & -1 & 1 & | & 0 \end{bmatrix}$$

with r.r.e.f.

$$\begin{bmatrix} 0 & \boxed{1} & -1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

The first and third vectors can be expressed as linear combinations of the remaining vector (the second), which forms a L.I. set Therefore, a basis for span S is formed by that vector: $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

b. dim span $S = 1$ (since in part a we found a basis for span S containing one vector)

c. a line passing through the origin

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2BFyyQZ

25. a. The corresponding homogeneous system has augmented matrix

$$\begin{bmatrix} 0 & 1 & 1 & 1 & | & 0 \\ 1 & 0 & 2 & 1 & | & 0 \\ -1 & 2 & 0 & 1 & | & 0 \end{bmatrix}$$

with r.r.e.f.

$$\left[\begin{array}{cccc|c} \boxed{1} & 0 & 2 & 1 & 0 \\ 0 & \boxed{1} & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The third and fourth vectors can be expressed as linear combinations of the remaining vectors (first and second), which are L.I. Therefore, a basis for span S is formed by the vectors $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$.

b. dim span $S = 2$ (since in part a we found a basis for span S containing two vectors)

c. a plane passing through the origin

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2GHpGQK

27. a. The corresponding homogeneous system has augmented matrix

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 2 & 4 & 2 & 0 \\ 1 & 3 & 3 & 0 \end{array}\right]$$

with r.r.e.f.

$$\left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \end{array}\right].$$

The three vectors are L.I; a basis for span S is formed by the vectors $\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

b. dim span $S = 3$ (since in part a we found a basis for span S containing three vectors)

c. the entire 3-space

29. a. The zero vector spans a subspace of R^3 which contains only that vector. This subspace has no basis.

b. dim span $S = 0$

c. a point (the origin)

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2CA8ybx

31. a. The corresponding homogeneous system has augmented matrix

$$\left[\begin{array}{cccccc|c} 1 & 0 & 2 & 2 & -1 & 1 & 0 \\ 0 & -1 & 2 & 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 & -3 & 0 & 0 \\ 0 & 3 & -6 & -3 & -3 & 2 & 0 \end{array}\right]$$

with r.r.e.f.

$$\left[\begin{array}{cccccc|c} \boxed{1} & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & \boxed{1} & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & \boxed{1} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 \end{array}\right].$$

The fourth and fifth vectors can be expressed as linear combinations of the remaining vectors (first, second, third, and sixth), which are L.I. Therefore, a basis for span S is formed by the vectors

$$\begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \\ -6 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}$$

b. dim span $S = 4$ (since in part a we found a basis for span S containing four vectors)

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2ENfZhG

33. a. The equation

$$c_1(-t) + c_2(1-t) + c_3(2+t) + c_4(t-t^2) = 0$$

is equivalent to the homogeneous system with the augmented matrix

$$\left[\begin{array}{rrrr|r} 0 & 1 & 2 & 0 & 0 \\ -1 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{array}\right].$$

The r.r.e.f. is $\left[\begin{array}{rrrr|r} \boxed{1} & 0 & -3 & 0 & 0 \\ 0 & \boxed{1} & 2 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 \end{array}\right]$.

The third vector can be expressed as a linear combination of the remaining vectors (first, second, and fourth), which are L.I. Therefore, a basis for span S is formed by the vectors $-t, 1-t, t-t^2$

b. $\dim$ span $S = 3$ (since in part a we found a basis for span S containing three vectors)

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2Vjhauu

35. a. The equation

$$c_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} + c_5 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

is equivalent to the homogeneous system with the augmented matrix

$$\left[\begin{array}{rrrrr|r} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 & 0 & 0 \end{array}\right].$$

The r.r.e.f. is $\left[\begin{array}{rrrrr|r} \boxed{1} & 0 & 1 & 0 & 1 & 0 \\ 0 & \boxed{1} & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$.

The third and fifth vectors can be expressed as linear combinations of the remaining vectors (first, second, and fourth), which are L.I. Therefore, a basis for span S is formed by the vectors $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$.

b. $\dim$ span $S = 3$ (since in part a we found a basis for span S containing three vectors)

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2BGIwl6

37. a. Appending the R^3 standard basis vectors to the given vector, we form the homogeneous system with augmented matrix

$$\left[\begin{array}{rrrr|r} 1 & 1 & 0 & 0 & 0 \\ -3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right].$$

The r.r.e.f. is $\left[\begin{array}{rrrr|r} \boxed{1} & 0 & -\frac{1}{3} & 0 & 0 \\ 0 & \boxed{1} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 \end{array}\right]$.

Answer: $\begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2BJQdHa

b. Appending the R^3 standard basis vectors to the given vectors, we form the homogeneous system with augmented matrix

$$\left[\begin{array}{rrrrr|r} 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -2 & -3 & 0 & 0 & 1 & 0 \end{array}\right].$$

Its r.r.e.f. is $\left[\begin{array}{rrrrr|r} \boxed{1} & 0 & -3 & 0 & -2 & 0 \\ 0 & \boxed{1} & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 & 0 \end{array}\right]$

Answer: $\left[\begin{array}{r} 1 \\ 0 \\ -2 \end{array}\right], \left[\begin{array}{r} 2 \\ 0 \\ -3 \end{array}\right], \left[\begin{array}{r} 0 \\ 1 \\ 0 \end{array}\right].$

39. FALSE

Any vector space (except for $\{\overrightarrow{0}\}$) has infinitely many different bases.

41. TRUE

Dimension 1 means any basis for the space has one vector in it, which spans the space.

43. TRUE

If the set S is L.I. then it is a basis for span S, which is a subspace of V.

4.5 Coordinates

1. a. $[\vec{v}]_S = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ where

$$c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}.$$

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2QPYHHp

This is equivalent to the linear system with augmented matrix

$$\begin{bmatrix} 1 & -1 & | & -3 \\ 2 & 3 & | & 4 \end{bmatrix}.$$

The r.r.e.f. $\begin{bmatrix} 1 & 0 & | & -1 \\ 0 & 1 & | & 2 \end{bmatrix}$ leads to the solution $[\vec{v}]_S = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

Check: $-1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} \overset{\checkmark}{=} \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

b. $\vec{w} = 5 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$

3. a. $[\vec{w}]_T = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ where

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}.$$

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2TeetIS

This is equivalent to the linear system with augmented matrix

$$\begin{bmatrix} 1 & -1 & 1 & | & 0 \\ 0 & 1 & 0 & | & 4 \\ 0 & 0 & 1 & | & 1 \end{bmatrix}.$$

The r.r.e.f. $\begin{bmatrix} 1 & 0 & 0 & | & 3 \\ 0 & 1 & 0 & | & 4 \\ 0 & 0 & 1 & | & 1 \end{bmatrix}$ leads to the solution $[\vec{w}]_T = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$.

Check: $3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \overset{\checkmark}{=} \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}$

b. $\vec{v} = 2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 7 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} - 3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -8 \\ 7 \\ -3 \end{bmatrix}$.

5. a. $[\overrightarrow{w}]_T = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$ where

$$c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} + c_4 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ -2 \\ 0 \end{bmatrix}$$

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2RhtKen

This is equivalent to the linear system with augmented matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 & | & 4 \\ -1 & 1 & 0 & 0 & | & -2 \\ 0 & 0 & 1 & 1 & | & -2 \\ 0 & 0 & -1 & 1 & | & 0 \end{bmatrix}.$$

The r.r.e.f. $\begin{bmatrix} 1 & 0 & 0 & 0 & | & 3 \\ 0 & 1 & 0 & 0 & | & 1 \\ 0 & 0 & 1 & 0 & | & -1 \\ 0 & 0 & 0 & 1 & | & -1 \end{bmatrix}$ leads to the solution $[\overrightarrow{w}]_T = \begin{bmatrix} 3 \\ 1 \\ -1 \\ -1 \end{bmatrix}$.

Check:

$$3 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - 1 \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} - 1 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \overset{\checkmark}{=} \begin{bmatrix} 4 \\ -2 \\ -2 \\ 0 \end{bmatrix}$$

b. $\overrightarrow{v} = 4 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - 2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ -1 \\ 3 \end{bmatrix}$

7. a. $[\overrightarrow{v}]_S = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ where

$$c_1 (t+1) + c_2 (t-1) = 3t - 1.$$

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2EPgvvD

The coefficients corresponding to the same powers of t on both sides must be equal. This results in the linear system

$$\begin{array}{rcrcr} c_1 & - & c_2 & = & -1 \\ c_1 & + & c_2 & = & 3 \end{array}$$

with augmented matrix

$$\begin{bmatrix} 1 & -1 & | & -1 \\ 1 & 1 & | & 3 \end{bmatrix}.$$

The r.r.e.f. $\begin{bmatrix} 1 & 0 & | & 1 \\ 0 & 1 & | & 2 \end{bmatrix}$ leads to the solution $[\overrightarrow{v}]_S = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Check: $1(t+1) + 2(t-1) \overset{\checkmark}{=} 3t - 1$

b. $\overrightarrow{w} = 6(t+1) + 4(t-1) = 10t + 2$

9. a. $[\overrightarrow{v}]_S = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$ where

$$c_1 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}$$

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2RhzTY4

corresponds to the system with the augmented matrix $\begin{bmatrix} 1 & 1 & 0 & 0 & \mid & 3 \\ 0 & 1 & 0 & 0 & \mid & 2 \\ 0 & 0 & 1 & 0 & \mid & -1 \\ -1 & -1 & 0 & 1 & \mid & 1 \end{bmatrix}$

The r.r.e.f. $\begin{bmatrix} 1 & 0 & 0 & 0 & \mid & 1 \\ 0 & 1 & 0 & 0 & \mid & 2 \\ 0 & 0 & 1 & 0 & \mid & -1 \\ 0 & 0 & 0 & 1 & \mid & 4 \end{bmatrix}$ leads to the solution $[\overrightarrow{v}]_S = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 4 \end{bmatrix}$.

Check:

$$1 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + 2 \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} - 1 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \overset{\checkmark}{=} \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}$$

b. $\overrightarrow{w} = -2 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + 1 \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} + 0 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 0 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$

11.

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2BFAvwW

a. if possible, find $[\overrightarrow{u}]_S$ for $\overrightarrow{u} = \begin{bmatrix} 2 \\ -2 \\ -11 \end{bmatrix}$:

$\begin{bmatrix} 1 & 0 & 2 \\ 2 & 2 & -2 \\ -1 & 3 & -11 \end{bmatrix}$ has r.r.e.f. $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix} \Longrightarrow [\overrightarrow{u}]_S = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2TaIXva

if possible, find $[\overrightarrow{v}]_S$ for $\overrightarrow{v} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$:

$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 2 & 1 \\ -1 & 3 & 1 \end{bmatrix}$ has r.r.e.f. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Longrightarrow [\overrightarrow{v}]_S$ cannot be found.

What does it say about a vector when its coordinate vector with respect to S cannot be found?
Answer: such a vector (e.g., $\overrightarrow{v}$) is not in the plane spanned by S.

b. find $\overrightarrow{w}$ such that $[\overrightarrow{w}]_S = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$.

$$4 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \\ -1 \end{bmatrix}$$

4.6 Rank and Nullity

Refer to the Linear Algebra Toolkit for details: http://bit.ly/2BFQv1T

1. a. A row echelon form is $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$.

i. A basis for the column space: $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

ii. A basis for the row space: $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$.

iii. 2

Refer to the Linear Algebra Toolkit for details: http://bit.ly/2GEU3qV

iv. x_3 is arbitrary; $x_1 = -2x_3$; $x_2 = -3x_3$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_3 \\ -3x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}.$$

Basis for null space of A : $\begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}$

v. 1.

Refer to the Linear Algebra Toolkit for details: http://bit.ly/2CBJeSr

b. A r.e.f. $\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$

i. A basis for the column space: $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

ii. A basis for the row space: $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

iii. 1

Refer to the Linear Algebra Toolkit for details: http://bit.ly/2GIHOd3

iv. x_2 is arbitrary; $x_1 = 2x_2$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Basis for null space: $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

v. 1.

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2LyEGzb

c. A r.e.f. is $\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

i. A basis for the column space: $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

ii. A basis for the row space: $\begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}$

iii. 1

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2TaIXva

iv. x_2, x_3, and x_4 are arbitrary; $x_1 = x_3 - 2x_4$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 - 2x_4 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Basis for null space of A : $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

v. 3.

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2BLPiG7

3. a. A r.e.f. is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

i. A basis for the column space: . $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 3 \\ 1 \end{bmatrix}$

ii. A basis for the row space: $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

iii. 3

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2GRuRgK

iv. The only solution is $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

No basis for null space of A.

v. 0.

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2Q7fksV

b. A r.e.f. is $\begin{bmatrix} 0 & 1 & -1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

i. A basis for the column space: $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}$

ii. A basis for the row space: $\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ -1 \end{bmatrix}$

iii. 3

Refer to the
Linear Algebra
Toolkit for details:
http://bit.ly/2QOUUKh

iv. The r.r.e.f. is $\begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

x_1, x_5, and x_6 are arbitrary; $x_2 = -x_5 - x_6$; $x_3 = -x_5 + x_6$; $x_4 = -x_5 + x_6$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_5 - x_6 \\ -x_5 + x_6 \\ -x_5 + x_6 \\ x_5 \\ x_6 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ -1 \\ -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Basis for null space of A : $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

v. 3.

5. a. rank $[A \,|\, \overrightarrow{b}\,] =$ rank $A + 1$ (the discrepancy corresponds to the row $[0...0|nonzero]$)
   b. rank $[A \,|\, \overrightarrow{b}\,] =$ rank $A = n$ (no row $[0...0|nonzero]$; every one of the n left hand side columns contains a leading entry)
   c. rank $[A \,|\, \overrightarrow{b}\,] =$ rank $A < n$ (no row $[0...0|nonzero]$; at least one of the left hand side columns does not contain a leading entry)

7. TRUE

9. TRUE

There are 7 columns in R^4 - no more than 4 vectors can be L.I. in R^4.

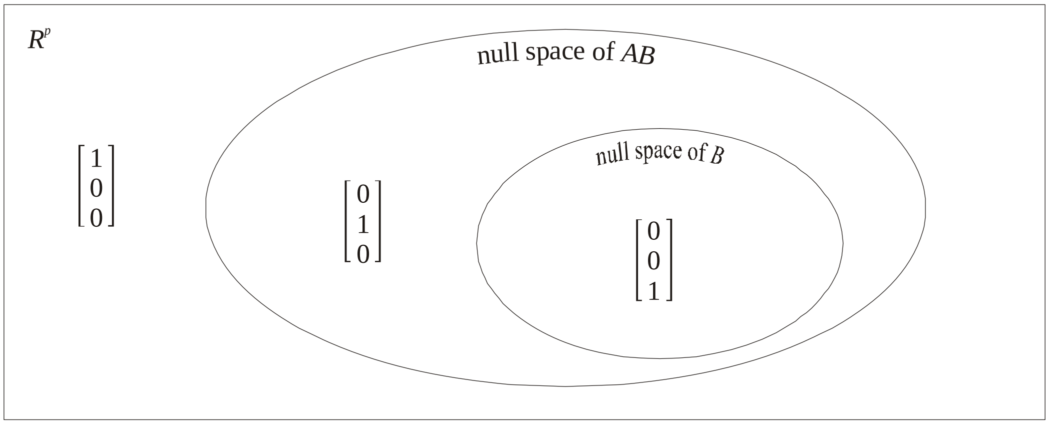
11. e.g., $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

13. Such matrix cannot exist -

rank+nullity=3, therefore, both must be no bigger than 3

17. a. If $\overrightarrow{x} \in$ (null space of B) then $B\overrightarrow{x} = \overrightarrow{0}$. Therefore, $AB\overrightarrow{x} = A\overrightarrow{0} = \overrightarrow{0}$, which implies $\overrightarrow{x} \in$ (null space of AB).
   b. e.g., $\overrightarrow{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \in$ (null space of AB) but $B\overrightarrow{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \neq \overrightarrow{0}$ so that $\overrightarrow{x} \notin$ (null space of B).

c.


R^p
null space of AB
null space of B
$\begin{bmatrix}1\\0\\0\end{bmatrix}$
$\begin{bmatrix}0\\1\\0\end{bmatrix}$
$\begin{bmatrix}0\\0\\1\end{bmatrix}$

5 Linear Transformations

5.1 Linear Transformations in General Vector Spaces

1. a. Yes

Cond. 1 $LHS = F(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x' \\ y' \end{bmatrix}) = F(\begin{bmatrix} x+x' \\ y+y' \end{bmatrix}) = \begin{bmatrix} 2(x+x') \\ 3(x+x') \end{bmatrix}$

$RHS = F(\begin{bmatrix} x \\ y \end{bmatrix}) + F(\begin{bmatrix} x' \\ y' \end{bmatrix}) = \begin{bmatrix} 2x \\ 3x \end{bmatrix} + \begin{bmatrix} 2x' \\ 3x' \end{bmatrix}$ ✓

Cond. 2 $LHS = F(c\begin{bmatrix} x \\ y \end{bmatrix}) = F(\begin{bmatrix} cx \\ cy \end{bmatrix}) = \begin{bmatrix} 2cx \\ 3cx \end{bmatrix}$

$RHS = cF(\begin{bmatrix} x \\ y \end{bmatrix}) = c\begin{bmatrix} 2x \\ 3x \end{bmatrix}$ ✓

b. Yes.

Cond. 1 $LHS = F(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x' \\ y' \end{bmatrix}) = F(\begin{bmatrix} x+x' \\ y+y' \end{bmatrix}) = \begin{bmatrix} 0 \\ x+x'-(y+y') \end{bmatrix}$

$RHS = F(\begin{bmatrix} x \\ y \end{bmatrix}) + F(\begin{bmatrix} x' \\ y' \end{bmatrix}) = \begin{bmatrix} 0 \\ x-y \end{bmatrix} + \begin{bmatrix} 0 \\ x'-y' \end{bmatrix}$ ✓

Cond. 2 $LHS = F(c\begin{bmatrix} x \\ y \end{bmatrix}) = F(\begin{bmatrix} cx \\ cy \end{bmatrix}) = \begin{bmatrix} 0 \\ cx-cy \end{bmatrix}$

$RHS = cF(\begin{bmatrix} x \\ y \end{bmatrix}) = c\begin{bmatrix} 0 \\ x-y \end{bmatrix}$ ✓

c. No.

e.g., Condition 2 does not hold for

$LHS = F(2\begin{bmatrix} 0 \\ 0 \end{bmatrix}) = F(\begin{bmatrix} 0 \\ 0 \end{bmatrix}) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$RHS = 2F(\begin{bmatrix} 0 \\ 0 \end{bmatrix}) = 2\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$ not equal

3. a. No.

e.g. Condition 2 does not hold for

$LHS = F(2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}) = F(\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}) = \begin{bmatrix} 2 \\ 24 \end{bmatrix}$

$RHS = 2F(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}) = 2\begin{bmatrix} 1 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 12 \end{bmatrix}$ not equal

b. Yes.

Cond. 1 $LHS = F(\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}) = F(\begin{bmatrix} x+x' \\ y+y' \\ z+z' \end{bmatrix}) = \begin{bmatrix} y+y' \\ 2(z+z') \end{bmatrix}$

$RHS = F(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) + F(\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}) = \begin{bmatrix} y \\ 2z \end{bmatrix} + \begin{bmatrix} y' \\ 2z' \end{bmatrix} \checkmark$

Cond. 2 $LHS = F(c\begin{bmatrix} x \\ y \\ z \end{bmatrix}) = F(\begin{bmatrix} cx \\ cy \\ cz \end{bmatrix}) = \begin{bmatrix} cy \\ 2cz \end{bmatrix}$

$RHS = cF(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) = c\begin{bmatrix} y \\ 2z \end{bmatrix} \checkmark$

c. No.

e.g. Condition 1 does not hold for

$LHS = F(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}) = F(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$RHS = F(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}) + F(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}) = \begin{bmatrix} 0 \\ -2 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ not equal

5. a. Yes.

After simplifying, the formula for G becomes

$G(\begin{bmatrix} x \\ y \end{bmatrix}) = \ln e^{x-y} = x - y.$

Therefore, both conditions of the definition are satisfied

Cond 1: $G(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x' \\ y' \end{bmatrix}) = G(\begin{bmatrix} x + x' \\ y + y' \end{bmatrix}) = (x + x') - (y + y') = (x - y) + (x' - y') = G(\begin{bmatrix} x \\ y \end{bmatrix}) + G(\begin{bmatrix} x' \\ y' \end{bmatrix})$

Cond 2: $G(c\begin{bmatrix} x \\ y \end{bmatrix}) = G(\begin{bmatrix} cx \\ cy \end{bmatrix}) = cx - cy = c(x - y) = cG(\begin{bmatrix} x \\ y \end{bmatrix})$

b. No.

e.g. Condition 1 does not hold for

$LHS = F(\begin{bmatrix} \pi/2 \\ 1 \end{bmatrix} + \begin{bmatrix} \pi/2 \\ 2 \end{bmatrix}) = F(\begin{bmatrix} \pi \\ 3 \end{bmatrix}) = \sin \pi = 0$

$RHS = F(\begin{bmatrix} \pi/2 \\ 1 \end{bmatrix}) + F(\begin{bmatrix} \pi/2 \\ 2 \end{bmatrix}) = \sin \frac{\pi}{2} + \sin \frac{\pi}{2} = 1 + 1 = 2$ not equal

c. For $x + y = 0$ (e.g. $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$), G is undefined. Therefore, this is not a linear transformation.

7. Yes

We use Theorem 1.3.

For all $m \times n$ matrices A_1 and A_2 Condition 1 holds:

$F(A_1 + A_2) = 2(A_1 + A_2) = 2A_1 + 2A_2 = F(A_1) + F(A_2).$

For all $m \times n$ matrices A and real numbers c Condition 2 holds:

$F(cA) = 2(cA) = c(2A) = cF(A)$.

9. Yes

   We use properties of matrix multiplication (Theorem 1.5).

   For all $n \times n$ matrices A_1 and A_2 Condition 1 holds:

   $H(A_1 + A_2) = B(A_1 + A_2) = BA_1 + BA_2 = H(A_1) + H(A_2)$.

   For all $n \times n$ matrices A and real numbers c Condition 2 holds:

   $H(cA) = B(cA) = c(BA) = cH(A)$

11. No

    e.g. $G(3\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}) = G(\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}) = \begin{bmatrix} 36 & 0 \\ 0 & 36 \end{bmatrix}$ does not equal

    $3G(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}) = 3\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 12 & 0 \\ 0 & 12 \end{bmatrix}$

    so Condition 2 does not generally hold.

13. No

    e.g. $H(3\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}) = H(\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}) = 2$ does not equal

    $3H(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}) = 3(2) = 6$

    so Condition 2 does not generally hold.

15. Yes

    Condition 1:

    $LHS = F(\left(a_0 + a_1t + a_2t^2 + a_3t^3\right) + \left(b_0 + b_1t + b_2t^2 + b_3t^3\right))$

    $= F((a_0 + b_0) + (a_1 + b_1)t + (a_2 + b_2)t^2 + (a_3 + b_3)t^3)$

    $= a_0 + b_0$

    $= F(a_0 + a_1t + a_2t^2 + a_3t^3) + F(b_0 + b_1t + b_2t^2 + b_3t^3) = RHS$

    Condition 2:

    $LHS = F(c\left(a_0 + a_1t + a_2t^2 + a_3t^3\right))$

    $= F(ca_0 + ca_1t + ca_2t^2 + ca_3t^3)$

    $= ca_0$

    $= cF(a_0 + a_1t + a_2t^2 + a_3t^3) = RHS$

17. Yes

    Condition 1:

    $LHS = F(\left(a_0 + a_1t + a_2t^2 + a_3t^3\right) + \left(b_0 + b_1t + b_2t^2 + b_3t^3\right))$

    $= F((a_0 + b_0) + (a_1 + b_1)t + (a_2 + b_2)t^2 + (a_3 + b_3)t^3)$

    $= 6(a_3 + b_3)$

    $= 6a_3 + 6b_3$

    $= F(a_0 + a_1t + a_2t^2 + a_3t^3) + F(b_0 + b_1t + b_2t^2 + b_3t^3) = RHS$

    Condition 2:

$LHS = F(c\left(a_0 + a_1t + a_2t^2 + a_3t^3\right))$

$= F(ca_0 + ca_1t + ca_2t^2 + ca_3t^3)$

$= 6(ca_3)$

$c(6a_3)$

$= cF(a_0 + a_1t + a_2t^2 + a_3t^3) = RHS$

19. No

Condition 1:

$LHS = F(\left(a_0 + a_1t + a_2t^2 + a_3t^3\right) + \left(b_0 + b_1t + b_2t^2 + b_3t^3\right))$

$= F((a_0 + b_0) + (a_1 + b_1)t + (a_2 + b_2)t^2 + (a_3 + b_3)t^3)$

$= a_0 + b_0 - 1$

does not match

$RHS = F(a_0 + a_1t + a_2t^2 + a_3t^3) + F(b_0 + b_1t + b_2t^2 + b_3t^3)$

$= a_0 - 1 + b_0 - 1$

21. No, violates condition 2, e.g.

$F(2\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}) = F(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}) = 1$

$2F(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}) = (2)(1) = 2$ do not equal

23. Yes

Cond. 1:

$LHS = H(\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} a' \\ b' \end{bmatrix}) = H(\begin{bmatrix} a + a' \\ b + b' \end{bmatrix}) = \begin{bmatrix} a + a' & a + a' + b + b' & a + a' + 2(b + b') \end{bmatrix}$

$RHS = H(\begin{bmatrix} a \\ b \end{bmatrix}) + H(\begin{bmatrix} a' \\ b' \end{bmatrix}) = \begin{bmatrix} a & a + b & a + 2b \end{bmatrix} + \begin{bmatrix} a' & a' + b' & a' + 2b' \end{bmatrix} \checkmark$

Cond. 2:

$LHS = H(c\begin{bmatrix} a \\ b \end{bmatrix}) = H(\begin{bmatrix} ca \\ cb \end{bmatrix}) = \begin{bmatrix} ca & ca + cb & ca + 2cb \end{bmatrix}$

$RHS = cH(\begin{bmatrix} a \\ b \end{bmatrix}) = c\begin{bmatrix} a & a + b & a + 2b \end{bmatrix} \checkmark$

5.2 Kernel and Range

1. a. • $F(\begin{bmatrix} 0 \\ 0 \end{bmatrix}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ therefore $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is in $\ker F$;

   • $F(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ therefore $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is not in $\ker F$;

   • $F(\begin{bmatrix} 1 \\ 2 \end{bmatrix}) = \begin{bmatrix} 5 \\ 10 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ therefore $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is not in $\ker F$;

   • $F(\begin{bmatrix} -2 \\ 1 \end{bmatrix}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ therefore $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is in $\ker F$.

   b. • $F(\begin{bmatrix} 0 \\ 0 \end{bmatrix}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ therefore $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is in $\ker F$;

   • $F(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ therefore $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is not in $\ker F$;

   • $F(\begin{bmatrix} 1 \\ 2 \end{bmatrix}) = \begin{bmatrix} 2 \\ 6 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ therefore $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is not in $\ker F$;

   • $F(\begin{bmatrix} -2 \\ 1 \end{bmatrix}) = \begin{bmatrix} -4 \\ 3 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ therefore $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is not in $\ker F$.

3. a. • $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is in range F since $F(\overrightarrow{0}) = \overrightarrow{0}$ for any linear transformation F,

   Refer to the Linear Algebra Toolkit for details: http://bit.ly/2At9Nb1

   • Setting $\begin{bmatrix} x+2y \\ 2x+4y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ yields a system with augmented matrix $\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 4 & 0 \end{array}\right]$ whose r.r.e.f. is $\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 1 \end{array}\right]$.

   The system is inconsistent since the r.r.e.f. contains a row $[0 \cdots 0 \mid \text{nonzero}]$ therefore $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is not in range F.

   Refer to the Linear Algebra Toolkit for details: http://bit.ly/2SFD7SL

   • Setting $\begin{bmatrix} x+2y \\ 2x+4y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ yields a system with augmented matrix $\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 4 & 2 \end{array}\right]$ whose r.r.e.f. is $\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 0 & 0 \end{array}\right]$.

   The system is consistent since the r.r.e.f. does not contain a row $[0 \cdots 0 \mid \text{nonzero}]$ therefore $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is in range F.

   Refer to the Linear Algebra Toolkit for details: http://bit.ly/2CWsHsn

   • Setting $\begin{bmatrix} x+2y \\ 2x+4y \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ yields a system with augmented matrix $\left[\begin{array}{cc|c} 1 & 2 & -2 \\ 2 & 4 & 1 \end{array}\right]$ whose r.r.e.f. is $\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 1 \end{array}\right]$.

   The system is inconsistent since the r.r.e.f. contains a row $[0 \cdots 0 \mid \text{nonzero}]$ therefore $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is not in range F.

   b. • $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is in range F since $F(\overrightarrow{0}) = \overrightarrow{0}$ for any linear transformation F,

- Setting $\begin{bmatrix}2x\\3y\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$ yields $\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}\frac{1}{2}\\0\end{bmatrix}$. $\begin{bmatrix}1\\0\end{bmatrix}$ is in range F.
- Setting $\begin{bmatrix}2x\\3y\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}$ yields $\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}\frac{1}{2}\\\frac{2}{3}\end{bmatrix}$. $\begin{bmatrix}1\\2\end{bmatrix}$ is in range F.
- Setting $\begin{bmatrix}2x\\3y\end{bmatrix}=\begin{bmatrix}-2\\1\end{bmatrix}$ yields $\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-1\\\frac{1}{3}\end{bmatrix}$. $\begin{bmatrix}-2\\1\end{bmatrix}$ is in range F.

5. a. • $F(\begin{bmatrix}0&0\\0&0\end{bmatrix})=0$ therefore $\begin{bmatrix}0&0\\0&0\end{bmatrix}$ is in $\ker F$;

- $F(\begin{bmatrix}0&4\\0&2\end{bmatrix})=0$ therefore $\begin{bmatrix}0&4\\0&2\end{bmatrix}$ is in $\ker F$;
- $F(\begin{bmatrix}1&1\\1&1\end{bmatrix})=t^2\neq 0$ therefore $\begin{bmatrix}1&1\\1&1\end{bmatrix}$ is not in $\ker F$;
- $F(\begin{bmatrix}4&0\\0&-1\end{bmatrix})=-4+4t^2\neq 0$ therefore $\begin{bmatrix}4&0\\0&-1\end{bmatrix}$ is not in $\ker F$;

b. • $F(\begin{bmatrix}0&0\\0&0\end{bmatrix})=0$ therefore $\begin{bmatrix}0&0\\0&0\end{bmatrix}$ is in $\ker F$;

- $F(\begin{bmatrix}0&4\\0&2\end{bmatrix})=4+8t-4t^2\neq 0$ therefore $\begin{bmatrix}0&4\\0&2\end{bmatrix}$ is not in $\ker F$;
- $F(\begin{bmatrix}1&1\\1&1\end{bmatrix})=1+5t\neq 0$ therefore $\begin{bmatrix}1&1\\1&1\end{bmatrix}$ is not in $\ker F$;
- $F(\begin{bmatrix}4&0\\0&-1\end{bmatrix})=0$ therefore $\begin{bmatrix}4&0\\0&-1\end{bmatrix}$ is in $\ker F$.

7. a. • $\begin{bmatrix}0&0\\0&0\end{bmatrix}$ is in range F since $F(\overrightarrow{0})=\overrightarrow{0}$ for any linear transformation F;

- Setting $\begin{bmatrix}a&b\\b&a\end{bmatrix}=\begin{bmatrix}0&4\\0&2\end{bmatrix}$ yields an inconsistent system ($a=0$, $b=4$, $b=0$, $a=2$). $\begin{bmatrix}0&4\\0&2\end{bmatrix}$ is not in range F.
- Setting $\begin{bmatrix}a&b\\b&a\end{bmatrix}=\begin{bmatrix}1&1\\1&1\end{bmatrix}$ yields a consistent system ($a=1$, $b=1$). $\begin{bmatrix}1&1\\1&1\end{bmatrix}$ is in range F.
- Setting $\begin{bmatrix}a&b\\b&a\end{bmatrix}=\begin{bmatrix}4&0\\0&-1\end{bmatrix}$ yields an inconsistent system ($a=4$, $b=0$, $b=0$, $a=-1$). $\begin{bmatrix}4&0\\0&-1\end{bmatrix}$ is not in range F.

b. • $\begin{bmatrix}0&0\\0&0\end{bmatrix}$ is in range F since $F(\overrightarrow{0})=\overrightarrow{0}$ for any linear transformation F;

- Setting $\begin{bmatrix}0&2c\\0&c\end{bmatrix}=\begin{bmatrix}0&4\\0&2\end{bmatrix}$ yields a consistent system ($c=2$). $\begin{bmatrix}0&4\\0&2\end{bmatrix}$ is in range F.
- Setting $\begin{bmatrix}0&2c\\0&c\end{bmatrix}=\begin{bmatrix}1&1\\1&1\end{bmatrix}$ yields an inconsistent system (including the equation $0=1$). $\begin{bmatrix}1&1\\1&1\end{bmatrix}$ is not in range F.

- Setting $\begin{bmatrix} 0 & 2c \\ 0 & c \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$ yields an inconsistent system (including the equation $0 = 4$). $\begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$ is not in range F.

9. a. The kernel of F is the set of all vectors in R^2 such that $\begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The only solution is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The kernel has no basis.

b. The range of F is the set of all images of F, i.e. $\{\begin{bmatrix} y \\ x \end{bmatrix} \mid x, y \in R\}$. Writing

$$\begin{bmatrix} y \\ x \end{bmatrix} = x \begin{bmatrix} 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

we have range F =span$\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\}$. The vectors $\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ are L.I., consequently, they form a basis for range F.

c. range $F = R^2 \Rightarrow F$ is onto.

d. ker $F = \{\overrightarrow{0}\} \Rightarrow F$ is one-to-one (by Theorem 5.4).

e. F is onto and one-to-one $\Rightarrow F$ is invertible (by Theorem 5.7).

11. a. The kernel of F is the set of all vectors in R^3 such that $\begin{bmatrix} x-y \\ y-z \\ z-x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

The corresponding linear system has the coefficient matrix $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$ whose r.r.e.f. is $\begin{bmatrix} \boxed{1} & 0 & -1 \\ 0 & \boxed{1} & -1 \\ 0 & 0 & 0 \end{bmatrix}$. There are many solutions:

$$\begin{aligned} x &= z \\ y &= z \\ z &= arbitrary \end{aligned}$$

Any solution has the form

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ z \\ z \end{bmatrix} = z \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

ker F =span$\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\}$. The vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ forms a basis for ker F.

b. The range of F is the set of all images of F,

$$\begin{aligned} \begin{bmatrix} x-y \\ y-z \\ z-x \end{bmatrix} &= \begin{bmatrix} x \\ 0 \\ -x \end{bmatrix} + \begin{bmatrix} -y \\ y \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -z \\ z \end{bmatrix} \\ &= x \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}. \end{aligned}$$

Based on the r.r.e.f. obtained in part a., the third vector is a linear combination of the first two,

$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, which are L.I. Therefore, $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ form a basis for range F.

c. range $F \neq R^3 \Rightarrow F$ is not onto.

d. ker $F \neq \{\overrightarrow{0}\} \Rightarrow F$ is not one-to-one (by Theorem 5.4).

e. F is neither onto nor one-to-one $\Rightarrow F$ is not invertible (by Theorem 5.7).

13. a. The kernel of F is the set of all vectors in R^3 such that

$$\begin{bmatrix} x+y \\ y+z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The corresponding homogeneous system has the coefficient matrix $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, whose r.r.e.f. is $\begin{bmatrix} \boxed{1} & 0 & -1 \\ 0 & \boxed{1} & 1 \end{bmatrix}$.

There are many solutions:

$$\begin{aligned} x &= z \\ y &= -z \\ z &= arbitrary \end{aligned}$$

Every vector in $\ker F$ has the form

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ -z \\ z \end{bmatrix} = z \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

ker F has a basis formed by the vector $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

b. The range of F consists of all images of F :

$$\begin{aligned} \begin{bmatrix} x+y \\ y+z \end{bmatrix} &= \begin{bmatrix} x \\ 0 \end{bmatrix} + \begin{bmatrix} y \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ z \end{bmatrix} \\ &= x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1 \\ 1 \end{bmatrix} + z \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \end{aligned}$$

Of the three vectors, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, based on the r.r.e.f. obtained in part a., the third vector is a linear combination of the first two, which are L.I. Therefore, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ form a basis for range F.

c. range $F = R^2 \Rightarrow F$ is onto (by part 2 of Theorem 4.17).

d. ker $F \neq \{\overrightarrow{0}\} \Rightarrow F$ is not one-to-one (by Theorem 5.4).

e. F is not one-to-one $\Rightarrow F$ is not invertible (by Theorem 5.7).

15. a. The kernel of F is the set of all vectors in R^3 such that

$$\begin{bmatrix} x-2z \\ y+z \\ x+2y \\ x+y-z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The corresponding homogeneous system has the coefficient matrix $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & -1 \end{bmatrix}$ has the r.r.e.f. $\begin{bmatrix} \boxed{1} & 0 & -2 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. According to this matrix, there are many solutions to the system:

$$\begin{aligned} x &= 2z \\ y &= -z \\ z &= arbitrary \end{aligned}$$

so that vectors in the kernel of F are

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2z \\ -z \\ z \end{bmatrix} = z \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}.$$

The vector $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ forms a basis for $\ker F$.

b. range F consists of vectors

$$\begin{aligned} \begin{bmatrix} x-2z \\ y+z \\ x+2y \\ x+y-z \end{bmatrix} &= \begin{bmatrix} x \\ 0 \\ x \\ x \end{bmatrix} + \begin{bmatrix} 0 \\ y \\ 2y \\ y \end{bmatrix} + \begin{bmatrix} -2z \\ z \\ 0 \\ -z \end{bmatrix} \\ &= x \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} + z \begin{bmatrix} -2 \\ 1 \\ 0 \\ -1 \end{bmatrix}. \end{aligned}$$

Based on the r.r.e.f. found in part a, the third vector is a linear combination of the first two, which are L.I. The vectors $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}$ form a basis for range F.

c. range $F \neq R^3 \Rightarrow F$ is not onto.

d. ker $F \neq \{\overrightarrow{0}\} \Rightarrow F$ is not one-to-one (by Theorem 5.4).

e. F is neither onto nor one-to-one $\Rightarrow$ F is not invertible (by Theorem 5.7).

17. a. The kernel of F is composed of all vectors $a_0 + a_1 t$ in P_1 such that

$$a_1 = 0,$$

i.e., ker $F = \{a_0 + a_1 t \mid a_1 = 0, a_0 \in R\} = \{a_0 \mid a_0 \in R\}$ =span$\{1\}$.
The polynomial $p(t) = 1$ is a basis for $\ker F$.

b. The range of F is composed of all vectors in P_1 which are images of F. Thus, range F =span$\{1\}$. The polynomial $p(t) = 1$ is a basis for range$F$.

c. range $F \neq P_1 \Rightarrow F$ is not onto.

d. ker $F \neq \{\overrightarrow{0}\} \Rightarrow F$ is not one-to-one (by Theorem 5.4).

e. F is neither onto nor one-to-one $\Rightarrow$ F is not invertible (by Theorem 5.7).

19. a. The kernel of F contains all vectors in P_2, $a_0 + a_1 t + a_2 t^2$,such that

$$-a_0 + a_1 + a_2 - a_1 t + (a_0 - 2a_1 + a_2)t^2 = 0$$

For this to be true, the coefficients in front of the same powers of t on both sides must be equal.

$$\begin{array}{rcrcrcl} -a_0 & + & a_1 & + & a_2 & = & 0 \\ & - & a_1 & & & = & 0 \\ a_0 & - & 2a_1 & + & a_2 & = & 0 \end{array}$$

The coefficient matrix of this homogeneous system, $\begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$ has r.r.e.f. $\begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix}$.

Therefore, $\ker F = \{\overrightarrow{0}\}$ - it has no basis.

b. The range of F is a set composed of vectors in P_2 of the form

$$\begin{aligned} & -a_0 + a_1 + a_2 - a_1 t + (a_0 - 2a_1 + a_2)t^2 \\ = \quad & (a_0)(-1 + t^2) + (a_1)(1 - t - 2t^2) + (a_2)(1 + t^2). \end{aligned}$$

From the r.r.e.f. in part a., $-1+t^2$, $1-t-2t^2$, and $1+t^2$ are L.I., so that they are a basis for range F.

c. range $F = P_2 \Rightarrow F$ is onto (by part 2 of Theorem 4.17).

d. ker $F = \{\overrightarrow{0}\} \Rightarrow F$ is one-to-one (by Theorem 5.4).

e. F is onto and one-to-one $\Rightarrow F$ is invertible (by Theorem 5.7).

21. a. $\ker F$ contains all vectors in M_{22} (i.e., 2×2 matrices $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$) for which $a_{11} + a_{22} = 0$. (The coefficient matrix, $\begin{bmatrix} \boxed{1} & 0 & 0 & 1 \end{bmatrix}$, is already in the r.r.e.f.) The solutions satisfy

$$\begin{array}{rcl} a_{11} & = & -a_{22} \\ a_{12} & = & arbitrary \\ a_{21} & = & arbitrary \\ a_{22} & = & arbitrary \end{array}$$

Vectors in $\ker F$ have the form

$$\begin{aligned} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} &= \begin{bmatrix} -a_{22} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \\ &= a_{12} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + a_{21} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + a_{22} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

A basis for the kernel of F is formed by $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, and $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

b. The range of F is composed of all real numbers that can be expressed as $a_{11} + a_{22}$. This means range $F = R$. A basis can be provided by any nonzero number (e.g., 1).

c. range $F = R \Rightarrow F$ is onto (by part 2 of Theorem 4.17).

d. ker $F \neq \{\overrightarrow{0}\} \Rightarrow F$ is not one-to-one (by Theorem 5.4).

e. F is not one-to-one $\Rightarrow F$ is not invertible (by Theorem 5.7).

23. a. ker $F = P_2$ - a basis: $1, t, t^2$

b. range $F = \{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\}$ - no basis

c. range $F \neq M_{22} \Rightarrow F$ is not onto.

d. ker $F \neq \{\overrightarrow{0}\} \Rightarrow F$ is not one-to-one (by Theorem 5.4).

e. F is neither onto nor one-to-one $\Rightarrow F$ is not invertible (by Theorem 5.7).

25. The degree of precision of the quadrature formula (67) is the largest integer k such that $P_k \subseteq \ker G$ where

$$G(f) = \int_a^b f(x)dx - \sum_{i=0}^{n} c_i f(x_i).$$

Because G is linear, $G(a_0 + a_1 x + \cdots + a_k x^k) = a_0 G(1) + a_1 G(x_1) + \cdots + a_k G(x^k)$. Therefore, the degree of precision is the largest integer k such that $G(1) = G(x) = \cdots G(x^k) = 0$ and $G(x^{k+1}) \neq 0$.

- $G(1) = \int_a^b 1dx - (b-a)\,(1) = b - a - (b-a) = 0$
- $G(x) = \int_a^b xdx - (b-a)\,(\frac{a+b}{2}) = \left[\frac{x^2}{2}\right]_a^b - \frac{b^2-a^2}{2} = \frac{b^2-a^2}{2} - \frac{b^2-a^2}{2} = 0$
- $G(x^2) = \int_a^b x^2dx - (b-a)\,(\frac{a+b}{2})^2 = \left[\frac{x^3}{3}\right]_a^b - (b-a)\,(\frac{a+b}{2})^2$
  $= \frac{b^3-a^3}{3} - (b-a)\,(\frac{a+b}{2})^2 = \frac{1}{12}b^3 - \frac{1}{12}a^3 + \frac{1}{4}a^2b - \frac{1}{4}ab^2 \neq 0$

Since $P_1 \subseteq \ker G$ and $P_2 \nsubseteq \ker G$, we conclude that the degree of precision of Midpoint Rule is 1.

5.3 Matrices of Linear Transformations

1. $\overrightarrow{u_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \overrightarrow{u_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$F(\overrightarrow{u_1}) = \begin{bmatrix} 3 \\ 5 \end{bmatrix}, F(\overrightarrow{u_2}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Refer to the
Linear Algebra
Toolkit for details:
http://bit.ly/2TyUlRH

$\begin{bmatrix} 1 & 0 & | & 3 & 1 \\ 2 & 1 & | & 5 & 1 \end{bmatrix}$ has r.r.e.f. $\begin{bmatrix} 1 & 0 & | & 3 & 1 \\ 0 & 1 & | & -1 & -1 \end{bmatrix}$

The matrix of F with respect to S is $A = \begin{bmatrix} 3 & 1 \\ -1 & -1 \end{bmatrix}$.

$F(\begin{bmatrix} -2 \\ 3 \end{bmatrix})$ obtained directly: $\begin{bmatrix} -2+3 \\ 3(-2)+3 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$.

Using the matrix A :

Refer to the
Linear Algebra
Toolkit for details:
http://bit.ly/2H0fMtD

$\begin{bmatrix} 1 & 0 & | & -2 \\ 2 & 1 & | & 3 \end{bmatrix}$ has r.r.e.f. $\begin{bmatrix} 1 & 0 & | & -2 \\ 0 & 1 & | & 7 \end{bmatrix}$

Therefore, $[\begin{bmatrix} -2 \\ 3 \end{bmatrix}]_S = \begin{bmatrix} -2 \\ 7 \end{bmatrix}$.

$$A[\begin{bmatrix} -2 \\ 3 \end{bmatrix}]_S = \begin{bmatrix} 3 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} -2 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$$

$$1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} - 5 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

3. $\overrightarrow{u_1} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \overrightarrow{u_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$F(\overrightarrow{u_1}) = \begin{bmatrix} -1 \\ -2 \end{bmatrix}, F(\overrightarrow{u_2}) = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$

Refer to the
Linear Algebra
Toolkit for details:
http://bit.ly/2RaYq1C

$\begin{bmatrix} 1 & 0 & | & -1 & 3 \\ 2 & 1 & | & -2 & 3 \end{bmatrix}$ has r.r.e.f. $\begin{bmatrix} 1 & 0 & | & -1 & 3 \\ 0 & 1 & | & 0 & -3 \end{bmatrix}$

The matrix of F with respect to S and T is $A = \begin{bmatrix} -1 & 3 \\ 0 & -3 \end{bmatrix}$.

$F(\begin{bmatrix} 6 \\ 0 \end{bmatrix})$ obtained directly: $\begin{bmatrix} 6+2(0) \\ 2(6)+0 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \end{bmatrix}$.

Using the matrix A :

Refer to the
Linear Algebra
Toolkit for details:
http://bit.ly/2SEWBqz

$\begin{bmatrix} -1 & 1 & | & 6 \\ 0 & 1 & | & 0 \end{bmatrix}$ has r.r.e.f. $\begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & 0 \end{bmatrix}$

Therefore, $[\begin{bmatrix} 6 \\ 0 \end{bmatrix}]_S = \begin{bmatrix} -6 \\ 0 \end{bmatrix}$.

$$A[\begin{bmatrix} 6 \\ 0 \end{bmatrix}]_S = \begin{bmatrix} -1 & 3 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} -6 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$$

$$6\begin{bmatrix}1\\2\end{bmatrix}+0\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}6\\12\end{bmatrix}$$

5. $\overrightarrow{u_1}=\begin{bmatrix}1\\3\end{bmatrix}$, $\overrightarrow{u_2}=\begin{bmatrix}3\\-1\end{bmatrix}$

$$F(\overrightarrow{u_1})=\begin{bmatrix}3\\4\\1\end{bmatrix},\ F(\overrightarrow{u_2})=\begin{bmatrix}-1\\2\\3\end{bmatrix}$$

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2Qvgu1e

$$\left[\begin{array}{ccc|cc}0&1&1&3&-1\\1&0&0&4&2\\0&1&-1&1&3\end{array}\right]\text{ has r.r.e.f. }\left[\begin{array}{ccc|cc}1&0&0&4&2\\0&1&0&2&1\\0&0&1&1&-2\end{array}\right]$$

The matrix of F with respect to S and T is $A=\begin{bmatrix}4&2\\2&1\\1&-2\end{bmatrix}$.

$F(\begin{bmatrix}2\\-4\end{bmatrix})$ obtained directly: $\begin{bmatrix}-4\\2-4\\2\end{bmatrix}=\begin{bmatrix}-4\\-2\\2\end{bmatrix}$.

Using the matrix A :

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2seS3LP

$$\left[\begin{array}{cc|c}1&3&2\\3&-1&-4\end{array}\right]\text{ has r.r.e.f. }\left[\begin{array}{cc|c}1&0&-1\\0&1&1\end{array}\right]$$

Therefore, $[\begin{bmatrix}2\\-4\end{bmatrix}]_S=\begin{bmatrix}-1\\1\end{bmatrix}$.

$$A[\begin{bmatrix}2\\-4\end{bmatrix}]_S=\begin{bmatrix}4&2\\2&1\\1&-2\end{bmatrix}\begin{bmatrix}-1\\1\end{bmatrix}=\begin{bmatrix}-2\\-1\\-3\end{bmatrix}$$

$$-2\begin{bmatrix}0\\1\\0\end{bmatrix}-1\begin{bmatrix}1\\0\\1\end{bmatrix}-3\begin{bmatrix}1\\0\\-1\end{bmatrix}=\begin{bmatrix}-4\\-2\\2\end{bmatrix}$$

7. $\overrightarrow{u_1}=\begin{bmatrix}1\\1\\0\end{bmatrix}$, $\overrightarrow{u_2}=\begin{bmatrix}0\\1\\1\end{bmatrix}$, $\overrightarrow{u_3}=\begin{bmatrix}1\\0\\1\end{bmatrix}$

$$F(\overrightarrow{u_1})=\begin{bmatrix}1\\2\\-1\end{bmatrix},\ F(\overrightarrow{u_2})=\begin{bmatrix}0\\1\\0\end{bmatrix},\ F(\overrightarrow{u_3})=\begin{bmatrix}1\\1\\-1\end{bmatrix}$$

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2sdUn5P

$$\left[\begin{array}{ccc|ccc}1&0&1&1&0&1\\1&1&0&2&1&1\\0&1&1&-1&0&-1\end{array}\right]\text{ has r.r.e.f. }\left[\begin{array}{ccc|ccc}1&0&0&2&\frac{1}{2}&\frac{3}{2}\\0&1&0&0&\frac{1}{2}&-\frac{1}{2}\\0&0&1&-1&-\frac{1}{2}&-\frac{1}{2}\end{array}\right]$$

The matrix of F with respect to S is $A=\begin{bmatrix}2&\frac{1}{2}&\frac{3}{2}\\0&\frac{1}{2}&-\frac{1}{2}\\-1&-\frac{1}{2}&-\frac{1}{2}\end{bmatrix}$.

$F(\begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix})$ obtained directly: $\begin{bmatrix} 2 \\ 2+4 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ -2 \end{bmatrix}$.

Using the matrix A :

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2LXEgCk

$$\begin{bmatrix} 1 & 0 & 1 & | & 2 \\ 1 & 1 & 0 & | & 4 \\ 0 & 1 & 1 & | & 4 \end{bmatrix} \text{ has r.r.e.f. } \begin{bmatrix} 1 & 0 & 0 & | & 1 \\ 0 & 1 & 0 & | & 3 \\ 0 & 0 & 1 & | & 1 \end{bmatrix}$$

Therefore, $[\begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}]_S = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$

$$A[\begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}]_S = \begin{bmatrix} 2 & \frac{1}{2} & \frac{3}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ -1 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -3 \end{bmatrix}$$

$$5\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - 3\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ -2 \end{bmatrix}.$$

9. $F(1) = 1$, $F(t) = 2t$, $F(t^2) = 4t^2$

To find $[F(1)]_T = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $[F(t)]_T = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ and $[F(t^2)]_T = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$ we must have

$$\begin{array}{ccccccc} (x_1)\left(1-2t+t^2\right) & + & (x_2)\left(2t-2t^2\right) & + & x_3t^2 & = & 1 \\ (y_1)\left(1-2t+t^2\right) & + & (y_2)\left(2t-2t^2\right) & + & y_3t^2 & = & 2t \\ (z_1)\left(1-2t+t^2\right) & + & (z_2)\left(2t-2t^2\right) & + & z_3t^2 & = & 4t^2 \end{array}$$

Rewriting the left hand sides, collecting the like terms (in powers of t) we get

$$\begin{array}{ccccccc} x_1 & + & (-2x_1+2x_2)\,t & + & (x_1-2x_2+x_3)\,t^2 & = & 1 \\ y_1 & + & (-2y_1+2y_2)\,t & + & (y_1-2y_2+y_3)\,t^2 & = & 2t \\ z_1 & + & (-2z_1+2z_2)\,t & + & (z_1-2z_2+z_3)\,t^2 & = & 4t^2 \end{array}$$

For these to hold for all values of t, the coefficients assigned to the same power of t on both sides of each equation must equal. This leads to three systems of equations, which can be represented using a single augmented matrix:

$$\begin{bmatrix} 1 & 0 & 0 & | & 1 & 0 & 0 \\ -2 & 2 & 0 & | & 0 & 2 & 0 \\ 1 & -2 & 1 & | & 0 & 0 & 4 \end{bmatrix}.$$

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2TzSKef

The r.r.e.f. is $\begin{bmatrix} 1 & 0 & 0 & | & 1 & 0 & 0 \\ 0 & 1 & 0 & | & 1 & 1 & 0 \\ 0 & 0 & 1 & | & 1 & 2 & 4 \end{bmatrix}$

The matrix of F with respect to S and T is

$$A = \begin{bmatrix} | & | & | \\ [F(1)]_T & [F(t)]_T & [F(t^2)]_T \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 4 \end{bmatrix}.$$

$F(-2+t+3t^2)$ obtained directly: $-2+2t+12t^2$

Using the matrix A :

$$\left[-2+t+3t^2\right]_S = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \text{ where}$$

$$d_1 + d_2 t + d_3 t^2 = -2 + t + 3t^2$$

Clearly, the solution is

$$\left[-2+t+3t^2\right]_S = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$$

$$A[-2+t+3t^2]_S = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 12 \end{bmatrix}$$

$$-2\left(1-2t+t^2\right) - 1\left(2t-2t^2\right) + 12t^2 = -2 + 2t + 12t^2$$

11. To find the coordinate-change matrix $P_{T \leftarrow S}$, we perform elementary row operations on the matrix

$$\begin{bmatrix} 1 & 1 & | & 1 & 0 \\ -1 & 1 & | & 0 & 1 \end{bmatrix}$$

Refer to the
Linear Algebra
Toolkit for details:
http://bit.ly/2LUpDQf

resulting in the r.r.e.f. $\begin{bmatrix} 1 & 0 & | & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & | & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$.

Therefore, $P_{T \leftarrow S} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$.

Clearly, $[\overrightarrow{u}]_S = \overrightarrow{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

Refer to the
Linear Algebra
Toolkit for details:
http://bit.ly/2CVx9Yc

To determine $[\overrightarrow{u}]_T$, we solve the linear system with the augmented matrix $\begin{bmatrix} 1 & 1 & | & 2 \\ -1 & 1 & | & 3 \end{bmatrix}$. The r.r.e.f. is $\begin{bmatrix} 1 & 0 & | & -\frac{1}{2} \\ 0 & 1 & | & \frac{5}{2} \end{bmatrix}$, therefore, $[\overrightarrow{u}]_T = \begin{bmatrix} -\frac{1}{2} \\ \frac{5}{2} \end{bmatrix}$.

$$P_{T \leftarrow S}\,[\overrightarrow{u}]_S = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{5}{2} \end{bmatrix} \overset{\checkmark}{=} [\overrightarrow{u}]_T$$

13. To find the coordinate-change matrix $P_{T \leftarrow S}$, we perform elementary row operations on the matrix

$$\begin{bmatrix} 1 & 3 & | & 1 & 0 \\ -1 & -1 & | & 3 & 1 \end{bmatrix}$$

Refer to the
Linear Algebra
Toolkit for details:
http://bit.ly/2QqVMzP

resulting in the r.r.e.f. $\begin{bmatrix} 1 & 0 & | & -5 & -\frac{3}{2} \\ 0 & 1 & | & 2 & \frac{1}{2} \end{bmatrix}$

Therefore, $P_{T \leftarrow S} = \begin{bmatrix} -5 & -\frac{3}{2} \\ 2 & \frac{1}{2} \end{bmatrix}$.

Refer to the
Linear Algebra
Toolkit for details:
http://bit.ly/2TC2FAj

To determine $[\overrightarrow{u}]_S$, we solve the linear system with the augmented matrix $\begin{bmatrix} 1 & 0 & | & 4 \\ 3 & 1 & | & -1 \end{bmatrix}$. The r.r.e.f. is $\begin{bmatrix} 1 & 0 & | & 4 \\ 0 & 1 & | & -13 \end{bmatrix}$ therefore, $[\overrightarrow{u}]_S = \begin{bmatrix} 4 \\ -13 \end{bmatrix}$.

Refer to the Linear Algebra Toolkit for details:

http://bit.ly/2sb0jws

To determine $[\overrightarrow{u}]_T$, we solve the linear system with the augmented matrix $\left[\begin{array}{rr|r} 1 & 3 & 4 \\ -1 & -1 & -1 \end{array}\right]$. The r.r.e.f. is $\left[\begin{array}{rr|r} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{3}{2} \end{array}\right]$, therefore, $[\overrightarrow{u}]_T = \left[\begin{array}{r} -\frac{1}{2} \\ \frac{3}{2} \end{array}\right]$.

$$P_{T\leftarrow S}\,[\overrightarrow{u}]_S = \left[\begin{array}{rr} -5 & -\frac{3}{2} \\ 2 & \frac{1}{2} \end{array}\right]\left[\begin{array}{r} 4 \\ -13 \end{array}\right] = \left[\begin{array}{r} -\frac{1}{2} \\ \frac{3}{2} \end{array}\right] \overset{\checkmark}{=} [\overrightarrow{u}]_T$$

15. To find the coordinate-change matrix $P_{T\leftarrow S}$, we perform elementary row operations on the matrix

$$\left[\begin{array}{rrr|rrr} 1 & 1 & -1 & 0 & 1 & 1 \\ -1 & 1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 1 & 1 & 0 \end{array}\right]$$

Refer to the Linear Algebra Toolkit for details:

http://bit.ly/2Qt1wsL

resulting in the r.r.e.f. $\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & 2 & 1 \\ 0 & 1 & 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 & 3 & 2 \end{array}\right]$

Therefore, $P_{T\leftarrow S} = \left[\begin{array}{rrr} 1 & 2 & 1 \\ 1 & 2 & 2 \\ 2 & 3 & 2 \end{array}\right]$.

Refer to the Linear Algebra Toolkit for details:

http://bit.ly/2C4H1gS

To determine $[\overrightarrow{u}]_S$, we solve the linear system with the augmented matrix $\left[\begin{array}{rrr|r} 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & -2 \\ 1 & 1 & 0 & 1 \end{array}\right]$. The r.r.e.f. is $\left[\begin{array}{rrr|r} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -2 \end{array}\right]$ therefore, $[\overrightarrow{u}]_S = \left[\begin{array}{r} -5 \\ 6 \\ -2 \end{array}\right]$.

To determine $[\overrightarrow{u}]_T$, we solve the linear system with the augmented matrix $\left[\begin{array}{rrr|r} 1 & 1 & -1 & 4 \\ -1 & 1 & 0 & -2 \\ 0 & -1 & 1 & 1 \end{array}\right]$.

Refer to the Linear Algebra Toolkit for details:

http://bit.ly/2TzUWCv

The r.r.e.f. is $\left[\begin{array}{rrr|r} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{array}\right]$ therefore, $[\overrightarrow{u}]_T = \left[\begin{array}{r} 5 \\ 3 \\ 4 \end{array}\right]$.

$$P_{T\leftarrow S}\,[\overrightarrow{u}]_S = \left[\begin{array}{rrr} 1 & 2 & 1 \\ 1 & 2 & 2 \\ 2 & 3 & 2 \end{array}\right]\left[\begin{array}{r} -5 \\ 6 \\ -2 \end{array}\right] = \left[\begin{array}{r} 5 \\ 3 \\ 4 \end{array}\right] \overset{\checkmark}{=} [\overrightarrow{u}]_T$$

17. We want to find $P_{S\leftarrow T}$ where $S = \{1, t, t^2\}$ and $T = \{(1-t)^2, 2t(1-t), t^2\}$. The desired matrix could be viewed as a matrix of the identity transformation with respect to T and S :

$$\begin{aligned} P_{S\leftarrow T} &= \left[\begin{array}{ccc} | & | & | \\ \left[1-2t+t^2\right]_S & \left[2t-2t^2\right]_S & \left[t^2\right]_S \\ | & | & | \end{array}\right] \\ &= \left[\begin{array}{rrr} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{array}\right] \end{aligned}$$

19. We want to find $P_{T\leftarrow S}$ where $S=\{1,t,t^2,t^3\}$ and $T=\{(1-t)^3,3t(1-t)^2,3t^2(1-t),t^3\}$.

$$P_{T\leftarrow S}=\begin{bmatrix} | & | & | & | \\ [1]_T & [t]_T & [t^2]_T & [t^3]_T \\ | & | & | & | \end{bmatrix}.$$

To find $[1]_T=\begin{bmatrix} a_1\\ a_2\\ a_3\\ a_4\end{bmatrix}$, we solve

$$a_1(1-t)^3+(a_2)(3t)(1-t)^2+(a_3)(3t^2)(1-t)+a_4t^3=1,$$

i.e.,

$$a_1(1-3t+3t^2-t^3)+a_2(3t-6t^2+3t^3)+a_3(3t^2-3t^3)+a_4t^3=1,$$

or

$$a_1+(-3a_1+3a_2)t+(3a_1-6a_2+3a_3)t^2+(-a_1+3a_2-3a_3+a_4)t^3=1+0t+0t^2+0t^3.$$

For the two sides to remain equal for all t, the coefficients associated with the same power of t on both sides must be equal. This yields the system

$$\begin{array}{rcrcrcrcl} a_1 & & & & & & & = & 1\\ -3a_1 & + & 3a_2 & & & & & = & 0\\ 3a_1 & - & 6a_2 & + & 3a_3 & & & = & 0\\ -a_1 & + & 3a_2 & - & 3a_3 & + & a_4 & = & 0 \end{array}$$

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2ABKDqH

whose augmented matrix $\left[\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & 1\\ -3 & 3 & 0 & 0 & 0\\ 3 & -6 & 3 & 0 & 0\\ -1 & 3 & -3 & 1 & 0\end{array}\right]$ has the r.r.e.f. $\left[\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & 1\\ 0 & 1 & 0 & 0 & 1\\ 0 & 0 & 1 & 0 & 1\\ 0 & 0 & 0 & 1 & 1\end{array}\right]$ therefore $[1]_T=\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$.

The remaining three columns of the matrix $P_{T\leftarrow S}$ can be obtained in the same manner:

- to find $[t]_T=\begin{bmatrix} b_1\\ b_2\\ b_3\\ b_4\end{bmatrix}$, we solve

  $$b_1(1-t)^3+(b_2)(3t)(1-t)^2+(b_3)(3t^2)(1-t)+b_4t^3=t,$$

  which is equivalent to a linear system with the augmented matrix $\left[\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & 0\\ -3 & 3 & 0 & 0 & 1\\ 3 & -6 & 3 & 0 & 0\\ -1 & 3 & -3 & 1 & 0\end{array}\right]$

  Refer to the
  Linear Algebra
  Toolkit for details:

  http://bit.ly/2sf0lTX

  whose r.r.e.f. is $\left[\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & \frac{1}{3}\\ 0 & 0 & 1 & 0 & \frac{2}{3}\\ 0 & 0 & 0 & 1 & 1\end{array}\right]$ so that $[t]_T=\begin{bmatrix}0\\ \frac{1}{3}\\ \frac{2}{3}\\ 1\end{bmatrix}$;

- to find $\left[t^2\right]_T = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$, we solve

$$c_1(1-t)^3 + (c_2)(3t)(1-t)^2 + (c_3)(3t^2)(1-t) + c_4t^3 = t^2,$$

which is equivalent to a linear system with the augmented matrix $\left[\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 & 0 \\ 3 & -6 & 3 & 0 & 1 \\ -1 & 3 & -3 & 1 & 0 \end{array}\right]$

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2TA0b59

whose r.r.e.f. is $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$

so that $\left[t^2\right]_T = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{3} \\ 1 \end{bmatrix}$;

- to find $\left[t^3\right]_T = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix}$, we solve

$$d_1(1-t)^3 + (d_2)(3t)(1-t)^2 + (d_3)(3t^2)(1-t) + d_4t^3 = t^3,$$

which is equivalent to a linear system with the augmented matrix $\left[\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 & 0 \\ 3 & -6 & 3 & 0 & 0 \\ -1 & 3 & -3 & 1 & 1 \end{array}\right]$

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2HcjCA3

whose r.r.e.f. is $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$ so that $\left[t^3\right]_T = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$;

We conclude that $P_{T\leftarrow S} = \begin{bmatrix} | & | & | & | \\ [1]_T & [t]_T & \left[t^2\right]_T & \left[t^3\right]_T \\ | & | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & \frac{1}{3} & 0 & 0 \\ 1 & \frac{2}{3} & \frac{1}{3} & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2FgtdmH

(Note that the matrix $P_{T\leftarrow S}$ could also be found by inverting the matrix

$$P_{S\leftarrow T} = \begin{bmatrix} | & | & | & | \\ \left[(1-t)^3\right]_S & \left[3t(1-t)^2\right]_S & \left[3t^2(1-t)\right]_S & \left[t^3\right]_S \\ | & | & | & | \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}.)$$

21. FALSE

If F is one-to-one then by Theorem 5.4, $\ker F = \{\overrightarrow{0}\}$ so that $\dim \ker F = 0$.

Consequently, rank $F = 3 - 0 = 3 \neq 4$.

23. TRUE

nullity $F = 0 \Rightarrow F$ is one-to-one (by Theorem 5.4)

rank $F = \dim R^4 -$nullity $F = 4 - 0 = 4 \Rightarrow F$ is onto.(by part 2 of Theorem 4.17)

Therefore, by Theorem 5.7 F is invertible.

6 Orthogonality and Projections

6.1 Orthogonality

1. a. (ii) orthogonal, but not orthonormal:

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 0 \text{ but}$$

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 2 \neq 1$$

b. (i) orthonormal:

$$\begin{bmatrix} \frac{-1}{3} \\ \frac{2}{3} \\ \frac{-2}{3} \end{bmatrix} \cdot \begin{bmatrix} \frac{2}{3} \\ \frac{-1}{3} \\ \frac{-2}{3} \end{bmatrix} = 0$$

$$\begin{bmatrix} \frac{-1}{3} \\ \frac{2}{3} \\ \frac{-2}{3} \end{bmatrix} \cdot \begin{bmatrix} \frac{-1}{3} \\ \frac{2}{3} \\ \frac{-2}{3} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{-1}{3} \\ \frac{-2}{3} \end{bmatrix} \cdot \begin{bmatrix} \frac{2}{3} \\ \frac{-1}{3} \\ \frac{-2}{3} \end{bmatrix} = 1$$

c. (iii) neither:

$$\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = 4 \neq 0$$

3. a. (i) orthonormal:

$$\begin{bmatrix} \frac{-1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{-1}{2} \end{bmatrix} \cdot \begin{bmatrix} \frac{-1}{2} \\ \frac{1}{2} \\ \frac{-1}{2} \\ \frac{1}{2} \end{bmatrix} = 0$$

$$\begin{bmatrix} \frac{-1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{-1}{2} \end{bmatrix} \cdot \begin{bmatrix} \frac{-1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{-1}{2} \end{bmatrix} = \begin{bmatrix} \frac{-1}{2} \\ \frac{1}{2} \\ \frac{-1}{2} \\ \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} \frac{-1}{2} \\ \frac{1}{2} \\ \frac{-1}{2} \\ \frac{1}{2} \end{bmatrix} = 1$$

b. (ii) orthogonal, but not orthonormal:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \neq 1$$

5. $$[\overrightarrow{v}]_S = \begin{bmatrix} \dfrac{\begin{bmatrix} 0 \\ -20 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \end{bmatrix}}{\begin{bmatrix} 3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \end{bmatrix}} \\ \dfrac{\begin{bmatrix} 0 \\ -20 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 6 \end{bmatrix}}{\begin{bmatrix} 2 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 6 \end{bmatrix}} \end{bmatrix} = \begin{bmatrix} \frac{20}{10} \\ \frac{-120}{40} \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

Check: $2\begin{bmatrix} 3 \\ -1 \end{bmatrix} - 3\begin{bmatrix} 2 \\ 6 \end{bmatrix} \overset{\checkmark}{=} \begin{bmatrix} 0 \\ -20 \end{bmatrix}$

7. $[\overrightarrow{w}]_S = \begin{bmatrix} \begin{bmatrix} 3 \\ -9 \\ 12 \end{bmatrix} \cdot \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{-1}{3} \end{bmatrix} \\ \begin{bmatrix} 3 \\ -9 \\ 12 \end{bmatrix} \cdot \begin{bmatrix} \frac{2}{3} \\ \frac{-1}{3} \\ \frac{2}{3} \end{bmatrix} \\ \begin{bmatrix} 3 \\ -9 \\ 12 \end{bmatrix} \cdot \begin{bmatrix} \frac{-1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} -8 \\ 13 \\ 1 \end{bmatrix}.$

Check: $-8\begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{-1}{3} \end{bmatrix} + 13\begin{bmatrix} \frac{2}{3} \\ \frac{-1}{3} \\ \frac{2}{3} \end{bmatrix} + 1\begin{bmatrix} \frac{-1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix} = \begin{bmatrix} 3 \\ -9 \\ 12 \end{bmatrix}$

9. a. Orthogonal matrix

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^T \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

b. Not orthogonal

$$\begin{bmatrix} 2 & 1 & -2 \\ 1 & 2 & 2 \\ 2 & -2 & 1 \end{bmatrix}^T \begin{bmatrix} 2 & 1 & -2 \\ 1 & 2 & 2 \\ 2 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & -2 \\ -2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & -2 \\ 1 & 2 & 2 \\ 2 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(Note that the columns are orthogonal, but are not orthonormal.)

c. Orthogonal matrix

$$\begin{bmatrix} \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & -1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}^T \begin{bmatrix} \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & -1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & -1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & -1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

11. TRUE

Zero vector is orthogonal to any vector in the same space.

13. FALSE

If T has fewer than 7 vectors then it cannot be a basis for R^7.

15. TRUE

If $A^{-1} = A^T$ then $\left(A^{-1}\right)^T = (A^T)^{-1} = \left(A^{-1}\right)^{-1}$ which means A^{-1} is orthogonal.

6.2 Orthogonal Projections and Orthogonal Complements

1\. a. $\cos\alpha = \dfrac{\begin{bmatrix}1\\4\end{bmatrix}\cdot\begin{bmatrix}2\\-3\end{bmatrix}}{\left\|\begin{bmatrix}1\\4\end{bmatrix}\right\|\left\|\begin{bmatrix}2\\-3\end{bmatrix}\right\|} = \frac{-10}{\sqrt{17}\sqrt{13}}$

Since $-1 < \cos\alpha < 0$, the vectors form an obtuse angle.

b. $\cos\alpha = \dfrac{\begin{bmatrix}-1\\2\\1\end{bmatrix}\cdot\begin{bmatrix}3\\1\\1\end{bmatrix}}{\left\|\begin{bmatrix}-1\\2\\1\end{bmatrix}\right\|\left\|\begin{bmatrix}3\\1\\1\end{bmatrix}\right\|} = \frac{0}{\sqrt{6}\sqrt{11}} = 0$

The vectors are perpendicular.

c. $\cos\alpha = \dfrac{\begin{bmatrix}1\\0\\2\end{bmatrix}\cdot\begin{bmatrix}0\\2\\1\end{bmatrix}}{\left\|\begin{bmatrix}1\\0\\2\end{bmatrix}\right\|\left\|\begin{bmatrix}0\\2\\1\end{bmatrix}\right\|} = \frac{2}{\sqrt{5}\sqrt{5}} = \frac{2}{5}$

Since $0 < \cos\alpha < 1$, the vectors form an acute angle.

3\. a. $\cos\alpha = \dfrac{\begin{bmatrix}1\\0\\2\\1\end{bmatrix}\cdot\begin{bmatrix}2\\0\\4\\2\end{bmatrix}}{\left\|\begin{bmatrix}1\\0\\2\\1\end{bmatrix}\right\|\left\|\begin{bmatrix}2\\0\\4\\2\end{bmatrix}\right\|} = \frac{12}{\sqrt{6}\sqrt{24}} = 1$

The vectors are in the same direction.

b. $\cos\alpha = \dfrac{\begin{bmatrix}0\\1\\1\\-1\\1\end{bmatrix}\cdot\begin{bmatrix}0\\-1\\1\\-1\\1\end{bmatrix}}{\left\|\begin{bmatrix}0\\1\\1\\-1\\1\end{bmatrix}\right\|\left\|\begin{bmatrix}0\\-1\\1\\-1\\1\end{bmatrix}\right\|} = \frac{2}{\sqrt{4}\sqrt{4}} = \frac{1}{2}$

Since $0 < \cos\alpha < 1$, the vectors form an acute angle.

5. Set up a matrix with the given vectors as rows

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2SHJVz4

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 \end{bmatrix} \text{ has the r.r.e.f. } \begin{bmatrix} \boxed{1} & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \boxed{1} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

The null space of the matrix consists of vectors

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = z \begin{bmatrix} \frac{1}{2} \\ \frac{-1}{2} \\ 1 \\ 0 \end{bmatrix} + w \begin{bmatrix} \frac{1}{2} \\ \frac{-1}{2} \\ 0 \\ 1 \end{bmatrix}$$

A basis is $\begin{bmatrix} \frac{1}{2} \\ \frac{-1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{-1}{2} \\ 0 \\ 1 \end{bmatrix}$.

7. Set up a matrix with the given vectors as rows

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2RczPto

$$\begin{bmatrix} -1 & 2 & 1 & -3 \\ 3 & 1 & -2 & 1 \\ 2 & 3 & -1 & -2 \end{bmatrix} \text{ has the r.r.e.f. } \begin{bmatrix} \boxed{1} & 0 & -\frac{5}{7} & \frac{5}{7} \\ 0 & \boxed{1} & \frac{1}{7} & -\frac{8}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The null space of the matrix consists of vectors

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = z \begin{bmatrix} \frac{5}{7} \\ \frac{-1}{7} \\ 1 \\ 0 \end{bmatrix} + w \begin{bmatrix} \frac{-5}{7} \\ \frac{8}{7} \\ 0 \\ 1 \end{bmatrix}$$

A basis is $\begin{bmatrix} \frac{5}{7} \\ \frac{-1}{7} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{-5}{7} \\ \frac{8}{7} \\ 0 \\ 1 \end{bmatrix}$.

9. a. $\overrightarrow{p} = \text{proj}_{\text{span}\{\overrightarrow{v}\}} \overrightarrow{u} = \dfrac{\begin{bmatrix} 3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 4 \end{bmatrix}}{\begin{bmatrix} -2 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 4 \end{bmatrix}} \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \frac{-10}{20} \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

Check: $\overrightarrow{u} - \overrightarrow{p} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$(\overrightarrow{u} - \overrightarrow{p}) \cdot \overrightarrow{v} = (2)(-2) + (1)(4) \overset{\checkmark}{=} 0$

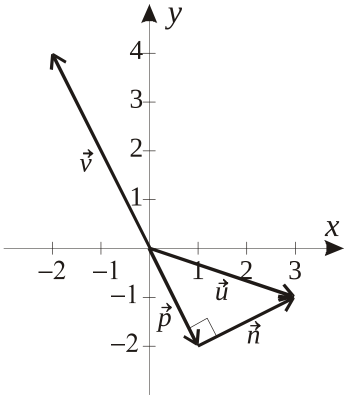


Illustration for Exercise 9a.

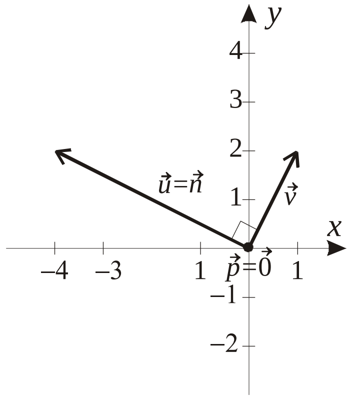


Illustration for Exercise 9b.

b. $\overrightarrow{p} = \text{proj}_{\text{span}\{\overrightarrow{v}\}}\overrightarrow{u} = \dfrac{\begin{bmatrix} -4 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}}{\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{0}{5}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Check: $\overrightarrow{u} - \overrightarrow{p} = \begin{bmatrix} -4 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$

$(\overrightarrow{u} - \overrightarrow{p}) \cdot \overrightarrow{v} = (-4)(1) + (2)(2) \overset{\checkmark}{=} 0$

($\overrightarrow{u}$ is orthogonal to $\overrightarrow{v}$)

11. a. $\overrightarrow{p} = \text{proj}_{\text{span}\{\overrightarrow{v}\}}\overrightarrow{u} = \dfrac{\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{3}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Check: $\overrightarrow{u} - \overrightarrow{p} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$(\overrightarrow{u} - \overrightarrow{p}) \cdot \overrightarrow{v} = (-1)(1) + (1)(1) + (0)(1) \overset{\checkmark}{=} 0$

b. $\overrightarrow{p} = \text{proj}_{\text{span}\{\overrightarrow{v}\}}\overrightarrow{u} = \dfrac{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}}{\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \frac{3}{9}\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{-2}{3} \\ \frac{1}{3} \end{bmatrix}$

Check: $\overrightarrow{u} - \overrightarrow{p} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{2}{3} \\ \frac{-2}{3} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}$

$(\overrightarrow{u} - \overrightarrow{p}) \cdot \overrightarrow{v} = (\frac{1}{3})(2) + (\frac{2}{3})(-2) + (\frac{2}{3})(1) \overset{\checkmark}{=} 0$

13. a. $\overrightarrow{p}$ $=\text{proj}_{\text{span}\{\overrightarrow{v}\}}\overrightarrow{u} = \dfrac{\begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} = \frac{-4}{4} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$

Check: $\overrightarrow{u} - \overrightarrow{p} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}$

$(\overrightarrow{u} - \overrightarrow{p}) \cdot \overrightarrow{v} = (1)(1) + (2)(1) + (1)(-1) + (2)(-1) \overset{\checkmark}{=} 0$

b. $\overrightarrow{p}$ $=\text{proj}_{\text{span}\{\overrightarrow{v}\}}\overrightarrow{u} = \dfrac{\begin{bmatrix} 3 \\ 0 \\ -1 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}}{\begin{bmatrix} 2 \\ 1 \\ -1 \\ -1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}} \begin{bmatrix} 2 \\ 1 \\ -1 \\ -1 \\ -1 \end{bmatrix} = \frac{4}{8} \begin{bmatrix} 2 \\ 1 \\ -1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$

Check: $\overrightarrow{u} - \overrightarrow{p} = \begin{bmatrix} 3 \\ 0 \\ -1 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 2 \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{3}{2} \\ \frac{5}{2} \end{bmatrix}$

$(\overrightarrow{u} - \overrightarrow{p}) \cdot \overrightarrow{v} \overset{\checkmark}{=} (2)(2) + (-\frac{1}{2})(1) + (-\frac{1}{2})(-1) + (\frac{3}{2})(-1) + (\frac{5}{2})(-1) \overset{\checkmark}{=} 0.$

15. Let $\overrightarrow{w_1} = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{bmatrix}$ and $\overrightarrow{w_2} = \begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ -\frac{1}{3} \end{bmatrix}$

We have

$\overrightarrow{w_1} \cdot \overrightarrow{w_2} = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{bmatrix} \cdot \begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ -\frac{1}{3} \end{bmatrix} = (\frac{1}{3})(\frac{2}{3}) + (\frac{2}{3})(-\frac{2}{3}) + (-\frac{2}{3})(-\frac{1}{3}) = 0,$

$\|\overrightarrow{w_1}\| = \sqrt{\frac{1}{9} + \frac{4}{9} + \frac{4}{9}} = 1$, and

$\|\overrightarrow{w_2}\| = \sqrt{\frac{4}{9} + \frac{4}{9} + \frac{1}{9}} = 1$ therefore S is an orthonormal set.

Using formula (84),

$$\begin{aligned}
\text{proj}_{\text{span}S}\overrightarrow{u} &= (\overrightarrow{u}\cdot\overrightarrow{w_1})\,\overrightarrow{w_1} + (\overrightarrow{u}\cdot\overrightarrow{w_2})\,\overrightarrow{w_2} = \\
&= \left(\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\cdot\begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{bmatrix}\right)\begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{bmatrix} + \left(\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\cdot\begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ -\frac{1}{3} \end{bmatrix}\right)\begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ -\frac{1}{3} \end{bmatrix} \\
&= \frac{-5}{3}\begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{bmatrix} + \frac{5}{3}\begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{5}{9} \\ -\frac{20}{9} \\ \frac{5}{9} \end{bmatrix}
\end{aligned}$$

17. Let $\overrightarrow{w_1} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$ and $\overrightarrow{w_2} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$

We have

$\overrightarrow{w_1}\cdot\overrightarrow{w_2} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}\cdot\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = (\frac{1}{2})(\frac{1}{2}) + (\frac{1}{2})(-\frac{1}{2}) + (\frac{1}{2})(-\frac{1}{2}) + (\frac{1}{2})(\frac{1}{2}) = 0,$

$\|\overrightarrow{w_1}\| = \sqrt{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}} = 1$, and

$\|\overrightarrow{w_2}\| = \sqrt{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}} = 1$ therefore S is an orthonormal set.

Using formula (84),

$$\begin{aligned}
\text{proj}_{\text{span}S}\overrightarrow{u} &= (\overrightarrow{u}\cdot\overrightarrow{w_1})\,\overrightarrow{w_1} + (\overrightarrow{u}\cdot\overrightarrow{w_2})\,\overrightarrow{w_2} = \\
&= \left(\begin{bmatrix} 1 \\ -2 \\ 1 \\ 1 \end{bmatrix}\cdot\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}\right)\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} + \left(\begin{bmatrix} 1 \\ -2 \\ 1 \\ 1 \end{bmatrix}\cdot\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}\right)\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} + \frac{3}{2}\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}
\end{aligned}$$

19. Let $\overrightarrow{v_1} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and $\overrightarrow{v_2} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

We have

$\overrightarrow{v_1}\cdot\overrightarrow{v_2} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}\cdot\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = (1)(-1) + (2)(1) + (-1)(1) = 0,$

therefore S is an orthogonal set.

Using formula (83),

$$\begin{aligned}
\text{proj}_{\text{span}S}\overrightarrow{u} &= \frac{\overrightarrow{u}\cdot\overrightarrow{v_1}}{\overrightarrow{v_1}\cdot\overrightarrow{v_1}}\overrightarrow{v_1}+\frac{\overrightarrow{u}\cdot\overrightarrow{v_2}}{\overrightarrow{v_2}\cdot\overrightarrow{v_2}}\overrightarrow{v_2}= \\
&= \frac{\begin{bmatrix}5\\2\\-2\end{bmatrix}\cdot\begin{bmatrix}1\\2\\-1\end{bmatrix}}{\begin{bmatrix}1\\2\\-1\end{bmatrix}\cdot\begin{bmatrix}1\\2\\-1\end{bmatrix}}\begin{bmatrix}1\\2\\-1\end{bmatrix}+\frac{\begin{bmatrix}5\\2\\-2\end{bmatrix}\cdot\begin{bmatrix}-1\\1\\1\end{bmatrix}}{\begin{bmatrix}-1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}-1\\1\\1\end{bmatrix}}\begin{bmatrix}-1\\1\\1\end{bmatrix} \\
&= \frac{11}{6}\begin{bmatrix}1\\2\\-1\end{bmatrix}-\frac{5}{3}\begin{bmatrix}-1\\1\\1\end{bmatrix}=\begin{bmatrix}\frac{7}{2}\\2\\-\frac{7}{2}\end{bmatrix}
\end{aligned}$$

21. Let $\overrightarrow{v_1}=\begin{bmatrix}1\\0\\1\\1\end{bmatrix}$, $\overrightarrow{v_2}=\begin{bmatrix}-1\\-1\\1\\0\end{bmatrix}$, and $\overrightarrow{v_3}=\begin{bmatrix}0\\-1\\-1\\1\end{bmatrix}$

We have

$$\overrightarrow{v_1}\cdot\overrightarrow{v_2}=\begin{bmatrix}1\\0\\1\\1\end{bmatrix}\cdot\begin{bmatrix}-1\\-1\\1\\0\end{bmatrix}=(1)(-1)+(0)(-1)+(1)(1)+(1)(0)=0,$$

$$\overrightarrow{v_1}\cdot\overrightarrow{v_3}=\begin{bmatrix}1\\0\\1\\1\end{bmatrix}\cdot\begin{bmatrix}0\\-1\\-1\\1\end{bmatrix}=(1)(0)+(0)(-1)+(1)(-1)+(1)(1)=0, \text{ and}$$

$$\overrightarrow{v_2}\cdot\overrightarrow{v_3}=\begin{bmatrix}-1\\-1\\1\\0\end{bmatrix}\cdot\begin{bmatrix}0\\-1\\-1\\1\end{bmatrix}=(-1)(0)+(-1)(-1)+(1)(-1)+(0)(1)=0,$$

therefore S is an orthogonal set.

Using formula (83),

$$\begin{aligned}
\text{proj}_{\text{span}S}\overrightarrow{u} &= \frac{\overrightarrow{u}\cdot\overrightarrow{v_1}}{\overrightarrow{v_1}\cdot\overrightarrow{v_1}}\overrightarrow{v_1}+\frac{\overrightarrow{u}\cdot\overrightarrow{v_2}}{\overrightarrow{v_2}\cdot\overrightarrow{v_2}}\overrightarrow{v_2}+\frac{\overrightarrow{u}\cdot\overrightarrow{v_3}}{\overrightarrow{v_3}\cdot\overrightarrow{v_3}}\overrightarrow{v_3}\\
&= \frac{\begin{bmatrix}2\\0\\1\\2\end{bmatrix}\cdot\begin{bmatrix}1\\0\\1\\1\end{bmatrix}}{\begin{bmatrix}1\\0\\1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\0\\1\\1\end{bmatrix}}\begin{bmatrix}1\\0\\1\\1\end{bmatrix}+\frac{\begin{bmatrix}2\\0\\1\\2\end{bmatrix}\cdot\begin{bmatrix}-1\\-1\\1\\0\end{bmatrix}}{\begin{bmatrix}-1\\-1\\1\\0\end{bmatrix}\cdot\begin{bmatrix}-1\\-1\\1\\0\end{bmatrix}}\begin{bmatrix}-1\\-1\\1\\0\end{bmatrix}+\frac{\begin{bmatrix}2\\0\\1\\2\end{bmatrix}\cdot\begin{bmatrix}0\\-1\\-1\\1\end{bmatrix}}{\begin{bmatrix}0\\-1\\-1\\1\end{bmatrix}\cdot\begin{bmatrix}0\\-1\\-1\\1\end{bmatrix}}\begin{bmatrix}0\\-1\\-1\\1\end{bmatrix}\\
&= \frac{5}{3}\begin{bmatrix}1\\0\\1\\1\end{bmatrix}-\frac{1}{3}\begin{bmatrix}-1\\-1\\1\\0\end{bmatrix}+\frac{1}{3}\begin{bmatrix}0\\-1\\-1\\1\end{bmatrix}=\begin{bmatrix}2\\0\\1\\2\end{bmatrix}
\end{aligned}$$

23. Taking $\overrightarrow{v_1}=\overrightarrow{v}$ in formula (88)

$$B=\frac{1}{\overrightarrow{v}\cdot\overrightarrow{v}}\overrightarrow{v}\overrightarrow{v}^T=\frac{1}{4+16}\begin{bmatrix}-2\\4\end{bmatrix}\begin{bmatrix}-2&4\end{bmatrix}=\frac{1}{20}\begin{bmatrix}4&-8\\-8&16\end{bmatrix}=\begin{bmatrix}\frac{1}{5}&-\frac{2}{5}\\-\frac{2}{5}&\frac{4}{5}\end{bmatrix}$$

$$B\overrightarrow{u}=\begin{bmatrix}\frac{1}{5}&-\frac{2}{5}\\-\frac{2}{5}&\frac{4}{5}\end{bmatrix}\begin{bmatrix}3\\-1\end{bmatrix}=\begin{bmatrix}1\\-2\end{bmatrix}$$

25. Taking $\overrightarrow{v_1}=\overrightarrow{v}$ in formula (88)

$$B=\frac{1}{\overrightarrow{v}\cdot\overrightarrow{v}}\overrightarrow{v}\overrightarrow{v}^T=\frac{1}{1+1+1}\begin{bmatrix}1\\1\\1\end{bmatrix}\begin{bmatrix}1&1&1\end{bmatrix}=\frac{1}{3}\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}=\begin{bmatrix}\frac{1}{3}&\frac{1}{3}&\frac{1}{3}\\\frac{1}{3}&\frac{1}{3}&\frac{1}{3}\\\frac{1}{3}&\frac{1}{3}&\frac{1}{3}\end{bmatrix}$$

$$B\overrightarrow{u}=\begin{bmatrix}\frac{1}{3}&\frac{1}{3}&\frac{1}{3}\\\frac{1}{3}&\frac{1}{3}&\frac{1}{3}\\\frac{1}{3}&\frac{1}{3}&\frac{1}{3}\end{bmatrix}\begin{bmatrix}0\\2\\1\end{bmatrix}=\begin{bmatrix}1\\1\\1\end{bmatrix}$$

27. Taking $\overrightarrow{v_1}=\overrightarrow{v}$ in formula (88)

$$\begin{aligned}
B &= \frac{1}{\overrightarrow{v}\cdot\overrightarrow{v}}\overrightarrow{v}\overrightarrow{v}^T=\frac{1}{1+1+1+1}\begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}\begin{bmatrix}1&1&-1&-1\end{bmatrix}\\
&= \frac{1}{4}\begin{bmatrix}1&1&-1&-1\\1&1&-1&-1\\-1&-1&1&1\\-1&-1&1&1\end{bmatrix}=\begin{bmatrix}\frac{1}{4}&\frac{1}{4}&-\frac{1}{4}&-\frac{1}{4}\\\frac{1}{4}&\frac{1}{4}&-\frac{1}{4}&-\frac{1}{4}\\-\frac{1}{4}&-\frac{1}{4}&\frac{1}{4}&\frac{1}{4}\\-\frac{1}{4}&-\frac{1}{4}&\frac{1}{4}&\frac{1}{4}\end{bmatrix}
\end{aligned}$$

$$B\overrightarrow{u} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

29. Using formula (89),

$$B = AA^T = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{5}{9} & -\frac{2}{9} & -\frac{4}{9} \\ -\frac{2}{9} & \frac{8}{9} & -\frac{2}{9} \\ -\frac{4}{9} & -\frac{2}{9} & \frac{5}{9} \end{bmatrix}$$

$$B\overrightarrow{u} = \begin{bmatrix} \frac{5}{9} & -\frac{2}{9} & -\frac{4}{9} \\ -\frac{2}{9} & \frac{8}{9} & -\frac{2}{9} \\ -\frac{4}{9} & -\frac{2}{9} & \frac{5}{9} \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{9} \\ -\frac{20}{9} \\ \frac{5}{9} \end{bmatrix}.$$

31. Using formula (88),

$$\begin{aligned} B &= \frac{1}{\overrightarrow{v_1} \cdot \overrightarrow{v_1}} \overrightarrow{v_1}\overrightarrow{v_1}^T + \frac{1}{\overrightarrow{v_2} \cdot \overrightarrow{v_2}} \overrightarrow{v_2}\overrightarrow{v_2}^T + \frac{1}{\overrightarrow{v_3} \cdot \overrightarrow{v_3}} \overrightarrow{v_3}\overrightarrow{v_3}^T \\ &= \frac{1}{3} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 1 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -1 & -1 & 1 & 0 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 0 \\ -1 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & -1 & 1 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 1 & 1 & -1 & 0 \\ 1 & 1 & -1 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & -1 & -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & 0 \\ \frac{1}{3} & -\frac{1}{3} & 0 & \frac{2}{3} \end{bmatrix} \end{aligned}$$

$$B\overrightarrow{u} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & 0 \\ \frac{1}{3} & -\frac{1}{3} & 0 & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 2 \end{bmatrix}.$$

6.3 Gram-Schmidt Process and Least Squares Approximation

1. a. $\overrightarrow{u_1} = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}, \overrightarrow{u_2} = \begin{bmatrix} -3 \\ 2 \\ 5 \end{bmatrix}$

Orthogonal basis:

$$\overrightarrow{v_1} = \overrightarrow{u_1} = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$$

$$\overrightarrow{v_2} = \overrightarrow{u_2} - \tfrac{\overrightarrow{u_2} \cdot \overrightarrow{v_1}}{\overrightarrow{v_1} \cdot \overrightarrow{v_1}} \overrightarrow{v_1} = \begin{bmatrix} -3 \\ 2 \\ 5 \end{bmatrix} - \tfrac{18}{9} \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Orthonormal basis:

$$\overrightarrow{w_1} = \tfrac{1}{\|\overrightarrow{v_1}\|} \overrightarrow{v_1} = \tfrac{1}{\sqrt{9}} \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$$

$$\overrightarrow{w_2} = \tfrac{1}{\|\overrightarrow{v_2}\|} \overrightarrow{v_2} = \tfrac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

b. $\overrightarrow{u_1} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \overrightarrow{u_2} = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$

Orthogonal basis:

$$\overrightarrow{v_1} = \overrightarrow{u_1} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

$$\overrightarrow{v_2} = \overrightarrow{u_2} - \tfrac{\overrightarrow{u_2} \cdot \overrightarrow{v_1}}{\overrightarrow{v_1} \cdot \overrightarrow{v_1}} \overrightarrow{v_1} = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} - \tfrac{9}{6} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} + \begin{bmatrix} -3 \\ \frac{-3}{2} \\ \frac{3}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{3}{2} \\ \frac{7}{2} \end{bmatrix}$$

We can replace $\overrightarrow{v_2}$ with the vector twice as long to avoid fractions: $\overrightarrow{v_2} = \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix}$

Orthonormal basis:

$$\overrightarrow{w_1} = \tfrac{1}{\|\overrightarrow{v_1}\|} \overrightarrow{v_1} = \tfrac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

$$\overrightarrow{w_2} = \frac{1}{\|\overrightarrow{v_2}\|}\overrightarrow{v_2} = \frac{1}{\sqrt{62}}\begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix}$$

c. $\overrightarrow{u_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \overrightarrow{u_2} = \begin{bmatrix} 6 \\ 4 \\ 0 \\ 2 \end{bmatrix}$

Orthogonal basis:

$$\overrightarrow{v_1} = \overrightarrow{u_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\overrightarrow{v_2} = \overrightarrow{u_2} - \frac{\overrightarrow{u_2}\cdot\overrightarrow{v_1}}{\overrightarrow{v_1}\cdot\overrightarrow{v_1}}\overrightarrow{v_1} = \begin{bmatrix} 6 \\ 4 \\ 0 \\ 2 \end{bmatrix} - \frac{12}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -3 \\ -1 \end{bmatrix}.$$

Orthonormal basis:

$$\overrightarrow{w_1} = \frac{1}{\|\overrightarrow{v_1}\|}\overrightarrow{v_1} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\overrightarrow{w_2} = \frac{1}{\|\overrightarrow{v_2}\|}\overrightarrow{v_2} = \frac{1}{\sqrt{20}}\begin{bmatrix} 3 \\ 1 \\ -3 \\ -1 \end{bmatrix}$$

3. a. $\overrightarrow{u_1} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \end{bmatrix}, \overrightarrow{u_2} = \begin{bmatrix} -1 \\ -3 \\ 2 \\ -1 \end{bmatrix}, \overrightarrow{u_3} = \begin{bmatrix} -2 \\ -1 \\ 3 \\ 2 \end{bmatrix}$

Orthogonal basis:

$$\overrightarrow{v_1} = \overrightarrow{u_1} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \end{bmatrix}$$

$$\overrightarrow{v_2} = \overrightarrow{u_2} - \frac{\overrightarrow{u_2}\cdot\overrightarrow{v_1}}{\overrightarrow{v_1}\cdot\overrightarrow{v_1}}\overrightarrow{v_1} = \begin{bmatrix} -1 \\ -3 \\ 2 \\ -1 \end{bmatrix} - \frac{-9}{9}\begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 2 \\ 1 \end{bmatrix}$$

$$\overrightarrow{v_3} = \overrightarrow{u_3} - \frac{\overrightarrow{u_3}\cdot\overrightarrow{v_1}}{\overrightarrow{v_1}\cdot\overrightarrow{v_1}}\overrightarrow{v_1} - \frac{\overrightarrow{u_3}\cdot\overrightarrow{v_2}}{\overrightarrow{v_2}\cdot\overrightarrow{v_2}}\overrightarrow{v_2}$$

$$= \begin{bmatrix} -2 \\ -1 \\ 3 \\ 2 \end{bmatrix} - \frac{0}{9} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \end{bmatrix} - \frac{9}{6} \begin{bmatrix} 0 \\ -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}$$

We can replace $\overrightarrow{v_3}$ with the vector twice as long to avoid fractions: $\overrightarrow{v_3} = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

Orthonormal basis:

$$\overrightarrow{w_1} = \frac{1}{||\overrightarrow{v_1}||}\overrightarrow{v_1} = \frac{1}{3} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \end{bmatrix}$$

$$\overrightarrow{w_2} = \frac{1}{||\overrightarrow{v_2}||}\overrightarrow{v_2} = \frac{1}{\sqrt{6}} \begin{bmatrix} 0 \\ -1 \\ 2 \\ 1 \end{bmatrix}$$

$$\overrightarrow{w_3} = \frac{1}{||\overrightarrow{v_3}||}\overrightarrow{v_3} = \frac{1}{\sqrt{18}} \begin{bmatrix} -4 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

b. $\overrightarrow{u_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \overrightarrow{u_2} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \\ 4 \end{bmatrix}, \overrightarrow{u_3} = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \\ 4 \end{bmatrix}$

Orthogonal basis:

$$\overrightarrow{v_1} = \overrightarrow{u_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

$$\overrightarrow{v_2} = \overrightarrow{u_2} - \frac{\overrightarrow{u_2} \cdot \overrightarrow{v_1}}{\overrightarrow{v_1} \cdot \overrightarrow{v_1}}\overrightarrow{v_1} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \\ 4 \end{bmatrix} - \frac{-4}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \\ 2 \end{bmatrix}$$

$$\overrightarrow{v_3} = \overrightarrow{u_3} - \frac{\overrightarrow{u_3} \cdot \overrightarrow{v_1}}{\overrightarrow{v_1} \cdot \overrightarrow{v_1}}\overrightarrow{v_1} - \frac{\overrightarrow{u_3} \cdot \overrightarrow{v_2}}{\overrightarrow{v_2} \cdot \overrightarrow{v_2}}\overrightarrow{v_2}$$

$$= \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \\ 4 \end{bmatrix} - \frac{-2}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} - \frac{12}{12} \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

Orthonormal basis:

$$\overrightarrow{w_1} = \frac{1}{\|\overrightarrow{v_1}\|}\overrightarrow{v_1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

$$\overrightarrow{w_2} = \frac{1}{\|\overrightarrow{v_2}\|}\overrightarrow{v_2} = \frac{1}{\sqrt{12}} \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$\overrightarrow{w_3} = \frac{1}{\|\overrightarrow{v_3}\|}\overrightarrow{v_3} = \frac{1}{\sqrt{8}} \begin{bmatrix} 1 \\ -2 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

5. The null space V of the plane equation consists of vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y+z \\ y \\ z \end{bmatrix} = y \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$,

so that the vectors $\overrightarrow{u_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\overrightarrow{u_2} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ form a basis for V.

a. To use formula (83), we need an orthogonal basis for V, which we shall find using the Gram-Schmidt process.

$$\overrightarrow{v_1} = \overrightarrow{u_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\overrightarrow{v_2} = \overrightarrow{u_2} - \frac{\overrightarrow{u_2}\cdot\overrightarrow{v_1}}{\overrightarrow{v_1}\cdot\overrightarrow{v_1}}\overrightarrow{v_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{-1}{2} \\ 1 \end{bmatrix}.$$

We can replace this vector with twice the same vector, $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$, to avoid fractions in the remaining computations.

The matrix of the transformation can be obtained as

$$\frac{1}{\overrightarrow{v_1}\cdot\overrightarrow{v_1}}\overrightarrow{v_1}\overrightarrow{v_1}^T + \frac{1}{\overrightarrow{v_2}\cdot\overrightarrow{v_2}}\overrightarrow{v_2}\overrightarrow{v_2}^T$$

$$= \tfrac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \end{bmatrix} + \tfrac{1}{6}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}\begin{bmatrix} 1 & -1 & 2 \end{bmatrix}$$

$$= \tfrac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \tfrac{1}{6}\begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \\ 2 & -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

Another way to perform the last step is to use (83) directly:

$$\begin{aligned}
F\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) &= \text{proj}_{\text{span}\{\overrightarrow{v_1},\overrightarrow{v_2}\}}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \\
&= \frac{\begin{bmatrix} x \\ y \\ z \end{bmatrix}\cdot\overrightarrow{v_1}}{\overrightarrow{v_1}\cdot\overrightarrow{v_1}}\overrightarrow{v_1} + \frac{\begin{bmatrix} x \\ y \\ z \end{bmatrix}\cdot\overrightarrow{v_2}}{\overrightarrow{v_2}\cdot\overrightarrow{v_2}}\overrightarrow{v_2} \\
&= \frac{x+y}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{x-y+2z}{6}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{2}{3}x+\frac{1}{3}y+\frac{1}{3}z \\ \frac{1}{3}x+\frac{2}{3}y-\frac{1}{3}z \\ \frac{1}{3}x-\frac{1}{3}y+\frac{2}{3}z \end{bmatrix} \\
&= \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{-1}{3} \\ \frac{1}{3} & \frac{-1}{3} & \frac{2}{3} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}.
\end{aligned}$$

(or, apply the projection to each of the standard basis vectors.)

b. Our plane is the column space of $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$. Using (98), we obtain the matrix of projection:

$$\begin{aligned}
A(A^TA)^{-1}A^T &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}
\end{aligned}$$

7. The null space V of the plane equation consists of vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -z \\ y \\ z \end{bmatrix} = y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$,

so that the vectors $\overrightarrow{u_1} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\overrightarrow{u_2} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ form a basis for V.

a. These vectors are already orthogonal, so that we can proceed directly to use formula (83) without using the Gram-Schmidt process, taking

$\overrightarrow{v_1} = \overrightarrow{u_1}$ and $\overrightarrow{v_2} = \overrightarrow{u_2}$.

The matrix of the transformation can be obtained as

$\frac{1}{\overrightarrow{v_1} \cdot \overrightarrow{v_1}} \overrightarrow{v_1} \overrightarrow{v_1}^T + \frac{1}{\overrightarrow{v_2} \cdot \overrightarrow{v_2}} \overrightarrow{v_2} \overrightarrow{v_2}^T$

$$= \frac{1}{1} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

Another way: according to (83) the transformation F can be expressed as

$$\begin{aligned}
F(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) &= \text{proj}_{\text{span}\{\overrightarrow{v_1},\overrightarrow{v_2}\}} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\
&= \frac{\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \overrightarrow{v_1}}{\overrightarrow{v_1} \cdot \overrightarrow{v_1}} \overrightarrow{v_1} + \frac{\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \overrightarrow{v_2}}{\overrightarrow{v_2} \cdot \overrightarrow{v_2}} \overrightarrow{v_2} \\
&= \frac{y}{1} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \frac{-x+z}{2} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2}x - \frac{1}{2}z \\ y \\ -\frac{1}{2}x + \frac{1}{2}z \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2} & 0 & \frac{-1}{2} \\ 0 & 1 & 0 \\ \frac{-1}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.
\end{aligned}$$

(The same matrix A can be obtained by applying the projection to each of the standard basis vectors.)

b.

Our plane is the column space of $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$. Using (98), we obtain the matrix of projection:

$$\begin{aligned}
A(A^TA)^{-1}A^T &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}
\end{aligned}$$

9. $\overrightarrow{x} = (A^TA)^{-1}A^T\overrightarrow{b} = \left(\begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$

$= \left(\begin{bmatrix} 2 & 3 \\ 3 & 6 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

11. $\overrightarrow{x} = (A^TA)^{-1}A^T\overrightarrow{b} = \left(\begin{bmatrix} 1 & 1 & -1 & 1 \\ -2 & -2 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 1 & -2 \\ -1 & 1 \\ 1 & -1 \end{bmatrix}\right)^{-1}\begin{bmatrix} 1 & 1 & -1 & 1 \\ -2 & -2 & 1 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 2 \\ 6 \end{bmatrix}$

$= \left(\begin{bmatrix} 4 & -6 \\ -6 & 10 \end{bmatrix}\right)^{-1}\begin{bmatrix} 1 & 1 & -1 & 1 \\ -2 & -2 & 1 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 2 \\ 6 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} & \frac{3}{2} \\ \frac{3}{2} & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & -1 & 1 \\ -2 & -2 & 1 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 2 \\ 6 \end{bmatrix} =$

$\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

13. The normal equation

$A^TA\overrightarrow{x} = A^T\overrightarrow{b}$

has $A^TA = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 2 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 10 \\ 0 & 0 & 0 \\ 10 & 0 & 20 \end{bmatrix}$

The right hand side becomes $A^T\overrightarrow{b} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix}$

The normal equation leads to a linear system with the augmented matrix $\begin{bmatrix} 5 & 0 & 10 & 3 \\ 0 & 0 & 0 & 0 \\ 10 & 0 & 20 & 6 \end{bmatrix}$ with the reduced row echelon form: $\begin{bmatrix} 1 & 0 & 2 & \frac{3}{5} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Therefore, the solutions satisfy

$$\begin{aligned} x_1 &= \frac{3}{5} - 2x_3 \\ x_2 &= arbitrary \\ x_3 &= arbitrary \end{aligned}$$

6.4 Introduction to Singular Value Decomposition

1. a. $2, 1$; b. $1, 1$; c.$2, 0$; d. $4, 1$.

3. a. $4, 2$; b. $5, 1$; . c. $3, 1$; d. $5, 3$.

5. a. iii; b. iv; c. i; d. ii..

7. a. ii; b. iv; c. i; d. iii..

9. a. i; b. iii; c. ii; d. iv

7 Eigenvalues and Singular Values

7.1 Eigenvalues and Eigenvectors

1. $A\overrightarrow{u_1} = \frac{-1}{2}\overrightarrow{u_1}$, therefore the correct answer is: (f) the given vector is an eigenvector of A associated with $\lambda = \frac{-1}{2}$.

3. $A\overrightarrow{u_3} = 2\overrightarrow{u_3}$, therefore the correct answer is: (c) the given vector is an eigenvector of A associated with $\lambda = 2$.

5. (a) iv; (b) i; (c) ii;

7.a. Characteristic polynomial:

$$\det(\lambda I_2 - A) = \det \begin{bmatrix} \lambda - 2 & -4 \\ -1 & \lambda + 1 \end{bmatrix}$$

$$= (\lambda - 2)(\lambda + 1) - (-4)(-1) = \lambda^2 - \lambda - 6 = (\lambda + 2)(\lambda - 3)$$

The eigenvalues are -2 and 3.

b. Characteristic polynomial:

$$\det(\lambda I_3 - A) = \det \begin{bmatrix} \lambda & -2 & 0 \\ -1 & \lambda + 1 & 0 \\ 0 & -2 & \lambda + 3 \end{bmatrix}$$

Expand along the third column

$$= (-1)^{3+3}(\lambda + 3)\det \begin{bmatrix} \lambda & -2 \\ -1 & \lambda + 1 \end{bmatrix} = (\lambda + 3)\left[\lambda(\lambda + 1) - (-2)(-1)\right]$$

$$= (\lambda + 3)(\lambda^2 + \lambda - 2) = (\lambda + 3)(\lambda + 2)(\lambda - 1)$$

The eigenvalues are 1, -2 and -3.

9. $A = \begin{bmatrix} -1 & 4 & 0 \\ 3 & -2 & 0 \\ 2 & 0 & 4 \end{bmatrix}$ has eigenvalues $\lambda_1 = 4$, $\lambda_2 = -5$, and $\lambda_3 = 2$.

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2LZ8uVU

For $\lambda_1 = 4$ the coefficient matrix of the homogeneous system is

$$(4)I_3 - A = \begin{bmatrix} 5 & -4 & 0 \\ -3 & 6 & 0 \\ -2 & 0 & 0 \end{bmatrix}$$

The r.r.e.f. is $\begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so that the corresponding eigenvectors are of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

with $x_3 \neq 0$.

The eigenspace has a basis formed by $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2FhX7qx

For $\lambda_2 = -5$ the coefficient matrix of the homogeneous system is

$$(-5)I_3 - A = \begin{bmatrix} -4 & -4 & 0 \\ -3 & -3 & 0 \\ -2 & 0 & -9 \end{bmatrix}$$

The r.r.e.f. is $\begin{bmatrix} \boxed{1} & 0 & \frac{9}{2} \\ 0 & \boxed{1} & -\frac{9}{2} \\ 0 & 0 & 0 \end{bmatrix}$ so that the corresponding eigenvectors are of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{9}{2}x_3 \\ \frac{9}{2}x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -\frac{9}{2} \\ \frac{9}{2} \\ 1 \end{bmatrix}$$

with $x_3 \neq 0$.

The eigenspace has a basis formed by $\begin{bmatrix} -\frac{9}{2} \\ \frac{9}{2} \\ 1 \end{bmatrix}$.

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2RBlyFV

For $\lambda_3 = 2$ the coefficient matrix of the homogeneous system is

$$(2)I_3 - A = \begin{bmatrix} 3 & -4 & 0 \\ -3 & 4 & 0 \\ -2 & 0 & -2 \end{bmatrix}$$

The r.r.e.f. is $\begin{bmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & \frac{3}{4} \\ 0 & 0 & 0 \end{bmatrix}$ so that the corresponding eigenvectors are of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ \frac{-3}{4}x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ \frac{-3}{4} \\ 1 \end{bmatrix}$$

with $x_3 \neq 0$.

The eigenspace has a basis formed by $\begin{bmatrix} -1 \\ \frac{-3}{4} \\ 1 \end{bmatrix}$.

11.a. Characteristic polynomial:

$$\det(\lambda I_2 - A) = \det \begin{bmatrix} \lambda - 4 & -2 \\ 0 & \lambda - 1 \end{bmatrix} = (\lambda - 4)(\lambda - 1)$$

The eigenvalues are 4 and 1 (both with algebraic multiplicity 1).

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2FmZngv

For $\lambda_1 = 4$ the coefficient matrix of the homogeneous system is

$$(4)I_2 - A = \begin{bmatrix} 0 & -2 \\ 0 & 3 \end{bmatrix}$$

The r.r.e.f. is $\begin{bmatrix} 0 & \boxed{1} \\ 0 & 0 \end{bmatrix}$ so that the corresponding eigenvectors are of the form

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

with $x_1 \neq 0$.

The eigenspace has a basis formed by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2SIxCCR

For $\lambda_1 = 1$ the coefficient matrix of the homogeneous system is

$$(1)I_2 - A = \begin{bmatrix} -3 & -2 \\ 0 & 0 \end{bmatrix}$$

The r.r.e.f. is $\begin{bmatrix} \boxed{1} & \frac{2}{3} \\ 0 & 0 \end{bmatrix}$ so that the corresponding eigenvectors are of the form

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3}x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix}$$

with $x_2 \neq 0$.

The eigenspace has a basis formed by $\begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix}$.

b. Characteristic polynomial:

$$\det(\lambda I_2 - A) = \det \begin{bmatrix} \lambda - 6 & -5 \\ -1 & \lambda - 2 \end{bmatrix} = (\lambda - 6)(\lambda - 2) - (-5)(-1)$$

$$= \lambda^2 - 8\lambda + 7 = (\lambda - 1)(\lambda - 7).$$

The eigenvalues are 1 and 7 (both with algebraic multiplicity 1).

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2Fk281R

For $\lambda_1 = 1$ the coefficient matrix of the homogeneous system is

$$(1)I_2 - A = \begin{bmatrix} -5 & -5 \\ -1 & -1 \end{bmatrix}$$

The r.r.e.f. is $\begin{bmatrix} \boxed{1} & 1 \\ 0 & 0 \end{bmatrix}$ so that the corresponding eigenvectors are of the form

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

with $x_2 \neq 0$.

The eigenspace has a basis formed by $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2H1N77u

For $\lambda_1 = 7$ the coefficient matrix of the homogeneous system is

$$(-2)I_2 - A = \begin{bmatrix} 1 & -5 \\ -1 & 5 \end{bmatrix}$$

The r.r.e.f. is $\begin{bmatrix} \boxed{1} & -5 \\ 0 & 0 \end{bmatrix}$ so that the corresponding eigenvectors are of the form

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

with $x_2 \neq 0$.

The eigenspace has a basis formed by $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$.

c. Characteristic polynomial:

$$\det(\lambda I_2 - A) = \det \begin{bmatrix} \lambda - 1 & 2 \\ -8 & \lambda - 9 \end{bmatrix} = (\lambda - 1)(\lambda - 9) - (2)(-8)$$

$= \lambda^2 - 10\lambda + 25 = (\lambda - 5)^2$

The only eigenvalue is 5, with algebraic multiplicity 2.

For this eigenvalue, the coefficient matrix of the homogeneous system is

$$(5)I_2 - A = \begin{bmatrix} 4 & 2 \\ -8 & -4 \end{bmatrix}$$

Refer to the
Linear Algebra
Toolkit for details:
http://bit.ly/2ADLeYX

The r.r.e.f. is $\begin{bmatrix} \boxed{1} & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$ so that the corresponding eigenvectors are of the form

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{-1}{2}x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} \frac{-1}{2} \\ 1 \end{bmatrix}$$

with $x_2 \neq 0$.

The eigenspace has a basis formed by $\begin{bmatrix} \frac{-1}{2} \\ 1 \end{bmatrix}$.

13.a. Characteristic polynomial (we use a cofactor expansion along the first row):

$$\det(\lambda I_3 - A) = \det \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda - 4 & -2 \\ 1 & 0 & \lambda + 3 \end{bmatrix} = (-1)^{1+1}(\lambda) \det \begin{bmatrix} \lambda - 4 & -2 \\ 0 & \lambda + 3 \end{bmatrix}$$

$= \lambda(\lambda - 4)(\lambda + 3)$.

The eigenvalues are 0, 4 and -3 (all with multiplicities 1).

Refer to the
Linear Algebra
Toolkit for details:
http://bit.ly/2CWQ5pz

For $\lambda_1 = 0$ the coefficient matrix of the homogeneous system is

$$(0)I_3 - A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -4 & -2 \\ 1 & 0 & 3 \end{bmatrix}$$

The r.r.e.f. is $\begin{bmatrix} \boxed{1} & 0 & 3 \\ 0 & \boxed{1} & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$ so that the corresponding eigenvectors are of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3x_3 \\ \frac{-1}{2}x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ \frac{-1}{2} \\ 1 \end{bmatrix}$$

with $x_3 \neq 0$.

The eigenspace has a basis formed by $\begin{bmatrix} -3 \\ \frac{-1}{2} \\ 1 \end{bmatrix}$.

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2FcyJYp

For $\lambda_2 = 4$ the coefficient matrix of the homogeneous system is

$$(4)I_3 - A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & -2 \\ 1 & 0 & 7 \end{bmatrix}$$

The r.r.e.f. is $\begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{bmatrix}$ so that the corresponding eigenvectors are of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

with $x_2 \neq 0$.

The eigenspace has a basis formed by $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2FmPQWm

For $\lambda_3 = -3$ the coefficient matrix of the homogeneous system is

$$(-3)I_3 - A = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -7 & -2 \\ 1 & 0 & 0 \end{bmatrix}$$

The r.r.e.f. is $\begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & \frac{2}{7} \\ 0 & 0 & 0 \end{bmatrix}$ so that the corresponding eigenvectors are of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{-2}{7}x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ \frac{-2}{7} \\ 1 \end{bmatrix}$$

with $x_3 \neq 0$.

The eigenspace has a basis formed by $\begin{bmatrix} 0 \\ \frac{-2}{7} \\ 1 \end{bmatrix}$.

b. Characteristic polynomial (we use a cofactor expansion along the first row):

$$\det(\lambda I_3 - A) = \det \begin{bmatrix} \lambda - 2 & 0 & 0 \\ 0 & \lambda + 2 & 2 \\ 0 & 2 & \lambda - 1 \end{bmatrix} = (-1)^{1+1}(\lambda) \det \begin{bmatrix} \lambda + 2 & 2 \\ 2 & \lambda - 1 \end{bmatrix}$$

$$= (\lambda - 2)\left[(\lambda + 2)(\lambda - 1) - (2)(2)\right] = (\lambda - 2)\left[\lambda^2 + \lambda - 6\right] = (\lambda - 2)^2(\lambda + 3)$$

The eigenvalues are 2 (with algebraic multiplicity 2) and -3 (with algebraic multiplicity 1).

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2RvLcfx

For $\lambda_1 = 2$ the coefficient matrix of the homogeneous system is

$$(2)I_3 - A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

The r.r.e.f. is $\begin{bmatrix} 0 & \boxed{1} & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so that the corresponding eigenvectors are of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ \frac{-1}{2}x_3 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ \frac{-1}{2} \\ 1 \end{bmatrix}$$

with $x_1, x_3 \neq 0$.

The eigenspace has a basis formed by $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{-1}{2} \\ 1 \end{bmatrix}$.

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2CdCTem

For $\lambda_2 = -3$ the coefficient matrix of the homogeneous system is

$$(-3)I_3 - A = \begin{bmatrix} -5 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & 2 & -4 \end{bmatrix}$$

The r.r.e.f. is $\begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & -2 \\ 0 & 0 & 0 \end{bmatrix}$ so that the corresponding eigenvectors are of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

with $x_3 \neq 0$.

The eigenspace has a basis formed by $\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$.

c. Characteristic polynomial (use Formula (30)):

$$\det(\lambda I_3 - A) = \det \begin{bmatrix} \lambda - 1 & 4 & -4 \\ -2 & \lambda - 2 & -4 \\ 0 & -2 & \lambda + 1 \end{bmatrix}$$

$$= (\lambda - 1)(\lambda - 2)(\lambda + 1) + (4)(-4)(0) + (-4)(-2)(-2)$$
$$-(\lambda - 1)(-4)(-2) - (4)(-2)(\lambda + 1) - (-4)(\lambda - 2)(0)$$
$$= \lambda^3 - 2\lambda^2 - \lambda + 2$$

You may be able to notice that this polynomial can be factored:

$\lambda^3 - 2\lambda^2 - \lambda + 2 = (\lambda - 2)(\lambda^2 - 1) = (\lambda - 2)(\lambda - 1)(\lambda + 1)$

If not, then search among the factors of the free term 2 $(1, -1, 2, -2)$ for a zero:

trying $\lambda = 1$ leads to $1^3 - 2\left(1^2\right) - 1 + 2 = 0$ so that $\lambda - 1$ must be a factor in the characteristic polynomial. Dividing

$$\begin{array}{rrrrrrrr}
 & \lambda^2 & - & \lambda & - & 2 & & \\
 & -- & - & -- & - & -- & - & -- \\
(\lambda-1)\ | & \lambda^3 & - & 2\lambda^2 & - & \lambda & + & 2 \\
 & -\lambda^3 & + & \lambda^2 & & & & \\
 & -- & - & -- & - & -- & - & -- \\
 & & - & \lambda^2 & - & \lambda & + & 2 \\
 & & + & \lambda^2 & - & \lambda & & \\
 & & & -- & - & -- & - & -- \\
 & & & & - & 2\lambda & + & 2 \\
 & & & & + & 2\lambda & - & 2 \\
 & & & & & -- & - & -- \\
 & & & & & & & 0
\end{array}$$

we obtain

$$\begin{aligned} \lambda^3 - 2\lambda^2 - \lambda + 2 &= (\lambda-1)\left(\lambda^2-\lambda-2\right) \\ &= (\lambda-1)(\lambda-2)(\lambda+1). \end{aligned}$$

Either way, we arrive at the eigenvalues $\lambda_1 = 1, \lambda_2 = 2$, and $\lambda_3 = -1$ (all with multiplicities 1).

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2sfCfZi

For $\lambda_1 = 1$ the coefficient matrix of the homogeneous system is

$$(1)I_3 - A = \begin{bmatrix} 0 & 4 & -4 \\ -2 & -1 & -4 \\ 0 & -2 & 2 \end{bmatrix}$$

The r.r.e.f. is $\begin{bmatrix} \boxed{1} & 0 & \frac{5}{2} \\ 0 & \boxed{1} & -1 \\ 0 & 0 & 0 \end{bmatrix}$ so that the corresponding eigenvectors are of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{-5}{2}x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \frac{-5}{2} \\ 1 \\ 1 \end{bmatrix}$$

with $x_3 \neq 0$.

The eigenspace has a basis formed by $\begin{bmatrix} \frac{-5}{2} \\ 1 \\ 1 \end{bmatrix}$.

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2Fl0qgz

For $\lambda_2 = 2$ the coefficient matrix of the homogeneous system is

$$(2)I_3 - A = \begin{bmatrix} 1 & 4 & -4 \\ -2 & 0 & -4 \\ 0 & -2 & 3 \end{bmatrix}$$

The r.r.e.f. is $\begin{bmatrix} \boxed{1} & 0 & 2 \\ 0 & \boxed{1} & -\frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$ so that the corresponding eigenvectors are of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_3 \\ \frac{3}{2}x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ \frac{3}{2} \\ 1 \end{bmatrix}$$

with $x_3 \neq 0$.

The eigenspace has a basis formed by $\begin{bmatrix} -2 \\ \frac{3}{2} \\ 1 \end{bmatrix}$.

Refer to the
Linear Algebra
Toolkit for details:
http://bit.ly/2RfCK4p

For $\lambda_3 = -1$ the coefficient matrix of the homogeneous system is

$$(-3)I_3 - A = \begin{bmatrix} -2 & 4 & -4 \\ -2 & -3 & -4 \\ 0 & -2 & 0 \end{bmatrix}$$

The r.r.e.f. is $\begin{bmatrix} \boxed{1} & 0 & 2 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so that the corresponding eigenvectors are of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

with $x_3 \neq 0$.

The eigenspace has a basis formed by $\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$.

d. Characteristic polynomial (use Formula (30)):

$\det(\lambda I_3 - A) = \det \begin{bmatrix} \lambda & -1 & 0 \\ -1 & \lambda & 2 \\ -1 & 1 & \lambda \end{bmatrix}$

$= (\lambda)(\lambda)(\lambda) + (-1)(2)(-1) + (0)(-1)(1) - (\lambda)(2)(1) - (-1)(-1)(\lambda) - (0)(\lambda)(-1)$

$= \lambda^3 - 3\lambda + 2$

Search among the factors of the free term 2: $1, -1, 2, -2$

trying $\lambda = 1$ leads to $1^3 - 3\,(1) + 2 = 0$ so that 1 is a root;

$\lambda - 1$ must be a factor in the characteristic polynomial. Dividing

$$\begin{array}{rrcrcrcr}
 & \lambda^2 & + & \lambda & - & 2 & & \\
 & -- & - & -- & - & -- & - & -- \\
(\lambda - 1)\ | & \lambda^3 & & & - & 3\lambda & + & 2 \\
 & -\lambda^3 & + & \lambda^2 & & & & \\
 & -- & - & -- & - & -- & - & -- \\
 & & & \lambda^2 & - & 3\lambda & + & 2 \\
 & & - & \lambda^2 & + & \lambda & & \\
 & & & -- & - & -- & - & -- \\
 & & & & - & 2\lambda & + & 2 \\
 & & & & + & 2\lambda & - & 2 \\
 & & & & & -- & - & -- \\
 & & & & & & & 0
\end{array}$$

we obtain

$$\begin{aligned} \lambda^3 - 3\lambda + 2 &= (\lambda - 1)\left(\lambda^2 + \lambda - 2\right) \\ &= (\lambda - 1)(\lambda - 1)(\lambda + 2). \end{aligned}$$

We arrive at one double (i.e., algebraic multiplicity 2) eigenvalue $\lambda_1 = 1$, and one single (algebraic multiplicity 1) eigenvalue $\lambda_2 = -2$.

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2CaMvXl

For $\lambda_1 = 1$ the coefficient matrix of the homogeneous system is

$$1I_3 - A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 2 \\ -1 & 1 & 1 \end{bmatrix}$$

The r.r.e.f. is $\begin{bmatrix} \boxed{1} & -1 & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{bmatrix}$ so that the corresponding eigenvectors are of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

with $x_2 \neq 0$.

The eigenspace has a basis formed by $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2sg5aMM

For $\lambda_2 = -2$ the coefficient matrix of the homogeneous system is

$$-2I_3 - A = \begin{bmatrix} -2 & -1 & 0 \\ -1 & -2 & 2 \\ -1 & 1 & -2 \end{bmatrix}$$

The r.r.e.f. is $\begin{bmatrix} \boxed{1} & 0 & \frac{2}{3} \\ 0 & \boxed{1} & -\frac{4}{3} \\ 0 & 0 & 0 \end{bmatrix}$ so that the corresponding eigenvectors are of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3}x_3 \\ \frac{4}{3}x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -\frac{2}{3} \\ \frac{4}{3} \\ 1 \end{bmatrix}$$

with $x_3 \neq 0$.

The eigenspace has a basis formed by $\begin{bmatrix} -\frac{2}{3} \\ \frac{4}{3} \\ 1 \end{bmatrix}$.

15. Characteristic polynomial:

$$\det(\lambda I_4 - A) = \det \begin{bmatrix} \lambda - 1 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ -3 & -3 & \lambda + 1 & -1 \\ 2 & -3 & 0 & \lambda \end{bmatrix}$$

Expand along the third column:

$$= (-1)^{3+3}(\lambda + 1) \det \begin{bmatrix} \lambda - 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 2 & -3 & \lambda \end{bmatrix}$$

expand the 3×3 determinant along the second row

$$= (-1)^{2+2}(\lambda + 1)\lambda \det \begin{bmatrix} \lambda - 1 & 0 \\ 2 & \lambda \end{bmatrix} = (\lambda + 1)\lambda^2(\lambda - 1)$$

The eigenvalues are

- $\lambda_1 = -1$ (algebraic multiplicity 1),
- $\lambda_2 = 0$, (algebraic multiplicity 2) and
- $\lambda_3 = 1$ (algebraic multiplicity 1).

Refer to the Linear Algebra Toolkit for details:

http://bit.ly/2TAZNUk

For $\lambda_1 = -1$ the coefficient matrix of the homogeneous system is

$$(-1)I_4 - A = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -3 & -3 & 0 & -1 \\ 2 & -3 & 0 & -1 \end{bmatrix}$$

The r.r.e.f. is $\begin{bmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{bmatrix}$ so that the corresponding eigenvectors are of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

with $x_3 \neq 0$.

The eigenspace has a basis formed by $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$.

Refer to the Linear Algebra Toolkit for details:

http://bit.ly/2Fh0LSx

For $\lambda_2 = 0$ the coefficient matrix of the homogeneous system is

$$0I_4 - A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -3 & -3 & 1 & -1 \\ 2 & -3 & 0 & 0 \end{bmatrix}$$

The r.r.e.f. is $\begin{bmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ so that the corresponding eigenvectors are of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ x_4 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

with $x_4 \neq 0$.

The eigenspace has a basis formed by $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$.

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2QwUEud

For $\lambda_3 = 1$ the coefficient matrix of the homogeneous system is

$$1I_4 - A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & -3 & 2 & -1 \\ 2 & -3 & 0 & 1 \end{bmatrix}$$

The r.r.e.f. is $\begin{bmatrix} \boxed{1} & 0 & 0 & \frac{1}{2} \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & \frac{1}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix}$ so that the corresponding eigenvectors are of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{-1}{2}x_4 \\ 0 \\ \frac{-1}{4}x_4 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} \frac{-1}{2} \\ 0 \\ \frac{-1}{4} \\ 1 \end{bmatrix}$$

with $x_4 \neq 0$.

The eigenspace has a basis formed by $\begin{bmatrix} \frac{-1}{2} \\ 0 \\ \frac{-1}{4} \\ 1 \end{bmatrix}$.

17. FALSE

Equivalent statements imply that A cannot have a zero eigenvalue, so it can have n nonzero eigenvalues.

19. TRUE

$$A \begin{bmatrix} 0 \\ y \end{bmatrix} = 2 \begin{bmatrix} 0 \\ y \end{bmatrix}$$

21. FALSE

$\lambda = -1$ is an eigenvalue of A since $\det((-1)I - A) = 0$ if and only if $\det(A + I) = 0$

23. TRUE

$\lambda I - A$ is also a scalar matrix, with all main diagonal entries $\lambda - c$

det($\lambda I - A$)=$(\lambda - c)^n$

7.2 Diagonalization

1.a. Characteristic polynomial:

$$\det(\lambda I_2 - A) = \det\begin{bmatrix} \lambda-6 & 6 \\ -6 & \lambda+7 \end{bmatrix} = (\lambda-6)(\lambda+7)-(6)(-6)$$

$$= \lambda^2+\lambda-6 = (\lambda+3)(\lambda-2)$$

Refer to the
Linear Algebra
Toolkit for details:
http://bit.ly/2FetAPw

For $\lambda=-3$, the coefficient matrix of the homogeneous system is $\begin{bmatrix} -9 & 6 \\ -6 & 4 \end{bmatrix}$, with r.r.e.f. $\begin{bmatrix} 1 & -\frac{2}{3} \\ 0 & 0 \end{bmatrix}$.

Eigenvectors are $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$ with $x_2 \neq 0$ (e.g., $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$)

Refer to the
Linear Algebra
Toolkit for details:
http://bit.ly/2FlgMWv

For $\lambda=2$, the coefficient matrix of the homogeneous system is $\begin{bmatrix} -4 & 6 \\ -6 & 9 \end{bmatrix}$, with r.r.e.f. $\begin{bmatrix} 1 & -\frac{3}{2} \\ 0 & 0 \end{bmatrix}$.

Eigenvectors are $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$ with $x_2 \neq 0$ (e.g., $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$).

A is diagonalizable: $P^{-1}AP = D$ with $P = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$ and $D = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$.

Verify: $AP = \begin{bmatrix} 6 & -6 \\ 6 & -7 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} -6 & 6 \\ -9 & 4 \end{bmatrix}$.

$PD = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -6 & 6 \\ -9 & 4 \end{bmatrix}$ ✓

b. Characteristic polynomial:

$$\det(\lambda I_2 - A) = \det\begin{bmatrix} \lambda+5 & 2 \\ -2 & \lambda+1 \end{bmatrix} = (\lambda+5)(\lambda+1)-(2)(-2)$$

$$= \lambda^2+6\lambda+9 = (\lambda+3)^2$$

$\lambda=-3$ is the only eigenvalue, of algebraic multiplicity 2.

Refer to the
Linear Algebra
Toolkit for details:
http://bit.ly/2SEdaTs

For $\lambda=-3$, the coefficient matrix of the homogeneous system is $\begin{bmatrix} 2 & 2 \\ -2 & -2 \end{bmatrix}$, with r.r.e.f. $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

Eigenvectors are $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ with $x_2 \neq 0$ (e.g., $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$)

This eigenspace has dimension 1, therefore, it cannot yield 2 linearly independent eigenvectors required for diagonalization.

A is not diagonalizable.

c. Characteristic polynomial:

$$\det(\lambda I_3 - A) = \det\begin{bmatrix} \lambda-3 & -2 & -2 \\ 1 & \lambda & -2 \\ 0 & 0 & \lambda \end{bmatrix}$$

Expand along the third row

$$= (-1)^{3+3}(\lambda)\det\begin{bmatrix} \lambda-3 & -2 \\ 1 & \lambda \end{bmatrix}$$

$$= \lambda\left[(\lambda-3)(\lambda)-(-2)(1)\right]=(\lambda)\left[\lambda^2-3\lambda+2\right]=(\lambda)(\lambda-1)(\lambda-2).$$

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2Fgemto

For $\lambda = 0$, the coefficient matrix of the homogeneous system is $\begin{bmatrix} -3 & -2 & -2 \\ 1 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$, with r.r.e.f. $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$. Eigenvectors are $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}$ with $x_3 \neq 0$ (e.g., $\begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}$)

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2AAfUui

For $\lambda = 1$, the coefficient matrix of the homogeneous system is $\begin{bmatrix} -2 & -2 & -2 \\ 1 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$, with r.r.e.f. $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Eigenvectors are $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ with $x_2 \neq 0$ (e.g., $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$).

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2Fh1ZMO

For $\lambda = 2$, the coefficient matrix of the homogeneous system is $\begin{bmatrix} -1 & -2 & -2 \\ 1 & 2 & -2 \\ 0 & 0 & 2 \end{bmatrix}$, with r.r.e.f. $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Eigenvectors are $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ with $x_2 \neq 0$ (e.g., $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$).

A is diagonalizable: $P^{-1}AP = D$ with $P = \begin{bmatrix} 2 & -1 & -2 \\ -4 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

Verify: $AP = \begin{bmatrix} 3 & 2 & 2 \\ -1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 & -2 \\ -4 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.

$$PD = \begin{bmatrix} 2 & -1 & -2 \\ -4 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \checkmark$$

3.a. Characteristic polynomial (we use Formula (30)):

$$\det(\lambda I_3 - A) = \det \begin{bmatrix} \lambda-2 & 4 & -3 \\ -1 & \lambda+2 & -1 \\ 4 & 0 & \lambda+6 \end{bmatrix}$$

$$= (\lambda-2)(\lambda+2)(\lambda+6) + (4)(-1)(4) + (-3)(-1)(0)$$
$$-(\lambda-2)(-1)(0) - (4)(-1)(\lambda+6) - (-3)(\lambda+2)(4)$$
$$= \lambda^3 + 6\lambda^2 + 12\lambda + 8$$

Test factors of the free term, 8 $(1, -1, 2, -1, 4, -4, 8, -8)$.

$\det(1I_3 - A) = 1 + 6 + 12 + 8 = 27 \neq 0 \Rightarrow$ 1 is not an eigenvalue of A

$\det(-1I_3 - A) = -1 + 6 - 12 + 8 = 1 \neq 0 \Rightarrow -1$ is not an eigenvalue of A

$\det(2I_3 - A) = 8 + 6(4) + 12(2) + 8 = 64 \neq 0 \Rightarrow$ 2 is not an eigenvalue of A

$\det(-2I_3 - A) = -8 + 6(4) + 12(-2) + 8 = 0 \Rightarrow -2$ is an eigenvalue of A.

Therefore, $(\lambda+2)$ is a factor of the characteristic polynomial. Use long division:

$$\begin{array}{rrcrcrcr}
 & \lambda^2 & + & 4\lambda & + & 4 & & \\
 & -- & - & -- & - & -- & - & -- \\
(\lambda+2)\ \mid & \lambda^3 & + & 6\lambda^2 & + & 12\lambda & + & 8 \\
 & -\lambda^3 & - & 2\lambda^2 & & & & \\
 & -- & - & -- & - & -- & - & -- \\
 & & & 4\lambda^2 & + & 12\lambda & + & 8 \\
 & & - & 4\lambda^2 & - & 8\lambda & & \\
 & & & -- & - & -- & - & -- \\
 & & & & & 4\lambda & + & 8 \\
 & & & & - & 4\lambda & - & 8 \\
 & & & & & -- & - & -- \\
 & & & & & & & 0
\end{array}$$

Since $\det(\lambda I_3 - A) = (\lambda+2)\left(\lambda^2+4\lambda+4\right) = (\lambda+2)^3$, it follows that $\lambda=-2$ is the only eigenvalue with algebraic multiplicity 3

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2VCNB79

For this eigenvalue, the coefficient matrix of the homogeneous system is $\begin{bmatrix} -4 & 4 & -3 \\ -1 & 0 & -1 \\ 4 & 0 & 4 \end{bmatrix}$, with r.r.e.f. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 \end{bmatrix}$. Eigenvectors are $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ \frac{-1}{4} \\ 1 \end{bmatrix}$ with $x_3 \neq 0$.

This eigenspace has dimension 1, therefore, it cannot yield 3 linearly independent eigenvectors required for diagonalization.

A is not diagonalizable.

b. Characteristic polynomial:

$$\det(\lambda I_4 - A) = \det \begin{bmatrix} \lambda & 2 & -1 & 1 \\ 0 & \lambda+1 & 0 & 0 \\ 0 & -1 & \lambda-2 & 2 \\ 0 & 0 & 0 & \lambda+2 \end{bmatrix}$$

Expand along the first column:

$$\det(\lambda I_4 - A) = (-1)^{1+1}\lambda \det \begin{bmatrix} \lambda+1 & 0 & 0 \\ -1 & \lambda-2 & 2 \\ 0 & 0 & \lambda+2 \end{bmatrix}$$

... now, expand along the first row

$$\det(\lambda I_4 - A) = \lambda(-1)^{1+1}(\lambda+1)\det \begin{bmatrix} \lambda-2 & 2 \\ 0 & \lambda+2 \end{bmatrix} = \lambda(\lambda+1)(\lambda-2)(\lambda+2)\,.$$

The matrix has four eigenvalues: 0, -1, 2, and -2 - each with algebraic multiplicity 1.

Refer to the
Linear Algebra
Toolkit for details:
http://bit.ly/2FkwbXd

For $\lambda = 0$, the coefficient matrix of the homogeneous system is $\begin{bmatrix} 0 & 2 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & -2 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ with r.r.e.f. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Eigenvectors are $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ with $x_1 \neq 0$ (e.g., $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$)

Refer to the
Linear Algebra
Toolkit for details:
http://bit.ly/2GZBBJL

For $\lambda = -1$, the coefficient matrix of the homogeneous system is $\begin{bmatrix} -1 & 2 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -3 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, with r.r.e.f. $\begin{bmatrix} 1 & 0 & 7 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Eigenvectors are $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -7 \\ -3 \\ 1 \\ 0 \end{bmatrix}$ with $x_3 \neq 0$ (e.g., $\begin{bmatrix} -7 \\ -3 \\ 1 \\ 0 \end{bmatrix}$).

Refer to the
Linear Algebra
Toolkit for details:
http://bit.ly/2shHKHa

For $\lambda = 2$, the coefficient matrix of the homogeneous system is $\begin{bmatrix} 2 & 2 & -1 & 1 \\ 0 & 3 & 0 & 0 \\ 0 & -1 & 0 & 2 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ with r.r.e.f. $\begin{bmatrix} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Eigenvectors are $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \\ 0 \end{bmatrix}$ with $x_3 \neq 0$ (e.g., $\begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}$)

Refer to the
Linear Algebra
Toolkit for details:
http://bit.ly/2LYzhl5

For $\lambda = -2$, the coefficient matrix of the homogeneous system is $\begin{bmatrix} -2 & 2 & -1 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, with r.r.e.f. $\begin{bmatrix} 1 & 0 & 0 & -\frac{1}{4} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Eigenvectors are $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} \frac{1}{4} \\ 0 \\ \frac{1}{2} \\ 1 \end{bmatrix}$ with $x_4 \neq 0$ (e.g., $\begin{bmatrix} 1 \\ 0 \\ 2 \\ 4 \end{bmatrix}$).

A is diagonalizable: $P^{-1}AP = D$ with $P = \begin{bmatrix} 1 & -7 & 1 & 1 \\ 0 & -3 & 0 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ and $D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$.

Verify: $AP = \begin{bmatrix} 0 & -2 & 1 & -1 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & -7 & 1 & 1 \\ 0 & -3 & 0 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 7 & 2 & -2 \\ 0 & 3 & 0 & 0 \\ 0 & -1 & 4 & -4 \\ 0 & 0 & 0 & -8 \end{bmatrix}$.

$$PD = \begin{bmatrix} 1 & -7 & 1 & 1 \\ 0 & -3 & 0 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 7 & 2 & -2 \\ 0 & 3 & 0 & 0 \\ 0 & -1 & 4 & -4 \\ 0 & 0 & 0 & -8 \end{bmatrix} \checkmark$$

5.a. The matrix is symmetric, therefore by Theorem 7.5 it is orthogonally diagonalizable.

Characteristic polynomial:

$$\det(\lambda I_2 - A) = \det \begin{bmatrix} \lambda+1 & 2 \\ 2 & \lambda-2 \end{bmatrix} = (\lambda+1)(\lambda-2) - (2)(2)$$

$$= \lambda^2 - \lambda - 6 = (\lambda+2)(\lambda-3).$$

Eigenvalues: $\lambda = -2$, $\lambda = 3$ (both have multiplicities 1)

Refer to the
Linear Algebra
Toolkit for details:
http://bit.ly/2Ca1xNa

For $\lambda = -2$, the homogeneous system has the coefficient matrix $\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix}$. Its r.r.e.f. is $\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$.

Solutions: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Basis for eigenspace: $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Orthonormal basis: $\frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Refer to the
Linear Algebra
Toolkit for details:
http://bit.ly/2GZ3MII

For $\lambda = 3$, the homogeneous system has the coefficient matrix $\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$, . Its r.r.e.f. is $\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$.

Solutions: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} \frac{-1}{2} \\ 1 \end{bmatrix}$. Basis for eigenspace: $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Orthonormal basis: $\frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

A can be orthogonally diagonalized, $Q^T A Q = D$ with the orthogonal matrix $Q = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$ and the diagonal matrix $D = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}$.

Check: $Q^T A Q = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} -1 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix} \stackrel{\checkmark}{=} D$

b. The matrix is symmetric, therefore by Theorem 7.5 it is orthogonally diagonalizable.

Characteristic polynomial (we use Formula (30)):

$$\det(\lambda I_3 - A) = \det \begin{bmatrix} \lambda & -1 & 1 \\ -1 & \lambda & -1 \\ 1 & -1 & \lambda \end{bmatrix}$$

$$= (\lambda)(\lambda)(\lambda) + (-1)(-1)(1) + (1)(-1)(-1) - (\lambda)(-1)(-1) - (-1)(-1)(\lambda) - (1)(\lambda)(1) =$$

$$= \lambda^3 - 3\lambda + 2$$

Test the factor of the free term, 2 ($1, -1, 2$, and -2)

$\det(1I_3 - A) = 1 - 3 + 2 = 0 \Rightarrow 1$ is an eigenvalue of A

Divide $\lambda^3 - 3\lambda + 2$ by $\lambda - 1$:

$$\begin{array}{rrrrrrrr}
 & \lambda^2 & + & \lambda & - & 2 & & \\
 & -- & - & -- & - & -- & - & -- \\
(\lambda - 1) \ \ | & \lambda^3 & & & - & 3\lambda & + & 2 \\
 & -\lambda^3 & + & \lambda^2 & & & & \\
 & -- & - & -- & - & -- & - & -- \\
 & & & \lambda^2 & - & 3\lambda & + & 2 \\
 & & - & \lambda^2 & + & \lambda & & \\
 & & & -- & - & -- & - & -- \\
 & & & & - & 2\lambda & + & 2 \\
 & & & & + & 2\lambda & - & 2 \\
 & & & & & -- & - & -- \\
 & & & & & & & 0
\end{array}$$

We have $\lambda^3 - 3\lambda + 2 = (\lambda - 1)\left(\lambda^2 + \lambda - 2\right) = (\lambda - 1)^2 (\lambda + 2)$ so that

- $\lambda = 1$ is an eigenvalue with algebraic multiplicity 2, and
- $\lambda = -2$ is an eigenvalue with algebraic multiplicity 1.

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2SEdnGo

For $\lambda = 1$ the homogeneous system has the coefficient matrix $\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$. Its reduced row echelon form is $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Solutions: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. To find an orthonormal basis for the eigenspace, use the Gram-Schmidt process:

$$\overrightarrow{v_1} = \overrightarrow{u_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\overrightarrow{v_2} = \overrightarrow{u_2} - \frac{\overrightarrow{u_2} \cdot \overrightarrow{v_1}}{\overrightarrow{v_1} \cdot \overrightarrow{v_1}} \overrightarrow{v_1} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{-1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}.$$

For simplicity, replace $\overrightarrow{v_2}$ with the vector twice as long, $\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$.

$$\overrightarrow{w_1} = \frac{1}{\|\overrightarrow{v_1}\|} \overrightarrow{v_1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

$$\overrightarrow{w_2} = \frac{1}{\|\overrightarrow{v_2}\|} \overrightarrow{v_2} = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}.$$

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2C9v4qh

For $\lambda = -2$, the homogeneous system has the coefficient matrix $\begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix}$ with r.r.e.f.

$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Solutions: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. Orthonormal basis for the eigenspace: $\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

A can be orthogonally diagonalized, $Q^T AQ = D$ with the orthogonal matrix $Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$

and the diagonal matrix $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$.

Check: $Q^T AQ = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \overset{\checkmark}{=} D.$

c. The matrix is symmetric, therefore by Theorem 7.5 it is orthogonally diagonalizable.

Characteristic polynomial:

$$\det(\lambda I_4 - A) = \det \begin{bmatrix} \lambda - 2 & 0 & 0 & 0 \\ 0 & \lambda - 1 & 1 & 1 \\ 0 & 1 & \lambda - 1 & 1 \\ 0 & 1 & 1 & \lambda - 1 \end{bmatrix}$$

expand along the first row then apply Formula (30) to evaluate the 3×3 determinant

$$= (-1)^{1+1} (\lambda - 2) \det \begin{bmatrix} \lambda - 1 & 1 & 1 \\ 1 & \lambda - 1 & 1 \\ 1 & 1 & \lambda - 1 \end{bmatrix}$$

$$= (\lambda - 2) \left[(\lambda - 1)(\lambda - 1)(\lambda - 1) + (1)(1)(1) + (1)(1)(1) \right.$$
$$\left. -(\lambda - 1)(1)(1) - (1)(1)(\lambda - 1) - (1)(\lambda - 1)(1) \right]$$

$$= (\lambda - 2) \left(\lambda^3 - 3\lambda^2 + 4 \right)$$

In $\lambda^3 - 3\lambda^2 + 4$, test the factors of the free term, 4 ($1, -1, 2, -2, 4$, and -4)

$\det(1I_4 - A) = 1 - 3 + 4 = 2 \neq 0 \Rightarrow 1$ is not an eigenvalue of A

$\det(-1I_4 - A) = -1 - 3 + 4 = 0 \Rightarrow -1$ is an eigenvalue of A

Divide $\lambda^3 - 3\lambda^2 + 4$ by $\lambda + 1$:

$$\begin{array}{rrrrrrrr}
 & \lambda^2 & - & 4\lambda & + & 4 & & \\
\cline{2-8}
(\lambda+1) \ | & \lambda^3 & - & 3\lambda^2 & & & + & 4 \\
 & -\lambda^3 & - & \lambda^2 & & & & \\
\cline{2-8}
 & & - & 4\lambda^2 & & & + & 4 \\
 & & + & 4\lambda^2 & + & 4\lambda & & \\
\cline{4-8}
 & & & & & 4\lambda & + & 4 \\
 & & & & - & 4\lambda & - & 4 \\
\cline{6-8}
 & & & & & & & 0
\end{array}$$

We have

$$\begin{aligned}
(\lambda-2)\left(\lambda^3 - 3\lambda^2 + 4\right) &= (\lambda-2)(\lambda+1)\left(\lambda^2 - 4\lambda + 4\right) \\
&= (\lambda+1)(\lambda-2)^3
\end{aligned}$$

so that

- $\lambda = -1$ is an eigenvalue with algebraic multiplicity 1, and
- $\lambda = 2$ is an eigenvalue with algebraic multiplicity 3.

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Linear Algebra
Toolkit for details:

http://bit.ly/2AvCrs3

For $\lambda = -1$ the homogeneous system has the coefficient matrix $\begin{bmatrix} -3 & 0 & 0 & 0 \\ 0 & -2 & 1 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 1 & 1 & -2 \end{bmatrix}$ Its reduced row echelon form is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Solutions: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. An orthonormal basis for the eigenspace is: $\frac{1}{\sqrt{3}} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

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http://bit.ly/2AA0DcQ

For $\lambda = 2$, the homogeneous system has the coefficient matrix $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$ with r.r.e.f. $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Solutions: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_3 - x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$.

To find an orthonormal basis for the eigenspace, use Gram-Schmidt process:

$$\overrightarrow{v_1} = \overrightarrow{u_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\overrightarrow{v_2} = \overrightarrow{u_2} - \frac{\overrightarrow{u_2}\cdot\overrightarrow{v_1}}{\overrightarrow{v_1}\cdot\overrightarrow{v_1}}\overrightarrow{v_1} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} - \frac{0}{1}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\overrightarrow{v_3} = \overrightarrow{u_3} - \frac{\overrightarrow{u_3}\cdot\overrightarrow{v_1}}{\overrightarrow{v_1}\cdot\overrightarrow{v_1}}\overrightarrow{v_1} - \frac{\overrightarrow{u_3}\cdot\overrightarrow{v_2}}{\overrightarrow{v_2}\cdot\overrightarrow{v_2}}\overrightarrow{v_2} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} - \frac{0}{1}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

For simplicity, we replace $\overrightarrow{v_3}$ with the vector that is twice as long: $\begin{bmatrix} 0 \\ -1 \\ -1 \\ 2 \end{bmatrix}$.

$$\overrightarrow{w_1} = \frac{1}{\|\overrightarrow{v_1}\|}\overrightarrow{v_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$\overrightarrow{w_2} = \frac{1}{\|\overrightarrow{v_2}\|}\overrightarrow{v_2} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}.$$

$$\overrightarrow{w_3} = \frac{1}{\|\overrightarrow{v_3}\|}\overrightarrow{v_3} = \frac{1}{\sqrt{6}}\begin{bmatrix} 0 \\ -1 \\ -1 \\ 2 \end{bmatrix}.$$

A can be orthogonally diagonalized, $Q^TAQ = D$ with the orthogonal matrix $Q = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{\sqrt{3}} & 0 & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & 0 & \frac{2}{\sqrt{6}} \end{bmatrix}$

and the diagonal matrix $D = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

Check: $Q^TAQ = \begin{bmatrix} 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 1 & 0 & 0 & 0 \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & -1 & 1 & -1 \\ 0 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{\sqrt{3}} & 0 & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & 0 & \frac{2}{\sqrt{6}} \end{bmatrix} =$

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \overset{\checkmark}{=} D.$$

d. The matrix is not symmetric, therefore by Theorem 7.5 it is not orthogonally diagonalizable.

7. a. Diagonalization yields $A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} \\ 1 & 1 \end{bmatrix}$

$$A^{11} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} (-1)^{11} & 0 \\ 0 & (2)^{11} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 682 & 683 \\ 1366 & 1365 \end{bmatrix}$$

b. Diagonalization yields

$$A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$$

$$A^{6} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1^6 & 0 \\ 0 & 4^6 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1366 & 2730 \\ 1365 & 2731 \end{bmatrix}$$

c. Diagonalization yields

$$A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$A^{10} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 0^{10} & 0 \\ 0 & (-2)^{10} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 512 & -512 \\ -512 & 512 \end{bmatrix}$$

11. FALSE

e.g., $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ has 2 linearly independent columns, but the eigenspace corresponding to its double (algebraic multiplicity 2) eigenvalue $\lambda = 1$ has dimension 1.

13. TRUE

Follows from the Spectral Theorem.

7.3 Applications of Eigenvalues and Eigenvectors

1. a. The matrix $A = \begin{bmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{bmatrix}$ is positive, therefore it is regular. A is stochastic, since each column has nonnegative entries adding up to one.

b.

n	0	1	2	3	4	5
$\begin{bmatrix} S^{(n)} \\ C^{(n)} \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0.9 \\ 0.1 \end{bmatrix}$	$\begin{bmatrix} 0.84 \\ 0.16 \end{bmatrix}$	$\begin{bmatrix} 0.804 \\ 0.196 \end{bmatrix}$	$\begin{bmatrix} 0.7824 \\ 0.2176 \end{bmatrix}$	$\begin{bmatrix} 0.76944 \\ 0.23056 \end{bmatrix}$

c.

n	0	1	2	3	4	5
$\begin{bmatrix} S^{(n)} \\ C^{(n)} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix}$	$\begin{bmatrix} 0.48 \\ 0.52 \end{bmatrix}$	$\begin{bmatrix} 0.588 \\ 0.412 \end{bmatrix}$	$\begin{bmatrix} 0.6528 \\ 0.3472 \end{bmatrix}$	$\begin{bmatrix} 0.69168 \\ 0.30832 \end{bmatrix}$

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http://bit.ly/2shXF8g

d. The matrix $I - A = \begin{bmatrix} 0.1 & -0.3 \\ -0.1 & 0.3 \end{bmatrix}$ has r.r.e.f. $\begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$ therefore the eigenspace corresponding to $\lambda_{\max} = 1$ is span$\{\begin{bmatrix} 3 \\ 1 \end{bmatrix}\}$. The stable vector is the eigenvector with entries adding up to 1: $\begin{bmatrix} 3/4 \\ 1/4 \end{bmatrix}$

Both sequences in b. and c. appear to approach this vector (although c. does so more slowly).

3. a. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. $A^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $A^3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, etc.

This matrix is not regular

b. $A = \begin{bmatrix} 0 & 2/5 & 1/4 \\ 2/3 & 0 & 3/4 \\ 1/3 & 3/5 & 0 \end{bmatrix}$; $A^2 = \begin{bmatrix} 0 & 2/5 & 1/4 \\ 2/3 & 0 & 3/4 \\ 1/3 & 3/5 & 0 \end{bmatrix}\begin{bmatrix} 0 & 2/5 & 1/4 \\ 2/3 & 0 & 3/4 \\ 1/3 & 3/5 & 0 \end{bmatrix} = \begin{bmatrix} \frac{7}{20} & \frac{3}{20} & \frac{3}{10} \\ \frac{1}{4} & \frac{43}{60} & \frac{1}{6} \\ \frac{2}{5} & \frac{2}{15} & \frac{8}{15} \end{bmatrix}$

A is regular

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http://bit.ly/2SKHko0

$I - A = \begin{bmatrix} 1 & -2/5 & -1/4 \\ -2/3 & 1 & -3/4 \\ -1/3 & -3/5 & 1 \end{bmatrix}$ has the r.r.e.f. $\begin{bmatrix} 1 & 0 & -\frac{3}{4} \\ 0 & 1 & -\frac{5}{4} \\ 0 & 0 & 0 \end{bmatrix}$ therefore the eigenspace corresponding to $\lambda_{\max} = 1$ is span$\{\begin{bmatrix} 3/4 \\ 5/4 \\ 1 \end{bmatrix}\}$. The stable vector $\overrightarrow{v}$ is the eigenvector whose entries add up to 1:

$$\overrightarrow{v} = \frac{1}{\frac{3}{4} + \frac{5}{4} + 1}\begin{bmatrix} 3/4 \\ 5/4 \\ 1 \end{bmatrix} = \frac{4}{12}\begin{bmatrix} 3/4 \\ 5/4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 5/12 \\ 1/3 \end{bmatrix}$$

5. a. $A = \begin{bmatrix} 0 & 1/3 & 0 & 0 \\ 1 & 0 & 1/3 & 1/3 \\ 0 & 1/3 & 0 & 2/3 \\ 0 & 1/3 & 2/3 & 0 \end{bmatrix}$;

$$A^2 = \begin{bmatrix} 0 & 1/3 & 0 & 0 \\ 1 & 0 & 1/3 & 1/3 \\ 0 & 1/3 & 0 & 2/3 \\ 0 & 1/3 & 2/3 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1/3 & 0 & 0 \\ 1 & 0 & 1/3 & 1/3 \\ 0 & 1/3 & 0 & 2/3 \\ 0 & 1/3 & 2/3 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & \frac{1}{9} & \frac{1}{9} \\ 0 & \frac{5}{9} & \frac{2}{9} & \frac{2}{9} \\ \frac{1}{3} & \frac{2}{9} & \frac{5}{9} & \frac{1}{9} \\ \frac{1}{3} & \frac{2}{9} & \frac{1}{9} & \frac{5}{9} \end{bmatrix}$$

$$A^3 = \begin{bmatrix} \frac{1}{3} & 0 & \frac{1}{9} & \frac{1}{9} \\ 0 & \frac{5}{9} & \frac{2}{9} & \frac{2}{9} \\ \frac{1}{3} & \frac{2}{9} & \frac{5}{9} & \frac{1}{9} \\ \frac{1}{3} & \frac{2}{9} & \frac{1}{9} & \frac{5}{9} \end{bmatrix} \begin{bmatrix} 0 & 1/3 & 0 & 0 \\ 1 & 0 & 1/3 & 1/3 \\ 0 & 1/3 & 0 & 2/3 \\ 0 & 1/3 & 2/3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{5}{27} & \frac{2}{27} & \frac{2}{27} \\ \frac{5}{9} & \frac{4}{27} & \frac{1}{3} & \frac{1}{3} \\ \frac{2}{9} & \frac{1}{3} & \frac{4}{27} & \frac{4}{9} \\ \frac{2}{9} & \frac{1}{3} & \frac{4}{9} & \frac{4}{27} \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 0 & \frac{5}{27} & \frac{2}{27} & \frac{2}{27} \\ \frac{5}{9} & \frac{4}{27} & \frac{1}{3} & \frac{1}{3} \\ \frac{2}{9} & \frac{1}{3} & \frac{4}{27} & \frac{4}{9} \\ \frac{2}{9} & \frac{1}{3} & \frac{4}{9} & \frac{4}{27} \end{bmatrix} \begin{bmatrix} 0 & 1/3 & 0 & 0 \\ 1 & 0 & 1/3 & 1/3 \\ 0 & 1/3 & 0 & 2/3 \\ 0 & 1/3 & 2/3 & 0 \end{bmatrix} = \begin{bmatrix} \frac{5}{27} & \frac{4}{81} & \frac{1}{9} & \frac{1}{9} \\ \frac{4}{27} & \frac{11}{27} & \frac{22}{81} & \frac{22}{81} \\ \frac{1}{3} & \frac{22}{81} & \frac{11}{27} & \frac{17}{81} \\ \frac{1}{3} & \frac{22}{81} & \frac{17}{81} & \frac{11}{27} \end{bmatrix}$$

Therefore A is regular.

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$I - A = \begin{bmatrix} 1 & -1/3 & 0 & 0 \\ -1 & 1 & -1/3 & -1/3 \\ 0 & -1/3 & 1 & -2/3 \\ 0 & -1/3 & -2/3 & 1 \end{bmatrix}$ has the r.r.e.f.: $\begin{bmatrix} 1 & 0 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. therefore the eigenspace corresponding to $\lambda_{\max} = 1$ is span$\{\begin{bmatrix} 1/3 \\ 1 \\ 1 \\ 1 \end{bmatrix}\}$. The stable vector $\overrightarrow{v}$ is the eigenvector whose entries add up to 1:

$$\overrightarrow{v} = \frac{1}{\frac{1}{3}+1+1+1} \begin{bmatrix} 1/3 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{3}{10} \begin{bmatrix} 1/3 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/10 \\ 3/10 \\ 3/10 \\ 3/10 \end{bmatrix}$$

b. $A = \begin{bmatrix} 0 & 1/3 & 1/3 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 2/3 & 2/3 & 0 \end{bmatrix}$;

$$A^2 = \begin{bmatrix} 0 & 1/3 & 1/3 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 2/3 & 2/3 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1/3 & 1/3 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 2/3 & 2/3 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{2}{3} & 0 & 0 & \frac{2}{3} \end{bmatrix}$$

$$A^3 = \begin{bmatrix} \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{2}{3} & 0 & 0 & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 0 & 1/3 & 1/3 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 2/3 & 2/3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{2}{3} & \frac{2}{3} & 0 \end{bmatrix} = A, \text{ etc.}$$

Therefore, A is not regular.

7. a. $A = \begin{bmatrix} 0 & 1 & 1 \\ 1/2 & 0 & 0 \\ 1/2 & 0 & 0 \end{bmatrix}$;

$$A^2 = \begin{bmatrix} 0 & 1 & 1 \\ 1/2 & 0 & 0 \\ 1/2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1/2 & 0 & 0 \\ 1/2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1/2 & 0 & 0 \\ 1/2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{bmatrix} = A, \text{ etc.}$$

A is not regular

b. $C = \frac{17}{20} \begin{bmatrix} 0 & 1 & 1 \\ 1/2 & 0 & 0 \\ 1/2 & 0 & 0 \end{bmatrix} + \frac{3}{20} \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} = \begin{bmatrix} \frac{1}{20} & \frac{9}{10} & \frac{9}{10} \\ \frac{19}{40} & \frac{1}{20} & \frac{1}{20} \\ \frac{19}{40} & \frac{1}{20} & \frac{1}{20} \end{bmatrix}$

the matrix $I - C = \begin{bmatrix} \frac{19}{20} & \frac{-9}{10} & \frac{-9}{10} \\ \frac{-19}{40} & \frac{19}{20} & \frac{-1}{20} \\ \frac{-19}{40} & \frac{-1}{20} & \frac{19}{20} \end{bmatrix}$ has the r.r.e.f. $\begin{bmatrix} 1 & 0 & -\frac{36}{19} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$. therefore the eigenspace corresponding to $\lambda_{\max} = 1$ is span$\left\{ \begin{bmatrix} 36/19 \\ 1 \\ 1 \end{bmatrix} \right\}$. The stable vector $\overrightarrow{v}$ is the eigenvector whose entries add up to 1:

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$$\overrightarrow{v} = \frac{1}{\frac{36}{19} + 1 + 1} \begin{bmatrix} 36/19 \\ 1 \\ 1 \end{bmatrix} = \frac{19}{74} \begin{bmatrix} 36/19 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 36/74 \\ 19/74 \\ 19/74 \end{bmatrix}$$

c. Page rank: page 1, followed by pages 2 and 3 (tied).

11. The matrix form (131) has $A = \begin{bmatrix} 2 & 6 \\ 3 & 5 \end{bmatrix}$ and $\overrightarrow{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The characteristic polynomial of A is

$\det(\lambda I - A) = (\lambda - 2)(\lambda - 5) - 18 = \lambda^2 - 7\lambda - 8 = (\lambda - 8)(\lambda + 1)$

The r.r.e.f. of $8I_2 - A = \begin{bmatrix} 6 & -6 \\ -3 & 3 \end{bmatrix}$ is $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ whereas the r.r.e.f. of $-1I_2 - A = \begin{bmatrix} -3 & -6 \\ -3 & -6 \end{bmatrix}$ equals $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$. Populating columns of P with eigenvectors we obtain $P = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$ which diagonalizes A :

$$P^{-1}AP = D = \begin{bmatrix} 8 & 0 \\ 0 & -1 \end{bmatrix}.$$

When substituting $\overrightarrow{y} = P^{-1}\overrightarrow{x}$ and $\overrightarrow{c} = P^{-1}\overrightarrow{b}$ we need to determine $\overrightarrow{c}$. We can do so by solving the system $P\overrightarrow{c} = \overrightarrow{b}$. Since the r.r.e.f. of $\left[P|\overrightarrow{b}\right] = \begin{bmatrix} 1 & 2 & 0 \\ 1 & -1 & 0 \end{bmatrix}$ is $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$, $\overrightarrow{c} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The system (133) now becomes

$$\begin{aligned} \frac{dy_1}{dt} &= 8y_1 \\ \frac{dy_2}{dt} &= -1y_2 \end{aligned}$$

and has a general solution

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} K_1 e^{8t} \\ K_2 e^{-t} \end{bmatrix}$$

so that the final solution is

$$\begin{aligned} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} K_1 e^{8t} \\ K_2 e^{-t} \end{bmatrix} \\ &= \begin{bmatrix} K_1 e^{8t} + 2K_2 e^{-t} \\ K_1 e^{8t} - K_2 e^{-t} \end{bmatrix} \end{aligned}$$

with arbitrary constants K_1 and K_2.

13. The matrix form (131) has $A=\begin{bmatrix}-3 & -3\\ -6 & 4\end{bmatrix}$ and $\overrightarrow{b}=\begin{bmatrix}-4\\ 1\end{bmatrix}$. The characteristic polynomial of A is

$\det(\lambda I-A)=(\lambda+3)(\lambda-4)-18=\lambda^2-\lambda-30=(\lambda+5)(\lambda-6)$

The r.r.e.f. of $-5I_2-A=\begin{bmatrix}-2 & 3\\ 6 & -9\end{bmatrix}$ is $\begin{bmatrix}1 & -3/2\\ 0 & 0\end{bmatrix}$ whereas the r.r.e.f. of $6I_2-A=\begin{bmatrix}9 & 3\\ 6 & 2\end{bmatrix}$ equals $\begin{bmatrix}1 & 1/3\\ 0 & 0\end{bmatrix}$. Populating columns of P with eigenvectors we obtain $P=\begin{bmatrix}3 & -1\\ 2 & 3\end{bmatrix}$ which diagonalizes A :

$$P^{-1}AP=D=\begin{bmatrix}-5 & 0\\ 0 & 6\end{bmatrix}.$$

When substituting $\overrightarrow{y}=P^{-1}\overrightarrow{x}$ and $\overrightarrow{c}=P^{-1}\overrightarrow{b}$ we need to determine $\overrightarrow{c}$. We can do so by solving the system $P\overrightarrow{c}=\overrightarrow{b}$. Since the r.r.e.f. of $\left[P|\overrightarrow{b}\right]=\begin{bmatrix}3 & -1 & -4\\ 2 & 3 & 1\end{bmatrix}$ is $\left[\begin{array}{cc|c}1 & 0 & -1\\ 0 & 1 & 1\end{array}\right]$, $\overrightarrow{c}=\begin{bmatrix}-1\\ 1\end{bmatrix}$. The system (133) now becomes

$$\begin{aligned}\frac{dy_1}{dt} &= -5y_1-1\\ \frac{dy_2}{dt} &= 6y_2+1\end{aligned}$$

and has a general solution

$$\begin{bmatrix}y_1\\ y_2\end{bmatrix}=\begin{bmatrix}K_1e^{-5t}-\frac{1}{5}\\ K_2e^{6t}-\frac{1}{6}\end{bmatrix}$$

so that the final solution is

$$\begin{aligned}\begin{bmatrix}x_1\\ x_2\end{bmatrix} &= \begin{bmatrix}3 & -1\\ 2 & 3\end{bmatrix}\begin{bmatrix}K_1e^{-5t}-\frac{1}{5}\\ K_2e^{6t}-\frac{1}{6}\end{bmatrix}\\ &= \begin{bmatrix}3K_1e^{-5t}-K_2e^{6t}-\frac{13}{30}\\ 2K_1e^{-5t}+3K_2e^{6t}-\frac{9}{10}\end{bmatrix}\end{aligned}$$

with arbitrary constants K_1 and K_2.

15. The matrix form (131) has $A=\begin{bmatrix}3 & 1 & 1\\ 1 & 3 & 1\\ 1 & 1 & 3\end{bmatrix}$ and $\overrightarrow{b}=\begin{bmatrix}2\\ 2\\ 5\end{bmatrix}$. The characteristic polynomial of A is

$\det(\lambda I-A)=\lambda^3-9\lambda^2+24\lambda-20=(\lambda-5)(\lambda-2)^2$

The r.r.e.f. of $5I_3-A=\begin{bmatrix}2 & -1 & -1\\ -1 & 2 & -1\\ -1 & -1 & 2\end{bmatrix}$ is $\begin{bmatrix}1 & 0 & -1\\ 0 & 1 & -1\\ 0 & 0 & 0\end{bmatrix}$ whereas the r.r.e.f. of $2I_3-A=\begin{bmatrix}-1 & -1 & -1\\ -1 & -1 & -1\\ -1 & -1 & -1\end{bmatrix}$ equals $\begin{bmatrix}1 & 1 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$. Populating columns of P with linearly independent eigenvectors we can, for instance, write $P=\begin{bmatrix}1 & -1 & -1\\ 1 & 1 & 0\\ 1 & 0 & 1\end{bmatrix}$ which diagonalizes A :

$$P^{-1}AP=D=\begin{bmatrix}5 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}.$$

When substituting $\overrightarrow{y}=P^{-1}\overrightarrow{x}$ and $\overrightarrow{c}=P^{-1}\overrightarrow{b}$ we need to determine $\overrightarrow{c}$. We can do so by solving the system $P\overrightarrow{c} = \overrightarrow{b}$. Since the r.r.e.f. of $\left[P|\overrightarrow{b}\right] = \begin{bmatrix} 1 & -1 & -1 & 2 \\ 1 & 1 & 0 & 2 \\ 1 & 0 & 1 & 5 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$,

$\overrightarrow{c} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$. The system (133) now becomes

$$\begin{aligned} \frac{dy_1}{dt} &= 5y_1 + 3 \\ \frac{dy_2}{dt} &= 2y_2 - 1 \\ \frac{dy_3}{dt} &= 2y_3 + 2 \end{aligned}$$

and has a general solution

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} K_1e^{5t} - \frac{3}{5} \\ K_2e^{2t} + \frac{1}{2} \\ K_3e^{2t} - 1 \end{bmatrix}$$

so that the final solution is

$$\begin{aligned} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} K_1e^{5t} - \frac{3}{5} \\ K_2e^{2t} + \frac{1}{2} \\ K_3e^{2t} - 1 \end{bmatrix} \\ &= \begin{bmatrix} K_1e^{5t} - K_2e^{2t} - K_3e^{2t} - \frac{1}{10} \\ K_1e^{5t} + K_2e^{2t} - \frac{1}{10} \\ K_1e^{5t} + K_3e^{2t} - \frac{8}{5} \end{bmatrix} \end{aligned}$$

with arbitrary constants K_1, K_2, and K_3.

(Note that the second and third column of P can contain any linearly independent eigenvectors corresponding to $\lambda = 2$. Choosing different vectors in these columns can result in the final solution being expressed in a different, but equivalent form. In particular, the solution in the text:
$x_1(t) = K_1e^{5t} + K_2e^{2t} - \frac{1}{10}$; $x_2(t) = K_1e^{5t} - K_2e^{2t} - K_3e^{2t} - \frac{1}{10}$; $x_3(t) = K_1e^{5t} + K_3e^{2t} - \frac{8}{5}$

corresponds to choosing $P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$ instead of the choice we made.)

17. The matrix form (131) has $A = \begin{bmatrix} 3 & -1 \\ -5 & 7 \end{bmatrix}$ and $\overrightarrow{b} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$. The characteristic polynomial of A is

$\det(\lambda I - A) = (\lambda - 3)(\lambda - 7) - 5 = \lambda^2 - 10\lambda + 16 = (\lambda - 8)(\lambda - 2)$

The r.r.e.f. of $8I_2 - A = \begin{bmatrix} 5 & 1 \\ 5 & 1 \end{bmatrix}$ is $\begin{bmatrix} 1 & 1/5 \\ 0 & 0 \end{bmatrix}$ whereas the r.r.e.f. of $2I_2 - A = \begin{bmatrix} -1 & 1 \\ 5 & -5 \end{bmatrix}$ equals $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$. Populating columns of P with eigenvectors we obtain $P = \begin{bmatrix} 1 & 1 \\ -5 & 1 \end{bmatrix}$ which diagonalizes A :

$$P^{-1}AP = D = \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix}.$$

When substituting $\overrightarrow{y} = P^{-1}\overrightarrow{x}$ and $\overrightarrow{c} = P^{-1}\overrightarrow{b}$ we need to determine $\overrightarrow{c}$. We can do so by solving the system $P\overrightarrow{c} = \overrightarrow{b}$. Since the r.r.e.f. of $\left[P|\overrightarrow{b}\right] = \begin{bmatrix} 1 & 1 & 0 \\ -5 & 1 & 6 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$, $\overrightarrow{c} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

The system (133) now becomes

$$\begin{aligned}\frac{dy_1}{dt} &= 8y_1 - 1\\ \frac{dy_2}{dt} &= 2y_2 + 1\end{aligned}$$

and has a general solution

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} K_1e^{8t} + \frac{1}{8} \\ K_2e^{2t} - \frac{1}{2} \end{bmatrix}$$

so that the general solution is

$$\begin{aligned}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ -5 & 1 \end{bmatrix}\begin{bmatrix} K_1e^{8t} + \frac{1}{8} \\ K_2e^{2t} - \frac{1}{2} \end{bmatrix}\\ &= \begin{bmatrix} K_1e^{8t} + K_2e^{2t} - \frac{3}{8} \\ -5K_1e^{8t} + K_2e^{2t} - \frac{9}{8} \end{bmatrix}.\end{aligned}$$

The values of constants K_1 and K_2 can now be determined by imposing the initial condition $x_1(0) = 0$, $x_2(0) = 1$ which leads to the system

$$\begin{aligned}K_1 + K_2 &= \frac{3}{8}\\ -5K_1 + K_2 &= \frac{17}{8}\end{aligned}$$

The augmented matrix $\begin{bmatrix} 1 & 1 & 3/8 \\ -5 & 1 & 17/8 \end{bmatrix}$ has the r.r.e.f. $\begin{bmatrix} 1 & 0 & -\frac{7}{24} \\ 0 & 1 & \frac{2}{3} \end{bmatrix}$ so that $K_1 = -\frac{7}{24}$, $K_2 = \frac{2}{3}$, and the final solution is

$$\begin{aligned}x_1(t) &= -\frac{7}{24}e^{8t} + \frac{2}{3}e^{2t} - \frac{3}{8}\\ x_2(t) &= \frac{35}{24}e^{8t} + \frac{2}{3}e^{2t} - \frac{9}{8}\end{aligned}$$

19. The matrix form (131) has $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ and $\overrightarrow{b} = \begin{bmatrix} -1 \\ 5 \\ -1 \end{bmatrix}$. The characteristic polynomial of A is

$\det(\lambda I - A) = \lambda^3 - 7\lambda^2 + 14\lambda - 8 = (\lambda - 4)(\lambda - 2)(\lambda - 1)$

The r.r.e.f. of $4I_3 - A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$, the r.r.e.f. of $2I_3 - A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, whereas the r.r.e.f. of $1I_3 - A = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$ equals $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$. Populating columns of P with eigenvectors we obtain $P = \begin{bmatrix} 1 & -1 & 1 \\ -2 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ which diagonalizes A :

$$P^{-1}AP = D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

When substituting $\overrightarrow{y} = P^{-1}\overrightarrow{x}$ and $\overrightarrow{c} = P^{-1}\overrightarrow{b}$ we need to determine $\overrightarrow{c}$. We can do so by solving the system $P\overrightarrow{c} = \overrightarrow{b}$. Since the r.r.e.f. of $\left[P|\overrightarrow{b}\right] = \begin{bmatrix} 1 & -1 & 1 & -1 \\ -2 & 0 & 1 & 5 \\ 1 & 1 & 1 & -1 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$,

$\overrightarrow{c} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$. The system (133) now becomes

$$\begin{aligned} \frac{dy_1}{dt} &= 4y_1 - 2 \\ \frac{dy_2}{dt} &= 2y_2 \\ \frac{dy_3}{dt} &= y_3 + 1 \end{aligned}$$

and has a general solution

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} K_1e^{4t} + \frac{1}{2} \\ K_2e^{2t} \\ K_3e^t - 1 \end{bmatrix}$$

so that the final solution is

$$\begin{aligned} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} 1 & -1 & 1 \\ -2 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} K_1e^{4t} + \frac{1}{2} \\ K_2e^{2t} \\ K_3e^t - 1 \end{bmatrix} \\ &= \begin{bmatrix} K_1e^{4t} - K_2e^{2t} + K_3e^t - \frac{1}{2} \\ -2K_1e^{4t} + K_3e^t - 2 \\ K_1e^{4t} + K_2e^{2t} + K_3e^t - \frac{1}{2} \end{bmatrix} \end{aligned}$$

The values of constants K_1, K_2, and K_3 can now be determined by imposing the initial condition $x_1(0) = 3;\ x_2(0) = -3;\ x_3(0) = 0$ which leads to the system

$$\begin{aligned} K_1 - K_2 + K_3 &= \frac{7}{2} \\ -2K_1 \qquad + K_3 &= -1 \\ K_1 + K_2 + K_3 &= \frac{1}{2} \end{aligned}$$

The augmented matrix $\begin{bmatrix} 1 & -1 & 1 & 7/2 \\ -2 & 0 & 1 & -1 \\ 1 & 1 & 1 & 1/2 \end{bmatrix}$ has the r.r.e.f. $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -\frac{3}{2} \\ 0 & 0 & 1 & 1 \end{bmatrix}$ so that $K_1 = 1$, $K_2 = -\frac{3}{2}$, $K_3 = 1$, and the final solution is

$$\begin{aligned} x_1(t) &= e^{4t} + \frac{3}{2}e^{2t} + e^t - \frac{1}{2} \\ x_2(t) &= -2e^{4t} + e^t - 2 \\ x_3(t) &= e^{4t} - \frac{3}{2}e^{2t} + e^t - \frac{1}{2} \end{aligned}$$

21 a. The following system of differential equations describes the situation:

rate of change of salt content in the 1st tank	$\frac{dx_1}{dt} = -\frac{x_1}{10}(20) + \frac{x_2}{10}(10)$	inflow of salt from 2nd tank minus the outflow
rate of change of salt content in the 2nd tank	$\frac{dx_2}{dt} = \frac{x_1}{10}(10) - \frac{x_2}{10}(20)$	inflow of salt from 1st tank minus the outflow

In the matrix form (131), this system has $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$ and $\overrightarrow{b} = \overrightarrow{0}$. The characteristic polynomial of A is

$\det(\lambda I - A) = (\lambda + 2)^2 - 1 = \lambda^2 + 4\lambda + 3 = (\lambda + 3)(\lambda + 1)$.

The r.r.e.f. of $-3I_2 - A = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$ is $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

The r.r.e.f. of $-1I_2 - A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ is . $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$.

Therefore, we have $P^{-1}AP = D$ with $P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix}$.

We substitute $\overrightarrow{y} = P^{-1}\overrightarrow{x}$ and let $\overrightarrow{c} = P^{-1}\overrightarrow{b} = \overrightarrow{0}$. The system (133) now becomes

$$\begin{aligned} \frac{dy_1}{dt} &= -3y_1 \\ \frac{dy_2}{dt} &= -y_2 \end{aligned}$$

and has a general solution

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} K_1e^{-3t} \\ K_2e^{-t} \end{bmatrix}$$

so that the general solution is

$$\begin{aligned} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} K_1e^{-3t} \\ K_2e^{-t} \end{bmatrix} \\ &= \begin{bmatrix} -K_1e^{-3t} + K_2e^{-t} \\ K_1e^{-3t} + K_2e^{-t} \end{bmatrix} \end{aligned}$$

The values of constants K_1, K_2 can now be determined by imposing the initial condition $x_1(0) = 2$, $x_2(0) = 1$ which leads to the system

$$\begin{aligned} -K_1 + K_2 &= 2 \\ K_1 + K_2 &= 1 \end{aligned}$$

The augmented matrix $\begin{bmatrix} -1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$ has the r.r.e.f. $\begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{3}{2} \end{bmatrix}$ so that $K_1 = -\frac{1}{2}$, $K_2 = \frac{3}{2}$, and the final solution is

$$\begin{aligned} x_1(t) &= \frac{1}{2}e^{-3t} + \frac{3}{2}e^{-t} \\ x_2(t) &= -\frac{1}{2}e^{-3t} + \frac{3}{2}e^{-t} \end{aligned}$$

b. The following system of differential equations describes the situation:

rate of change of salt content in the 1st tank	$\dfrac{dx_1}{dt} = -\dfrac{x_1}{10}(20)$	current salt concentration in the first tank times the (out)flow rate
rate of change of salt content in the 2nd tank	$\dfrac{dx_2}{dt} = \dfrac{x_1}{10}(10) - \dfrac{x_2}{10}(20) + \dfrac{x_3}{10}(10)$	inflow of salt minus the outflow
rate of change of salt content in the 3rd tank	$\dfrac{dx_3}{dt} = \dfrac{x_1}{10}(10) + \dfrac{x_2}{10}(10) - \dfrac{x_3}{10}(20)$	inflow of salt minus the outflow

In the matrix form (131), this system has $A = \begin{bmatrix} -2 & 0 & 0 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$ and $\overrightarrow{b} = \overrightarrow{0}$. The characteristic polynomial of A is

$\det(\lambda I - A) = \lambda^3 + 6\lambda^2 + 11\lambda + 6 = (\lambda + 3)(\lambda + 2)(\lambda + 1)$

The r.r.e.f. of $-3I_3 - A = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

The r.r.e.f. of $-2I_3 - A = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$.

The r.r.e.f. of $-1I_3 - A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$ is . $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$.

Therefore, we have $P^{-1}AP = D$ with $P = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

We substitute $\overrightarrow{y} = P^{-1}\overrightarrow{x}$ and let $\overrightarrow{c} = P^{-1}\overrightarrow{b} = \overrightarrow{0}$. The system (133) now becomes

$$\begin{aligned} \frac{dy_1}{dt} &= -3y_1 \\ \frac{dy_2}{dt} &= -2y_2 \\ \frac{dy_3}{dt} &= -y_3 \end{aligned}$$

and has a general solution

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} K_1e^{-3t} \\ K_2e^{-2t} \\ K_3e^{-t} \end{bmatrix}$$

so that the general solution is

$$\begin{aligned} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} &= \begin{bmatrix} 0 & -1 & 0 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} K_1e^{-3t} \\ K_2e^{-2t} \\ K_3e^{-t} \end{bmatrix} \\ &= \begin{bmatrix} -K_2e^{-2t} \\ -K_1e^{-3t} + K_2e^{-2t} + K_3e^{-t} \\ K_1e^{-3t} + K_2e^{-2t} + K_3e^{-t} \end{bmatrix} \end{aligned}$$

The values of constants K_1, K_2, K_3 can now be determined by imposing the initial condition $x_1(0) = 1$, $x_2(0) = 2, x_3(0) = 1$ which leads to the system

$$\begin{aligned} -K_2 &= 1 \\ -K_1 + K_2 + K_3 &= 2 \\ K_1 + K_2 + K_3 &= 1 \end{aligned}$$

The augmented matrix $\begin{bmatrix} 0 & -1 & 0 & 1 \\ -1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ has the r.r.e.f. $\begin{bmatrix} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & \frac{5}{2} \end{bmatrix}$ so that $K_1 = -\frac{1}{2}$, $K_2 = -1,\ K_3 = \frac{5}{2}$, and the final solution is

$$\begin{aligned} x_1(t) &= e^{-2t} \\ x_2(t) &= \frac{1}{2}e^{-3t} - e^{-2t} + \frac{5}{2}e^{-t} \\ x_3(t) &= -\frac{1}{2}e^{-3t} - e^{-2t} + \frac{5}{2}e^{-t} \end{aligned}$$

7.4 Singular Value Decomposition

1. Step 1.

$$A^TA = \begin{bmatrix} -3 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} -3 & -1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix}$$

The characteristic polynomial is $\det(\begin{bmatrix} \lambda-10 & -6 \\ -6 & \lambda-10 \end{bmatrix}) = \lambda^2 - 20\lambda + 64 = (\lambda-4)(\lambda-16)$

Since the eigenvalues are $\lambda_1 = 16$ and $\lambda_2 = 4$, the singular values of A are $\sigma_1 = 4$ and $\sigma_2 = 2$.

Refer to the Linear Algebra Toolkit for details: http://bit.ly/2QwUEud

For $\lambda = 16$, the system $(\lambda I - A^TA)\overrightarrow{x} = \overrightarrow{0}$ has coefficient matrix $\begin{bmatrix} 6 & -6 \\ -6 & 6 \end{bmatrix}$ with r.r.e.f. $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$.

The eigenspace is span$\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$. The first right singular vector $\overrightarrow{v_1} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$.

Refer to the Linear Algebra Toolkit for details: http://bit.ly/2QwUEud

For $\lambda = 4$, the system $(\lambda I - A^TA)\overrightarrow{x} = \overrightarrow{0}$ has coefficient matrix $\begin{bmatrix} -6 & -6 \\ -6 & -6 \end{bmatrix}$ with r.r.e.f. $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

The eigenspace is span$\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$. The second right singular vector $\overrightarrow{v_2} = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$.

We obtain $V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ and $\Sigma = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$.

Step 2.

Columns of $\widehat{U} = U$ are

$$\overrightarrow{u_1} = \frac{1}{\sigma_1}A\overrightarrow{v_1} = \frac{1}{4}\begin{bmatrix} -3 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} \frac{-4}{\sqrt{2}} \\ \frac{4}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\overrightarrow{u_2} = \frac{1}{\sigma_2}A\overrightarrow{v_2} = \frac{1}{2}\begin{bmatrix} -3 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \frac{2}{\sqrt{2}} \\ \frac{2}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

We found a following singular value decomposition of our matrix:

$$\underbrace{\begin{bmatrix} -3 & -1 \\ 1 & 3 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}}_{U}\underbrace{\begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}}_{\Sigma}\underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}}_{V^T}.$$

3. Step 1.

$$A^TA = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}$$

The characteristic polynomial is $\det(\begin{bmatrix} \lambda-9 & 0 \\ 0 & \lambda-1 \end{bmatrix}) = (\lambda-9)(\lambda-1)$

Since the eigenvalues are $\lambda_1 = 9$ and $\lambda_2 = 1$, the singular values of A are $\sigma_1 = 3$ and $\sigma_2 = 1$.

Refer to the Linear Algebra Toolkit for details: http://bit.ly/2QwUEud

For $\lambda = 9$, the system $(\lambda I - A^TA)\overrightarrow{x} = \overrightarrow{0}$ has coefficient matrix $\begin{bmatrix} 0 & 0 \\ 0 & 8 \end{bmatrix}$ with r.r.e.f. $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

The eigenspace is span$\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$. The first right singular vector $\overrightarrow{v_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2QwUEud

For $\lambda = 1$, the system $(\lambda I - A^T A)\overrightarrow{x} = \overrightarrow{0}$ has coefficient matrix $\begin{bmatrix} -8 & 0 \\ 0 & 0 \end{bmatrix}$ with r.r.e.f. $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

The eigenspace is span$\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$. The second right singular vector $\overrightarrow{v_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

We obtain $V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$.

Step 2.

Columns of $\widehat{U} = U$ are

$$\overrightarrow{u_1} = \frac{1}{\sigma_1} A\overrightarrow{v_1} = \frac{1}{3} \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\overrightarrow{u_2} = \frac{1}{\sigma_2} A\overrightarrow{v_2} = \frac{1}{1} \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

We found a following singular value decomposition of our matrix:

$$\underbrace{\begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{V^T}.$$

5. Step 1.

$$A^T A = \begin{bmatrix} -2 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} -2 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ -2 & 8 \end{bmatrix}$$

The characteristic polynomial is $\det(\begin{bmatrix} \lambda - 5 & 2 \\ 2 & \lambda - 8 \end{bmatrix}) = (\lambda - 5)(\lambda - 8) - 4 = \lambda^2 - 13\lambda + 36 = (\lambda - 9)(\lambda - 4)$.

Since the eigenvalues are $\lambda_1 = 9$ and $\lambda_2 = 4$, the singular values of A are $\sigma_1 = 3$ and $\sigma_2 = 2$.

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2QwUEud

For $\lambda = 9$, the system $(\lambda I - A^T A)\overrightarrow{x} = \overrightarrow{0}$ has coefficient matrix $\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$ with r.r.e.f. $\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$.

The eigenspace is span$\left\{ \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} \right\}$. The first right singular vector $\overrightarrow{v_1} = \begin{bmatrix} \frac{-1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$.

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2QwUEud

For $\lambda = 4$, the system $(\lambda I - A^T A)\overrightarrow{x} = \overrightarrow{0}$ has coefficient matrix $\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix}$ with r.r.e.f. $\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$

The eigenspace is span$\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$. The second right singular vector $\overrightarrow{v_2} = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$.

We obtain $V = \begin{bmatrix} \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$ and $\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$.

Step 2

Columns of $\widehat{U} = U$ are

$$\overrightarrow{u_1} = \frac{1}{\sigma_1} A\overrightarrow{v_1} = \frac{1}{3} \begin{bmatrix} -2 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix} = \frac{1}{3\sqrt{5}} \begin{bmatrix} -2 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \frac{1}{3\sqrt{5}} \begin{bmatrix} 6 \\ 3 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\overrightarrow{u_2} = \frac{1}{\sigma_2} A\overrightarrow{v_2} = \frac{1}{2} \begin{bmatrix} -2 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix} = \frac{1}{2\sqrt{5}} \begin{bmatrix} -2 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{2\sqrt{5}} \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

We found the following singular value decomposition of our matrix:

$$\underbrace{\begin{bmatrix} -2 & 2 \\ 1 & 2 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}}_{V^T}.$$

7. Step 1.

$$A^T A = \begin{bmatrix} 4 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 17 & 6 \\ 6 & 8 \end{bmatrix}$$

The characteristic polynomial is $\det(\begin{bmatrix} \lambda - 17 & -6 \\ -6 & \lambda - 8 \end{bmatrix}) = (\lambda-17)(\lambda-8) - 36 = \lambda^2 - 25\lambda + 100 = (\lambda - 20)(\lambda - 5)$.

Since the eigenvalues are $\lambda_1 = 20$ and $\lambda_2 = 5$, the singular values of A are $\sigma_1 = \sqrt{20} = 2\sqrt{5}$ and $\sigma_2 = \sqrt{5}$.

Refer to the
Linear Algebra
Toolkit for details:
http://bit.ly/2QwUEud

For $\lambda = 20$, the system $(\lambda I - A^T A)\overrightarrow{x} = \overrightarrow{0}$ has coefficient matrix $\begin{bmatrix} 3 & -6 \\ -6 & 12 \end{bmatrix}$ with r.r.e.f. $\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$.

The eigenspace is span$\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$. The first right singular vector $\overrightarrow{v_1} = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$.

Refer to the
Linear Algebra
Toolkit for details:
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For $\lambda = 5$, the system $(\lambda I - A^T A)\overrightarrow{x} = \overrightarrow{0}$ has coefficient matrix $\begin{bmatrix} -12 & -6 \\ -6 & -3 \end{bmatrix}$ with r.r.e.f. $\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$

The eigenspace is span$\left\{ \begin{bmatrix} \frac{-1}{2} \\ 1 \end{bmatrix} \right\}$. The second right singular vector $\overrightarrow{v_2} = \begin{bmatrix} \frac{-1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$.

We obtain $V = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$ and $\Sigma = \begin{bmatrix} 2\sqrt{5} & 0 \\ 0 & \sqrt{5} \end{bmatrix}$.

Step 2

Columns of $\widehat{U} = U$ are

$$\overrightarrow{u_1} = \frac{1}{\sigma_1} A\overrightarrow{v_1} = \frac{1}{2\sqrt{5}} \begin{bmatrix} 4 & 2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 4 & 2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 10 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\overrightarrow{u_2} = \frac{1}{\sigma_2} A\overrightarrow{v_2} = \frac{1}{\sqrt{5}} \begin{bmatrix} 4 & 2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 4 & 2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

We found the following singular value decomposition of our matrix:

$$\underbrace{\begin{bmatrix} 4 & 2 \\ -1 & 2 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 2\sqrt{5} & 0 \\ 0 & \sqrt{5} \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}}_{V^T}.$$

9. Step 1.

$$A^T A = \begin{bmatrix} -2 & 0 & -4 \\ 0 & 2 & -4 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & 2 \\ -4 & -4 \end{bmatrix} = \begin{bmatrix} 20 & 16 \\ 16 & 20 \end{bmatrix}$$

The characteristic polynomial is $\det(\begin{bmatrix} \lambda - 20 & -16 \\ -16 & \lambda - 20 \end{bmatrix}) = (\lambda - 20)^2 - 16^2 = (\lambda - 20 - 16)(\lambda - 20 + 16) = (\lambda - 36)(\lambda - 4)$.

Since the eigenvalues are $\lambda_1 = 36$ and $\lambda_2 = 4$, the singular values of A are $\sigma_1 = 6$.and $\sigma_2 = 2$.

Refer to the
Linear Algebra
Toolkit for details:
http://bit.ly/2QwUEud

For $\lambda = 36$, the system $(\lambda I - A^T A)\overrightarrow{x} = \overrightarrow{0}$ has coefficient matrix $\begin{bmatrix} 16 & -16 \\ -16 & 16 \end{bmatrix}$ with r.r.e.f. $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$. The eigenspace is span$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$. The corresponding right singular vector is $\overrightarrow{v_1} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$.

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Linear Algebra
Toolkit for details:
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For $\lambda = 4$, the system $(\lambda I - A^T A)\overrightarrow{x} = \overrightarrow{0}$ has coefficient matrix $\begin{bmatrix} -16 & -16 \\ -16 & -16 \end{bmatrix}$ with r.r.e.f. $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ The eigenspace is span$\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$. The second right singular vector $\overrightarrow{v_2} = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$.

We obtain $V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ and $\Sigma = \begin{bmatrix} 6 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$.

Step 2

$\hat{U}$ has two columns:

$$\overrightarrow{u_1} = \frac{1}{\sigma_1} A\overrightarrow{v_1} = \frac{1}{6} \begin{bmatrix} -2 & 0 \\ 0 & 2 \\ -4 & -4 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{6\sqrt{2}} \begin{bmatrix} -2 & 0 \\ 0 & 2 \\ -4 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{6\sqrt{2}} \begin{bmatrix} -2 \\ 2 \\ -8 \end{bmatrix} = \begin{bmatrix} \frac{-1}{3\sqrt{2}} \\ \frac{1}{3\sqrt{2}} \\ \frac{-4}{3\sqrt{2}} \end{bmatrix}$$

$$\overrightarrow{u_2} = \frac{1}{\sigma_2} A\overrightarrow{v_2} = \frac{1}{2} \begin{bmatrix} -2 & 0 \\ 0 & 2 \\ -4 & -4 \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{2\sqrt{2}} \begin{bmatrix} -2 & 0 \\ 0 & 2 \\ -4 & -4 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{1}{2\sqrt{2}} \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

Step 3

The remaining column of U, $\overrightarrow{u_3}$, should be a unit vector spanning the 1-dimensional space $(\text{span}\{\overrightarrow{u_1}, \overrightarrow{u_2}\})^\perp$

This space is also the solution space of the homogeneous system with coefficient matrix $\begin{bmatrix} \overrightarrow{u_1}^T \\ \overrightarrow{u_2}^T \end{bmatrix} = \begin{bmatrix} \frac{-1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{-4}{3\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$

After scaling the rows by their common denominators $\begin{bmatrix} -1 & 1 & -4 \\ 1 & 1 & 0 \end{bmatrix}$ we obtain r.r.e.f.: $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \end{bmatrix}$.

The solution space is

span$\left\{ \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \right\}$ Therefore, we can take $\overrightarrow{u_3} = \begin{bmatrix} \frac{-2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$.

We found the following singular value decomposition of our matrix:

$$\underbrace{\begin{bmatrix} -2 & 0 \\ 0 & 2 \\ -4 & -4 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} \frac{-1}{3\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{-2}{3} \\ \frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{2}{3} \\ \frac{-4}{3\sqrt{2}} & 0 & \frac{1}{3} \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 6 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}}_{V^T}.$$

13. Let us obtain the SVD for the transpose of $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & -2 \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & -2 \end{bmatrix}$.

Step 1.

$$AA^T = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 9 \end{bmatrix}$$

The characteristic polynomial is $\det(\begin{bmatrix} \lambda - 2 & 0 \\ 0 & \lambda - 9 \end{bmatrix}) = (\lambda - 9)(\lambda - 2)$.

Since the eigenvalues are $\lambda_1 = 9$ and $\lambda_2 = 2$, the singular values of A^T (and of A) are $\sigma_1 = 3$.and $\sigma_2 = \sqrt{2}$.

Refer to the Linear Algebra Toolkit for details: http://bit.ly/2QwUEud

For $\lambda = 9$, the system $(\lambda I - AA^T)\overrightarrow{x} = \overrightarrow{0}$ has coefficient matrix $\begin{bmatrix} 7 & 0 \\ 0 & 0 \end{bmatrix}$ with r.r.e.f. $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

The eigenspace is span$\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$. The corresponding right singular vector is $\overrightarrow{v_1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Refer to the Linear Algebra Toolkit for details: http://bit.ly/2QwUEud

For $\lambda = 2$, the system $(\lambda I - AA^T)\overrightarrow{x} = \overrightarrow{0}$ has coefficient matrix $\begin{bmatrix} 0 & 0 \\ 0 & -7 \end{bmatrix}$ with r.r.e.f. $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

The eigenspace is span$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$. The second right singular vector $\overrightarrow{v_2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

We obtain $V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}$.

Step 2

$\hat{U}$ has two columns:

$$\overrightarrow{u_1} = \frac{1}{\sigma_1} A^T \overrightarrow{v_1} = \frac{1}{3} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

$$\overrightarrow{u_2} = \frac{1}{\sigma_2} A \overrightarrow{v_2} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Step 3

The remaining column of U, $\overrightarrow{u_3}$, should be a unit vector spanning the 1-dimensional space $(\text{span}\{\overrightarrow{u_1}, \overrightarrow{u_2}\})^\perp$

This space is also the solution space of the homogeneous system with coefficient matrix $\begin{bmatrix} \overrightarrow{u_1}^T \\ \overrightarrow{u_2}^T \end{bmatrix} =$

$$\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

After scaling the rows by their common denominators $\begin{bmatrix} 2 & 1 & -2 \\ 1 & 0 & 1 \end{bmatrix}$ we obtain r.r.e.f.: $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -4 \end{bmatrix}$.

The solution space is

span$\left\{\begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}\right\}$ Therefore, we can take $\overrightarrow{u_3} = \begin{bmatrix} \frac{-1}{3\sqrt{2}} \\ \frac{4}{3\sqrt{2}} \\ \frac{1}{3\sqrt{2}} \end{bmatrix}$.

We found the following singular value decomposition of our matrix:

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & -2 \end{bmatrix}}_{A^T} = \underbrace{\begin{bmatrix} \frac{2}{3} & \frac{1}{2}\sqrt{2} & -\frac{1}{6}\sqrt{2} \\ \frac{1}{3} & 0 & \frac{2}{3}\sqrt{2} \\ -\frac{2}{3} & \frac{1}{2}\sqrt{2} & \frac{1}{6}\sqrt{2} \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 3 & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{V^T}.$$

Consequently, the original matrix has SVD:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{2}\sqrt{2} & 0 & \frac{1}{2}\sqrt{2} \\ -\frac{1}{6}\sqrt{2} & \frac{2}{3}\sqrt{2} & \frac{1}{6}\sqrt{2} \end{bmatrix}.$$

15. Let us obtain the SVD for the transpose of $A = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 2 & -1 & 0 \\ 1 & 0 & -1 & 2 \end{bmatrix}$, $A^T = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & -1 & -1 \\ -1 & 0 & 2 \end{bmatrix}$.

Step 1.

$$AA^T = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 2 & -1 & 0 \\ 1 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & -1 & -1 \\ -1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 2 & -2 \\ 2 & 6 & 2 \\ -2 & 2 & 6 \end{bmatrix}$$

The characteristic polynomial is $\det(\begin{bmatrix} \lambda-2 & -2 & 2 \\ -2 & \lambda-6 & -2 \\ 2 & -2 & \lambda-6 \end{bmatrix}) = \lambda^3 - 14\lambda^2 + 48\lambda$

$= \lambda(\lambda-6)(\lambda-8)$Since the eigenvalues are $\lambda_1 = 8$ and $\lambda_2 = 6$, the singular values of A^T (and of A) are $\sigma_1 = 2\sqrt{2}$.and $\sigma_2 = \sqrt{6}$.

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2QwUEud

For $\lambda = 8$, the system $(\lambda I - AA^T)\overrightarrow{x} = \overrightarrow{0}$ has coefficient matrix $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 2 & -2 \\ 2 & -2 & 2 \end{bmatrix}$,,

with r.r.e.f. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$. The eigenspace is span$\left\{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right\}$. The corresponding

right singular vector is $\overrightarrow{v_1} = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$.

Refer to the
Linear Algebra
Toolkit for details:

http://bit.ly/2QwUEud

For $\lambda = 6$, the system $(\lambda I - AA^T)\overrightarrow{x} = \overrightarrow{0}$ has coefficient matrix $\begin{bmatrix} 4 & -2 & 2 \\ -2 & 0 & -2 \\ 2 & -2 & 0 \end{bmatrix}$,

with r.r.e.f. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ The eigenspace is span$\left\{\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\right\}$. The second right singular vector $\overrightarrow{v_2} = \begin{bmatrix} -1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$.

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Linear Algebra
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For $\lambda = 0$, the system $(\lambda I - AA^T)\overrightarrow{x} = \overrightarrow{0}$ has coefficient matrix $\begin{bmatrix} -2 & -2 & 2 \\ -2 & -6 & -2 \\ 2 & -2 & -6 \end{bmatrix}$,

with r.r.e.f. $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ The eigenspace is span$\left\{\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}\right\}$. The second right

singular vector $\overrightarrow{v_2} = \begin{bmatrix} 2/\sqrt{6} \\ -1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$.

We obtain $V = \begin{bmatrix} 0 & -1/\sqrt{3} & 2/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & -1/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$ and $\Sigma = \begin{bmatrix} 2\sqrt{2} & 0 & 0 \\ 0 & \sqrt{6} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Step 2

$\hat{U}$ has two columns:

$$\overrightarrow{u_1} = \frac{1}{\sigma_1} A^T \overrightarrow{v_1} = \frac{1}{2\sqrt{2}} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & -1 & -1 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \frac{1}{2\sqrt{2}} \begin{bmatrix} 2/\sqrt{2} \\ 2/\sqrt{2} \\ -2/\sqrt{2} \\ 2/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$$

$$\overrightarrow{u_2} = \frac{1}{\sigma_2} A^T \overrightarrow{v_2} = \frac{1}{\sqrt{6}} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & -1 & -1 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} -1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} = \frac{1}{\sqrt{6}} \begin{bmatrix} 0 \\ -3/\sqrt{3} \\ 0 \\ 3/\sqrt{3} \end{bmatrix} = \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

Step 3

The remaining columns of U, $\overrightarrow{u_3}$ and $\overrightarrow{u_4}$ should be orthonormal vectors spanning the 2-dimensional space $(\text{span}\{\overrightarrow{u_1}, \overrightarrow{u_2}\})^\perp$

This space is also the solution space of the homogeneous system with coefficient matrix $\left[\overrightarrow{u_1}^T\right] = \begin{bmatrix} 1/2 & 1/2 & -1/2 & 1/2 \\ 0 & -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$

After scaling the rows by their common denominators $\begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix}$, we obtain reduced row echelon form: $\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 0 & -1 \end{bmatrix}$

.. Every solution satisfies

$$\begin{aligned} x_1 &= x_3 - 2x_4 \\ x_2 &= x_4 \\ x_3 &= x_3 \\ x_4 &= x_4 \end{aligned}$$

The solution space is

$\text{span}\left\{\begin{bmatrix}1\\0\\1\\0\end{bmatrix},\begin{bmatrix}-2\\1\\0\\1\end{bmatrix}\right\}$ Since the two vectors are not orthogonal, we apply Gram-Schmidt process.

$$\overrightarrow{w_3}=\begin{bmatrix}1\\0\\1\\0\end{bmatrix};$$

$$\overrightarrow{w_4}=\begin{bmatrix}-2\\1\\0\\1\end{bmatrix}-\frac{\begin{bmatrix}-2\\1\\0\\1\end{bmatrix}\cdot\begin{bmatrix}1\\0\\1\\0\end{bmatrix}}{\begin{bmatrix}1\\0\\1\\0\end{bmatrix}\cdot\begin{bmatrix}1\\0\\1\\0\end{bmatrix}}\begin{bmatrix}1\\0\\1\\0\end{bmatrix}=\begin{bmatrix}-1\\1\\1\\1\end{bmatrix}$$

Unit vectors in the directions of $\overrightarrow{w_3}$ and $\overrightarrow{w_4}$ are

$$\overrightarrow{u_3}=\begin{bmatrix}1/\sqrt{2}\\0\\1/\sqrt{2}\\0\end{bmatrix}\text{ and }\overrightarrow{u_4}=\begin{bmatrix}-1/2\\1/2\\1/2\\1/2\end{bmatrix}$$

We found the following singular value decomposition of A^T

$$\underbrace{\begin{bmatrix}0&1&1\\1&2&0\\0&-1&-1\\-1&0&2\end{bmatrix}}_{A^T}=\underbrace{\begin{bmatrix}1/2&0&1/\sqrt{2}&-1/2\\1/2&-1/\sqrt{2}&0&1/2\\-1/2&0&1/\sqrt{2}&1/2\\1/2&1/\sqrt{2}&0&1/2\end{bmatrix}}_{U}\underbrace{\begin{bmatrix}2\sqrt{2}&0&0\\0&\sqrt{6}&0\\0&0&0\\0&0&0\end{bmatrix}}_{\Sigma}\underbrace{\begin{bmatrix}0&\frac{1}{2}\sqrt{2}&\frac{1}{2}\sqrt{2}\\-\frac{1}{3}\sqrt{3}&-\frac{1}{3}\sqrt{3}&\frac{1}{3}\sqrt{3}\\\frac{1}{3}\sqrt{6}&-\frac{1}{6}\sqrt{6}&\frac{1}{6}\sqrt{6}\end{bmatrix}}_{V^T}.$$

Consequently, the original matrix has SVD:

$$A=\begin{bmatrix}0&1&0&-1\\1&2&-1&0\\1&0&-1&2\end{bmatrix}=$$

$$\begin{bmatrix}0&-1/\sqrt{3}&2/\sqrt{6}\\1/\sqrt{2}&-1/\sqrt{3}&-1/\sqrt{6}\\1/\sqrt{2}&1/\sqrt{3}&1/\sqrt{6}\end{bmatrix}\begin{bmatrix}2\sqrt{2}&0&0&0\\0&\sqrt{6}&0&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}\frac{1}{2}&\frac{1}{2}&-\frac{1}{2}&\frac{1}{2}\\0&-\frac{1}{2}\sqrt{2}&0&\frac{1}{2}\sqrt{2}\\\frac{1}{2}\sqrt{2}&0&\frac{1}{2}\sqrt{2}&0\\-\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&\frac{1}{2}\end{bmatrix}$$

17. In Exercise 13, we found an SVD of $A=\begin{bmatrix}1&0&1\\2&1&-2\end{bmatrix}$

$$A=\begin{bmatrix}1&0&1\\2&1&-2\end{bmatrix}=\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}3&0&0\\0&\sqrt{2}&0\end{bmatrix}\begin{bmatrix}\frac{2}{3}&\frac{1}{3}&-\frac{2}{3}\\\frac{1}{2}\sqrt{2}&0&\frac{1}{2}\sqrt{2}\\-\frac{1}{6}\sqrt{2}&\frac{2}{3}\sqrt{2}&\frac{1}{6}\sqrt{2}\end{bmatrix}.$$

The pseudoinverse of A is

$$\begin{aligned} A^{+} &= V\Sigma^{+}U^{T} \\ &= \begin{bmatrix} \frac{2}{3} & \frac{1}{2}\sqrt{2} & -\frac{1}{6}\sqrt{2} \\ \frac{1}{3} & 0 & \frac{2}{3}\sqrt{2} \\ -\frac{2}{3} & \frac{1}{2}\sqrt{2} & \frac{1}{6}\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/3 & 0 \\ 0 & 1/\sqrt{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} & \frac{2}{9} \\ 0 & \frac{1}{9} \\ \frac{1}{2} & -\frac{2}{9} \end{bmatrix} \end{aligned}$$

19. a. $A^{+} = V\Sigma^{+}U^{T}$

$$= \begin{bmatrix} 1/2 & 0 & 1/\sqrt{2} & -1/2 \\ 1/2 & -1/\sqrt{2} & 0 & 1/2 \\ -1/2 & 0 & 1/\sqrt{2} & 1/2 \\ 1/2 & 1/\sqrt{2} & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1/(2\sqrt{2}) & 0 & 0 \\ 0 & 1/\sqrt{6} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ -\frac{1}{3}\sqrt{3} & -\frac{1}{3}\sqrt{3} & \frac{1}{3}\sqrt{3} \\ \frac{1}{3}\sqrt{6} & -\frac{1}{6}\sqrt{6} & \frac{1}{6}\sqrt{6} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{6} & \frac{7}{24} & -\frac{1}{24} \\ 0 & -\frac{1}{8} & -\frac{1}{8} \\ -\frac{1}{6} & -\frac{1}{24} & \frac{7}{24} \end{bmatrix}$$

b. $A^{+}\overrightarrow{b} = \begin{bmatrix} 0 & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{6} & \frac{7}{24} & -\frac{1}{24} \\ 0 & -\frac{1}{8} & -\frac{1}{8} \\ -\frac{1}{6} & -\frac{1}{24} & \frac{7}{24} \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{8} \\ \frac{7}{24} \\ -\frac{1}{8} \\ -\frac{1}{24} \end{bmatrix}$

c. Corresponds to the general solution with $x_3 = -\frac{1}{8}, x_4 = -\frac{1}{24}$,

$$\begin{aligned} x_1 &= \frac{1}{6} - \frac{1}{8} + \frac{2}{24} = \frac{1}{8} \\ x_2 &= \frac{1}{3} - \frac{1}{24} = \frac{7}{24} \end{aligned}$$