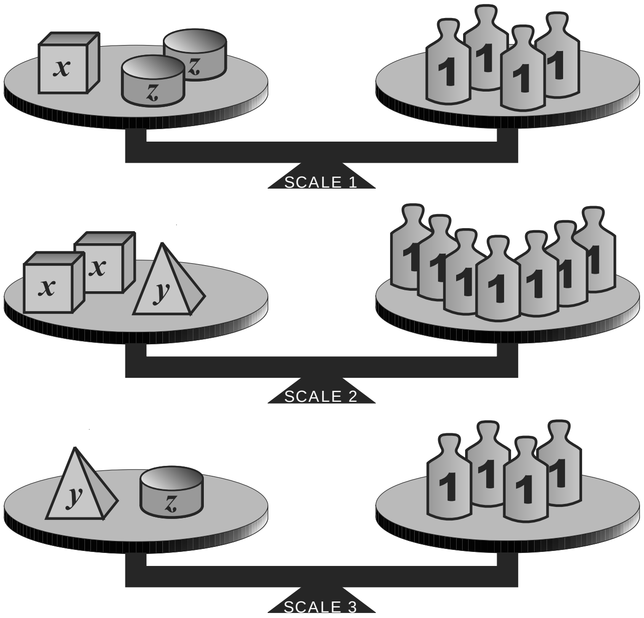
2 Linear Systems

In high school algebra, systems of two or three linear equations in two or three unknowns are frequently discussed, along with methods for solving them, e.g., substitution, or elimination by subtraction. Such systems often arise in real life situations, but they may involve more equations and/or unknowns. In this chapter, we will introduce methods for solving such systems (regardless of their size). Matrices and vectors, which were studied in the first chapter, will play a prominent role here as well.

Consider a scenario involving three kinds of containers, shaped as cubes, prisms, and cylinders. All cubes weigh exactly the same; the same can be said about all prisms and all cylinders, but their respective weights are unknown. The only information available to us is that each of the following three scales is in the state of equilibrium.

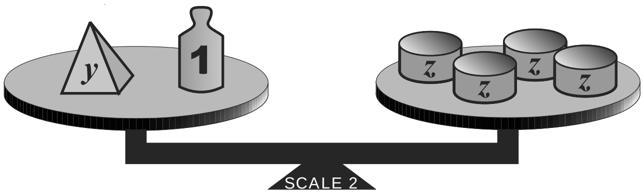


$$x + 2z = 4$$

$$2x + y = 7$$

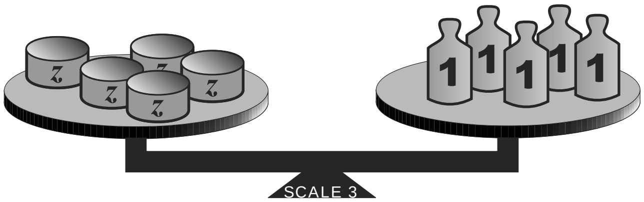
$$y + z = 4$$

Let us eliminate the cubes from the second scale, incorporating the information from the first scale. For each cube we remove from the left side of scale 2, we must also remove two cylinders from the left and four 1kg weights from the right. Since we are removing two cubes, we must remove twice as much: four cylinders from the left and eight 1kg weights from the right. Note, however, that there are no cylinders to remove on the left side – instead, to keep the scale balanced, we can add the four cylinders to the other side. Likewise, after removing the seven 1kg weights from the right, the eighth one is added on the other side. Here is the result on the second scale (the other two are unchanged):



$y - 4z = -1$

We can now proceed to use the current state of scale 2 to remove the prism from scale 3. To do so, along with removing the prism from the left side, we must remove a single 1kg weight from the left side (or, equivalently, add it on the right side) and remove four cylinders from the right side (or add them on the other side). The result:



$5z = 5$

The third scale now contains sufficient information to determine the weight of a cylinder: since five of them weigh 5kg, one must weigh 1kg.

Using this information in conjunction with the equilibrium contained by the second scale, we can see that a prism must weigh 3kg. Finally, the first scale helps us determine that a cube weighs 2kg.

We have managed to solve the system of equations

$$\begin{aligned} x + 2z &= 4 \\ 2x + y &= 7 \\ y + z &= 4, \end{aligned}$$

obtaining the solution

$$\begin{aligned} x &= 2 \\ y &= 3 \\ z &= 1. \end{aligned}$$

In this chapter, we shall develop procedures for solving similar systems, referring directly to the algebraic notation.

2.1 Systems of Linear Equations

DEFINITION (Linear System)

A *system* of m linear equations in n unknowns (or *linear system*) can be written as

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array} \tag{18}$$

with real numbers a_{ij} (the coefficient in the ith equation associated with the jth unknown) and b_i (the right-hand side value in the ith equation). The n-vector $\begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$ is called a *solution* of the linear system if substituting $x_1 = s_1, \ldots, x_n = s_n$ into all equations results in each equation becoming a true equality. The set containing all such solutions is called the *solution set* of the system.

Let us begin by illustrating one of the methods for solving linear systems you are likely to have encountered in your study of algebra.

EXAMPLE 2.1 To solve the linear system

$$\begin{array}{rcl} 2x + y & = & 5 \\ -4x + 3y & = & 5 \end{array}$$

using the method of elimination by addition (or subtraction), we want to multiply one or both equations (on both sides) by numbers such that when the resulting equations are added (or subtracted), one of the unknowns is eliminated. One way to accomplish this in our example is to multiply the first equation by 2 :

$$\begin{array}{rcl} 4x + 2y & = & 10 \\ -4x + 3y & = & 5. \end{array}$$

Adding the two equations results in

$$5y = 15;$$

therefore, $y = 3$ and, substituting into either one of the original equations, $x = 1$.

Check your answer!

Solving systems of equations (especially larger ones) might become quite involved; however, verifying the legitimacy of the solution we obtained is often quite easy: simply substitute the solution into the left-hand sides of the original system and check that they agree with the corresponding right-hand sides:

$$\begin{array}{rcl} 2(1) + 3 & = & 2 + 3 \overset{\checkmark}{=} 5 \\ -4(1) + 3(3) & = & -4 + 9 \overset{\checkmark}{=} 5 \end{array}$$

The main objective of this section will be to develop a systematic method for solving linear systems whose number of equations and/or unknowns can be considerably larger than 2 or 3. However, writing such large systems in their full form can be time- and space-consuming; therefore, we will first introduce a more compact way to represent linear systems.

Matrix representation of linear systems

Consider an $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \tag{19}$$

and an n-vector (or $n \times 1$ matrix)

$$\overrightarrow{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

The product $A\overrightarrow{x}$ is an m-vector (or $m \times 1$ matrix)

$$A\overrightarrow{x} = \begin{bmatrix} \text{row}_1 A \cdot \overrightarrow{x} \\ \text{row}_2 A \cdot \overrightarrow{x} \\ \vdots \\ \text{row}_m A \cdot \overrightarrow{x} \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}.$$

Setting this vector equal to the m-vector

$$\overrightarrow{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

we obtain a vector equation

$$A\overrightarrow{x} = \overrightarrow{b},$$

which is equivalent to the linear system (18).

The matrix A is called the *coefficient matrix* of the linear system. We shall also introduce the *augmented matrix* of the linear system, defined to be the following $m \times (n+1)$ matrix:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right].$$

EXAMPLE 2.2 The linear system

$$\begin{array}{rcrcrcr} 9x & & & - & z & = & 5 \\ & - & 2y & + & 2z & = & 0 \end{array}$$

can be represented using the coefficient matrix $A = \begin{bmatrix} 9 & 0 & -1 \\ 0 & -2 & 2 \end{bmatrix}$, the unknown vector $\overrightarrow{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, and the right-hand side vector $\overrightarrow{b} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$ in the matrix form

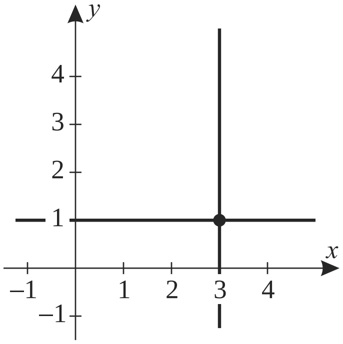
$$A\overrightarrow{x} = \overrightarrow{b}.$$

Its augmented matrix is

$$\left[\begin{array}{rrr|r} 9 & 0 & -1 & 5 \\ 0 & -2 & 2 & 0 \end{array}\right].$$

The augmented matrix can be considered as a data structure representing a linear system; of the three ingredients of the system $A\overrightarrow{x} = \overrightarrow{b}$, this matrix only omits the unknown vector, which contains just the names of the unknowns. These names are inconsequential to the solution itself.

Reduced row echelon form



EXAMPLE 2.3 Consider the system

$$\begin{array}{rcrcr} x & & & = & 3 \\ & & y & = & 1 \end{array}$$

If all linear systems were this easy to solve, this book would probably be a lot shorter!

Here is another "nice" system:

EXAMPLE 2.4

$$\begin{array}{rrrcrcr} x & & & + & 2w & = & 1 \\ & y & & - & w & = & 0 \\ & & z & + & 3w & = & -2 \end{array}$$

You should have no trouble seeing that this system possesses infinitely many solutions: letting w be an arbitrary number, we can calculate $x = 1 - 2w$, $y = w$, and $z = -2 - 3w$.

Both these systems have augmented matrices

$$\left[\begin{array}{rr|r} 1 & 0 & 3 \\ 0 & 1 & 1 \end{array}\right] \text{ and } \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 3 & -2 \end{array}\right]$$

that adhere to the conditions listed in the following definition.

DEFINITION (Reduced Row Echelon Form)

A matrix A is said to be in *reduced row echelon form* (r.r.e.f.) if it satisfies the following conditions:

1. If there are any zero rows in A, these are positioned below all other (nonzero) rows.
2. Every nonzero row of A must have its first nonzero entry equal to 1. It is called the *leading entry* of that row.
3. For any two nonzero rows of A, the leading entry of the row below is located to the right of the leading entry of the row above. (We can describe this by saying that the leading entries form a "staircase pattern".)
4. In any column that contains a leading entry, all remaining entries must equal zero.

If a matrix satisfies conditions 1–3, then we say it is in *row echelon form* (r.e.f.).

A column that contains a leading entry is called a *leading column*. Likewise, a column without a leading entry is referred to as a *nonleading column*.

EXAMPLE 2.5 Let us consider the following matrices:

$$A=\begin{bmatrix}1&0&1\\0&1&0\\0&0&0\end{bmatrix}, B=\begin{bmatrix}1&0\\0&0\\0&1\end{bmatrix}, C=\begin{bmatrix}1&2&3\\0&0&1\end{bmatrix}, D=\begin{bmatrix}1&0&0&0\\0&0&1&5\\0&0&0&0\end{bmatrix},$$

$$E=\begin{bmatrix}0&0&1&100\end{bmatrix}, F=\begin{bmatrix}1&0&0&0\\0&2&0&0\\0&0&1&0\end{bmatrix}, G=\begin{bmatrix}0&0\\0&0\end{bmatrix}, H=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}.$$

- The matrices A, D, E, and G satisfy all four conditions above. These matrices are in reduced row echelon form (and, also, in row echelon form).
- The matrix C satisfies conditions 1–3, but the entry $c_{13} = 3 \neq 0$ violates condition 4. Consequently, this matrix is in row echelon form but not in the reduced row echelon form.
- The remaining three matrices are neither in reduced row echelon form nor in row echelon form:
  - B violates condition 1 (its second row – zero row – should not be positioned above a nonzero row);
  - F violates condition 2 (the first nonzero entry in the second row is not 1);
  - H violates condition 3 (leading entries in first two rows do not form a staircase pattern).

Here is another example of a matrix satisfying all four conditions.

EXAMPLE 2.6 The matrix in reduced row echelon form: $\left[\begin{array}{cc|c}1&0&0\\0&1&0\\0&0&1\\0&0&0\end{array}\right]$ is the augmented matrix of the linear system

$$\begin{aligned} x \qquad &= 0 \\ y &= 0 \\ 0 &= 1 \\ 0 &= 0 \end{aligned}$$

which has no solution because of the contradiction $0 = 1$ contained in the third equation.

An augmented matrix in reduced row echelon form immediately reveals all information about the solutions of the underlying system.

THEOREM 2.1 Suppose the linear system $A\vec{x} = \vec{b}$ has the augmented matrix $[A|\vec{b}\,]$ in reduced row echelon form.

(a) If $[A|\vec{b}\,]$ contains a row $[0 \cdots 0 \mid 1]$, then the system is *inconsistent* (has **no solutions**).

(b) If $[A|\vec{b}\,]$ does not contain a row $[0 \cdots 0 \mid 1]$, then the system is *consistent* (has at least one solution). In this case:

(b1) If every column of A contains a leading entry, then the system has a **unique solution**.
(b2) If one or more columns of A do not contain leading entries, then the system has **infinitely many solutions**.

PROOF

Part (a) is true since the row $[0 \cdots 0 \mid 1]$ corresponds to the equation $0 = 1$. Because this equation has no solution, any system that contains it cannot have a solution either.

(b) If $[A|\vec{b}\,]$ does not contain the row $[0 \cdots 0 \mid 1]$, then it follows that $\vec{b}$ cannot contain a leading entry – instead, each nonzero row of $[A|\vec{b}\,]$ must contain a leading entry in A. Let k denote the number of nonzero rows of $[A|\vec{b}\,]$, let the k leading columns of A be numbered $i_1, \ldots, i_k$, and let the $n-k$ nonleading columns be numbered $j_1, \ldots, j_{n-k}$. The pth leading column is the pth column of I_n; i.e., $\mathrm{col}_{i_p} A = \vec{e_p}$. Therefore, the equation $A\vec{x} = \vec{b}$ is equivalent to

$$\begin{matrix} \\ \\ \\ \\ \begin{matrix} m-k \\ \text{zeros} \end{matrix}\left\{\begin{matrix} \\ \\ \\ \end{matrix}\right. \end{matrix} \begin{bmatrix} x_{i_1} \\ x_{i_2} \\ \vdots \\ x_{i_k} \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \\ 0 \\ \vdots \\ 0 \end{bmatrix} - x_{j_1} \overbrace{\begin{bmatrix} a_{1,j_1} \\ a_{2,j_1} \\ \vdots \\ a_{k,j_1} \\ 0 \\ \vdots \\ 0 \end{bmatrix}}^{\mathrm{col}_{j_1} A} - \cdots - x_{j_{n-k}} \overbrace{\begin{bmatrix} a_{1,j_{n-k}} \\ a_{2,j_{n-k}} \\ \vdots \\ a_{k,j_{n-k}} \\ 0 \\ \vdots \\ 0 \end{bmatrix}}^{\mathrm{col}_{j_{n-k}} A} . \tag{20}$$

(b1) If every column of A has a leading entry, then $k = n$ and the system becomes $x_1 = b_1$, $x_2 = b_2$, ..., $x_n = b_n$, explicitly specifying the unique solution. Each unknown is specified in exactly one equation, guaranteeing the solution exists and is unique.

(b2) If one or more columns of A do not contain leading entries, then $k < n$ so that $n - k\,(> 0)$ unknowns $x_{j_1}, \ldots, x_{j_{n-k}}$ are arbitrary. The remaining k leading columns correspond to unknowns $x_{i_1}, \ldots, x_{i_k}$ whose values can be determined using formula (20). ■

When an unknown corresponds to a leading column, we shall refer to it as a leading unknown. Note that for consistent systems (b1) and (b2) with augmented matrix in reduced row echelon form (r.r.e.f.), each leading unknown is specified using the information contained in that leading entry's row of the augmented matrix.

Row equivalence

While it is clear that linear systems with augmented matrices in reduced row echelon form are very easy to solve, you are probably beginning to wonder just how likely we are to continue to run into such systems throughout our study of linear algebra. Well, the bad news is that the answer is "not very likely." However, the good news is that it is possible to transform any linear system into an equivalent one (i.e., with the same solution set) whose augmented matrix is in r.r.e.f. by performing a sequence of the following operations on the augmented matrix (in these operations, we often denote the ith row by r_i):

DEFINITION (Elementary Row Operations)

Operations of the following three types are called *elementary row operations*:

1. add a multiple of one row to another row:
$$r_i + kr_j \to r_i \text{ where } i \neq j;$$
2. multiply a row by a nonzero number:
$$kr_i \to r_i \text{ where } k \neq 0;$$
3. interchange two rows:
$$r_i \leftrightarrow r_j.$$

If a matrix B is obtained from a matrix A by a finite sequence of elementary row operations, then we say that A is *row equivalent* to B.

The reader may notice that elementary row operations appear to be akin to the elimination approach used to solve smaller systems (e.g., in Example 2.1). Indeed, **performing any elementary row operation of one of these three types will never alter the solution set of the system** (we shall completely justify this statement in the next section). Therefore, we arrive at the following procedure to solve a linear system:

- Transform the system's augmented matrix, by a finite sequence of elementary row operations, to a reduced row echelon form.
- Obtain the solution set, as discussed in the previous subsection.

Let us illustrate this procedure in the following annotated example.

EXAMPLE 2.7 Solve the linear system

$$\begin{array}{rcrcrcrcr}
x & & & + & 2z & + & 3w & = & 5 \\
2x & & & + & 4z & & & = & 4 \\
 & & y & + & z & + & w & = & 4 \\
x & + & 3y & + & 5z & + & 2w & = & 13
\end{array}$$

We begin by forming the augmented matrix of the system:

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 5 \\ 2 & 0 & 4 & 0 & 4 \\ 0 & 1 & 1 & 1 & 4 \\ 1 & 3 & 5 & 2 & 13 \end{array}\right]$$

At any point during this procedure, one column will be distinguished as the pivotal column.

During the first part of the procedure, the location of the pivotal column will move from left to right (1st, 2nd, ..., 5th column).

In each pivotal column, a single entry called the *pivot* is selected according to the following pivoting strategy:

1. Each pivot must contain 1.
2. The first pivot must be in the first row.
3. Each subsequent pivot must reside in the row immediately following the row of the previously selected pivot.
4. If the desired pivot entry and all entries below it contain 0, the column must be skipped – it will not be a pivotal column.
5. If the desired pivot entry contains 0, but there is at least one nonzero entry below it, then the two rows should be interchanged.
6. If the desired pivot entry contains a nonzero value not equal to 1, then the row should be multiplied by the reciprocal of that value.

During the first part, all entries below each pivot must be eliminated.

During the second part, the same columns will be revisited in reverse order (some columns may be skipped, as shown below), and the pivots chosen during the first part are used to eliminate entries above them.

At the end, the matrix is in reduced row echelon form, and each pivot becomes its leading entry.

In the discussion below we will use a rectangle and an oval to designate the pivot and the entries to be eliminated, respectively:

pivot

$$\left[\begin{array}{cccc|c} \boxed{1} & 0 & 2 & 3 & 5 \\ 2 & 0 & 4 & 0 & 4 \\ 0 & 1 & 1 & 1 & 4 \\ 1 & 3 & 5 & 2 & 13 \end{array}\right]$$

entries to be eliminated

Column 1 ⇛ At the beginning, the first column is the pivotal column, its first entry (1) is the pivot, and the three entries below it (2, 0, and 1) are to be made into zeros. This requires the following two row operations (rather than three, since the (3,1) entry is already 0):

Step 1.
$r_2 - 2r_1 \to r_2$

$$\left[\begin{array}{cccc|c} \boxed{1} & 0 & 2 & 3 & 5 \\ 2 & 0 & 4 & 0 & 4 \\ 0 & 1 & 1 & 1 & 4 \\ 1 & 3 & 5 & 2 & 13 \end{array}\right] \xrightarrow{\cdot(-2)\ \oplus} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 5 \\ 0 & 0 & 0 & -6 & -6 \\ 0 & 1 & 1 & 1 & 4 \\ 1 & 3 & 5 & 2 & 13 \end{array}\right]$$

Step 2.
$r_4 - r_1 \to r_4$

$$\left[\begin{array}{cccc|c} \boxed{1} & 0 & 2 & 3 & 5 \\ 0 & 0 & 0 & -6 & -6 \\ 0 & 1 & 1 & 1 & 4 \\ 1 & 3 & 5 & 2 & 13 \end{array}\right] \xrightarrow{\cdot(-1)\ \oplus} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 5 \\ 0 & 0 & 0 & -6 & -6 \\ 0 & 1 & 1 & 1 & 4 \\ 0 & 3 & 3 & -1 & 8 \end{array}\right]$$

At this point, all of the entries inside of the oval under the pivot have been eliminated.

⇛ Column 2 ⇛

We now advance to the next column. Since the $(2,2)$ entry equals zero, we interchange the second and third rows to establish a nonzero pivot there:

Step 3.
$r_2 \leftrightarrow r_3$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 5 \\ 0 & 0 & 0 & -6 & -6 \\ 0 & 1 & 1 & 1 & 4 \\ 0 & 3 & 3 & -1 & 8 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 5 \\ 0 & 1 & 1 & 1 & 4 \\ 0 & 0 & 0 & -6 & -6 \\ 0 & 3 & 3 & -1 & 8 \end{array}\right]$$

and then we can eliminate the entries under the pivot:

Step 4.
$r_4 - 3r_2 \to r_4$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 5 \\ 0 & \boxed{1} & 1 & 1 & 4 \\ 0 & 0 & 0 & -6 & -6 \\ 0 & 3 & 3 & -1 & 8 \end{array}\right] \xrightarrow{\cdot(-3)\ \oplus} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 5 \\ 0 & 1 & 1 & 1 & 4 \\ 0 & 0 & 0 & -6 & -6 \\ 0 & 0 & 0 & -4 & -4 \end{array}\right]$$

⇛ Column 3 ⇛

Advancing to the third column, we realize that not only is the $(3,3)$ entry zero, but so is the rest of that column under it. Consequently, the third column will not be used as a pivotal column.

⇛ Column 4 ⇛

Skipping to the fourth column, to create the pivot in the $(3,4)$ entry, we multiply the third row by the reciprocal of the value currently there, then use the pivot to eliminate the entry below it.

Step 5.
$\frac{-1}{6} r_3 \to r_3$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 5 \\ 0 & 1 & 1 & 1 & 4 \\ 0 & 0 & 0 & \boxed{-6} & -6 \\ 0 & 0 & 0 & -4 & -4 \end{array}\right] \xrightarrow{\cdot(-1/6)} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 5 \\ 0 & 1 & 1 & 1 & 4 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -4 & -4 \end{array}\right]$$

Step 6.
$r_4 + 4r_3 \to r_4$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 5 \\ 0 & 1 & 1 & 1 & 4 \\ 0 & 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & -4 & -4 \end{array}\right] \xrightarrow{\cdot 4\ \oplus} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 5 \\ 0 & 1 & 1 & 1 & 4 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

⇛ Column 5

The fifth column is not pivotal as the $(4,5)$ entry is zero, and there are no entries below it.

In the second part, we go back through the list of the pivotal columns.

⇚ Column 5

Column 5 was not pivotal in the previous part; therefore, we skip it again.

⇚ Column 4 ⇚

In column 4, we want to eliminate all entries above the pivot:

Step 7.
$r_2 - r_3 \to r_2$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 5 \\ 0 & 1 & 1 & 1 & 4 \\ 0 & 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\cdot(-1)\ \oplus} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 5 \\ 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Step 8.
$r_1 - 3r_3 \to r_1$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 5 \\ 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\cdot(-3)\ \oplus} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 2 \\ 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\Lleftarrow$ Column 3 $\Lleftarrow$
$\Lleftarrow$ Column 2 $\Lleftarrow$
Column 1 $\Lleftarrow$

Column 3 is not pivotal; therefore, it is skipped. Column 2 already has the $(1, 2)$ entry above the pivot equal zero. Column 1 has no entries above the pivot.

The procedure of transforming our augmented matrix to a reduced row echelon form is now complete. Note that the leading entries in this matrix (enclosed in boxes) are the same as the pivots identified above.

$$\left[\begin{array}{cccc|c} \boxed{1} & 0 & 2 & 0 & 2 \\ 0 & \boxed{1} & 1 & 0 & 3 \\ 0 & 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The absence of a row $[0 \cdots 0 \mid 1\,]$ means the system is consistent. Since the third column does not contain a leading entry, the corresponding unknown z is arbitrary. The solution can now be obtained easily by rewriting this matrix in the form of a linear system

$$\begin{array}{rcrcrcrcl} x & & & + & 2z & & & = & 2 \\ & & y & + & z & & & = & 3 \\ & & & & & & w & = & 1 \\ & & & & & & 0 & = & 0 \end{array}$$

$x = 2 - 2z, y = 3 - z, z$ is arbitrary, $w = 1$.

To check this general solution, let us substitute it into the original system:

$$\begin{array}{rcrcrcrcr} (2-2z) & & & + & 2z & + & 3 & \overset{\checkmark}{=} & 5 \\ 2\,(2-2z) & & & + & 4z & & & \overset{\checkmark}{=} & 4 \\ & & (3-z) & + & z & + & 1 & \overset{\checkmark}{=} & 4 \\ (2-2z) & + & 3\,(3-z) & + & 5z & + & 2 & \overset{\checkmark}{=} & 13 \end{array}$$

Checking solutions

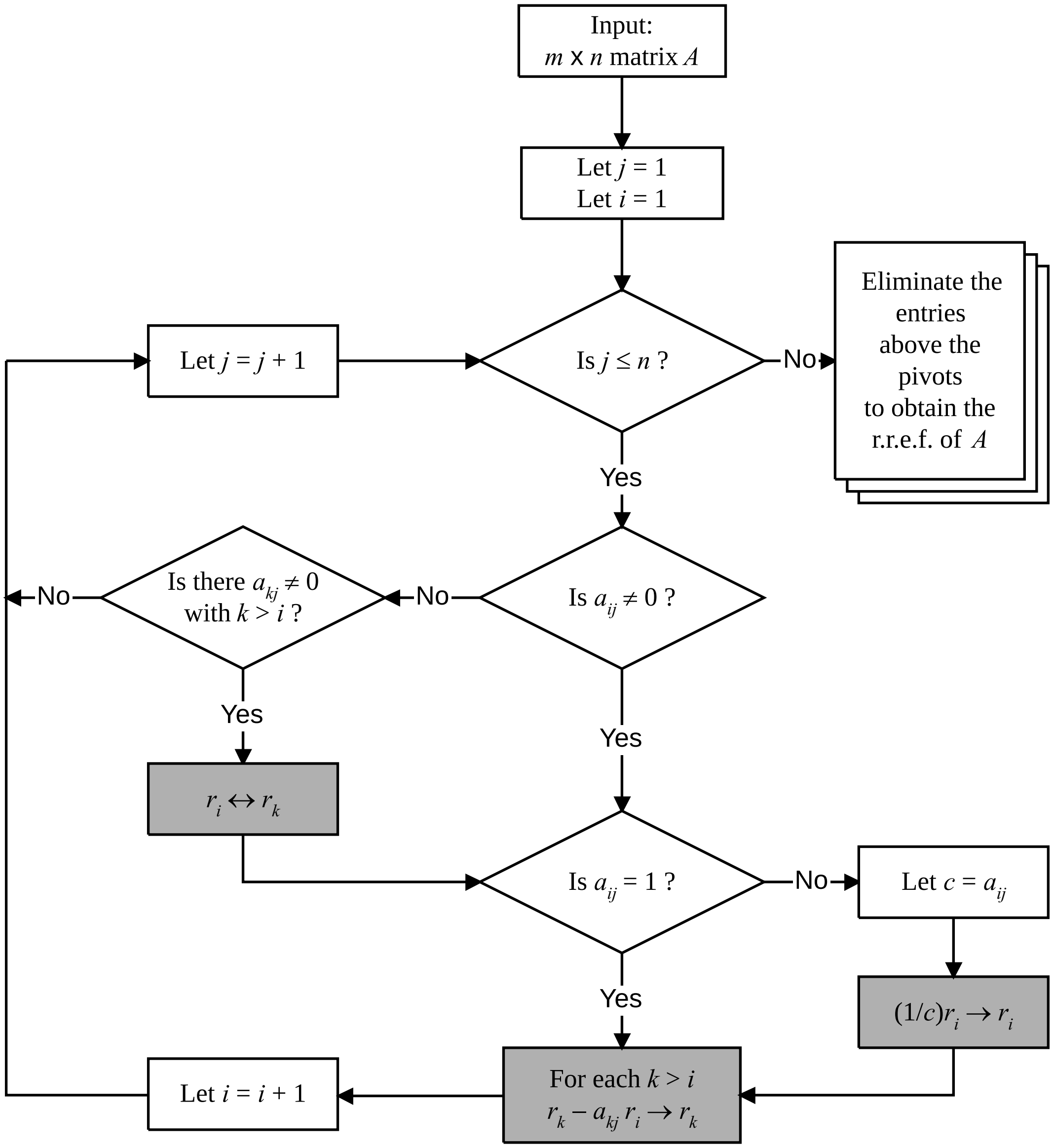
In the remainder of this book, we will engage in solving numerous linear systems. To save space (and some trees!), we will not go through the process of checking our answers after each of those solutions; however, we recommend that you do so as often as reasonably possible. (E.g., when you are solving some systems while taking an exam, it makes perfect sense to check your answers before turning the paper in!) When you do so, note that the extent to which you can check your answer depends upon the size of the solution set (no solution, one solution, infinitely many solutions).

- Clearly, if there is **no solution**, then there is nothing to check. (One could imagine trying random numbers just to make sure they *don't* satisfy the system, but such an approach would be grossly inefficient.)
- If there is **one solution**, it is usually an easy matter to check whether it satisfies each of the equations. The case of **infinitely many solutions** has a general solution involving arbitrary constant(s), which can be verified similarly (as shown at the end of the example above), possibly requiring some algebra.

A word of caution is in order: successfully checking a solution (or solutions) does not mean you found them all! (E.g., if you miss an arbitrary constant in your solution and erroneously claim the solution is unique, you will be unable to detect that mistake by checking your solution.)

Algorithm for reducing a matrix to r.r.e.f.

In Example 2.7, we have shown a complete sequence of elementary row operations which transformed the given matrix to a reduced row echelon form. Every operation performed in that example was fully explained, so that you should be able to perform this procedure on different matrices. Refer to the following flowchart for additional guidance on the operations necessary to identify the pivots and eliminate the entries below them:



Gauss-Jordan reduction and Gaussian elimination

The procedure for solving a linear system that was illustrated in Example 2.7 involved the following steps:

- form the augmented matrix of the linear system;
- transform the augmented matrix to a reduced row echelon form;
- using the linear system obtained from the r.r.e.f. of the augmented matrix, find the solution.

This procedure is called *Gauss-Jordan reduction*. Other methods for solving linear systems include *Gaussian elimination*:

- form the augmented matrix of the linear system;
- transform the augmented matrix to a row echelon form;
- solve the resulting system by backsubstitution.

EXAMPLE 2.8 Solve the system of Example 2.7 by Gaussian elimination.

Steps 1–6 lead to a row echelon form of the augmented matrix

$$\left[\begin{array}{cccc|c} \boxed{1} & 0 & 2 & 3 & 5 \\ 0 & \boxed{1} & 1 & 1 & 4 \\ 0 & 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Rather than continuing row operations (to obtain r.r.e.f.), we write the corresponding linear system

$$\begin{array}{ccccccccc} x & & & + & 2z & + & 3w & = & 5 \\ & & y & + & z & + & w & = & 4 \\ & & & & & & w & = & 1 \\ & & & & & & 0 & = & 0 \end{array}$$

In the process of backsubstitution, we solve for the unknowns corresponding to the leading variables, from the bottom to the top.

$$\begin{aligned} w &= 1 \\ y &= 4 - z - w \\ &= 3 - z \\ x &= 5 - 2z - 3w \\ &= 2 - 2z \end{aligned}$$

where z is arbitrary.

This solution matches the one obtained in Example 2.7.

Linear Algebra Toolkit

The author of this text has created a web-based interactive tool designed specifically to help students like yourself to improve their skill and understanding of many procedures we will be discussing here. Of those procedures, elementary row operations, Gauss-Jordan reduction, and Gaussian elimination occupy a central place in a linear algebra curriculum.

The name of this tool is Linear Algebra Toolkit. To access it, point your web browser to the URL

`latoolkit.com`

Of the modules listed in the main directory, you should find the following ones relevant to the material in this section:

- row operation calculator,
- transforming a matrix to row echelon form,
- transforming a matrix to reduced row echelon form,
- solving a system of linear equations.

The row operation calculator module allows you to specify which elementary row operations should be performed on the given matrix and see them performed by the Toolkit. It can be particularly useful while you are learning the mechanics of these operations, e.g., to verify the arithmetic at each step.

EXERCISES

In Exercises 1–4, write the augmented matrix and the coefficient matrix corresponding to each linear system. (Do not solve the system.)

1. The system:

$$\begin{array}{rcrcr} x_1 & + & 6x_2 & = & 0 \\ & & 3x_2 & = & 1 \end{array}$$

2. The system:

$$\begin{array}{rcrcrcr} 2x_1 & - & x_2 & + & x_3 & = & 5 \\ & & 4x_2 & & & = & 0 \\ x_1 & & & - & x_3 & = & 1 \end{array}$$

3. The system:

$$\begin{array}{rcrcrcr} & & & & z & = & 4 \\ 7x & & & + & 3z & = & 5 \\ & & 5y & & & = & -6 \\ x & - & y & & & = & 3 \end{array}$$

4. The system:

$$\begin{array}{rcrcr} 3x & + & 4y & = & 2 \\ & & 2y & = & -1 \\ & & 0 & = & 0 \\ & & 0 & = & 0 \end{array}$$

In Exercises 5–6, write the linear system corresponding to each augmented matrix. (Do not solve the system.)

5. a. $\left[\begin{array}{rr|r} 2 & -3 & 4 \\ 6 & 0 & 0 \end{array}\right]$; b. $\left[\begin{array}{rrr|r} 1 & 2 & 1 & 2 \\ 3 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$.

6. a. $\left[\begin{array}{ccc|c} 0 & 1 & -1 & 3 \\ 1 & 0 & 2 & 4 \end{array}\right]$; b. $\left[\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$.

In Exercises 7–8, determine if the given matrix is in

(i) both reduced row echelon form and row echelon form,
(ii) row echelon form, but not in reduced row echelon form,
(iii) neither.

7. a. $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$; b. $\begin{bmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$; c. $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$; d. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

8. a. $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$; b. $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$; c. $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$; d. $\begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Each of the Exercises 9–12 contains an augmented matrix of a linear system in reduced row echelon form.

i. Mark each leading entry in the matrix with a box.
ii. Determine if the system corresponding to the given augmented matrix in r.r.e.f. has no solution, one solution, or infinitely many solutions.
iii. If the system is consistent, find all solutions.

Do not use technology.

9. a. $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 2 \end{array}\right]$; b. $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$; c. $\left[\begin{array}{ccc|c} 1 & 0 & -3 & 4 \\ 0 & 1 & 2 & 5 \end{array}\right]$.

10. a. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 5 \end{array}\right]$; b. $\left[\begin{array}{cc|c} 1 & 0 & 6 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$; c. $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$.

11. a. $\left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$; b. $\left[\begin{array}{ccccc|c} 0 & 1 & 0 & -5 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$.

12. a. $\left[\begin{array}{ccccc|c} 1 & 6 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 4 & 3 \\ 0 & 0 & 0 & 1 & -1 & 2 \end{array}\right]$; b. $\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$.

In Exercises 13–26:

a. Write an augmented matrix for the given system.
b. Use elementary row operations to transform the augmented matrix to an r.e.f. and r.r.e.f.
c. Use the r.e.f. and backsubstitution (Gaussian elimination) to solve the system.
d. Use the r.r.e.f. (Gauss-Jordan reduction) to solve the system.
e. Make sure the solutions obtained in parts c and d agree. Check your solutions by substituting them back into the original system.

You should not use technology while working on steps a–e, but it is a good idea to refer to the Linear Algebra Toolkit and compare its results to your solutions.

13.
$$\begin{array}{rcrcr} x & + & 5y & = & -2 \\ 5x & + & 2y & = & 13 \end{array}$$

14.
$$\begin{array}{rcrcr} -2x & + & 4y & = & -6 \\ x & + & 3y & = & 8 \end{array}$$

15.
$$\begin{array}{rcrcr} 2x & + & 4y & = & 3 \\ x & + & 2y & = & -1 \end{array}$$

16.
$$\begin{array}{rcrcr} & & 2y & = & -4 \\ x & - & 3y & = & 7 \\ 2x & + & y & = & 0 \end{array}$$

17.
$$\begin{array}{rcrcrcrcr} 2x_1 & + & x_2 & + & 3x_3 & - & x_4 & = & 7 \\ -x_1 & + & 3x_2 & + & 2x_3 & + & 4x_4 & = & 0 \end{array}$$

18.
$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & + & 2x_3 & = & -2 \\ x_1 & - & x_2 & - & 2x_3 & = & 3 \end{array}$$

19.
$$\begin{array}{rcrcrcr} 3x & & & + & z & = & 2 \\ -x & + & y & & & = & 1 \\ 4x & + & 2y & + & z & = & 4 \end{array}$$

20.
$$\begin{array}{rcrcrcr} x & & & + & 2z & = & 3 \\ & & 2y & & & = & -4 \\ x & & & + & 3z & = & 4 \end{array}$$

21.
$$\begin{array}{rcrcrcrcr} x & + & 2y & + & 3z & - & 3w & = & 0 \\ 2x & + & y & + & 3z & & & = & 6 \\ -x & + & y & & & - & 3w & = & -6 \end{array}$$

22.
$$\begin{array}{rcrcrcr} & & y & - & z & = & 2 \\ x & + & 2y & - & z & = & 0 \\ -x & - & y & & & = & -1 \\ x & + & 2y & - & z & = & 3 \end{array}$$

23. $$\begin{array}{rcrcrcrcr} x & & & & & + & 5w & = & 1\\ x & + & y & & & + & w & = & 0\\ 2x & + & 3y & + & z & & & = & -3\\ & - & y & + & 2z & & & = & -3 \end{array}$$

24. $$\begin{array}{rcrcrcrcr} & & y & + & 2z & & & = & -1\\ & & 2y & & & + & w & = & 5\\ x & + & 3y & & & + & w & = & 6\\ & & & & z & + & 2w & = & 5 \end{array}$$

25. $$\begin{array}{rcrcrcrcrcr} & & x_2 & + & x_3 & + & x_4 & - & 2x_5 & = & 0\\ & - & x_2 & - & x_3 & - & x_4 & + & 2x_5 & = & 0\\ x_1 & - & x_2 & & & - & 3x_4 & + & 3x_5 & = & 0\\ x_1 & & & + & x_3 & - & 2x_4 & + & x_5 & = & 0 \end{array}$$

26. $$\begin{array}{rcrcrcrcr} -x_1 & & & - & x_3 & - & x_4 & = & 1\\ 3x_1 & + & x_2 & + & 2x_3 & + & 5x_4 & = & -1\\ 2x_1 & + & 3x_2 & - & x_3 & + & 8x_4 & = & 4\\ & & x_2 & - & x_3 & + & 2x_4 & = & 2 \end{array}$$

∃?

In Exercises 27–32, show an example of an augmented matrix in r.r.e.f. of a system that matches each description or explain why such a system cannot exist.

27. A system of 2 equations in 3 unknowns with a unique solution.

28. A system of 2 equations in 3 unknowns with no solution.

29. A system of 2 equations in 3 unknowns with infinitely many solutions.

30. A system of 3 equations in 2 unknowns with a unique solution.

31. A system of 3 equations in 2 unknowns with no solution.

32. A system of 3 equations in 2 unknowns with infinitely many solutions.

T/F?

In Exercises 33–38, decide whether each statement is true or false. Justify your answer.

33. If A and B are $m \times n$ matrices in r.r.e.f., then $A + B$ is also in r.r.e.f.

34. The matrix I_n is in r.r.e.f.

35. The system whose augmented matrix is I_4 has no solution.

36. If A is a square matrix in r.e.f., then A is upper triangular.

37. Matrices A and $3A$ are row equivalent.

38. For all $m \times n$ matrices A, C and $m \times p$ matrices B, D, if $[A|B]$ is row equivalent to $[C|D]$, then A is row equivalent to C.

2.2 Elementary Matrices and the Geometry of Linear Systems

The following theorem, while easy to prove, is of great importance to justify the procedure adopted in the previous section for solving linear systems.

THEOREM 2.2 Let A be an $m \times n$ matrix, $\overrightarrow{b}$ be an m-vector, and let C be a $p \times m$ matrix.

(a) If $\overrightarrow{x} = \overrightarrow{s}$ is a solution of the linear system $A\overrightarrow{x} = \overrightarrow{b}$, then it is also a solution of the system $(CA)\overrightarrow{x} = C\overrightarrow{b}$.

(b) If there exists an $m \times p$ matrix B such that[7] $BC = I_m$, then $\overrightarrow{x} = \overrightarrow{s}$ is a solution of the linear system $A\overrightarrow{x} = \overrightarrow{b}$ if and only if it is also a solution of the system $(CA)\overrightarrow{x} = C\overrightarrow{b}$.

PROOF

If

$$A\overrightarrow{s} = \overrightarrow{b}, \tag{21}$$

then premultiplying both sides by C yields

$$C(A\overrightarrow{s}) = C\overrightarrow{b}$$

and, by applying property 1 of Theorem 1.5,

$$(CA)\overrightarrow{s} = C\overrightarrow{b}, \tag{22}$$

proving part (a).

To prove part (b), premultiply both sides by B and apply property 1 of Theorem 1.5 to obtain

$$(BC)A\overrightarrow{s} = (BC)\overrightarrow{b}. \tag{23}$$

Because of the assumption $BC = I_m$ and by property 6 of Theorem 1.5, this is equivalent to (21). Consequently, from part (a) we have the following implications:

$$\begin{aligned}(\overrightarrow{s} \text{ is a solution of } A\overrightarrow{x} = \overrightarrow{b}) &\Rightarrow (\overrightarrow{s} \text{ is a solution of } CA\overrightarrow{x} = C\overrightarrow{b})\\ &\Rightarrow (\overrightarrow{s} \text{ is a solution of } A\overrightarrow{x} = \overrightarrow{b}),\end{aligned}$$

which proves part (b).

■

According to part (a) of the theorem above, any linear transformation applied to the columns of the augmented matrix results in a system whose solution set contains all of the solutions of the original system (and, possibly, additional ones); by part (b), if that linear transformation can be reversed, then the solution sets of the original and transformed systems are identical. We shall demonstrate that elementary row operations correspond to the transformations of the latter kind, therefore leaving the solution set unchanged.

THEOREM 2.3 For any elementary row operation applied to a matrix A to obtain a matrix B, there exists a matrix E, called an *elementary matrix*, such that postmultiplying it by A yields the same B (i.e., $EA = B$). Furthermore, each elementary row operation can be reversed by performing another elementary row operation.

PROOF

The following table contains the elementary matrices corresponding to the three types of the elementary row operations, as well as the matrices associated with reversing these (we leave it for the reader to verify the details – in particular, see Exercise 46 on p. 35 in Section 1.3). ■

---

[7] In some books, when $BC = I$, B is referred to as a left inverse of C, and C is called a right inverse of B.

Elementary row operations for transformation $A \to B$ and the corresponding elementary matrices E ($B = EA$)

$\text{row}_i A + k\ \text{row}_j A \to \text{row}_i B$	$k\ \text{row}_i A \to \text{row}_i B$	$\text{row}_i A \to \text{row}_j B$ $\text{row}_j A \to \text{row}_i B$
$\left[\begin{array}{cccccccccc} \boxed{1} & 0 & \ddots & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & \boxed{\ddots} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ \ddots & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & \\ 0 & 0 & \boxed{k} & 0 & 0 & \boxed{1} & 0 & 0 & 0 & \Leftarrow \text{ith row} \\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & \ddots & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{\ddots} & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 0 & \boxed{1} & \\ & & \Uparrow \text{jth column} & & & & & & & \end{array}\right]$	$\left[\begin{array}{cccccccccc} \boxed{1} & 0 & \ddots & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & \boxed{\ddots} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ \ddots & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & \boxed{k} & 0 & 0 & 0 & 0 & 0 & \Leftarrow \text{ith row} \\ 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & \ddots & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{\ddots} & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 0 & \boxed{1} & \\ & & & \Uparrow \text{ith column} & & & & & & \end{array}\right]$	$\left[\begin{array}{cccccccccc} \boxed{1} & 0 & \ddots & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & \boxed{\ddots} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ \ddots & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & \Leftarrow \text{ith row} \\ 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & \\ 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & \ddots & \Leftarrow \text{jth row} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{\ddots} & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 0 & \boxed{1} & \\ & & \Uparrow \text{ith column} & & & & \Uparrow \text{jth column} & & & \end{array}\right]$

Elementary row operations and elementary matrices reversing the operations above

$\text{row}_i B - k\ \text{row}_j B \to \text{row}_i A$	$\frac{1}{k}\ \text{row}_i B \to \text{row}_i A$	$\text{row}_i B \to \text{row}_j A$ $\text{row}_j B \to \text{row}_i A$
$\left[\begin{array}{cccccccccc} \boxed{1} & 0 & \ddots & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & \boxed{\ddots} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ \ddots & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & \\ 0 & 0 & \boxed{-k} & 0 & 0 & \boxed{1} & 0 & 0 & 0 & \Leftarrow \text{ith row} \\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & \ddots & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{\ddots} & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 0 & \boxed{1} & \\ & & \Uparrow \text{jth column} & & & & & & & \end{array}\right]$	$\left[\begin{array}{cccccccccc} \boxed{1} & 0 & \ddots & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & \boxed{\ddots} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ \ddots & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & \boxed{1/k} & 0 & 0 & 0 & 0 & 0 & \Leftarrow \text{ith row} \\ 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & \ddots & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{\ddots} & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 0 & \boxed{1} & \\ & & & \Uparrow \text{ith column} & & & & & & \end{array}\right]$	$\left[\begin{array}{cccccccccc} \boxed{1} & 0 & \ddots & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & \boxed{\ddots} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ \ddots & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & \Leftarrow \text{ith row} \\ 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & \\ 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & \ddots & \Leftarrow \text{jth row} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{\ddots} & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 0 & \boxed{1} & \\ & & \Uparrow \text{ith column} & & & & \Uparrow \text{jth column} & & & \end{array}\right]$

Note that because of Theorem 2.3, if A is row equivalent to B, then B is also row equivalent to A. Consequently, we can simply refer to A and B being row equivalent from now on.

COROLLARY 2.4 Linear systems with row equivalent augmented matrices have identical solution sets.

Performing an elementary row operation on a vector $\overrightarrow{x}$ is a linear transformation $F(\overrightarrow{x}) = E\overrightarrow{x}$ (by Theorem 1.8). Therefore, by (9), any elementary matrix can be constructed by applying the corresponding elementary row operation to the identity matrix of the appropriate size. We shall illustrate this in the following two examples.

Example 2.9 Consider the elementary matrices corresponding to the row operations conducted in Example 2.7. Denote the original augmented matrix $\left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 5 \\ 2 & 0 & 4 & 0 & 4 \\ 0 & 1 & 1 & 1 & 4 \\ 1 & 3 & 5 & 2 & 13 \end{array}\right]$ by A_0 – we shall refer to a matrix resulting from step i by A_i. All elementary matrices in this example must be $4{\times}4$. (Only a matrix of this size when multiplied by a 4×5 matrix results in another 4×5 matrix.).

- Step 1 of the example involved the row operation $r_2 - 2r_1 \to r_2$. The corresponding elementary matrix can be obtained by applying this operation to I_4 (or, equivalently, replacing its $(2,1)$ entry with -2): $E_1 = \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$. Multiplying produces the correct matrix:

$$\underbrace{\left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]}_{E_1}\underbrace{\left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 5 \\ 2 & 0 & 4 & 0 & 4 \\ 0 & 1 & 1 & 1 & 4 \\ 1 & 3 & 5 & 2 & 13 \end{array}\right]}_{A_0} = \underbrace{\left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 5 \\ 0 & 0 & 0 & -6 & -6 \\ 0 & 1 & 1 & 1 & 4 \\ 1 & 3 & 5 & 2 & 13 \end{array}\right]}_{A_1}.$$

- In Step 2, another row operation of type 1 was performed: $r_4 - r_1 \to r_4$. Using an elementary matrix,

$$\underbrace{\left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{array}\right]}_{E_2}\underbrace{\left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 5 \\ 0 & 0 & 0 & -6 & -6 \\ 0 & 1 & 1 & 1 & 4 \\ 1 & 3 & 5 & 2 & 13 \end{array}\right]}_{A_1} = \underbrace{\left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 5 \\ 0 & 0 & 0 & -6 & -6 \\ 0 & 1 & 1 & 1 & 4 \\ 0 & 3 & 3 & -1 & 8 \end{array}\right]}_{A_2}.$$

- Step 3 involved a row operation of type 3: interchanging rows 2 and 3. The same thing can be accomplished when multiplying by the elementary matrix E_3 as follows:

$$\underbrace{\left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]}_{E_3}\underbrace{\left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 5 \\ 0 & 0 & 0 & -6 & -6 \\ 0 & 1 & 1 & 1 & 4 \\ 0 & 3 & 3 & -1 & 8 \end{array}\right]}_{A_2} = \underbrace{\left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 5 \\ 0 & 1 & 1 & 1 & 4 \\ 0 & 0 & 0 & -6 & -6 \\ 0 & 3 & 3 & -1 & 8 \end{array}\right]}_{A_3}.$$

- The fourth step implemented another row operation of type 1, $r_4 - 3r_2 \to r_4$.

$$\underbrace{\left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{array}\right]}_{E_4}\underbrace{\left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 5 \\ 0 & 1 & 1 & 1 & 4 \\ 0 & 0 & 0 & -6 & -6 \\ 0 & 3 & 3 & -1 & 8 \end{array}\right]}_{A_3} = \underbrace{\left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 5 \\ 0 & 1 & 1 & 1 & 4 \\ 0 & 0 & 0 & -6 & -6 \\ 0 & 0 & 0 & -4 & -4 \end{array}\right]}_{A_4}.$$

- To create a pivot in the $(3,4)$ entry, Step 5 scaled the third row by performing an operation of type 2: $-\frac{1}{6}r_3 \to r_3$. The corresponding elementary matrix is obtained from I_4, replacing its $(3,3)$ entry by $-\frac{1}{6}$:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{6} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{E_5}\underbrace{\left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 5 \\ 0 & 1 & 1 & 1 & 4 \\ 0 & 0 & 0 & -6 & -6 \\ 0 & 0 & 0 & -4 & -4 \end{array}\right]}_{A_4} = \underbrace{\left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 5 \\ 0 & 1 & 1 & 1 & 4 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -4 & -4 \end{array}\right]}_{A_5}.$$

We leave it as an exercise for the reader to perform the remaining multiplications by elementary matrices to parallel the operations carried out in Example 2.7. After finding three additional E matrices, it will be possible to obtain the reduced row echelon form:

$$E_8E_7E_6E_5E_4E_3E_2E_1\left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 5 \\ 2 & 0 & 4 & 0 & 4 \\ 0 & 1 & 1 & 1 & 4 \\ 1 & 3 & 5 & 2 & 13 \end{array}\right] = \underbrace{\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 2 \\ 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]}_{A_8}.$$

EXAMPLE 2.10 Find the elementary matrix that performs the given elementary row operation on a 3×7 matrix:

a. $r_2 - \frac{3}{2}r_3 \to r_2$;

b. $r_2 \leftrightarrow r_3$.

SOLUTION

In both cases, the elementary matrix we are seeking must be 3×3. (Only a matrix of this size when multiplied on the right by a 3×7 matrix results in another 3×7 matrix.) We apply the given elementary row operation to I_3.

a. Applying $r_2 - \frac{3}{2}r_3 \to r_2$ to $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ yields the elementary matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{3}{2} \\ 0 & 0 & 1 \end{bmatrix}$.

b. Interchanging the last two rows in I_3 results in the elementary matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

Properties of linear systems

One might ask: if we performed a different sequence of elementary row operations, would the resulting reduced row echelon form be guaranteed to be the same? As it turns out, the answer to this question is “yes”.

THEOREM 2.5 Any $m \times n$ matrix A is row equivalent to a unique matrix in reduced row echelon form.

PROOF

The pivoting strategy on p. 68 provides a constructive proof of the existence of the r.r.e.f. A is row equivalent to.

To show the uniqueness of the r.r.e.f., let us assume that A is row equivalent to both B and C in r.r.e.f. Consider B and C to be coefficient matrices of linear systems $B\overrightarrow{x} = \overrightarrow{0}$ and $C\overrightarrow{x} = \overrightarrow{0}$. The solutions sets of these systems are guaranteed to be identical by Corollary 2.4 – each of these solutions is also guaranteed to be a solution of the linear system $(B - C)\overrightarrow{x} = \overrightarrow{0}$ (but not vice versa).

Let j be the first nonzero column of $B - C$.

It is not possible to have both jth columns of B and C as leading columns since the equality of all previous columns would require that there be the same number of leading columns (say, i) among them and the next leading column in both matrices be $\overrightarrow{e_{i+1}}$ (making their difference $\overrightarrow{0}$).

Therefore, if j is the first nonzero column of $B - C$, then the jth columns of the matrices B or C cannot both be leading columns. Consequently, the solution set of at least one of the systems $B\overrightarrow{x} = \overrightarrow{0}$ and $C\overrightarrow{x} = \overrightarrow{0}$ is guaranteed to contain a vector

$$\overrightarrow{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_{j-1} \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \\ \\ \\ \longleftarrow j\text{th position.} \\ \\ \\ \\ \end{matrix}$$

Since both solution sets must be identical, they must both contain $\overrightarrow{y}$; therefore, so does the solution set of $(B - C)\overrightarrow{x} = \overrightarrow{0}$. On the other hand, the jth column must be the first leading column of $B - C$; thus $(B - C)\overrightarrow{y} = \text{col}_j(B - C) \neq \overrightarrow{0}$ so that $\overrightarrow{y}$ is **not** a solution of $(B - C)\overrightarrow{x} = \overrightarrow{0}$. Assuming that $B - C$ contains a nonzero column leads us to a contradiction.

We conclude that $B = C$.

■

Geometry of linear systems

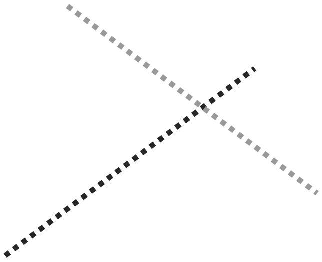
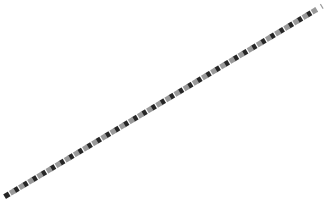
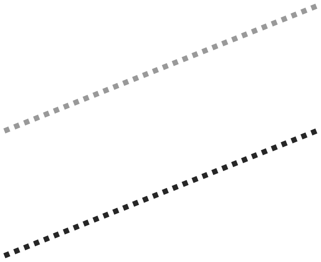
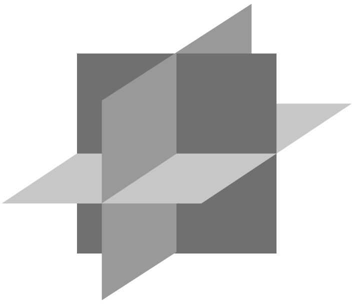
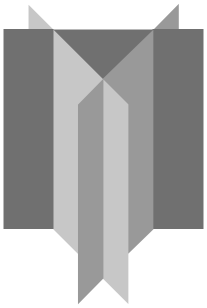
By Theorem 2.1 and Corollary 2.4, every linear system has either

1. one solution,
2. infinitely many solutions, or
3. no solution.

The following table contains examples of each of these three possibilities for systems

- of two linear equations in two unknowns and
- of three linear equations in three unknowns.

Note that the configurations of the lines and planes shown in this table do not exhaust all of the possibilities (e.g., another way in which a system of three equations in three unknowns can have no solution is when the three planes are parallel).

	one solution	infinitely many solutions	no solution
linear system of two equations in two unknowns; each equation corresponds to a line			
linear system of three equations in three unknowns; each equation corresponds to a plane			

Homogeneous linear systems

A linear system whose right-hand side equals $\overrightarrow{0}$ is called a *homogeneous system*. Every homogeneous system $A\overrightarrow{x} = \overrightarrow{0}$ is guaranteed to have at least one solution $\overrightarrow{x} = \overrightarrow{0}$, which we will refer to as the *trivial solution*. Additionally, some homogeneous systems may possess *nontrivial solutions*, $\overrightarrow{x} \neq \overrightarrow{0}$.

Example 2.11 Solve the following homogeneous linear systems.

a.
$$\begin{array}{rcrcrcrcr} x & - & 2y & + & z & + & 2w & = & 0 \\ 2x & - & 3y & + & 3z & + & w & = & 0 \\ & - & 3y & + & z & - & w & = & 0 \end{array}$$

and

b.
$$\begin{array}{rcrcr} -2x & + & 3y & = & 0 \\ x & + & 2y & = & 0 \\ 2x & + & y & = & 0 \\ 3x & - & y & = & 0. \end{array}$$

SOLUTION

a. The augmented matrix of the system, $\left[\begin{array}{cccc|c} 1 & -2 & 1 & 2 & 0 \\ 2 & -3 & 3 & 1 & 0 \\ 0 & -3 & 1 & -1 & 0 \end{array}\right]$, has the r.r.e.f.

$$\left[\begin{array}{cccc|c} \boxed{1} & 0 & 0 & \frac{7}{2} & 0 \\ 0 & \boxed{1} & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & \boxed{1} & -\frac{5}{2} & 0 \end{array}\right].$$

The column corresponding to the unknown w contains no leading entry; therefore, w is arbitrary, whereas the remaining unknowns can be solved for (in terms of w):

$$\begin{aligned} x &= \frac{-7}{2}w \\ y &= \frac{1}{2}w \\ z &= \frac{5}{2}w. \end{aligned}$$

This system has infinitely many solutions, which include

- the trivial solution – taking $w = 0$ leads to $x = y = z = w = 0$, and
- nontrivial solutions; e.g., if $w = 2$, then $x = -7, y = 1, z = 5$.

You can use the Linear Algebra Toolkit (latoolkit.com) to reproduce the elementary row operations used in parts (a) and (b) of this example.

b. The augmented matrix of the second system is $\left[\begin{array}{cc|c} -2 & 3 & 0 \\ 1 & 2 & 0 \\ 2 & 1 & 0 \\ 3 & -1 & 0 \end{array}\right]$. It follows from its reduced row echelon form, $\left[\begin{array}{cc|c} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$, that the system has a unique solution, $x = y = 0$ (the trivial solution).

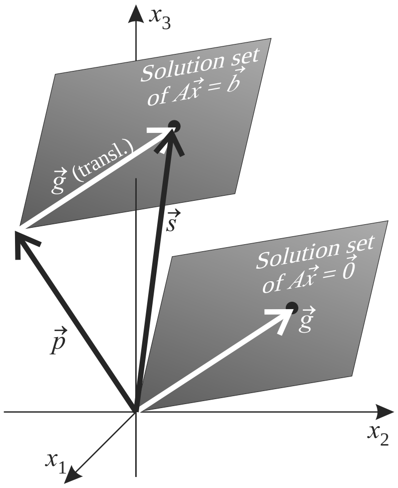
THEOREM 2.6 If a homogeneous system of m linear equations in n unknowns has only the trivial solution, then $m \geq n$.

PROOF

For the linear homogeneous system to have a unique solution, it must have a leading entry in every left-hand side column of the r.r.e.f. of the augmented matrix. Each of the n leading entries must be located in a different row; consequently, the number of rows, m, must not be less than n.

■

Our next result provides a connection between solutions of a general linear system and solutions of the homogeneous system with the same coefficient matrix.



THEOREM 2.7 Let A be an $m \times n$ matrix and let $\overrightarrow{b}$ be an m-vector. If $\overrightarrow{p}$ is a solution of the system $A\overrightarrow{x} = \overrightarrow{b}$, then every solution $\overrightarrow{s}$ of that system can be expressed as

$$\overrightarrow{s} = \overrightarrow{p} + \overrightarrow{g}$$

where $\overrightarrow{g}$ is some solution of the homogeneous system $A\overrightarrow{x} = \overrightarrow{0}$.

PROOF

Let $\overrightarrow{p}$ be a solution of $A\overrightarrow{x} = \overrightarrow{b}$. For every solution $\overrightarrow{s}$ of $A\overrightarrow{x} = \overrightarrow{b}$, we can define $\overrightarrow{g} = \overrightarrow{s} - \overrightarrow{p}$ which is a solution of $A\overrightarrow{x} = \overrightarrow{0}$ since

$$A\overrightarrow{g} = A(\overrightarrow{s} - \overrightarrow{p}) = A\overrightarrow{s} - A\overrightarrow{p} = \overrightarrow{b} - \overrightarrow{b} = \overrightarrow{0}.$$

■

The vector $\overrightarrow{p}$ in our last theorem is sometimes called a *particular solution* of the system $A\overrightarrow{x} = \overrightarrow{b}$. Geometrically the solution set of a linear system can be viewed as a translation of the solution set of the associated homogeneous system by the particular solution of the original system.

A word of caution is in order: this does not imply that the two solution sets are identical in size. For instance, check that replacing 0 with 1 on the right-hand side of the first equation in part b of Example 2.11 will result in an inconsistent system – the translation described above cannot occur if a particular solution does not exist.

The following result focuses on the relationship between solution sets of $A\overrightarrow{x} = \overrightarrow{b}$ and $A\overrightarrow{x} = \overrightarrow{0}$ when A is a square matrix.

THEOREM 2.8 Given an $n \times n$ matrix A, the following three statements are equivalent:

A. A is row equivalent to I_n.

B. For every n-vector $\overrightarrow{b}$, the system $A\overrightarrow{x} = \overrightarrow{b}$ has a unique solution.

C. The system $A\overrightarrow{x} = \overrightarrow{0}$ has only the trivial solution.

PROOF

The equivalence of these statements means A⇔B, B⇔C, and A⇔C. However, it is sufficient to prove A⇒B, B⇒C, and C⇒A:

- Part I (A⇒B)
  From the assumption that A is row equivalent to I_n, it follows that for every n-vector $\overrightarrow{b}$, there exists an n-vector $\overrightarrow{d}$ such that the augmented matrix $[A|\overrightarrow{b}]$ is row equivalent to $[I_n|\overrightarrow{d}]$. This matrix is in reduced row echelon form, which, by Theorem 2.5, is unique. This means $\overrightarrow{d}$ is unique, and from $I_n\overrightarrow{x} = \overrightarrow{d}$ we conclude that the system has a unique solution $\overrightarrow{x} = \overrightarrow{d}$.
- Part II (B⇒C)
  We assume $A\overrightarrow{x} = \overrightarrow{b}$ has a unique solution for every n-vector $\overrightarrow{b}$. Therefore, it also has a unique solution for the specific n-vector $\overrightarrow{b} = \overrightarrow{0}$.
- Part III (C⇒A)
  Every column of the r.r.e.f. of A must contain a leading entry (otherwise the column without a leading entry would correspond to an arbitrary variable, leading to many solutions).
  The only way for an $n \times n$ matrix in r.r.e.f. to have leading entries in all n columns is when that matrix equals I_n. ■

EXERCISES

In Exercises 1–10:

a. Find the elementary matrix that performs each row operation.

b. Multiply this matrix by the given matrix A and verify that the result is consistent with the desired row operation.

1. Apply $r_3 + 4r_1 \rightarrow r_3$ to $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & -2 & 2 \\ -4 & 3 & 1 & 2 \end{bmatrix}$.

2. Apply $r_4 - 3r_2 \rightarrow r_4$ to $\begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \\ 0 & 3 & 1 \end{bmatrix}$.

3. Apply $r_2 - 2r_3 \rightarrow r_2$ to $\begin{bmatrix} 1 & -3 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.

4. Apply $r_1 + r_3 \rightarrow r_1$ to $\begin{bmatrix} 1 & 4 & -1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$.

5. Apply $r_2 - r_1 \rightarrow r_2$ to $\begin{bmatrix} 2 & 3 & 1 & -2 \\ 2 & 4 & 5 & 2 \end{bmatrix}$.

6. Apply $r_1 - 2r_2 \rightarrow r_1$ to $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 6 \end{bmatrix}$.

7. Apply $\frac{1}{2}r_3 \rightarrow r_3$ to $\begin{bmatrix} 1 & 3 & 1 & -2 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 2 & -6 \\ 0 & 0 & 3 & 7 \end{bmatrix}$.

8. Apply $2r_1 \rightarrow r_1$ to $\begin{bmatrix} \frac{1}{2} & \frac{3}{2} & 1 & -1 \\ 3 & 2 & 1 & 1 \\ 4 & 0 & 1 & 2 \end{bmatrix}$.

9. Apply $r_1 \leftrightarrow r_3$ to $\begin{bmatrix} 0 & 2 \\ 0 & 3 \\ 2 & 5 \\ -7 & 1 \end{bmatrix}$.

10. Apply $r_2 \leftrightarrow r_3$ to $\begin{bmatrix} 1 & 2 & 0 & -2 & 0 \\ 0 & 0 & 5 & 0 & 1 \\ 0 & 3 & 1 & 1 & 0 \end{bmatrix}$.

11. Find elementary matrices that perform the following row operations on a 4×5 matrix:

    a. $r_4 + 5r_1 \to r_4$; b. $r_1 \leftrightarrow r_4$; c. $3r_2 \to r_2$.

12. Find elementary matrices that perform the following row operations on a 3×8 matrix:

    a. $r_1 \leftrightarrow r_2$; b. $\frac{1}{5}r_1 \to r_1$; c. $r_3 - r_1 \to r_3$.

13. Find elementary matrices that perform the following row operations on a 5×3 matrix:

    a. $-6r_5 \to r_5$; b. $r_3 \leftrightarrow r_5$; c. $r_2 - \frac{1}{2}r_5 \to r_2$.

14. Find elementary matrices that perform the following row operations on a 2×4 matrix:

    a. $r_2 + 4r_1 \to r_2$; b. $r_1 \leftrightarrow r_2$; c. $-r_2 \to r_2$.

In Exercises 15 and 16, consider a linear system $A\overrightarrow{x} = \overrightarrow{b}$ with an $m \times n$ matrix A, and let $\overrightarrow{e_i}$ denote the ith column of the $m \times m$ identity matrix I_m.

15. a. * Show that $E = I_m + k\overrightarrow{e_i}\,\overrightarrow{e_j}^T$ is the elementary matrix corresponding to the operation $\text{row}_i + k\ \text{row}_j \to \text{row}_i$.
    b. * Prove that premultiplying by $E^* = I_m - k\overrightarrow{e_i}\,\overrightarrow{e_j}^T$ completely reverses the operation performed in part a. (Hint: Show that $E^*E = \left(I_m - k\overrightarrow{e_i}\,\overrightarrow{e_j}^T\right)\left(I_m + k\overrightarrow{e_i}\,\overrightarrow{e_j}^T\right) = I_m$.)

16. a. * Show that $E = I_m - (\overrightarrow{e_i} - \overrightarrow{e_j})(\overrightarrow{e_i} - \overrightarrow{e_j})^T$ is the elementary matrix corresponding to the operation $r_i \leftrightarrow r_j$.
    b. * Prove that premultiplying by the same E completely reverses this operation.

T/F?

In Exercises 17–19, decide whether each statement is true or false. Justify your answer.

17. A linear system has infinitely many solutions if and only if the r.r.e.f. of its augmented matrix contains a row $[0\ \cdots\ 0 \mid 0]$.

18. A linear system has no solution if and only if the r.r.e.f. of its augmented matrix contains a row $[0\ \cdots\ 0 \mid 1]$.

19. Suppose the linear system is consistent and has the augmented matrix with r.r.e.f. $[C|\overrightarrow{d}]$. The system has a unique solution if and only if every column of C contains a leading entry.

20. * Perform different sequences of elementary row operations to solve the system of Example 2.7 on p. 67 beginning with

    a. $r_2 \leftrightarrow r_1$; then $\frac{1}{2}r_1 \to r_1$ to create a pivot in the $(1,1)$ entry, and then continue with the usual pivoting strategy;
    b. $r_1 \leftrightarrow r_4$, and then continue with the usual pivoting strategy.

    Show that the r.r.e.f. of the augmented matrix obtained in either case is identical to the one obtained at the end of the example.

21. * Modify the table of figures from p. 82 to illustrate solutions of

    - a system of 2 equations in 3 unknowns, and
    - a system of 3 equations in 2 unknowns.

    (You may want to refer to Exercises 27–32 on p. 76.)

2.3 Matrix Inverse

In Section 1.2, some algebraic operations on matrices were defined, including matrix addition, scalar multiplication, and subtraction. Later on, in Section 1.3, we have introduced a product of two matrices. As these operations share many properties with their counterparts for real numbers (although not all: e.g., matrix multiplication is not commutative), you might wonder whether it's also possible to *divide* two matrices.

Before we tackle this question, let us take a step back to the arithmetic of real numbers and take a close look at the operation of division among such numbers.

Saying that a^{-1} is the reciprocal of a nonzero number a is equivalent to saying that $aa^{-1} = a^{-1}a = 1$, where 1 is the neutral element of real number multiplication $(1a = a(1) = a$ for all $a)$. Dividing a by a nonzero number b amounts to multiplying a by the reciprocal of b : $a/b = ab^{-1}$.

In matrix multiplication, the identity matrix plays the role of the neutral element (see property 6 in Theorem 1.5). If A is an $m \times n$ matrix, then

$$AI_n = I_m A = A.$$

Let us introduce a counterpart of a real number reciprocal for matrices.

You might ask whether $AB = I_m$ and $BA = I_n$ can be satisfied by $m \times n$ matrices A and $n \times m$ matrices B with $m \neq n$. It turns out to be impossible, as will be shown in Exercise 23 on p. 215. This is the reason why our definition admits square matrices only.

DEFINITION (Matrix Inverse)

An *inverse* of an $n \times n$ matrix A is a matrix B such that

$$AB = BA = I_n.$$

If a matrix A has an inverse, then A is said to be *invertible*, or *nonsingular*.

Otherwise, we say A is *noninvertible*, or *singular*.

THEOREM 2.9 If an inverse of a matrix exists, then it is unique.

PROOF

Let B and C both be inverses of an $n \times n$ matrix A. From the definition, we have

$$AC = I_n \qquad (*)$$

and

$$BA = I_n. \qquad (**)$$

Then

$$B \overset{\substack{\text{Th. 1.5}\\ \text{part 6}}}{=} BI_n \overset{(*)}{=} B(AC) \overset{\substack{\text{Th. 1.5}\\ \text{part 1}}}{=} (BA)C \overset{(**)}{=} I_nC \overset{\substack{\text{Th. 1.5}\\ \text{part 6}}}{=} C.$$

■

According to the above theorem, we can refer to *the* inverse of a nonsingular matrix A. We shall denote the inverse of A by A^{-1}.

EXAMPLE 2.12

a. Is $\begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}$ the inverse of $\begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix}$?

b. Is $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 2 \\ 0 & 1 & 1 \end{bmatrix}$ the inverse of $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$?

SOLUTION

a. Performing the matrix multiplications $\begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$ and $\begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$, we conclude that $\begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}$ is the inverse of $\begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix}$.

b. Multiplying the matrices yields $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 2 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 5 & 8 \\ 6 & 7 & 11 \\ 2 & 2 & 3 \end{bmatrix} \neq I_3$.

Consequently, $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 2 \\ 0 & 1 & 1 \end{bmatrix}$ is not the inverse of $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$.

Our next objective is to find a method to invert a given matrix, if possible. Before we can accomplish this, we need to develop some additional theory.

LEMMA 2.10 If A and B are $n \times n$ matrices such that $AB = I_n$, then A is row equivalent to I_n.

PROOF

Let us assume that A is **not** row equivalent to I_n. Because of this assumption, there must exist a finite number of elementary matrices $E_1, \ldots, E_k$ such that the matrix $E_k \cdots E_1 A$ contains a zero row. Therefore, so does the matrix $E_k \cdots E_1 AB$. However, since we assumed $AB = I_n$, it follows that

$$E_k \cdots E_1 AB = E_k \cdots E_1 I_n,$$

which means we reached a contradiction: a product of elementary matrices and an identity matrix on the right-hand side cannot have a zero row.

We conclude that A is row equivalent to I_n.

■

LEMMA 2.11 If A is row equivalent to I_n, then there exists a matrix C such that $CA = I_n$.

PROOF

Since A is row equivalent to I_n, there exist elementary matrices $E_1, \ldots, E_k$ such that

$$E_k \cdots E_1 A = I_n.$$

Consequently, $C = E_k \cdots E_1$ satisfies $CA = I_n$.

■

The lemmas above serve as stepping stones which yield important results that follow.

THEOREM 2.12 If A and B are $n \times n$ matrices such that $AB = I_n$, then $BA = I_n$.

PROOF

Let us assume that A and B are $n \times n$ matrices such that $AB = I_n$. From Lemma 2.10, it follows that A is row equivalent to I_n. Furthermore, from Lemma 2.11, there exists a matrix C such that $CA = I_n$. By following steps similar to those in the proof of Theorem 2.9, we can show that $C = B$; therefore, $BA = I_n$.

■

Recall that in part a of Example 2.12, we verified that $AB = I_n$ and $BA = I_n$ as well. According to Theorem 2.12, the latter follows automatically from the former.

Theorem 2.9 and Lemmas 2.10 and 2.11 lead us directly to our next theorem.

THEOREM 2.13 An $n \times n$ matrix is nonsingular if and only if it is row equivalent to I_n.

Consequently, once we see that a square matrix cannot be row reduced to identity, we can conclude that it must be singular. For instance, if A has its ith row filled completely with zeros, then so does AB, making it impossible for $AB = I_n$ to hold true.

THEOREM 2.14 An $n \times n$ matrix containing a row of zeros must be singular.

We are now ready to formulate an efficient method to decide whether a given $n \times n$ matrix A is invertible and, if so, to find its inverse. We begin by forming an $n \times 2n$ matrix $[A|I_n]$.

- If A is invertible, then following Theorem 2.13 as well as the reasoning in our proof of Lemma 2.11, we have
$$E_k \cdots E_1 A = I_n \text{ and } E_k \cdots E_1 I_n = A^{-1};$$
therefore,
$$E_k \cdots E_1 [\, A \mid I_n \,] = [\, I_n \mid A^{-1} \,]$$
so that $[A|I_n]$ is row equivalent to $[\, I_n \mid A^{-1} \,]$, which is guaranteed to be in a reduced row echelon form (check!).
- If A is singular, then by Theorem 2.13 the r.r.e.f. of A is not I_n – instead, it must be a matrix whose nth row (and possibly others) is completely made up of zeros. Note that by Theorem 2.14 we can conclude that A is singular as soon as we see it become row equivalent to a matrix with a zero row (without having to obtain the r.r.e.f.).

Let us summarize our findings.

Procedure for finding the inverse of an $n \times n$ matrix A:

Form the $n \times 2n$ matrix $[\, A \mid I_n \,]$ and perform elementary row operations to obtain $[\, C \mid D \,]$ (C and D are $n \times n$ matrices) where either $C = I_n$ or C contains a row of zeros.

· If $C = I_n$, then $D = A^{-1}$.

· If C contains a row of zeros, then the matrix A is singular.

EXAMPLE 2.13 Find the inverse of $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix}$.

$$[\, A \mid I_3 \,] = \left[\begin{array}{rrr|rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ -6 & 2 & 3 & 0 & 0 & 1 \end{array}\right]$$

$r_2 - r_1 \to r_2$

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ -6 & 2 & 3 & 0 & 0 & 1 \end{array}\right]$$

$r_3 + 6r_1 \to r_3$

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & -4 & 3 & 6 & 0 & 1 \end{array}\right]$$

$r_3 + 4r_2 \to r_3$

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 2 & 4 & 1 \end{array}\right]$$

$-1r_3 \to r_3$

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right]$$

$r_2 + r_3 \to r_2$

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right]$$

$r_1 + r_2 \to r_1$

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -2 & -3 & -1 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right] = [\, I_3 \mid A^{-1} \,]$$

Answer: $A^{-1} = \begin{bmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{bmatrix}$.

The inverse obtained in the example above can be checked by multiplying it by the matrix A :

$$\begin{bmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix} \overset{\checkmark}{=} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

EXAMPLE 2.14 Find the inverse of $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ -2 & 3 & -2 \end{bmatrix}$.

$$[\, A \mid I_3 \,] = \left[\begin{array}{rrr|rrr} 1 & 2 & 0 & 1 & 0 & 0 \\ 3 & -1 & 2 & 0 & 1 & 0 \\ -2 & 3 & -2 & 0 & 0 & 1 \end{array}\right]$$

$r_2 - 3r_1 \to r_2$

$$\left[\begin{array}{rrr|rrr} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -7 & 2 & -3 & 1 & 0 \\ -2 & 3 & -2 & 0 & 0 & 1 \end{array}\right]$$

$r_3 + 2r_1 \to r_3$

$$\left[\begin{array}{rrr|rrr} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -7 & 2 & -3 & 1 & 0 \\ 0 & 7 & -2 & 2 & 0 & 1 \end{array}\right]$$

$r_3 + r_2 \to r_3$

$$\left[\begin{array}{rrr|rrr} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -7 & 2 & -3 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right] = [\, C \mid D \,]$$

Since C contains a row of zeros, we conclude that A is singular.

This procedure is implemented by the matrix inverse module included in the Linear Algebra Toolkit (entitled "Calculating the inverse using row operations"). You can use it to invert matrices up to size 6 × 6.

Properties of matrix inverse

THEOREM 2.15 (Inverse of a Matrix Product)

If A and B are $n \times n$ nonsingular matrices, then AB is also a nonsingular matrix and

$$(AB)^{-1} = B^{-1}A^{-1}.$$

PROOF

According to the definition of the inverse, for $n \times n$ matrices C and D,

$$C^{-1} = D \quad \text{if} \quad CD = I_n \quad \text{and} \quad DC = I_n.$$

Taking $C = AB$ and $D = B^{-1}A^{-1}$ and using properties 1 and 6 of Theorem 1.5 as well as the definition of the inverse, we obtain

$$CD = (AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AI_nA^{-1} = AA^{-1} = I_n.$$

Theorem 2.12 yields $DC = I_n$; therefore, we conclude that $D = B^{-1}A^{-1}$ is the inverse of $C = AB$.

■

Two additional properties of matrix inverse are stated in the theorem below. Their proofs are left as exercises (25 and 26 on p. 97) and can be conducted similarly to the proof of Theorem 2.15.

THEOREM 2.16 If A is a nonsingular matrix, then

1. A^{-1} is also a nonsingular matrix and $(A^{-1})^{-1} = A$;
2. A^T is also a nonsingular matrix and $(A^T)^{-1} = (A^{-1})^T$.

Four equivalent statements

According to Theorem 2.13, an $n \times n$ matrix A is nonsingular if and only if it is row equivalent to I_n. On the other hand, Theorem 2.8 established the equivalence of the latter statement to two others:

- the linear system $A\overrightarrow{x} = \overrightarrow{b}$ has a unique solution for any n-vector $\overrightarrow{b}$, and
- the system $A\overrightarrow{x} = \overrightarrow{0}$ has a unique solution.

These two theorems can be “repackaged” more neatly in the following form.

4 Equivalent Statements

For an $n \times n$ matrix A, the following statements are equivalent.

1. A is nonsingular.
2. A is row equivalent to I_n.
3. For every n-vector $\overrightarrow{b}$, the system $A\overrightarrow{x} = \overrightarrow{b}$ has a unique solution.
4. The homogeneous system $A\overrightarrow{x} = \overrightarrow{0}$ has only the trivial solution.

Throughout this book, we will be adding more and more statements to this list. The more we add, the more we will appreciate the advantage offered by this “multilateral” format as opposed to the “bilateral” theorems like Theorem 2.13.

Note that the equivalence means that if one of the statements is not satisfied, none of the others are satisfied either.

4 Equivalent “Negative” Statements

For an $n \times n$ matrix A, the following statements are equivalent.

-1. A is singular.

-2. A is not row equivalent to I_n.

-3. For some n-vector $\overrightarrow{b}$, the system $A\overrightarrow{x} = \overrightarrow{b}$ has either no solution or many solutions.

-4. The homogeneous system $A\overrightarrow{x} = \overrightarrow{0}$ has nontrivial solutions.

Inverse of the coefficient matrix and solution of a system

According to the equivalent conditions introduced above, a linear system of n equations in n unknowns with a nonsingular coefficient matrix A is guaranteed to have a unique solution. If A^{-1} is actually known, then it can be used to determine this unique solution. Multiplying both sides of

$$A\overrightarrow{x} = \overrightarrow{b}$$

from the left by A^{-1} we obtain

$$A^{-1}A\overrightarrow{x} = A^{-1}\overrightarrow{b}.$$

The left-hand side equals $I_n\overrightarrow{x}$, so that

$$\overrightarrow{x} = A^{-1}\overrightarrow{b}. \tag{24}$$

This is an explicit formula for the unique solution $\overrightarrow{x}$.

EXAMPLE 2.15 The linear system

$$\begin{array}{rcrcrcr} x_1 & - & x_2 & & & = & -2 \\ x_1 & & & - & x_3 & = & 2 \\ -6x_1 & + & 2x_2 & + & 3x_3 & = & -3 \end{array}$$

can be written in the form

$$A\overrightarrow{x} = \overrightarrow{b} \text{ with } A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix} \text{ and } \overrightarrow{b} = \begin{bmatrix} -2 \\ 2 \\ -3 \end{bmatrix}.$$

In Example 2.13, we found the inverse of the coefficient matrix A (at the same time showing that A is nonsingular): $A^{-1} = \begin{bmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{bmatrix}$. Equation (24) yields

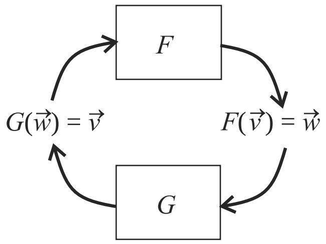
$$\overrightarrow{x} = \underbrace{\begin{bmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{bmatrix}}_{A^{-1}} \underbrace{\begin{bmatrix} -2 \\ 2 \\ -3 \end{bmatrix}}_{\overrightarrow{b}} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}.$$

(Check that $A\overrightarrow{x} = \overrightarrow{b}$. You may also want to verify that the same solution is obtained using Gauss-Jordan reduction or Gaussian elimination.)

Suppose we are asked to solve multiple linear systems that share the same coefficient matrix, but for different right-hand side vectors: $\overrightarrow{Ax^{(1)} = b^{(1)}}$, $\overrightarrow{Ax^{(2)} = b^{(2)}}$, etc. Rather than performing elementary row operations on $[A|\overrightarrow{b^{(1)}}]$, $[A|\overrightarrow{b^{(2)}}]$, etc., an attractive alternative would appear to be to find the inverse of A, then multiply it by each right-hand side vector. Unfortunately, when using finite precision arithmetic, this approach can potentially introduce large errors into the computation.

However, an efficient solution can be found that does not have the numerical drawbacks associated with inverting matrices – it will involve matrix factorizations discussed in the following section.

Invertible transformations



In Section 1.4, we have introduced linear transformations and have shown that if $F : R^n \to R^n$ is a linear transformation, then an $n \times n$ matrix A exists such that $F(\overrightarrow{x}) = A\overrightarrow{x}$ for all n-vectors $\overrightarrow{x}$.

If A is nonsingular, then we can define another transformation $G : R^n \to R^n$ by taking $G(\overrightarrow{x}) = A^{-1}\overrightarrow{x}$ for all $\overrightarrow{x}$ in R^n. The transformation G is called an *inverse transformation* of F because

$$G(F(\overrightarrow{x})) = \overrightarrow{x} \text{ and } F(G(\overrightarrow{y})) = \overrightarrow{y} \text{ for all } \overrightarrow{x} \text{ and } \overrightarrow{y} \text{ in } R^n.$$

A transformation F is said to be invertible if it has an inverse transformation; otherwise, it is called noninvertible.

EXAMPLE 2.16 The transformation $F : R^2 \to R^2$ defined in Example 1.22 performed a counterclockwise rotation by 90 degrees:

$$F(\begin{bmatrix} x \\ y \end{bmatrix}) = \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix}.$$

To invert A, we set up

$$\left[\begin{array}{rr|rr} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right]$$

and perform elementary row operations ($r_1 \leftrightarrow r_2$; $-r_2 \to r_2$) to obtain

$$\left[\begin{array}{rr|rr} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{array}\right].$$

The transformation F is invertible – its inverse transformation,

$$G(\begin{bmatrix} x \\ y \end{bmatrix}) = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{A^{-1}} \begin{bmatrix} x \\ y \end{bmatrix},$$

corresponds to clockwise rotation by 90 degrees (check by referring to Example 1.23).

EXAMPLE 2.17 One of the transformations listed in the table on p. 42 is $F(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ – projection of vectors in R^3 onto the xy-plane. There are two very good reasons why this transformation is not invertible:

- the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is singular (check!) and
- it is impossible to find any function (not just a linear transformation) that will "recover" the original vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ from its "shadow" in the xy-plane $\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$ (the z component of the original vector is irretrievably lost).

Note that linear transformations from R^n to R^m with $n \neq m$ cannot be invertible, as their matrices are not square (only square matrices can possibly be inverted).

EXERCISES

1. Is $\begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}$ the inverse of $\begin{bmatrix} 1 & 3 \\ 1 & 6 \end{bmatrix}$?

2. Is $\begin{bmatrix} 3 & 8 \\ 2 & 5 \end{bmatrix}$ the inverse of $\begin{bmatrix} -5 & 8 \\ 2 & -3 \end{bmatrix}$?

3. Is $\begin{bmatrix} -4 & 0 & -3 \\ 0 & 1 & 2 \\ 7 & 0 & 5 \end{bmatrix}$ the inverse of $\begin{bmatrix} 5 & 0 & 3 \\ 14 & 1 & 8 \\ -7 & 0 & -4 \end{bmatrix}$?

4. Is $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 4 \\ 3 & 2 & 1 \end{bmatrix}$ the inverse of $\begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$?

In Exercises 5–12, follow the procedure on p. 89 to find the inverse of each matrix if possible. If you obtained the inverse, verify it multiplying it by the original matrix. (Do not use technology.)

5. a. $\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$; b. $\begin{bmatrix} 2 & 3 \\ 2 & 5 \end{bmatrix}$; c. $\begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$.

6. a. $\begin{bmatrix} -2 & 1 \\ -1 & 1 \end{bmatrix}$; b. $\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix}$; c. $\begin{bmatrix} \frac{1}{2} & -2 \\ \frac{3}{2} & 4 \end{bmatrix}$.

7. a. $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; b. $\begin{bmatrix} -2 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix}$; c. $\begin{bmatrix} -3 & 2 & 2 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}$.

8. a. $\begin{bmatrix} 3 & 0 & -2 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$; b. $\begin{bmatrix} 1 & 0 & 0 \\ 3 & -2 & 0 \\ -2 & -3 & 1 \end{bmatrix}$; c. $\begin{bmatrix} 0 & 1 & 1 \\ 3 & -1 & 2 \\ 2 & 1 & 3 \end{bmatrix}$.

9. a. $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$; b. $\begin{bmatrix} 0 & 1 & -1 & 0 \\ 2 & 0 & 1 & 1 \\ 2 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$.

10. a. $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$; b. $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 1 & 1 \end{bmatrix}$.

11. a. $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$; b. $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

12. a. $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$; b. $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

In Exercises 13–16, solve each system by using the inverse of the coefficient matrix.

13.

$$\begin{array}{rcrcr} x & - & y & = & 1 \\ -4x & + & 3y & = & -3 \end{array}$$

14.

$$\begin{array}{rcrcr} 2x & + & 3y & = & 3 \\ 2x & + & 4y & = & 2 \end{array}$$

15.

$$\begin{array}{rcrcrcr} & & 3y & + & 2z & = & 0 \\ -x & + & 2y & + & z & = & -1 \\ x & & & & & = & -1 \end{array}$$

16.

$$\begin{array}{rcrcrcr} 2x & - & y & & & = & 1 \\ x & & & + & z & = & 0 \\ x & + & y & + & 2z & = & -4 \end{array}$$

In Exercises 17–20, determine if the given linear transformation is invertible. If so, describe the inverse transformation geometrically and find its matrix.

17. $F : R^2 \to R^2$ performing reflection with respect to the y-axis.

18. $F : R^2 \to R^2$ performing the dilation by the factor 3.

19. $F : R^3 \to R^3$ performing the projection onto the z-axis.

20. $F : R^3 \to R^3$ performing the reflection with respect to the xz-plane.

In Exercises 21–24, you are given inverses A^{-1} and B^{-1}; all problems in parts a–d can be solved by relying on properties of the matrix inverse, *without* actually *inverting any matrices* (in particular, without inverting A^{-1} to obtain A or inverting B^{-1} to obtain B).

21. Given $A^{-1} = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 0 & 3 & 1 \\ 3 & 1 & 3 \\ 1 & 3 & 0 \end{bmatrix}$

a. evaluate $(AB)^{-1}$, b. evaluate $\left(A^T\right)^{-1}$,

c. find $\overrightarrow{x}$ such that $A\overrightarrow{x} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$,

d. are the matrices A and B row equivalent? (Justify your answer.)

22. Given $A^{-1} = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 4 & 0 & 3 & 1 \end{bmatrix}$

a. evaluate $(BA)^{-1}$, b. evaluate $\left(A^T\right)^{-1}$,

c. find $\overrightarrow{x}$ such that $A\overrightarrow{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$,

d. are the matrices A and B row equivalent? (Justify your answer.)

23. Given $A^{-1} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 2 & 0 & 2 & 0 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

a. evaluate $(B^T A)^{-1}$, b. evaluate $((A^{-1})^{-1})^{-1}$,

c. evaluate $\left(B^2\right)^{-1}$, d. evaluate $(AB^{-1})^{-1}(BA^{-1})^{-1}$.

24. Given $A^{-1} = \begin{bmatrix} 5 & 4 & 2 \\ 2 & 3 & 1 \\ 1 & 0 & 3 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ 2 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix}$

a. evaluate $(A^T B)^{-1}$, b. evaluate $((B^T)^{-1})^T$,

c. evaluate $\left(A^2\right)^{-1}$, d. evaluate $B^{-1}(AB^{-1})^{-1}$.

In Exercises 25 and 26, you may find it helpful to adopt an approach similar to the one used in the proof of Theorem 2.15.

25. * Prove part 1 of Theorem 2.16.

26. * Prove part 2 of Theorem 2.16.

T/F?

In Exercises 27–34, decide whether each statement is true or false. Justify your answer.

27. For all $n \times n$ nonsingular matrices A and B, $(A^{-1}B^{-1})^T = (A^T B^T)^{-1}$.

28. For all $n \times n$ nonsingular matrices A, $\left(A^3\right)^{-1} = \left(A^{-1}\right)^3$.

29. If A is a 2×2 matrix such that the linear system $A\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ has no solution, then A is nonsingular.

30. If $A \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = A \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, then A is invertible.

31. If A is row equivalent to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then $A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \\ 3 \end{bmatrix}$ is consistent.

32. If A is a nonsingular matrix, then its r.r.e.f. contains at least one zero row.

33. All nonsingular 5×5 matrices are row equivalent.

34. There exist some invertible linear transformations $F : R^2 \to R^3$.

35. * If matrices A, B, C, and D have sizes $m \times m$, $m \times n$, $n \times m$, and $n \times n$, respectively, and A is nonsingular, show that

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} I_m & 0 \\ \hline CA^{-1} & I_n \end{array}\right] \left[\begin{array}{c|c} A & B \\ \hline 0 & D - CA^{-1}B \end{array}\right].$$

2.4 Applications of Linear Systems and Matrix Factorizations

Application: Alloys

In the following example, a linear system is set up using the alloy vectors introduced in Example 1.5.

EXAMPLE 2.18 Suppose a supply of the following three alloys of gold is available:

- alloy 1 (22 karat), made up of 22 parts gold, 1 part silver, and 1 part copper;
- alloy 2 (14 karat), made up of 14 parts gold, 6 parts silver, and 4 parts copper;
- alloy 3 (18 karat), made up of 18 parts gold and 6 parts copper.

How can these three alloys be combined to obtain:

- alloy 4 (18 karat), made up of 18 parts gold, 3 parts silver, and 3 parts copper?

To solve this problem, we set up a system of three linear equations in three unknowns:

$$\begin{array}{rcrcrcl} \frac{22}{24}x_1 & + & \frac{14}{24}x_2 & + & \frac{18}{24}x_3 & = & \frac{18}{24} \\ \frac{1}{24}x_1 & + & \frac{6}{24}x_2 & & & = & \frac{3}{24} \\ \frac{1}{24}x_1 & + & \frac{4}{24}x_2 & + & \frac{6}{24}x_3 & = & \frac{3}{24} \end{array}$$

In this system, the unknowns $x_1, x_2,$ and x_3 represent the amounts of the first, second, and third alloy used to mix together and obtain the fourth alloy – we must insist on these unknowns being nonnegative in order for the solution to make sense. The first equation expresses the requirement that the amount of gold in the mix (left-hand side) matches the required amount of gold (right-hand side). The second and the third equations express the analogous requirements with respect to silver and copper, respectively.

Using the augmented matrix of the system (and referring to the Linear Algebra Toolkit for details)

$$\left[\begin{array}{ccc|c} \frac{22}{24} & \frac{14}{24} & \frac{18}{24} & \frac{18}{24} \\ \frac{1}{24} & \frac{6}{24} & 0 & \frac{3}{24} \\ \frac{1}{24} & \frac{4}{24} & \frac{6}{24} & \frac{3}{24} \end{array}\right] \begin{array}{c} \to\to\to\to\to\to \\ \text{a sequence} \\ \text{of elem.} \\ \text{row ops.} \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{3}{7} \\ 0 & 1 & 0 & \frac{3}{7} \\ 0 & 0 & 1 & \frac{1}{7} \end{array}\right]$$

we can conclude that the system has a unique solution:

$$\begin{array}{rcl} x_1 & = & \frac{3}{7} \\ x_2 & = & \frac{3}{7} \\ x_3 & = & \frac{1}{7} \end{array}$$

Therefore, alloy 4 can be obtained by mixing three parts of alloy 1, three parts of alloy 2, and one part of alloy 3.

Here is a vector interpretation of this solution

$$\frac{3}{7}\begin{bmatrix} \frac{22}{24} \\ \frac{1}{24} \\ \frac{1}{24} \end{bmatrix} + \frac{3}{7}\begin{bmatrix} \frac{14}{24} \\ \frac{6}{24} \\ \frac{4}{24} \end{bmatrix} + \frac{1}{7}\begin{bmatrix} \frac{18}{24} \\ 0 \\ \frac{6}{24} \end{bmatrix} = \begin{bmatrix} \frac{18}{24} \\ \frac{3}{24} \\ \frac{3}{24} \end{bmatrix},$$

where each alloy is represented by a vector: $\begin{bmatrix} \text{gold content} \\ \text{silver content} \\ \text{copper content} \end{bmatrix}$.

Application: Balancing chemical reaction equations

Chemical equations are often used to describe chemical reactions. For example, the reaction taking place when methane is burning in the air is represented by the equation

$$CH_4 + 2O_2 \to CO_2 + 2H_2O.$$

Methane (CH_4) and oxygen (O_2) are the reactants, positioned on the left side of the arrow. On the right side, the products of the reaction are listed: carbon dioxide (CO_2) and water (H_2O).

During such a reaction, **atoms can be neither created nor destroyed**. For example, there are four oxygen atoms on the reactant side (since $2O_2$ denotes two molecules, each containing two oxygen atoms) and four oxygen atoms on the product side.

EXAMPLE 2.19 Let us balance the reaction $x_1\mathrm{NH_3} + x_2\mathrm{O_2} \to x_3\mathrm{NO} + x_4\mathrm{H_2O}$.

We form a system of equations, each corresponding to a different element.

Each molecule of both ammonia ($\mathrm{NH_3}$) and nitrogen oxide (NO) contains one atom of nitrogen. Therefore, if reactants include x_1 molecules of ammonia and products include x_3 molecules of nitrogen oxide, then we must have

$$x_1 = x_3.$$

Likewise, we can balance the number of hydrogen atoms (keeping in mind that each ammonia molecule contains three hydrogen atoms, while each molecule of water has two):

$$3x_1 = 2x_4.$$

Finally, for oxygen we have

$$2x_2 = x_3 + x_4.$$

These three equations can now be rewritten in the standard form (18)

$$\begin{array}{rcrcrcrcl} x_1 & & & - & x_3 & & & = & 0 \\ 3x_1 & & & & & - & 2x_4 & = & 0 \\ & & 2x_2 & - & x_3 & - & x_4 & = & 0 \end{array} \qquad (25)$$

The augmented matrix of this linear system, $\left[\begin{array}{rrrr|r} 1 & 0 & -1 & 0 & 0 \\ 3 & 0 & 0 & -2 & 0 \\ 0 & 2 & -1 & -1 & 0 \end{array}\right]$, has the reduced row echelon form: $\left[\begin{array}{rrrr|r} 1 & 0 & 0 & -\frac{2}{3} & 0 \\ 0 & 1 & 0 & -\frac{5}{6} & 0 \\ 0 & 0 & 1 & -\frac{2}{3} & 0 \end{array}\right]$. (Check this by hand, or use the Linear Algebra Toolkit.)

Since the fourth column has no leading entry, x_4 is arbitrary, whereas the remaining variables can be expressed in terms of x_4 :

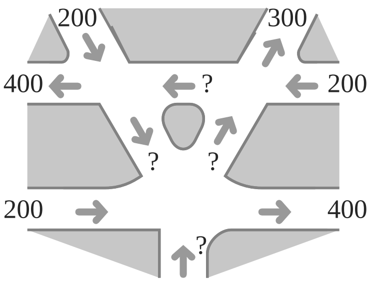
$$\begin{aligned} x_1 &= \frac{2}{3}x_4 \\ x_2 &= \frac{5}{6}x_4 \\ x_3 &= \frac{2}{3}x_4. \end{aligned}$$

While mathematically this system has infinitely many solutions, including these three
(a) $x_1 = \frac{2}{3}$, $x_2 = \frac{5}{6}$, $x_3 = \frac{2}{3}$, $x_4 = 1$,
(b) $x_1 = -8$, $x_2 = -10$, $x_3 = -8$, $x_4 = -12$,
(c) $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $x_4 = 0$,
none of these are considered acceptable solutions to the problem of balancing the given chemical equation for various reasons: (a) it doesn't make sense to consider a fraction (e.g., $2/3$) of a molecule, (b) the coefficients should not be negative (otherwise a reactant would become a product, and vice versa), and (c) an equation with all coefficients equal to zero represents **no** reaction.

What we really want is to make sure that all the coefficients are positive integers, using as small values as possible. Taking $x_4 = 6$ yields $x_1 = 4$, $x_2 = 5$, and $x_3 = 4$, and this corresponds to the balanced reaction

$$4\mathrm{NH_3} + 5\mathrm{O_2} \to 4\mathrm{NO} + 6\mathrm{H_2O}.$$

Application: Network flow

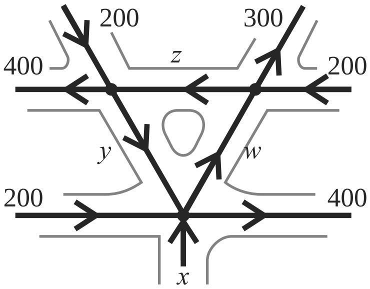


EXAMPLE 2.20 Consider the network of one-way streets depicted in the margin. Each number indicates the traffic flow, in cars per hour, measured along the given street. Our objective is to determine the unknown traffic flow figures (indicated with question marks).

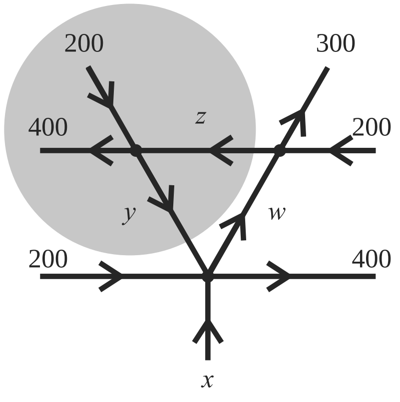
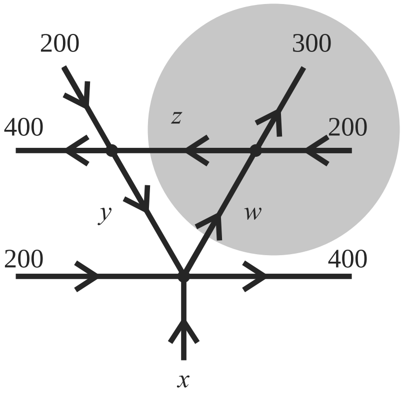
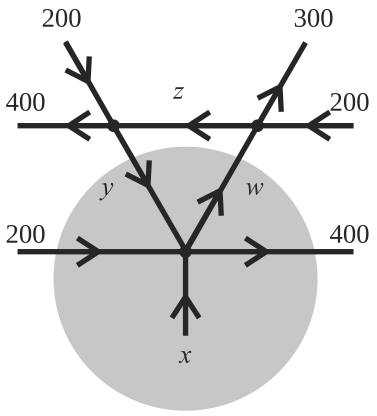
Let us superimpose a network of oriented line segments on our street network so that we can clearly observe the connections between various quantities. Furthermore, we denote the four unknown quantities by x, y, z, and w and then proceed to set up equations reflecting the relationships between the known and the unknown quantities.

The intersections are designated with solid dots in our diagram. Each intersection yields exactly one equation, based on the principle

$$\left(\begin{array}{c}\text{total traffic}\\ \text{arriving at}\\ \text{the intersection}\end{array}\right) = \left(\begin{array}{c}\text{total traffic}\\ \text{leaving}\\ \text{the intersection}\end{array}\right).$$



In the following table, we apply this principle to obtain the equations corresponding to the highlighted intersection.

		
$z + 200 = y + 400$	$w + 200 = z + 300$	$x + y + 200 = w + 400$

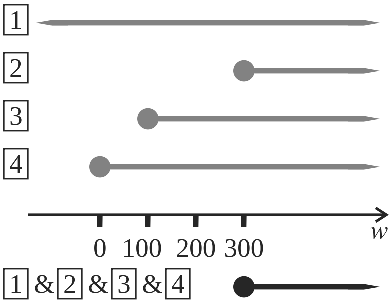
The three linear equations can now be rewritten in the standard form, with the unknown terms on the left-hand sides:

$$\begin{array}{rcrcrcrcl} & - & y & + & z & & & = & 200 \\ & & & - & z & + & w & = & 100 \\ x & + & y & & & - & w & = & 200. \end{array}$$

The augmented matrix of this system, $\left[\begin{array}{rrrr|r} 0 & -1 & 1 & 0 & 200 \\ 0 & 0 & -1 & 1 & 100 \\ 1 & 1 & 0 & -1 & 200 \end{array}\right]$, has the reduced row echelon form $\left[\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & 500 \\ 0 & 1 & 0 & -1 & -300 \\ 0 & 0 & 1 & -1 & -100 \end{array}\right]$. (Check!)

Therefore, there are infinitely many solutions to our problem

$$\begin{aligned} x &= 500 \\ y &= w - 300 \\ z &= w - 100 \end{aligned}$$



where w can have an arbitrary value.

To avoid reversing the directions of the one-way streets in our network, we must choose w to make sure that the inequalities

$$\boxed{1}\ x \geq 0;\ \boxed{2}\ y \geq 0;\ \boxed{3}\ z \geq 0;\ \boxed{4}\ w \geq 0$$

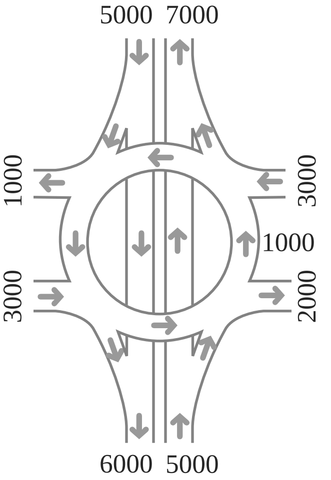
hold true – we shall refer to the solutions of our system that satisfy these inequalities as *feasible solutions*.

To determine the w values leading to the feasible solutions rewrite the four inequalities in terms of w :

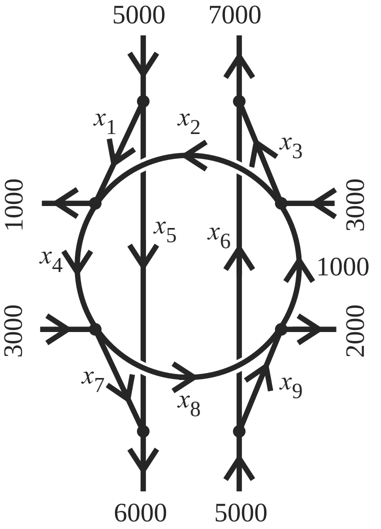
$$\boxed{1}\ 500 \geq 0;\ \boxed{2}\ w - 300 \geq 0;\ \boxed{3}\ w - 100 \geq 0;\ \boxed{4}\ w \geq 0.$$

Refer to the figure in the margin to see how we can use the w-axis to graphically find the set of values of w that satisfies all four inequalities simultaneously – in our case all such solutions correspond to $w \geq 300$.

For example, if $w = 350$, then $x = 500$, $y = 50$, and $z = 250$.



Example 2.21 A two-level intersection is shown in the margin, including traffic volumes per hour during the evening commute. Our objective is to determine all remaining traffic volumes throughout this intersection.



We begin by redrawing the diagram using one-way street segments and then denoting each unknown quantity by a name $x_1, \ldots, x_9$. There are eight intersections involving these one-way segments. Each of them is marked by a solid dot – make sure to distinguish these intersections from the situations where one segment passes over another one at a different level!

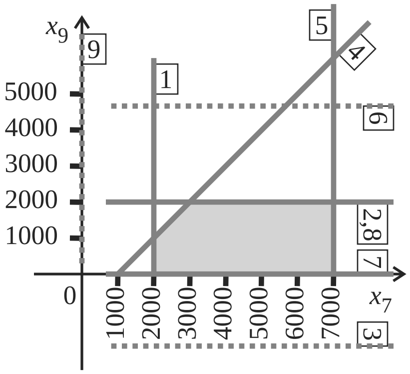
Similarly to the previous example, each intersection leads to a single equation. Verify that the system has an augmented matrix

$$\left[\begin{array}{ccccccccc|c} -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -5000 \\ 1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1000 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 7000 \\ 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -4000 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & 0 & -3000 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 7000 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2000 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & -5000 \end{array}\right]$$

whose reduced row echelon form is

$$\left[\begin{array}{cccccccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -2000 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2000 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 2000 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & -1000 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 7000 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 5000 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2000 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

(Use the Linear Algebra Toolkit to verify this.)



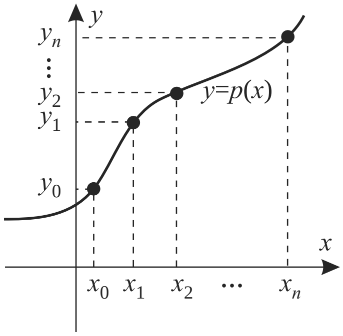
Two of the nine unknowns, x_7 and x_9, are arbitrary, subject to the restriction that they lead to nonnegative values of all unknowns. To graphically determine feasible solutions, we plot an intersection of half-planes corresponding to the inequalities $\boxed{1}\ x_1 \geq 0; \ldots;\ \boxed{9}\ x_9 \geq 0$. We plot the line for each equation, along with the number tag on the side of it where the inequality holds (lines not adjacent to the feasible region are plotted as dotted lines).

An example of a feasible solution can be obtained by taking $x_7 = 5000$, $x_9 = 1000$, leading to $x_1 = 3000$, $x_2 = 1000$, $x_3 = 3000$, $x_4 = 3000$, $x_5 = 2000$, $x_6 = 4000$, $x_8 = 1000$.

Determining nonnegative feasible solutions in cases involving three or more arbitrary variables is considerably more complex and will not be required in this book (geometrically, the corresponding feasible region will be an intersection of half-spaces in higher-dimensional space).[8]

Application: Polynomial interpolation

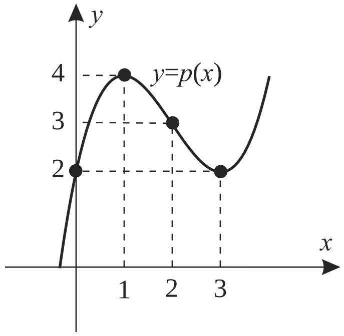
Consider $n+1$ given points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ such that the values $x_0, x_1, \ldots, x_n$ are distinct. A polynomial of degree n or less, $p(x) = a_0 + a_1 x + \cdots + a_n x^n$, that passes through all the given points, i.e., $p(x_i) = y_i$ for $i = 0, 1, \ldots, n$, is called the *Lagrange interpolating polynomial*. Its coefficients satisfy the following linear system:



$$\begin{array}{ccccccccc} a_0 & + & x_0 a_1 & + & \cdots & + & x_0^n a_n & = & y_0 \\ a_0 & + & x_1 a_1 & + & \cdots & + & x_1^n a_n & = & y_1 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_0 & + & x_n a_1 & + & \cdots & + & x_n^n a_n & = & y_n \end{array}$$

It can be shown that this linear system has a unique solution.

Example 2.22 Find the Lagrange interpolating polynomial of degree 3 or less passing through $(0,2)$, $(1,4)$, $(2,3)$, and $(3,2)$.



The polynomial we are seeking has the form $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ with the coefficients a_0, a_1, a_2, and a_3 satisfying the linear system

$$\begin{array}{ccccccccc} a_0 & & & & & & & = & 2 \\ a_0 & + & a_1 & + & a_2 & + & a_3 & = & 4 \\ a_0 & + & 2a_1 & + & 2^2 a_2 & + & 2^3 a_3 & = & 3 \\ a_0 & + & 3a_1 & + & 3^2 a_2 & + & 3^3 a_3 & = & 2 \end{array}$$

---

[8] A related problem of great practical importance is a linear programming problem – it involves maximizing a linear function of the unknowns over the feasible region.

Since the augmented matrix $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 1 & 1 & 1 & 1 & 4 \\ 1 & 2 & 4 & 8 & 3 \\ 1 & 3 & 9 & 27 & 2 \end{array}\right]$ has the r.r.e.f. $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & \frac{9}{2} \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \end{array}\right]$ we conclude that $p(x) = 2 + \frac{9}{2}x - 3x^2 + \frac{1}{2}x^3$ is the desired polynomial.

Linear systems with parameters

The linear systems we have solved so far in this chapter involved only given real numbers as coefficients and right-hand side values. However, in some problems it will become desirable to study such systems in which at least some of these values are not explicitly specified but are left as parameters instead.

EXAMPLE 2.23 Consider the linear system with the augmented matrix $\left[\begin{array}{cc|c} a & 4 & -6 \\ 1 & a & 3 \end{array}\right]$.

The first elementary row operation we will perform is $r_1 \leftrightarrow r_2$. This is done to avoid having to divide by a, since we cannot guarantee $a \neq 0$.

$$\left[\begin{array}{cc|c} 1 & a & 3 \\ a & 4 & -6 \end{array}\right]$$

Now, we can safely eliminate the $(2,1)$ entry using the operation $r_2 - ar_1 \to r_2$.

$$\left[\begin{array}{cc|c} 1 & a & 3 \\ 0 & 4-a^2 & -6-3a \end{array}\right]$$

If $4 - a^2 \neq 0$, i.e., a is neither -2 nor 2, then the system has a *unique solution* (each of the first two columns will contain a leading entry).

If $a = -2$, then the matrix becomes $\left[\begin{array}{cc|c} 1 & -2 & 3 \\ 0 & 0 & 0 \end{array}\right]$, corresponding to *infinitely many solutions* (the second unknown is arbitrary since there is no leading entry in its column).

If $a = 2$, then the matrix $\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 0 & -12 \end{array}\right]$ indicates the system has *no solution* (the second equation, $0 = -12$, is inconsistent).

LU decomposition

EXAMPLE 2.24 Consider a sequence of elementary row operations transforming the coefficient matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 0 & -1 \\ 2 & 2 & 3 \end{bmatrix}$ to an upper triangular matrix:

$$A \xrightarrow{r_2 - \frac{3}{2}r_1 \to r_2} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -\frac{3}{2} & -\frac{5}{2} \\ 2 & 2 & 3 \end{bmatrix} \xrightarrow{r_3 - r_1 \to r_3} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -\frac{3}{2} & -\frac{5}{2} \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{r_3 + \frac{2}{3}r_2 \to r_3} \underbrace{\begin{bmatrix} 2 & 1 & 1 \\ 0 & -\frac{3}{2} & -\frac{5}{2} \\ 0 & 0 & \frac{1}{3} \end{bmatrix}}_{U}.$$

(Note that the matrix U is not in row echelon form.) The same transitions can be expressed in terms of elementary matrices:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{2}{3} & 1 \end{bmatrix}}_{E_3}\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}}_{E_2}\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_1}\underbrace{\begin{bmatrix} 2 & 1 & 1 \\ 3 & 0 & -1 \\ 2 & 2 & 3 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 2 & 1 & 1 \\ 0 & -\frac{3}{2} & -\frac{5}{2} \\ 0 & 0 & \frac{1}{3} \end{bmatrix}}_{U}. \tag{26}$$

Recall from the table on p. 78 that the result of the operation $r_i + kr_j \to r_i$ is completely reversed by $r_i - kr_j \to r_i$ (also see Exercise a in Section 2.2 on p. 86); therefore,

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{2}{3} & 1 \end{bmatrix}.$$

Premultiplying both sides of the equation (26) by $E_1^{-1}E_2^{-1}E_3^{-1}$ yields

$$A = E_1^{-1}E_2^{-1}E_3^{-1}U.$$

The product $E_1^{-1}E_2^{-1}E_3^{-1}$ is unit lower triangular, with 1's on the main diagonal (since it is a product of unit lower triangular matrices E_i^{-1} – see Exercise 40 in Section 1.3 on p. 34). It can be thought of as a result of performing a sequence of elementary row operations on the identity matrix:

- $r_3 - \frac{2}{3}r_2 \to r_3$ applied to I_3 yields $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{2}{3} & 1 \end{bmatrix} = E_3^{-1}$,
- $r_3 + r_1 \to r_3$ applied to the above yields $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -\frac{2}{3} & 1 \end{bmatrix} = E_2^{-1}E_3^{-1}$,
- then $r_2 + \frac{3}{2}r_1 \to r_2$ applied to the above result leads to $\begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ 1 & -\frac{2}{3} & 1 \end{bmatrix} = E_1^{-1}E_2^{-1}E_3^{-1}$.

Notice how each subsequent row operation (and the corresponding matrix multiplication) changes precisely one entry in the resulting matrix. More specifically, $r_i + kr_j \to r_i$ introduces the number k at the (i,j) entry. This works because the sequence of the j values is nonincreasing (in our case, $j = 2$, 1, 1), so that in $r_i + kr_j$ we always have $r_j = \overrightarrow{e_j}^T$. [9]

Denoting $L = E_1^{-1}E_2^{-1}E_3^{-1}$, we obtain an LU decomposition (also called LU factorization) of A :

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ 1 & -\frac{2}{3} & 1 \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} 2 & 1 & 1 \\ 0 & -\frac{3}{2} & -\frac{5}{2} \\ 0 & 0 & \frac{1}{3} \end{bmatrix}}_{U}.$$

---

[9] To convince yourself about the importance of the j sequence being nondecreasing, compare this to the product $E_3E_2E_1$ – the j values used are 1, 1, 2; therefore, $E_3E_2E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{2}{3} & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\neq \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 1 & 0 \\ -1 & \frac{2}{3} & 1 \end{bmatrix}$.

Once an LU decomposition of the coefficient matrix A is known, the system

$$A\overrightarrow{x} = \overrightarrow{b}$$

can be rewritten as

$$L\underbrace{(U\overrightarrow{x})}_{\overrightarrow{y}} = \overrightarrow{b}$$

and solved in two easy steps:

- first, solve
$$L\overrightarrow{y} = \overrightarrow{b},$$
- then
$$U\overrightarrow{x} = \overrightarrow{y}.$$

Since both coefficient matrices, L and U, are triangular, solving each of these is as simple as the backsubstitution discussed in Example 2.8.

In particular, for the system $A\overrightarrow{x} = \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}$, we have

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ 1 & -\frac{2}{3} & 1 \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}}_{\overrightarrow{y}} = \underbrace{\begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}}_{\overrightarrow{b}}$$

or

$$\begin{array}{rcrcrcl} y_1 & & & & & = & 1 \\ \frac{3}{2}y_1 & + & y_2 & & & = & 5 \\ y_1 & - & \frac{2}{3}y_2 & + & y_3 & = & 2 \end{array}$$

which is easily solved from top to bottom:

$$\begin{aligned} y_1 &= 1 \\ y_2 &= 5 - \frac{3}{2}y_1 = \frac{7}{2} \\ y_3 &= 2 - y_1 + \frac{2}{3}y_2 = \frac{10}{3} \end{aligned}$$

and

$$\underbrace{\begin{bmatrix} 2 & 1 & 1 \\ 0 & -\frac{3}{2} & -\frac{5}{2} \\ 0 & 0 & \frac{1}{3} \end{bmatrix}}_{U}\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\overrightarrow{x}} = \underbrace{\begin{bmatrix} 1 \\ \frac{7}{2} \\ \frac{10}{3} \end{bmatrix}}_{\overrightarrow{y}},$$

i.e.,

$$\begin{array}{rcrcrcl} 2x_1 & + & x_2 & + & x_3 & = & 1 \\ & - & \frac{3}{2}x_2 & - & \frac{5}{2}x_3 & = & \frac{7}{2} \\ & & & & \frac{1}{3}x_3 & = & \frac{10}{3} \end{array}$$

for which backsubstitution yields

$$\begin{aligned} x_3 &= \frac{10/3}{1/3} = 10 \\ x_2 &= \frac{\frac{7}{2} + \frac{5}{2}x_3}{-3/2} = -19 \\ x_1 &= \frac{1 - x_2 - x_3}{2} = 5 \end{aligned}$$

PA = *LU* decomposition

The elementary row operations executed at the beginning of Example 2.24 did not include any operations of the type $kr_i \to r_i$. The resulting pivots (and leading entries) were not made to equal 1, as the standard pivoting strategy would. Consequently, the matrix U was upper triangular, but not in row echelon form.

We can think of this arrangement as postponing the row-scaling elementary operations $kr_i \to r_i$ – they could be performed next, to result in a row echelon form, if we so desired. Considering a sequence of operations of the type $r_i + kr_j \to r_i$ guaranteed that the product of the corresponding elementary matrices, as well as its inverse, is a unit lower triangular matrix.

This plan will work well as long as each pivotal column contains a nonzero pivot at the correct position. However, if a row interchange is required, then our plan needs to be modified, as demonstrated in the following example.

EXAMPLE 2.25 In order to obtain a pivot at the (1,1) entry of the matrix

$$A = \begin{bmatrix} 0 & 0 & 2 \\ -4 & 3 & 2 \\ 2 & 1 & 5 \end{bmatrix},$$

the first row must be interchanged with one of the other two rows. If we interchange it with the second row, then creating a pivot in the second column will require an additional row interchange:

$$A \overset{r_1 \leftrightarrow r_2}{\to} \begin{bmatrix} -4 & 3 & 2 \\ 0 & 0 & 2 \\ 2 & 1 & 5 \end{bmatrix} \overset{r_3 + \frac{1}{2}r_1 \to r_3}{\to} \begin{bmatrix} -4 & 3 & 2 \\ 0 & 0 & 2 \\ 0 & \frac{5}{2} & 6 \end{bmatrix} \overset{r_2 \leftrightarrow r_3}{\to} \underbrace{\begin{bmatrix} -4 & 3 & 2 \\ 0 & \frac{5}{2} & 6 \\ 0 & 0 & 2 \end{bmatrix}}_{U}.$$

Using elementary matrices, this can be written as

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{E_3} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{bmatrix}}_{E_2} \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_1} A = U.$$

The matrices E_1 and E_3 are not unit lower triangular; therefore, we will be unable to claim a decomposition $A = LU$ of the type discussed in the last subsection.

Here is how we can accomplish something comparable by modifying the original sequence of elementary row operations:

- Perform all the row interchanges (operations of type $r_i \leftrightarrow r_j$) first, before any other row operations are performed. (These are exactly the same row interchanges performed in the original sequence.)
- Follow with the sequence of operations of the type $r_i + kr_j \to r_i$. The numbers of rows i and j may need to be adjusted to reflect the new positions of rows.

Implementing this for our matrix, we have

$$A \overset{r_1 \leftrightarrow r_2}{\to} \begin{bmatrix} -4 & 3 & 2 \\ 0 & 0 & 2 \\ 2 & 1 & 5 \end{bmatrix} \overset{r_2 \leftrightarrow r_3}{\to} \begin{bmatrix} -4 & 3 & 2 \\ 2 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix} \overset{r_2 + \frac{1}{2}r_1 \to r_2}{\to} U,$$

i.e.,

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_3^*}\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{E_2^*}\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_1^*} A = U.$$

Letting $P = E_2^* E_1^* = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ and $L = (E_3^*)^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ we arrive at the decomposition $PA = LU$.

DEFINITION

A matrix obtained by multiplying elementary $n \times n$ matrices corresponding to row interchanges $r_i \leftrightarrow r_j$ is called a *permutation matrix*.

Such a matrix properly "stores" all of the row interchanges involved. E.g., it can be easily verified that in the last example, $PA = \begin{bmatrix} \text{row}_2 A \\ \hline \text{row}_3 A \\ \hline \text{row}_1 A \end{bmatrix}$, correctly reflecting the operations $r_1 \leftrightarrow r_2;\ r_2 \leftrightarrow r_3$.

Similarly to $A = LU$, the $PA = LU$ decomposition is helpful when solving a linear system $A\overrightarrow{x} = \overrightarrow{b}$.

EXAMPLE 2.26 Solve the system

$$\begin{array}{rcrcrcr} & & & & 2x_3 & = & 4 \\ -4x_1 & + & 3x_2 & + & 2x_3 & = & 13 \\ 2x_1 & + & x_2 & + & 5x_3 & = & 3 \end{array}$$

The coefficient matrix of this system is the matrix A of Example 2.25. Using the decomposition obtained therein,

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}}_{P}\underbrace{\begin{bmatrix} 0 & 0 & 2 \\ -4 & 3 & 2 \\ 2 & 1 & 5 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} -4 & 3 & 2 \\ 0 & \frac{5}{2} & 6 \\ 0 & 0 & 2 \end{bmatrix}}_{U},$$

we can solve the system $A\overrightarrow{x} = \overrightarrow{b}$ (with $\overrightarrow{b} = \begin{bmatrix} 4 \\ 13 \\ 3 \end{bmatrix}$) by first premultiplying both sides by P:

$$PA\overrightarrow{x} = P\overrightarrow{b}$$

and then using the decomposition

$$LU\overrightarrow{x} = P\overrightarrow{b}.$$

Denoting $\overrightarrow{y} = U\overrightarrow{x}$, it should be clear that the system can be, again, solved in two steps, each involving simple backsubstitution:

- Solve the system

$$L\overrightarrow{y} = P\overrightarrow{b}.$$

In our case, $P\overrightarrow{b} = \begin{bmatrix} 13 \\ 3 \\ 4 \end{bmatrix}$. A process similar to that in Example 2.24 yields $\overrightarrow{y} = \begin{bmatrix} 13 \\ \frac{19}{2} \\ 4 \end{bmatrix}$ (check!).

- To finish, solve the system
$$U\overrightarrow{x} = \overrightarrow{y}.$$
Once again, we ask the reader to verify that backsubstitution yields $\overrightarrow{x} = \begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix}$.

Numerical considerations

In this book, we generally assume all computations are performed exactly. However, many of the real-life problems involve data for which only approximate values are available. Moreover, it is common for problems of practical importance to involve massive data sets so that the corresponding matrices and vectors tend to be very large.

These are among the reasons why it may not be practical to insist on performing calculations exactly, but rather use the tools of numerical linear algebra, dealing with approximate data using finite precision computer arithmetic. The detailed study of numerical linear algebra is outside the scope of this text, but we shall occasionally find it appropriate to refer to some numerical aspects of the topics we cover.

One of the most frequently used numerical procedures for (approximately) solving linear systems is known as Gaussian elimination with partial pivoting. It is based on a modification of the Gaussian elimination we introduced in Section 2.1.

In partial pivoting, each time we select a pivot in a pivotal column, we do so by selecting the value of the largest magnitude among the eligible entries (i.e., at or below the pivot location) – this is done to minimize the potential for error growth in the computation. The solution is typically carried out using the $PA = LU$ decomposition. Refer to Exercise 45 for an example.

EXERCISES

In Exercises 1–4, balance each chemical reaction using the smallest positive integers.

1. x_1 $N_2O_5 \rightarrow x_2$ $NO_2 + x_3O_2$
2. $x_1Fe_2O_3 + x_2CO \rightarrow x_3FeO + x_4CO_2$
3. $x_1C_{10}H_{16} + x_2Cl_2 \rightarrow x_3C + x_4HCl$
4. $x_1As + x_2NaOH \rightarrow x_3Na_3AsO_3 + x_4H_2$

In Exercises 5–8, some reaction equations contain errors, making them impossible to balance. If possible, balance each reaction; if it is not possible, state so.

5. $x_1C_6H_{12}O_6 \rightarrow x_2C_2H_5OH + x_3CO$

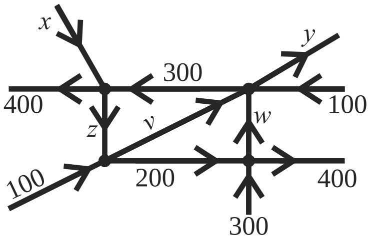


Figure for Exercise 9

6. $x_1\mathrm{C_2H_6} + x_2\mathrm{O_2} \to x_3\mathrm{CO_2} + x_4\mathrm{H_2O}$

7. $x_1\mathrm{Rb_3PO_4} + x_2\mathrm{CrCl_3} \to x_3\mathrm{RbCl} + x_4\mathrm{CrPO_4}$

8. $x_1\mathrm{AgI} + x_2\mathrm{Fe_2(CO_3)_3} \to x_3\mathrm{FeI_2} + x_4\mathrm{Ag_2CO_3}$

In Exercises 9–12, consider the given network of one-way streets along with the traffic flow figures (in vehicles per hour). Solve for the traffic flow at all one-way street segments that are not given. There are infinitely many solutions with one or two arbitrary values – describe the feasible set of these values corresponding to nonnegative rates of flow. Provide one specific example involving only positive numbers.

9. The one-way street network involving 4 intersections (equations) and five unknowns.

10. The one-way street network involving 6 intersections (equations) and six unknowns. (Note that to simplify matters, we are using x twice – why is this legitimate?)

11. The one-way street network involving 4 intersections and 6 unknowns (you need to label them first, then draw an arrow diagram).

12. The given interstate interchange can be viewed as a one-way street network. Draw an appropriate arrow diagram (make sure you correctly identify all six "intersection" points where lanes either split or merge).

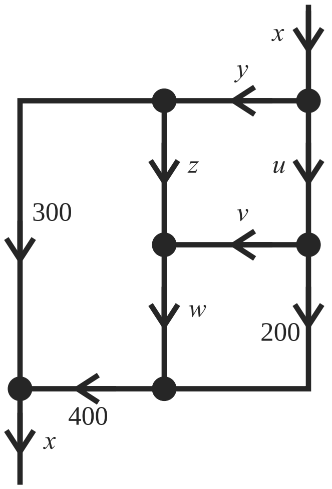


Figure for Exercise 10

In Exercises 13–17, consider the following five alloys:

	Alloy I	Alloy II	Alloy III	Alloy IV	Alloy V
gold	22/24	14/24	18/24	18/24	18/24
silver	1/24	6/24	0	3/24	2/24
copper	1/24	4/24	6/24	3/24	4/24

13. How can the alloys I, II, and III be mixed to obtain alloy V?

14. How can the alloys II, III, and IV be mixed to obtain alloy V?

15. How can the alloys I, II, III, and V be mixed to obtain alloy IV?

16. How can the alloys I, III, IV, and V be mixed to obtain alloy II?

17. Of the set of the three alloys III, IV, and V, which one can be obtained by mixing the remaining ones in the set?

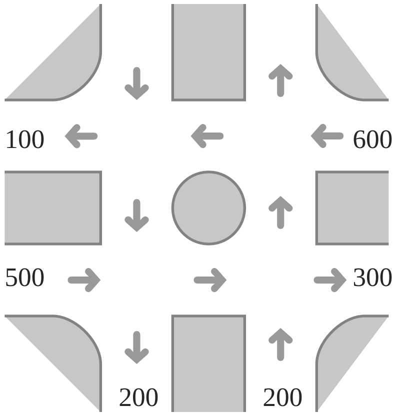


Figure for Exercise 11

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18. Most of the coins presently in circulation in the United States are made of alloys of copper and nickel. The following table specifies approximate nickel and copper content in one dollar's worth of the following three coins:

	20 nickels (=\$1)	10 dimes (=\$1)	1 Susan B. Anthony Dollar
nickel [grams]	25	2	1
copper [grams]	75	21	7

Coins that are no longer fit for circulation (worn or mutilated) are melted at the United States Mint and reused for manufacturing new coins.

Which one of the three denominations can be obtained by melting coins of the remaining two? In what proportion should these two be mixed together? What is the net "gain" or "loss" realized in processing \$1 worth of old coins (excluding production costs)?

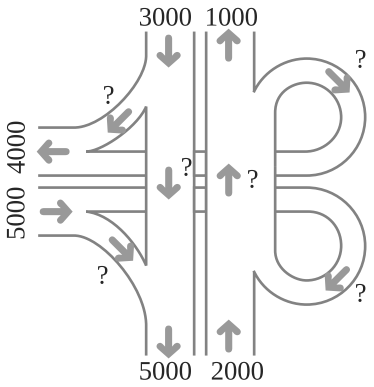

Figure for Exercise 12

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19. Find the Lagrange interpolating polynomial of degree 2 or less that passes through $(0, 2)$, $(1, 4)$, and $(2, 0)$.

20. Find the Lagrange interpolating polynomial of degree 2 or less that passes through $(0, 1)$, $(2, 1)$, and $(3, 7)$.

21. Find the Lagrange interpolating polynomial of degree 3 or less that passes through $(0, 0)$, $(1, 2)$, $(2, 2)$, and $(3, 0)$.

22. Find the Lagrange interpolating polynomial of degree 3 or less that passes through $(-1, 0)$, $(0, 6)$, $(1, 6)$, and $(2, 12)$.

23. The *Hermite interpolating polynomial* $p(x)$ of degree $2n + 1$ satisfies the conditions

$$p(x_i) = y_i \quad \text{and} \quad p'(x_i) = d_i \quad \text{for } i = 0, 1, \ldots, n,$$

where the values x_i, y_i, and d_i are given (and x_i's are distinct). Use a linear system to determine $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ such that

$$p(0) = 2,\ p'(0) = 1,\ p(1) = 4,\ \text{and } p'(1) = 0.$$

24. Repeat Exercise 23 for

$$p(1) = 1,\ p'(1) = 1,\ p(2) = 1,\ \text{and } p'(2) = -1.$$

25. If possible, find a polynomial $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ such that

$$p(0) = 1,\ p'(0) = 1,\ p'(2) = 0,\ \text{and } p(3) = 2.$$

Note that this is not a Hermite interpolating polynomial – e.g., at $x = 2$, the first derivative is specified, but the value is not. This is an example of a *Hermite-Birkhoff interpolation* problem, and it may or may not have a solution (even if it does, the solution need not be unique).

26. (Another Hermite-Birkhoff interpolation problem) If possible, find a polynomial $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ such that

$$p(0) = -1,\ p'(0) = 2,\ p'(1) = 2,\ \text{and } p(2) = -1.$$

27. Given the values $x_0 < x_1 < \cdots < x_n$ and the corresponding values $y_0, y_1, \ldots, y_n$, a *natural cubic spline* is defined by

$$s(x) = \begin{cases} p_0(x) = y_0 + b_0(x - x_0) + c_0(x - x_0)^2 + d_0(x - x_0)^3 & \text{if } x_0 \le x < x_1 \\ p_1(x) = y_1 + b_1(x - x_1) + c_1(x - x_1)^2 + d_1(x - x_1)^3 & \text{if } x_1 \le x < x_2 \\ \vdots & \\ p_{n-1}(x) = y_{n-1} + b_{n-1}(x - x_{n-1}) + c_{n-1}(x - x_{n-1})^2 + d_{n-1}(x - x_{n-1})^3 & \\ \qquad\qquad\qquad \text{if } x_{n-1} \le x \le x_n & \end{cases}$$

where

$$\begin{aligned} p_i(x_{i+1}) &= y_{i+1} \text{ for } i = 0, \ldots, n - 1, \\ p'_{i-1}(x_i) &= p'_i(x_i) \text{ for } i = 1, \ldots, n - 1, \\ p''_{i-1}(x_i) &= p''_i(x_i) \text{ for } i = 1, \ldots, n - 1, \\ p''_0(x_0) &= p''_{n-1}(x_n) = 0. \end{aligned} \tag{27}$$

For $n = 2$, set up and solve a system of 6 linear equations in 6 unknowns $b_0, c_0, d_0, b_1, c_1, d_1$ to determine the natural cubic spline passing through the points $(0, 2)$, $(1, 3)$, and $(2, 0)$. After forming the polynomials p_0 and p_1, verify that the six conditions in (27) hold true.

28. * Use the formulas of the previous exercise to set up a system of 9 equations in 9 unknowns to determine the natural cubic spline passing through the points $(0, 0)$, $(1, 1)$, $(2, 5)$, and $(3, 6)$. Use the Linear Algebra Toolkit to solve this system. After forming the polynomials p_0, p_1, and p_2, verify that the nine conditions in (27) hold true.

In Exercises 29–32, for the system with the given augmented matrix, find all values of a that correspond to

(i) no solution, (ii) one solution, (iii) many solutions.

29. $\left[\begin{array}{cc|c} 1 & 3 & 1 \\ a & 6 & 2 \end{array}\right]$

30. $\left[\begin{array}{cc|c} 2 & 4 & 8 \\ 3 & a-1 & 1 \end{array}\right]$

31. $\left[\begin{array}{cc|c} a & 1 & 0 \\ -1 & a & 0 \end{array}\right]$

32. $\left[\begin{array}{cc|c} a-1 & -2 & 0 \\ -2 & a-1 & 0 \end{array}\right]$

In Exercises 33–36, for the system with the given augmented matrix, find all values of a and b that correspond to

(i) no solution, (ii) one solution, (iii) many solutions.

33. $\left[\begin{array}{cc|c} 1 & a+1 & 0 \\ a-1 & 3 & b \end{array}\right]$

34. $\left[\begin{array}{cc|c} a & 0 & b \\ 1 & b & 0 \end{array}\right]$

35. $\left[\begin{array}{cc|c} a & b & 0 \\ b & a & 0 \end{array}\right]$

36. $\left[\begin{array}{cc|c} 1 & 1 & 2 \\ a & b & 0 \\ a^2 & b^2 & \frac{2}{3} \\ a^3 & b^3 & 0 \end{array}\right]$

In Exercises 37–39, prove the following properties of an $n \times n$ permutation matrix P.

37. * Each row of P and each column of P contains exactly one nonzero entry, which equals 1.

38. * If the (i, j) entry of P, p_{ij} is 1, then $\text{row}_i(PA) = \text{row}_j A$.

39. * $P^T P = P P^T = I_n$.

40. After the first three elementary row operations in Example 2.13, the matrix $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix}$ was transformed to $U = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix}$. Determine the unit lower triangular matrix L such that $A = LU$. Verify that multiplying LU yields A.

In Exercises 41–43, for the given linear system:

a. Find an LU decomposition of the coefficient matrix of the system A (verify).

b. Use the decomposition found in part a to solve the given system: first solve $L\overrightarrow{y} = \overrightarrow{b}$, then $U\overrightarrow{x} = \overrightarrow{y}$. Verify that your solution agrees with the answer printed in Appendix A for each original exercise.

41. The linear system of Exercise 13 on p. 75 in Section 2.1.

42. The linear system of Exercise 19 on p. 75 in Section 2.1.

43. The linear system of Exercise 23 on p. 76 in Section 2.1.

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44. Check that when the matrix $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 2 \\ -1 & 2 & 1 \end{bmatrix}$ undergoes the elementary row operations $r_2 - r_1 \to r_2$, $r_3 + r_1 \to r_3$, $r_2 \leftrightarrow r_3$, and $r_3 - 3r_2 \to r_3$, it is transformed to $U = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -5 \end{bmatrix}$. Determine the unit lower triangular matrix L and the permutation matrix P such that $PA = LU$. Verify this equality.

45. For the linear system of Exercise 42 above, find a $PA = LU$ decomposition that avoids introducing fractions into the matrices L and U (apply $r_1 \leftrightarrow r_2$ before any other elementary row operations). Use this decomposition to solve the system.

2.5 Chapter Review

System of *m* linear equations in *n* unknowns

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{array}$$

- Coefficient matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- Matrix representation

$$A\vec{x} = \vec{b}$$

$$\text{with } \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

- Augmented matrix

$$[A|\vec{b}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

Elementary row operations and elementary matrices

to perform this row operation $A\vec{x}=\vec{b}$	premultiply $[A\mid\vec{b}]$ by ...
1. Add a multiple of one equation (row) to another $r_i + k\,r_j \rightarrow r_i$	$\begin{array}{ccccccl} 1&0&0&0&0&0& \\ 0&1&0&0&0&0& \\ 0&0&1&0&0&0& \\ 0&k&0&1&0&0& i\text{th row} \\ 0&0&0&0&1&0& \\ 0&0&0&0&0&1& \\ & j\text{th column} &&&&& \end{array}$
2. Multiply an equation (row) by a nonzero number $k\,r_i \rightarrow r_i$	$\begin{array}{ccccccl} 1&0&0&0&0&0& \\ 0&1&0&0&0&0& \\ 0&0&k&0&0&0& i\text{th row} \\ 0&0&0&1&0&0& \\ 0&0&0&0&1&0& \\ 0&0&0&0&0&1& \\ && i\text{th column} &&&& \end{array}$
3. Interchange two equations (rows) $r_i \longleftrightarrow r_j$	$\begin{array}{ccccccl} 1&0&0&0&0&0& \\ 0&0&0&0&1&0& i\text{th row} \\ 0&0&1&0&0&0& \\ 0&0&0&1&0&0& \\ 0&1&0&0&0&0& j\text{th row} \\ 0&0&0&0&0&1& \\ & i\text{th column} &&& j\text{th column} && \end{array}$

Row equivalence

A is said to be row equivalent to *B* if *B* can be obtained from *A* by a finite sequence of elementary row operations.

Properties

- If *A* is row equivalent to *B*, then *B* is row equivalent to *A*.
- If *A* is row equivalent to *B* and *B* is row equivalent to *C*, then *A* is row equivalent to *C*.
- Linear systems whose augmented matrices are row equivalent have the same solution sets.

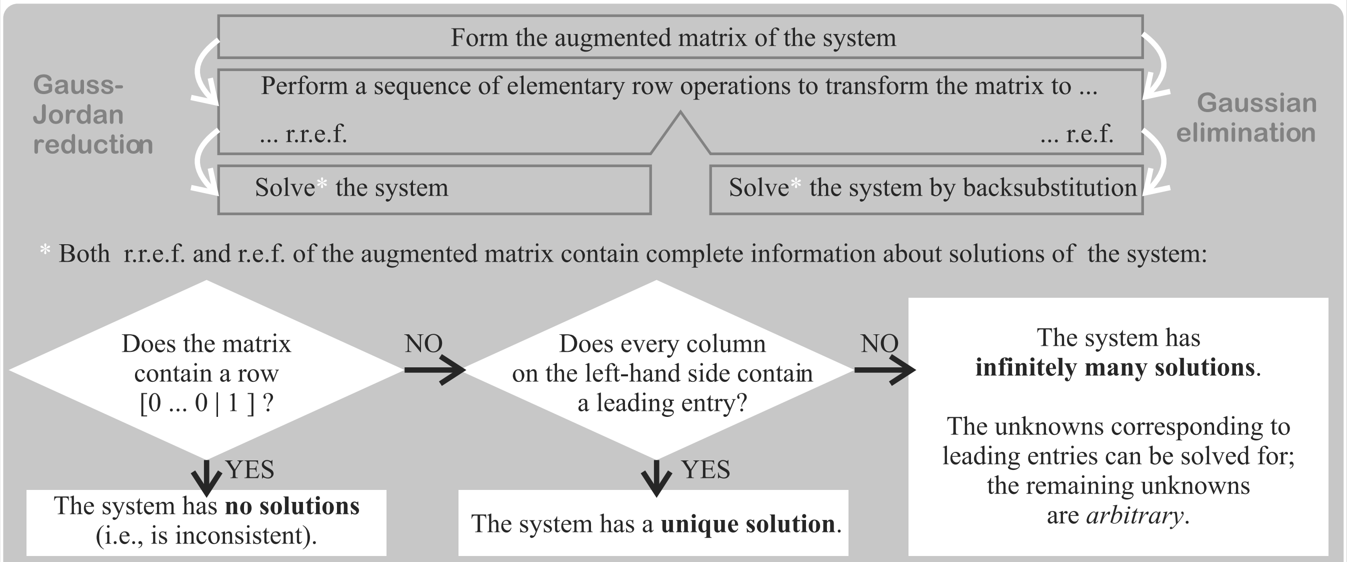
Reduced row echelon form (r.r.e.f.) and row echelon form (r.e.f.)

r.r.e.f.

r.e.f.

1. If there are any zero rows, these are positioned below all nonzero rows.
2. Every nonzero row must have its first nonzero entry equal to 1 (called the leading entry of that row).
3. For any two nonzero rows, the leading entry of the row below is located to the right of the leading entry of the row above (a "staircase pattern").
4. In any column with a leading entry, all remaining entries must equal zero.

Solving a linear system by Gauss-Jordan reduction and Gaussian elimination



Form the augmented matrix of the system

Perform a sequence of elementary row operations to transform the matrix to ...

... r.r.e.f.

... r.e.f.

Solve* the system

Solve* the system by backsubstitution

* Both r.r.e.f. and r.e.f. of the augmented matrix contain complete information about solutions of the system:

Does the matrix contain a row [0 ... 0 | 1] ?

The system has **no solutions** (i.e., is inconsistent).

Does every column on the left-hand side contain a leading entry?

The system has a **unique solution**.

The system has **infinitely many solutions**.

The unknowns corresponding to leading entries can be solved for; the remaining unknowns are *arbitrary*.

SPECIAL CASE: HOMOGENEOUS SYSTEM $A\vec{x}=\vec{0}$

Homogeneous system $A\vec{x}=\vec{0}$ <u>cannot be inconsistent</u> ($\vec{x}=\vec{0}$ always solves it).

If the solution of $A\vec{x}=\vec{0}$ is unique, it must be $\vec{x}=\vec{0}$ – the **trivial** solution.

If $A\vec{x}=\vec{0}$ has many solutions, it must include the trivial solution $\vec{x}=\vec{0}$ – all others ($\vec{x}\neq\vec{0}$) are called **nontrivial**.

Matrix inverse definition

- B is an **inverse** of A if $AB = BA = I$.
- A is called **invertible** (nonsingular) if it has an inverse.

  Otherwise, it is called **singular** (noninvertible).

Properties of inverse

- Inverse of A is unique; it is denoted by A^{-1}.
- A is nonsingular ⇔ A is row equivalent to I.
- $(AB)^{-1} = B^{-1}A^{-1}$.

Procedure to invert the nxn matrix A

- Form the n x $2n$ matrix $[\,A\,|\,I_n\,]$ and perform elementary row operations to obtain its r.r.e.f. $[\,C\,|\,D\,]$.
- If $C = I_n$, then $D = A^{-1}$.
- If $C \neq I_n$, then A is singular.